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Article

Invariance Property of Cauchy–Stieltjes Kernel Families Under Free and Boolean Multiplicative Convolutions

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Abstract: This article delves into some properties of free and Boolean multiplicative convolutions, in connection with the theory of Cauchy–Stieltjes kernel (CSK) families and their respective variance functions (VFs). Consider $\mathcal{K}_-(\mu) = \{Q_m^\mu(ds) : \ m \in (m_-^\mu, m_0^\mu)\}$, a CSK family induced by a non-degenerate probability measure μ on the positive real line with a finite first-moment m_0^μ. For $\gamma > 1$, we introduce a new family of measures: $(\mathcal{K}_-(\mu))^{\boxtimes\gamma} = \left\{\left(Q_m^\mu\right)^{\boxtimes\gamma}(ds) : \ m \in (m_-^\mu, m_0^\mu)\right\}$. We show that if $(\mathcal{K}_-(\mu))^{\boxtimes\gamma}$ represents a re-parametrization of the CSK family $\mathcal{K}_-(\mu)$, then μ is characterized by its corresponding VF $\mathcal{V}_\mu(m) = cm^2\ln(m)$, with $c > 0$. We also prove that if $(\mathcal{K}_-(\mu))^{\boxtimes\gamma}$ is a re-parametrization of $\mathcal{K}_-(D_{1/\gamma}(\mu^{\boxplus\gamma}))$ (where $\boxplus$ is the additive free convolution and $D_a(\mu)$ denotes the dilation μ by a number $a \neq 0$), then μ is characterized by its corresponding VF $\mathcal{V}_\mu(m) = c_1(m\ln(m))^2$, with $c_1 > 0$. Similar results are obtained if we substitute the free multiplicative convolution $\boxtimes$ with the Boolean multiplicative convolution $\uplus$.

Keywords: Cauchy transform; multiplicative Boolean and free convolutions; variance function

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1. Introduction

In probability theory, the basic concept of convolution is important in understanding interactions between random variables, particularly when they are independent. However, when extending the concept of probability to noncommutative structures, such as free probability and Boolean probability, the classical convolution operation is no longer suitable. The study of free and Boolean convolutions in noncommutative probability offers rich new perspectives for understanding the behavior of random elements in these settings, providing insights into the algebraic and statistical properties of the systems beyond the classical framework.

Free probability, introduced by Dan Voiculescu in the early 1980s [1], generalizes classical probability theory to the context of random variables associated with noncommutative algebras; see [2–5]. In this context, free multiplicative convolution appears as a natural extension of classical multiplicative convolution, where instead of independent random variables, we consider random elements in a noncommutative algebra that are free (i.e., they satisfy some form of noncommutative independence). This convolution is essential to the study of the behavior of free random variables, particularly in the analysis of their distributional properties and spectral behaviors, and has found applications in areas such as random matrices, operator algebras, and statistical mechanics.

Boolean independence was introduced in the context of noncommutative probability as an alternative to classical and free independence. It is characterized by the Boolean convolution, which defines the addition of Boolean-independent random variables. The key property of Boolean independence is that mixed moments factorize in a way distinct from classical or free independence. The Boolean independence and the Boolean convolution was introduced in [6] as a fundamental operation in noncommutative probability. The combinatorial structure was explored using moment–cumulant relations and connections to free probability. For a deeper perspective, the authors of [7] expanded on this framework by studying Boolean cumulants. Further results on the Boolean convolution can be found in [8–10]. These works laid the foundation for Boolean probability theory, influencing later developments in noncommutative probability.

Both free and Boolean multiplicative convolutions expand our understanding of convolution operations by adapting them to noncommutative probability settings. The free multiplicative convolution provides a framework for exploring the correlations between free random variables, while the Boolean multiplicative convolution offers a means of analyzing the dependencies between Boolean-valued random variables. Together, these operations highlight the versatility and depth of noncommutative probability theory, revealing novel insights into the behavior of systems with complex dependence structures.

