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Review

A Survey of Approximation Algorithms for the Universal Facility Location Problem

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Abstract: The facility location problem is a classical combinatorial optimization problem with extensive applications spanning communication technology, economic management, traffic governance, and public services. The facility location problem is to assign a set of clients to a set of facilities such that each client connects to a facility and the total cost (open cost and connection cost) is as low as possible. Among its various models, the uncapacitated facility location (UFL) problem is the most fundamental and widely studied. However, in real-world scenarios, resource constraints often make the UFL problem insufficient, necessitating more generalized models. This investigation primarily focuses on the universal facility location (Uni-FL) problem, a generalized framework encompassing both capacitated facility location problems (with hard and soft capacity constraints) and the UFL problem. Through a systematic analysis, we examine the Uni-FL problem alongside its specialized variants: the hard capacitated facility location (HCFL) problem and soft capacitated facility location (SCFL) problem. A comprehensive survey is conducted of existing approximation algorithms and theoretical results. The relevant results of their important variants are also discussed. In addition, we propose some open questions and future research directions for this problem based on existing research.

Keywords: universal; hard capacity; soft capacity; facility location problem; unsplittable; splittable

MSC: 90B35

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Academic Editor: Janez Žerovnik

Received: 9 February 2025
Revised: 9 March 2025
Accepted: 19 March 2025
Published: 21 March 2025

Citation: Xiao, H.; Zhang, J.; Zhang, Z.; Li, W. A Survey of Approximation Algorithms for the Universal Facility Location Problem. *Mathematics* **2025**, *13*, 1023. https://doi.org/10.3390/math13071023

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1. Introduction

In the field of combinatorial optimization, there are numerous problems, such as the vertex cover problem [1,2], the set cover problem [3,4], the machine scheduling problem [5,6], the allocation problem [7], the mobile crowdsensing services problem [8], the cloud-edge collaborative computation offloading problem [9], and the facility location problem, among others. The facility location problem has been reviewed in various aspects in recent years, such as the various applications of location models [10], applications and methods for facility location models [11], the synthesis and survey of location analysis [12], facility location models in the context of supply chain management [13], the service facility location problems [14], facility location problems for drone (uncrewed vehicle) delivery [15], and so on.

The facility location problem, as a classical problem in the field of combinatorial optimization, is widely applied in various areas such as communication technology, economic management, traffic governance, and public services, among others. For example, the location problems, such as location decision making for development zones and industrial

parks, the positioning of public service sites like logistics centers and gas stations, as well as the selection of locations for facilities like communication base stations and self-service facilities, are typical location problems encountered in everyday life. In real-world scenarios, common facilities generally include factories, base stations, schools, hospitals, supermarkets, post offices, warehouses, proxy servers, express delivery stations, sensors, and other entities [16–19].

There is a long history for the facility location problem. In ancient times, facility siting decisions predominantly relied on institutional frameworks and empirical knowledge rather than rigorous scientific methodologies. A pivotal transition occurred in 1909 with the first scientific treatise on the facility location problem by German scholars [20], marking its emergence as a rigorous research domain. As a typical NP-hard problem in the field of combinatorial optimization, the facility location problem has since profoundly influenced management science, operational research, and computational intelligence, attracting sustained scholarly attention globally. A classic application scenario is the facility location problem in a logistics network. The decision maker needs to determine the optimal warehouse construction solution from multiple candidate locations to achieve service coverage for decentralized demand points. The problem has a dual cost structure: on the one hand, each candidate warehouse location involves a corresponding fixed construction cost; on the other hand, the transportation of goods between a facility and a demand node incurs a variable transportation cost, the value of which is closely related to the spatial distribution of customer points. The goal is to select locations for building warehouses and allocating client demands such that construction and transportation costs are minimized.