This article delves into some properties of free and Boolean multiplicative convolutions, in connection with the recently introduced theory of Cauchy–Stieltjes kernel (CSK) families in the setting of free probability. To better present the outcomes to be proven in this study, we first present some basic concepts around free and Boolean multiplicative convolutions and also some preliminaries on CSK families. $\mathcal{P}$ ($\mathcal{P}_+$, respectively) denotes the set of (non-degenerate) probabilities on $\mathbb{R}$ ($\mathbb{R}_+$, respectively). For $\sigma \in \mathcal{P}$, the Cauchy transform $\mathbf{G}_\sigma(\cdot)$ is given by

$$\mathbf{G}_\sigma(z) = \int \frac{\sigma(d\xi)}{z-\xi}, \quad z \in \mathbb{C} \setminus \operatorname{supp}(\sigma)$$

and the free cumulant transform of σ, denoted $\mathcal{R}_\sigma(\cdot)$, is given by [11]

$$\mathcal{R}_\sigma(\mathbf{G}_\sigma(z)) = z - 1/\mathbf{G}_\sigma(z), \quad z \text{ close to } \infty.$$

For $\tau \in \mathcal{P}_+$, $(\tau \neq \delta_0)$, the $\mathbf{S}$-transform is introduced by the relation

$$\mathcal{R}_\tau(w\mathbf{S}_\tau(w)) = \frac{1}{\mathbf{S}_\tau(w)} \quad w \text{ in a neighborhood of } 0.$$

The operation of free multiplicative convolution $\varrho \boxtimes \tau$ of ϱ and $\tau \in \mathcal{P}_+$ is defined by $\mathbf{S}_{\varrho\boxtimes\tau}(w) = \mathbf{S}_\varrho(w)\mathbf{S}_\tau(w)$. Free multiplicative convolution powers $\tau^{\boxtimes\gamma}$ are defined at least $\forall\, \gamma \geq 1$ (see Theorem 2.17, [3]) by $\mathbf{S}_{\tau^{\boxtimes\gamma}}(w) = \mathbf{S}_\tau(w)^\gamma$.

We now introduce the notion of Boolean multiplicative convolution [12]. For $\tau \in \mathcal{P}_+$, consider

$$\Psi_\tau(\beta) = \int_0^{+\infty} \frac{\beta y}{1-\beta y}\tau(dy), \quad \beta \in \mathbb{C} \setminus \mathbb{R}_+.$$

For $\tau \in \mathcal{P}_+$, consider the η-transform [13]:

$$\begin{aligned} \eta_\tau : \mathbb{C}\backslash\mathbb{R}_+ &\rightarrow \mathbb{C}\backslash\mathbb{R}_+ \\ \beta &\mapsto \eta_\tau(\beta) = \frac{\Psi_\tau(\beta)}{1+\Psi_\tau(\beta)}. \end{aligned}$$

The transform

$$\mathbf{B}_\tau(\beta) = \frac{\beta}{\eta_\tau(\beta)}$$

is defined for $\beta \in \mathbb{C}\backslash\mathbb{R}_+$. The multiplicative Boolean convolution $\varrho \uplus \tau$ of ϱ and $\tau \in \mathcal{P}_+$ is defined by the relation

$$\mathbf{B}_{\varrho \uplus \tau}(\beta) = \mathbf{B}_\varrho(\beta)\mathbf{B}_\tau(\beta), \qquad \beta \in \mathbb{C}\backslash\mathbb{R}_+.$$

For $\varrho,\ \tau \in \mathcal{P}_+$ verifies the following:

(i) $\arg(\eta_\varrho(\beta)) + \arg(\eta_\tau(\beta)) - \arg(\beta) < \pi \quad \beta \in (-\infty, 0) \cup \mathbb{C}^+$;

(ii) One of the measures ϱ or τ (at least) has a finite first moment.

So, $\varrho \uplus \tau \in \mathcal{P}_+$ is well defined. It was demonstrated in [12] that the operation $\tau^{\uplus \kappa} \in \mathcal{P}_+$ is defined for $\kappa \in [0,1]$.

On the other hand, within the framework of noncommutative probability, the theory of CSK families has recently been defined. These families of probability measures are defined in a manner similar to natural exponential families (NEFs), but with the Cauchy–Stieltjes kernel $\frac{1}{1-\vartheta s}$ replacing the exponential kernel $\exp(\vartheta s)$. The CSK families (called also free exponential families) were examined in [14] for compactly supported measures. In [15–18], the authors extended the study of CSK families to include measures with support bounded on one side (for example, from above). $\mathcal{P}_{ba}$ (respectively, $\mathcal{P}_c$) refers to a subset of non-degenerate probability measures having support bounded from above (respectively, having compact support).

Let $\rho \in \mathcal{P}_{ba}$. Then,

$$\mathcal{M}_\rho(\vartheta) = \int \frac{\rho(dl)}{1-\vartheta l}$$

is defined for $0 < \vartheta < \vartheta_+^\rho$ were $\frac{1}{\vartheta_+^\rho} = \max\{0, \sup \operatorname{supp}(\rho)\}$. The family

$$\mathcal{K}_+(\rho) = \left\{ P_\vartheta^\rho(dl) = \frac{\rho(dl)}{\mathcal{M}_\rho(\vartheta)(1-\vartheta l)} :\ 0 < \vartheta < \vartheta_+^\rho \right\}$$

is said to be the CSK family generated by ρ.

The application $\vartheta \mapsto k_\rho(\vartheta) = \int l P_\vartheta^\rho(dl)$ is bijective from $(0, \vartheta_+^\rho)$ to $k_\mu((0, \vartheta_+^\rho)) = (m_0^\rho, m_+^\rho)$, which is the mean domain of $\mathcal{K}_+(\rho)$; see (pp. 579–580, [15]). We then obtain the re-parametrization with the mean of $\mathcal{K}_+(\rho)$: The inverse of $k_\rho(\cdot)$ is represented by $\psi_\rho(\cdot)$. For $m_0^\rho < m < m_+^\rho$, denote $Q_m^\rho(dl) = P_{\psi_\rho(m)}^\rho(dl)$. Then,

$$\mathcal{K}_+(\rho) = \{Q_m^\rho(dl) :\ m_0^\rho < m < m_+^\rho\}.$$

It is proved in Proposition 3.4 [15] that $m_0^\rho = \lim_{\vartheta \to 0^+} k_\rho(\vartheta)$ and $m_+^\rho = B_\rho - \lim_{z \to B_\rho^+} \frac{1}{\mathbf{G}_\rho(z)}$, where $B_\rho = \frac{1}{\vartheta_+^\rho}$.

The CSK family is represented by $\mathcal{K}_-(\rho)$ if the support of ρ is bounded from below. The family $\mathcal{K}_-(\rho)$ is defined analogously to $\mathcal{K}_+(\rho)$ except with negative values of ϑ, that is, $\vartheta \in (\vartheta_-^\rho, 0)$, where ϑ_-^ρ can either be $1/A_\rho$ or $-\infty$, with $A_\rho = \min\{0, \inf \operatorname{supp}(\rho)\}$. The mean domain for $\mathcal{K}_-(\rho)$ is given by (m_-^ρ, m_0^ρ), where $m_-^\rho = A_\rho - 1/\mathbf{G}_\rho(A_\rho)$. If $\rho \in \mathcal{P}_+$. Then, $A_\rho = 0$, and so

$$m_-^\rho = 0 - 1/\mathbf{G}_\rho(0) = \left(\int_{[0,+\infty)} \frac{1}{l} \rho(dl) \right)^{-1},$$

so that $0 < m_-^\rho \le m_0^\rho$. If $\rho \in \mathcal{P}_c$, then $\vartheta_-^\rho < \vartheta < \vartheta_+^\rho$, and $\mathcal{K}(\rho) = \mathcal{K}_-(\rho) \cup \{\rho\} \cup \mathcal{K}_+(\rho)$ is the two-sided CSK family.

Let $\rho \in \mathcal{P}_{ba}$ with a finite first moment $m_0^\rho = \int l \rho(dl)$. The variance function (VF) parameterizes the variance in terms of the mean. It is defined by (see ([14], Equation (2.5)))

$$m \mapsto \mathcal{V}_\rho(m) = \int (l-m)^2 Q_m^\rho(dl).$$

If the first moment of ρ does not exists, then in $\mathcal{K}_+(\rho)$, all measures have infinite variance. In Definition 3.1, [15], the pseudo-variance function (PVF) $\mathbb{V}_\rho(\cdot)$ is presented as a substitute. The results proved in this paper are based on expanding the **S**-transform (or the **B**-transform) as we will see in the next section. For this reason, we consider measure ρ with a finite first moment, which ensures the existence of the VF. So, there is no need to introduce the concept of the PVF in this paper.