Formally, the classical facility location problem is defined over a set $\mathcal{F}$ of facilities and a set $\mathcal{D}$ of clients. Each client $j \in \mathcal{D}$ has a demand $d_j \geq 0$, and each facility $i \in \mathcal{F}$ has an open cost $f_i \geq 0$. The unit connection cost between client $j \in \mathcal{D}$ and facility $i \in \mathcal{F}$ is c_{ij}. The problem is to open a subset $F \subseteq \mathcal{F}$ of facilities and connect each client $j \in \mathcal{D}$ to an open facility $i \in F$ such that the total cost, including open cost and connection cost, is minimized. This problem is also known as the uncapacitated facility location (UFL) problem.

Approximation algorithms constitute the principal methodologies for addressing NP-hard problems such as the facility location problem. The precise notion of approximation was initially proposed in the context of multiprocessor scheduling and bin packing [21].

Definition 1 ([22]). *Suppose Π is a minimization problem, which consists of instances $\mathcal{I}$ and feasible solutions $\mathcal{S}$ to these instances, and $\mathcal{M}$ is a polynomial-time algorithm for solving Π. The algorithm $\mathcal{M}$ is a ρ-factor approximation for Π, where ρ is referred to as the approximation ratio if, for every instance $I \in \mathcal{I}$, the solution $\mathcal{S}_{\mathcal{M}}$ returned by $\mathcal{M}$ satisfies*

$$c(\mathcal{S}_{\mathcal{M}}) \leq \rho \min_{S \in \mathcal{S}} c(S),$$

where $c(\cdot)$ denotes the cost function and $\rho \geq 1$.

The approximation ratio can objectively evaluate the quality of solutions provided by approximation algorithms, that is, the degree of closeness between the approximation solution and the optimal solution. The closer the approximation ratio ρ is to 1, the better.

Recall that, in the common facility location problem, the objective function value of a feasible solution is usually composed of two parts: open cost and connection cost. Based on this, the concept of the bifactor is introduced, which enables the precise quantification of the approximation algorithm performance.

Definition 2 ([23]). *Suppose Π is a facility location problem and $\mathcal{M}$ is a polynomial-time algorithm for solving Π. The algorithm $\mathcal{M}$ is a (r_f, r_c)-factor approximation algorithm where (r_f, r_c) is*

referred to as the bifactor if, for every instance $\mathcal{I}$, the open cost F_{ALG} and the connection cost C_{ALG} of the solution returned by $\mathcal{M}$ satisfies

$$F_{ALG} + C_{ALG} \leq r_f F^* + r_c C^*,$$

where F^* and C^* are the open cost and connection cost of the optimal solution for instance $\mathcal{I}$, respectively.

The concepts of approximation schemes, the polynomial-time approximation scheme (PTAS) and fully polynomial-time approximation scheme (FPTAS), are introduced below, and will be referenced in subsequent sections.

Definition 3 ([24]). *Suppose Π is a minimization problem.*

- *An approximation scheme for problem Π is a family A_ε of $(1+\varepsilon)$-approximation algorithms for problem Π over all $0 < \varepsilon < 1$.*
- *A polynomial-time approximation scheme (PTAS) for problem Π is an approximation scheme whose time complexity is polynomial in the input size.*
- *A fully polynomial-time approximation scheme (FPTAS) for problem Π is an approximation scheme whose time complexity is polynomial in the input size and also polynomial in $\frac{1}{\varepsilon}$.*

The notions of O, Ω, and Θ are introduced below. O provides an upper bound on the order of growth of a function, Ω provides a lower bound on the order of growth of a function, and Θ provides both an upper and lower bound on the order of growth of a function. The formal definitions are as follows.