Now, let us introduce the objective of this article in more detail: Consider $\mathcal{K}_-(\mu) = \{Q_m^\mu(ds) : \; m \in (m_-^\mu, m_0^\mu)\}$ as the CSK family induced by a (non-degenerate) $\mu \in \mathcal{P}_+$ with a finite first moment m_0^μ. For $\gamma > 1$, introduce a novel set of probabilities:

$$(\mathcal{K}_-(\mu))^{\boxtimes\gamma} = \left\{ \left(Q_m^\mu\right)^{\boxtimes\gamma}(ds) : \; m \in (m_-^\mu, m_0^\mu) \right\}. \tag{1}$$

In Theorem 1, we prove that if $(\mathcal{K}_-(\mu))^{\boxtimes\gamma}$ is a re-parametrization of $\mathcal{K}_-(\mu)$, then μ is characterized by its corresponding VF $\mathcal{V}_\mu(m) = cm^2\ln(m)$, with $c > 0$. We also prove that if $(\mathcal{K}_-(\mu))^{\boxtimes\gamma}$ is a re-parametrization of $\mathcal{K}_-(D_{1/\gamma}(\mu^{\boxplus\gamma}))$ (where $\boxplus$ is the additive free convolution and $D_a(\mu)$ represents the dilation of a measure μ by a number $a \neq 0$), then the measure μ is characterized by its corresponding VF $\mathcal{V}_\mu(m) = c_1(m\ln(m))^2$, with $c_1 > 0$. Similar results are obtained in Theorem 2 if we replace in (1) the free multiplicative convolution $\boxtimes$ with the Boolean multiplicative convolution $\uplus$. This will yield a new property for the Marchenko–Pastur law as we will see in Theorem 2(ii).

The following remark concludes this part by presenting some helpful information that supports the major findings of this article.

Remark 1. *Let $\mu \in \mathcal{P}_{ba}$ with finite first moment m_0^μ.*

(i) *The CSK families are different from NEFs in the fact that we can recover the generating measure μ without knowing the mean domain. According to Theorem 3.1 [14], the generating measure μ is characterized by $\mathcal{V}_\mu(\cdot)$ and m_0^μ: Denote $\Delta = \Delta(m) = m + \frac{\mathcal{V}_\mu(m)}{m - m_0^\mu}$, then*

$$\mathbf{G}_\mu(\Delta) = \frac{m - m_0^\mu}{\mathcal{V}_\mu(m)}. \tag{2}$$

(ii) *Consider $f : s \mapsto \iota s + \upsilon$, where $\iota \neq 0$ and $\upsilon \in \mathbb{R}$. For m close sufficiently to $m_0^{f(\mu)} = f(m_0^\mu) = \iota m_0^\mu + \upsilon$,*

$$\mathcal{V}_{f(\mu)}(m) = \iota^2 \mathcal{V}_\mu\left(\frac{m - \upsilon}{\iota}\right). \tag{3}$$

See Section 3.3 [15] for more details.

(iii) *The Marchenko–Pastur law is provided by*

$$\boldsymbol{MP}_a(ds) = \frac{\sqrt{((a+1)^2 - s)(s - (a-1)^2)}}{2\pi a^2 s} \mathbf{1}_{((a-1)^2,(a+1)^2)}(s)ds + (1 - 1/a^2)^+ \delta_0 \tag{4}$$

for $a \neq 0$, with $m_0^{\boldsymbol{MP}_a} = 1$. We have $\mathcal{V}_{\boldsymbol{MP}_a}(m) = a^2 m$.

2. Main Results

We present some properties related to free and Boolean multiplicative convolutions. The following finding is related to free multiplicative convolution.

Theorem 1. *Let $\rho \in \mathcal{P}_+$ with a finite first moment m_0^ρ. For $\gamma > 1$, consider the family of probability measures $(\mathcal{K}_-(\rho))^{\boxtimes\gamma}$ defined by (1).*

(i) Assume that $\forall\, r \in (m_-^\rho, m_0^\rho)$, there is $p = p(r,\gamma) \in (m_-^\rho, m_0^\rho)$ so that $Q_r^\rho = \left(Q_p^\rho\right)^{\boxtimes\gamma}$. Then, $p = r^{1/\gamma}$, and ρ is characterized by its corresponding VF given by

$$\mathcal{V}_\rho(r) = cr^2 \ln(r), \quad \forall\, r \in (m_-^\rho, m_0^\rho), \quad c > 0.$$

(ii) Assume that $\forall\, r \in \left(m_-^{D_{1/\gamma}(\rho^{\boxplus\gamma})}, m_0^{D_{1/\gamma}(\rho^{\boxplus\gamma})}\right)$, there is a $p = p(r,\gamma) \in (m_-^\rho, m_0^\rho)$ so that $Q_r^{D_{1/\gamma}(\rho^{\boxplus\gamma})} = \left(Q_p^\rho\right)^{\boxtimes\gamma}$. Then, $p = r^{1/\gamma}$, and ρ is characterized by its corresponding VF given by

$$\mathcal{V}_\rho(r) = c_1(r\ln(r))^2, \quad \forall\, r \in \left(m_-^{D_{1/\gamma}(\rho^{\boxplus\gamma})}, m_0^{D_{1/\gamma}(\rho^{\boxplus\gamma})}\right), \quad c_1 > 0.$$

Proof. (i) Assume that $\forall\, r \in (m_-^\rho, m_0^\rho)$, there is a $p = p(r,\gamma) \in (m_-^\rho, m_0^\rho)$ so that $Q_r^\rho = \left(Q_p^\rho\right)^{\boxtimes\gamma}$. This implies that

$$\mathbf{S}_{Q_r^\rho}(\zeta) = \left(\mathbf{S}_{Q_p^\rho}(\zeta)\right)^\gamma, \quad \forall\, \zeta \text{ close to } 0. \tag{5}$$

We know from Theorem 3.3 [19] that the $\mathbf{S}$-transform of Q_r^ρ may be given as

$$\mathbf{S}_{Q_r^\rho}(\zeta) = \frac{1}{m_0^{Q_r^\rho}} - \frac{\mathrm{Var}(Q_r^\rho)}{(m_0^{Q_r^\rho})^3}\zeta + \zeta\varepsilon(\zeta) = \frac{1}{r} - \frac{\mathcal{V}_\rho(r)}{r^3}\zeta + \zeta\varepsilon(\zeta), \text{ where } \varepsilon(\zeta) \xrightarrow{\zeta\to 0} 0. \tag{6}$$

Also we have

$$\left(\mathbf{S}_{Q_p^\rho}(\zeta)\right)^\gamma = \left(\frac{1}{p} - \frac{\mathcal{V}_\rho(p)}{p^3}\zeta + \zeta\varepsilon(\zeta)\right)^\gamma = \frac{1}{p^\gamma}\left(1 - \frac{\gamma\mathcal{V}_\rho(p)}{p^2}\zeta + \zeta\varepsilon_1(\zeta)\right), \text{ where } \varepsilon_1(\zeta) \xrightarrow{\zeta\to 0} 0. \tag{7}$$

Combining (5)–(7), we obtain

$$\frac{1}{r} - \frac{\mathcal{V}_\rho(r)}{r^3}\zeta + \zeta\varepsilon(\zeta) = \frac{1}{p^\gamma}\left(1 - \frac{\gamma\mathcal{V}_\rho(p)}{p^2}\zeta + \zeta\varepsilon_1(\zeta)\right). \tag{8}$$

This implies that $p = r^{1/\gamma}$, and then,

$$\frac{\mathcal{V}_\rho(r)}{r^2} = \frac{\gamma\mathcal{V}_\rho(r^{1/\gamma})}{r^{2/\gamma}}, \quad \forall\ r \in (m_-^\rho, m_0^\rho) \quad \text{and} \quad \forall\, \gamma > 1. \tag{9}$$

The solution of functional Equation (9) is $\frac{\mathcal{V}_\rho(r)}{r^2} = c\ln(r)$, for some $c > 0$. This concludes the proof of Theorem 1(i).