Definition 4 ([25]). *Suppose $f(n)$ and $g(n)$ are two positive valued functions.*

- *$f(n) = \Omega(g(n))$ if there exist positive constants $c > 0$ and $n_0 > 0$ such that $f(n) \geq cg(n)$ for all $n \geq n_0$;*
- *$f(n) = O(g(n))$ if there is constant $c > 0$ and $n_0 > 0$ such that $f(n) \leq cg(n)$ for all $n \geq n_0$;*
- *$f(n) = \Theta(g(n))$ if there are constants $c_1 > 0$, $c_2 > 0$ and $n_0 > 0$ such that $c_1 g(n) \leq f(n) \leq c_2 g(n)$ for all $n \geq n_0$.*

Hochbaum [26] reduced the UFL problem to the set cover problem and proposed a greedy algorithm achieving an approximation ratio of $O(\ln(|\mathcal{D}|))$. Based on Feige's results [4], it has been established that, under the assumption that P $\neq$ NP, the lower bound for the approximation ratio of this problem is $\Omega(\ln(|\mathcal{D}|))$.

To achieve enhanced approximation guarantees, the facility location problem is studied in metric space, where the connection cost c_{ij} between facility i and client j is metric.

Definition 5. *The connection cost $c_{\mathcal{D}\times\mathcal{F}}$ is metric, if it satisfies the following properties:*

(1) c_{ij} is non-negative for any $i \in \mathcal{F}$, $j \in \mathcal{D}$, i.e., $c_{ij} \geq 0$;
(2) c_{ij} is symmetric for any $i \in \mathcal{F}$, $j \in \mathcal{D}$, i.e., $c_{ij} = c_{ji}$;
(3) c_{ij} satisfies triangle inequality in $\mathcal{F} \cup \mathcal{D}$, i.e., for any $i, j, k \in \mathcal{F} \cup \mathcal{D}$, it holds that

$$c_{ij} \leq c_{ik} + c_{kj}.$$

In fact, for property (3), the following quadrilateral inequality is often used in most algorithm analyses (see Figure 1), i.e., for any facilities $i_1, i_2 \in \mathcal{F}$ and clients $j_1, j_2 \in \mathcal{D}$, it holds that $c_{i_1 j_2} \leq c_{i_1 j_1} + c_{i_2 j_1} + c_{i_2 j_2}$.

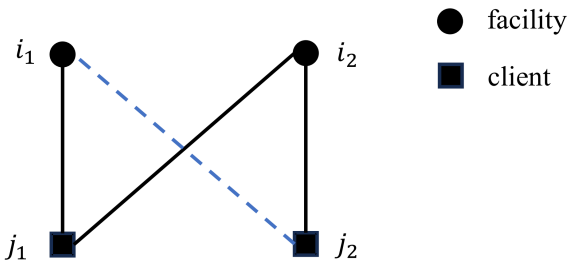


Figure 1. The quadrilateral inequality case.

In the following, unless explicitly stated otherwise, all subsequent discussions are confined to the metric space.

For the metric UFL problem, Shmoys et al. [27] established the first constant-factor approximation algorithm achieving a 3.16-approximation ratio using the same techniques of Lin and Vitter [28]. Since then, the ratio has been continually improved by the LP-rounding technique, local search technique, dual-fitting technique, and primal–dual technique [23,29–38]. Currently, the best approximation ratio is 1.488 [39], achieved by combining the LP-rounding and dual-fitting techniques. For the lower bound of the approximation ratio for the metric UFL problem, Guha and Khuller [33] demonstrated the inapproximability below a 1.463 factor, unless $NP \subseteq DTIME(n^{O(\ln \ln n)})$. Subsequently, Sviridenko [40] strengthened this conclusion to the condition that $NP \subseteq P$.

Furthermore, a general concept of metric, the p-th power metric space, is introduced in [41]. Specifically, the mathematical definition is presented as follows.

Definition 6 ([41]). *For any positive integer p, the distances $c_{\mathcal{D}\times\mathcal{F}}$ form a p-th power metric, if they are non-negative, symmetric, and obeying the following p-th power relaxed triangle inequality. That is, for any facilities $i_1, i_2 \in \mathcal{F}$ and clients $j_1, j_2 \in \mathcal{D}$, it holds that*

$$\sqrt[p]{c_{i_1 j_2}} \leq \sqrt[p]{c_{i_1 j_1}} + \sqrt[p]{c_{i_2 j_1}} + \sqrt[p]{c_{i_2 j_2}}.$$

It is easy to know that the metric is a particular case when $p = 1$, and the squared metric is another particular case when $p = 2$.

In recent years, a multitude of variants on the facility location problem have emerged, and several important ones are introduced below.

The first important variant of the facility location problem is the prize-collecting UFL problem, in which a penalty function $h : 2^{\mathcal{D}} \to \mathbb{R}_+$ is defined over the set of clients. The term "penalty" originates from the consideration of the impact of distant clients on the solution structure, whereby certain clients may remain unserved at the cost of incurring a specific penalty. Formally, we are given a set $\mathcal{F}$ of facilities and a set $\mathcal{D}$ of clients. Each client $j \in \mathcal{D}$ has a demand $d_j \geq 0$, and each facility $i \in \mathcal{F}$ has an open cost $f_i \geq 0$. The unit connection cost for connecting client $j \in \mathcal{D}$ to facility $i \in \mathcal{F}$ is c_{ij}, which is metric. For any client subset $S \subseteq \mathcal{D}$, there exists a penalty function $h(S)$ satisfying the following conditions: (1) $h(\emptyset) = 0$; (2) $h(\cdot)$ is nondecreasing. The function $h(\cdot)$ is termed submodular if it satisfies the following inequality:

$$h(S \cap T) + h(S \cup T) \leq h(S) + h(T), \quad \forall S, T \subseteq \mathcal{D}. \tag{1}$$

The prize-collecting UFL problem is to open a subset of facilities such that each client is either served by an open facility or penalized, with the objective of minimizing the total cost

which includes open cost, connection cost, and penalty cost. When the penalty function $h(\cdot)$ is submodular, the prize-collecting UFL problem is referred to as the UFL problem with submodular penalties (UFLPSP). This problem was first proposed by Hayrapetyan et al. [42], who demonstrated that, if there exists an LP-based α-approximation algorithm for the UFL problem, then a $(1+\alpha)$-approximation algorithm can be obtained for UFLPSP. Consequently, according to [39], a 2.488-approximation algorithm can be obtained. However, the algorithm proposed in [42] suffered from high computational complexity due to its reliance on the convexity of this problem and the use of an ellipsoid algorithm. Subsequently, Chudak and Nagano [43] developed an efficient $(1+\varepsilon)(1+\alpha)$-approximation algorithm for UFLPSP by employing convex relaxation instead of linear program relaxation, where α is the approximation factor of an LP-based approximation for the UFL problem. Consequently, a faster $(2.488+\varepsilon)$-approximation algorithm can be obtained. Additionally, Du et al. [44] proposed a 3-approximation algorithm based on the primal–dual technique. Li et al. [45] further improved the approximation ratio to 2.375 by integrating the primal–dual technique with a greedy augmentation approach. The best approximation ratio for UFLPSP is 2, achieved by Li et al. [46] using the LP-rounding technique. They provided a general framework for addressing the submodular penalty covering problem. Moreover, Li et al. [47] investigated the k-level UFL problem with submodular penalties (kL-UFLPSP) and proposed a 6-approximation algorithm based on the primal–dual technique. Later, Li et al. [48] developed an LP-based approximation algorithm with the ratio of $(1+\frac{2}{1-e^{-2}}) \approx 3.314$ for kL-UFLPSP. Zhang et al. [49] proposed a primal–dual greedy augmentation approximation algorithm with a ratio of 2.9444, which is the best result for kL-UFLPSP. Xu et al. [50] extended the problem to the stochastic UFL problem with submodular penalties (SUFLPSP) and gave a 3-approximation algorithm based on the primal–dual technique.