(ii) Assume that $\forall\, r \in \left(m_-^{D_{1/\gamma}(\rho^{\boxplus\gamma})}, m_0^{D_{1/\gamma}(\rho^{\boxplus\gamma})}\right)$, there is a $p = p(r,\gamma) \in (m_-^\rho, m_0^\rho)$ so that $Q_r^{D_{1/\gamma}(\rho^{\boxplus\gamma})} = \left(Q_p^\rho\right)^{\boxtimes\gamma}$. This implies that

$$\left(\mathbf{S}_{Q_p^\rho}(\zeta)\right)^\gamma = \mathbf{S}_{Q_r^{D_{1/\gamma}(\rho^{\boxplus\gamma})}}(\zeta), \quad \forall\, \zeta \text{ close to } 0. \tag{10}$$

Based on (6), the $\mathbf{S}$-transform of $Q_r^{D_{1/\gamma}(\rho^{\boxplus\gamma})}$ may be given as

$$\mathbf{S}_{Q_r^{D_{1/\gamma}(\rho^{\boxplus\gamma})}}(\zeta) = \frac{1}{m_0^{Q_r^{D_{1/\gamma}(\rho^{\boxplus\gamma})}}} - \frac{\mathrm{Var}\left(Q_r^{D_{1/\gamma}(\rho^{\boxplus\gamma})}\right)}{\left(m_0^{Q_r^{D_{1/\gamma}(\rho^{\boxplus\gamma})}}\right)^3}\zeta + \zeta\varepsilon(\zeta)$$
$$= \frac{1}{r} - \frac{\mathcal{V}_{D_{1/\gamma}(\rho^{\boxplus\gamma})}(r)}{r^3}\zeta + \zeta\varepsilon(\zeta), \quad \text{where} \quad \varepsilon(\zeta) \xrightarrow{\zeta\to 0} 0. \tag{11}$$

Based on (3) and knowing from [15] that $\mathcal{V}_{\rho^{\boxplus\gamma}}(r) = \gamma\mathcal{V}_\rho(r/\gamma)$, from (11), we have

$$\mathbf{S}_{Q_r^{D_{1/\gamma}(\rho^{\boxplus\gamma})}}(\zeta) = \frac{1}{r} - \frac{\mathcal{V}_\rho(r)/\gamma}{r^3}\zeta + \zeta\varepsilon(\zeta), \text{ where } \varepsilon(\zeta) \xrightarrow{\zeta\to 0} 0. \tag{12}$$

Combining (7), (10), and (12), we obtain

$$\frac{1}{p^\gamma}\left(1 - \frac{\gamma\mathcal{V}_\rho(p)}{p^2}\zeta + \zeta\varepsilon_1(\zeta)\right) = \frac{1}{r} - \frac{\mathcal{V}_\rho(r)/\gamma}{r^3}\zeta + \zeta\varepsilon(\zeta). \tag{13}$$

This gives that $p = r^{1/\gamma}$, and so,

$$\frac{\mathcal{V}_\rho(r)}{r^2} = \frac{\gamma^2\mathcal{V}_\rho(r^{1/\gamma})}{r^{2/\gamma}}, \quad \forall\ r \in \left(m_-^{D_{1/\gamma}(\rho^{\boxplus\gamma})}, m_0^{D_{1/\gamma}(\rho^{\boxplus\gamma})}\right) \text{ and } \ \forall\ \gamma > 1. \tag{14}$$

The solution of functional Equation (14) is $\frac{\mathcal{V}_\rho(r)}{r^2} = c_1(\ln(r))^2$, for some $c_1 > 0$. This concludes the proof of Theorem 1(ii). □

Remark 2. *Note that, up to affine transformations, we may suppose that $(m_-^\rho, m_0^\rho) \subset (1, +\infty)$ in Theorem 1(i) and $\left(m_-^{D_{1/\gamma}(\rho^{\boxplus\gamma})}, m_0^{D_{1/\gamma}(\rho^{\boxplus\gamma})}\right) \subset (1, +\infty)$ in Theorem 1(ii).*

The following finding is related to Boolean multiplicative convolution.

Theorem 2. *Let $\rho \in \mathcal{P}_+$ with a finite first moment m_0^ρ. For $\kappa \in (0,1)$, consider the family of probability measures*

$$(\mathcal{K}_-(\rho))^{\uplus\kappa} = \left\{\left(Q_m^\rho\right)^{\uplus\kappa}(ds) : \ m \in (m_-^\rho, m_0^\rho)\right\}.$$

(i) Assume that $\forall\ m \in (m_-^\rho, m_0^\rho)$, we have $\left(Q_m^\rho\right)^{\uplus\kappa} = Q_r^\rho$ for some $r = r(m,\kappa) \in (m_-^\rho, m_0^\rho)$. Then, $r = m^\kappa$, and ρ is characterized by its corresponding VF given by

$$\mathcal{V}_\rho(m) = dm\ln(m), \quad \forall\ m \in (m_-^\rho, m_0^\rho), \quad d > 0.$$

(ii) Assume that $\forall\ m \in (m_-^\rho, m_0^\rho)$, we have $\left(Q_m^\rho\right)^{\uplus\kappa} = Q_r^{D_\kappa(\rho^{\boxplus 1/\kappa})}$ for some $r = r(m,\kappa) \in \left(m_-^{D_\kappa(\rho^{\boxplus 1/\kappa})}, m_0^{D_\kappa(\rho^{\boxplus 1/\kappa})}\right)$. Then, $r = m^\kappa$, and ρ is a Marchenko–Pastur law scaled up.

Proof. (i) Assume that $\forall\ m \in (m_-^\rho, m_0^\rho)$, we have $\left(Q_m^\rho\right)^{\uplus\kappa} = Q_r^\rho$ for some $r = r(m,\kappa) \in (m_-^\rho, m_0^\rho)$. This implies that

$$\left(\mathbf{B}_{Q_m^\rho}(\zeta)\right)^\kappa = \mathbf{B}_{Q_r^\rho}(\zeta), \quad \forall\ \zeta \text{ close to } 0. \tag{15}$$

We know from Equation (24) [20] that

$$\frac{1}{\mathbf{B}_{Q_r^\rho}(\zeta)} = r + \mathcal{V}_\rho(r)\zeta + \zeta\epsilon(\zeta), \text{ where } \epsilon(\zeta) \xrightarrow{\zeta\to 0} 0. \tag{16}$$

Based on (16), we obtain

$$\left(\frac{1}{\mathbf{B}_{Q_m^\rho}(\zeta)}\right)^\kappa = (m + \mathcal{V}_\rho(m)\zeta + \zeta\epsilon(\zeta))^\kappa = m^\kappa\left(1 + \kappa\frac{\mathcal{V}_\rho(m)}{m}\zeta + \zeta\epsilon_1(\zeta)\right), \text{ with } \epsilon_1(\zeta) \xrightarrow{\zeta\to 0} 0. \tag{17}$$

Combining (15)–(17), we obtain

$$r + \mathcal{V}_\rho(r)\zeta + \zeta\epsilon(\zeta) = m^\kappa\left(1 + \kappa\frac{\mathcal{V}_\rho(m)}{m}\zeta + \zeta\epsilon_1(\zeta)\right).$$

This gives that $r = m^\kappa$, and so,

$$\frac{\mathcal{V}_\rho(m^\kappa)}{m^\kappa} = \frac{\kappa\mathcal{V}_\rho(m)}{m}, \quad \forall\ m \in (m_-^\rho, m_0^\rho) \ \text{ and } \ \forall\, \kappa \in (0,1). \tag{18}$$

The solution of functional Equation (18) is $\frac{\mathcal{V}_\rho(m)}{m} = d\ln(m)$, for some $d > 0$. This concludes the proof of Theorem 2(i).