While the penalty function $h(\cdot)$ is modular when we replace the inequality "$\leq$" with the equality "$=$" in (1). It is evident that the modular function is a special case of the submodular function. When the penalty function $h(\cdot)$ is modular, the prize-collecting UFL problem is referred to as the UFL problem with linear penalties (UFLPLP). This problem was first proposed by Charikar et al. [51], who developed a 3-approximation algorithm by the primal–dual technique. Xu and Xu [52] proposed an algorithm achieving an approximation ratio of $2+\frac{2}{e} \approx 2.736$ through the LP-rounding technique. Geunes et al. [53] established a theoretical connection between the UFL problem and UFLPLP, demonstrating that, if an α-approximation algorithm exists for UFL problem, then a $(\frac{1}{1-e^{-\frac{1}{\alpha}}})$-approximation algorithm can be derived for UFLPLP. They further obtained a 2.056-approximation algorithm for UFLPLP. Jain et al. [23] integrated dual-fitting with factor-revealing techniques to obtain a 2-approximation algorithm. Xu and Xu [54] combined the primal–dual and local search techniques to propose a 1.8526-approximation algorithm. The best approximation ratio currently is 1.5148, achieved by Li et al. [46] through an LP-rounding technique based on non-uniform distribution parameters. Moreover, Bumb [55] investigated the k-level UFL problem with linear penalties (kL-UFLPLP) and devised a 6-approximation algorithm-based primal–dual technique. Later, Asadi et al. [56] proposed an LP-based 4-approximation algorithm for kL-UFLPLP. Li et al. [48] further improved the approximation ratio to 3 using the LP-rounding technique. Wang et al. [57] studied the k-UFL problem with linear penalties (k-UFLPLP) and developed a local search-based approximation algorithm with a ratio of $2+\sqrt{3}+\varepsilon \approx 3.732+\varepsilon$. San et al. [58] introduced the online prize-collecting UFL (OPC-UFL) problem and proposed a primal–dual $O(\log n)$-competitive algorithm. Wu et al. [59] investigated the two-stage stochastic UFL problem with linear penalties and presented an LP-rounding algorithm with a per-scenario constant approximation bound 3.0294. Wang et al. [60] studied the k-level UFL game with penalties, formulated a

cost-sharing scheme for this problem, and demonstrated the approximate cost recovery was 6.

For ease of understanding, we organize the approximation ratios of the algorithms for prize-collecting UFL problem (and its variants) and the methods used in Table 1.

Table 1. The related results of prize-collecting UFL problem and its variants.

Problem	Ratio	Reference	Methods
UFLPSP	2.488	Using the conclusion of [42] and the algorithm of [39]	
	$2.488 + \varepsilon$	Using the conclusion of [43] and the algorithm of [39]	
	3	Du et al. [44]	Primal–dual
	2.375	Li et al. [45]	Combinatorial method
	2	Li et al. [46]	LP rounding
kL-UFLPSP	6	Li et al. [47]	Primal–dual
	3.314	Li et al. [48]	LP rounding
	2.9444	Zhang et al. [49]	Combinatorial method
SUFLPSP	3	Xu et al. [50]	Primal–dual
UFLPLP	3	Charikar et al. [51]	Primal–dual
	2.736	Xu and Xu [52]	LP rounding
	2.056	Geunes et al. [53]	Combinatorial method
	2	Jain et al. [23]	Combinatorial method
	1.8526	Xu and Xu [54]	Combinatorial method
	1.5148	Li et al. [46]	LP rounding
kL-UFLPLP	6	Bumb [55]	Primal–dual
	4	Asadi et al. [56]	LP rounding
	3	Li et al. [48]	LP rounding
k-UFLPLP	$3.732 + \varepsilon$	Wang et al. [57]	Local search
OPC-UFL	$O(\log n)$ [1]	San et al. [58]	Primal–dual

[1] n is the number of clients, i.e., $n = |\mathcal{D}|$.