(ii) Assume that $\forall\ m \in (m_-^\rho, m_0^\rho)$, we have $\left(Q_m^\rho\right)^{\uplus\kappa} = Q_r^{D_\kappa(\rho^{\boxplus 1/\kappa})}$ for some $r = r(m,\kappa) \in \left(m_-^{D_\kappa(\rho^{\boxplus 1/\kappa})}, m_0^{D_\kappa(\rho^{\boxplus 1/\kappa})}\right)$. This implies that

$$\left(\mathbf{B}_{Q_m^\rho}(\zeta)\right)^\kappa = \mathbf{B}_{Q_r^{D_\kappa(\rho^{\boxplus 1/\kappa})}}(\zeta), \quad \forall\, \zeta \text{ close to } 0. \tag{19}$$

Combining (3) and (16) and knowing from [15] that $\mathcal{V}_{\rho^{\boxplus 1/\kappa}}(r) = \mathcal{V}_\rho(\kappa r)/\kappa$, we obtain

$$\frac{1}{\mathbf{B}_{Q_r^{D_\kappa(\rho^{\boxplus 1/\kappa})}}(\zeta)} = r + \mathcal{V}_{D_\kappa(\rho^{\boxplus 1/\kappa})}(r)\zeta + \zeta\epsilon(\zeta) = r + \kappa\mathcal{V}_\rho(r)\zeta + \zeta\epsilon(\zeta) \ \text{ where } \epsilon(\zeta) \xrightarrow{\zeta\to 0} 0. \tag{20}$$

Combining (17), (19), and (20), we obtain

$$r + \kappa\mathcal{V}_\rho(r)\zeta + \zeta\epsilon(\zeta) = m^\kappa\left(1 + \kappa\frac{\mathcal{V}_\rho(m)}{m}\zeta + \zeta\epsilon_1(\zeta)\right).$$

This gives that $r = m^\kappa$, and so,

$$\frac{\mathcal{V}_\rho(m^\kappa)}{m^\kappa} = \frac{\mathcal{V}_\rho(m)}{m}, \quad \forall\ m \in (m_-^\rho, m_0^\rho), \ \text{ and } \ \forall\, \kappa \in (0,1). \tag{21}$$

The solution of functional Equation (21) is $\frac{\mathcal{V}_\rho(m)}{m} = d_1$, for some $d_1 > 0$. Thus, the measure ρ is a Marchenko–Pastur distribution scaled up. This concludes the proof of Theorem 2(ii). □

Remark 3. *Note that, up to affine transformations, we may suppose in Theorem 2(i) that* $(m_-^\rho, m_0^\rho) \subset (1, +\infty)$.

3. Conclusions

This paper studies the relationship of CSK families with multiplicative (free and Boolean) convolutions. Let $\rho \in \mathcal{P}_+$ with a finite first moment m_0^ρ. For $\kappa \in (0,1)$, we considered a new family of probability measures

$$(\mathcal{K}_-(\rho))^{\uplus\kappa} = \left\{\left(Q_m^\rho\right)^{\uplus\kappa}(ds) : \ m \in (m_-^\rho, m_0^\rho)\right\}.$$

We proved that if $(\mathcal{K}_-(\rho))^{\uplus\kappa}$ is a re-parametrization of $\mathcal{K}_-(\rho)$, then ρ is characterized by its corresponding VF $\mathcal{V}_\rho(m) = dm\ln(m), \quad \forall\ m \in (m_-^\rho, m_0^\rho), \quad d > 0$. We also proved that if $(\mathcal{K}_-(\rho))^{\uplus\kappa}$ is a re-parametrization of $\mathcal{K}_-(D_\kappa(\rho^{\boxplus 1/\kappa}))$, then ρ is a Marchenko–Pastur measure scaled up. Similar results are obtained if we replace the Boolean multiplicative convolution $\uplus$ with the free multiplicative convolution $\boxtimes$. The study of CSK families in relation to free and Boolean multiplicative convolutions provides a powerful analytic framework for understanding fundamental objects in noncommutative probability. It helps unify different independence structures, aids in explicit calculations, and connects probability theory with deep areas of complex analysis and functional analysis. The study of invariance properties of CSK families under free and Boolean multiplicative convolutions provides a deep structural understanding of noncommutative probability and free harmonic analysis. It helps classify fundamental distributions, connects probability with functional analysis, and reveals rich algebraic and analytic structures underlying multiplicative convolutions.

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