The second important variant of the facility location problem is the robust UFL (R-UFL) problem, also referred to as the UFL with outliers. In the classical facility location problem, serving a small number of distant clients may disproportionately affect the solution cost. By contrast, excluding these clients from service can substantially reduce the total cost, which motivates the formulation of the robust UFL problem. Unlike the prize-collecting UFL problem, the robust UFL problem imposes a strict constraint that at most L clients may remain unserved (without incurring penalty costs), and these L clients are termed outliers. Formally, in the robust UFL problem, we are given a set $\mathcal{F}$ of facilities and a set $\mathcal{D}$ of clients. Each client $j \in \mathcal{D}$ has a demand $d_j \geq 0$, and each facility $i \in \mathcal{F}$ has an open cost $f_i \geq 0$. The unit connection cost for connecting client $j \in \mathcal{D}$ to facility $i \in \mathcal{F}$ is c_{ij}, which is metric. Given an integer $L \geq 0$, the robust UFL problem is to open a subset $F \subseteq \mathcal{F}$ of facilities and find an outlier subset $O \subseteq \mathcal{D}$ with $|O| \leq L$, such that each client is either served by an open facility or assigned to the outlier set, with the objective of minimizing the sum of the open cost and connection cost. For the robust UFL problem, Charikar et al. [51] first proposed this problem, showing that the natural linear program has an unbounded integrality gap. They further devised a 3-approximation algorithm based on the primal–dual technique. Jain et al. [23] advanced this line of research by integrating dual-fitting and factor-revealing techniques to achieve a 2-approximation algorithm. For the uniform costs case, Friggstad et al. [61] established that a multiswap simple local search heuristic yields a PTAS for the doubling metric (including fixed-dimensional Euclidean metrics) or the shortest path metrics of graphs from a minor-closed family of graphs. Moreover, Luo et al. [62] introduced the priority UFL problem with outliers (PUFLPO), which generalizes both the robust UFL problem and the priority facility location problem

(like [63]). They proposed a 3-approximation algorithm using the primal–dual technique. Han et al. [64] extended the framework to the k-level UFL problem with outliers (kL-UFLO) and developed a 6-approximation algorithm by the primal–dual technique. Inspired by the study of [64], Zhang [65] investigated the k-level squared metric UFL problem with outliers (kL-SMUFLPO) and derived a 32-approximation algorithm based on the primal–dual technique.

For ease of understanding, we organize the approximation ratios of the algorithms for robust UFL problem (and its variants) and the methods used in Table 2.

Table 2. The related results of robust UFL problem and its variants.

Problem	Ratio	Reference	Methods
R-UFL	3	Charikar et al. [51]	Primal–dual
	2	Jain et al. [23]	Combinatorial method
R-UFL with uniform costs case	PTAS [1]	Friggstad et al. [61]	Local search
PUFLPO	3	Luo et al. [62]	Primal–dual
kL-UFLO	6	Han et al. [64]	Primal–dual
kL-SMUFLPO	32	Zhang [65]	Primal–dual

[1] On the doubling metric or the shortest path metrics of graphs from a minor-closed family of graphs.

The third important variant of the facility location problem is the k-UFL problem, also referred to as the uncapacitated k-facility location problem, where the number of facilities that can be opened is constrained to at most $k \geq 0$. The primary distinction between the UFL problem and the k-UFL problem is the number of open facilities. Intuitively, when $k = \infty$, the k-UFL problem is exactly the UFL problem. Formally, in the k-UFL problem, we are given a set $\mathcal{F}$ of facilities and a set $\mathcal{D}$ of clients. Each client $j \in \mathcal{D}$ has a demand $d_j \geq 0$, and each facility $i \in \mathcal{F}$ has an open cost $f_i \geq 0$. The unit connection cost between client $j \in \mathcal{D}$ and facility $i \in \mathcal{F}$ is denoted by c_{ij}, which is metric. Given an integer $k \geq 0$, the k-UFL problem is to open at most k facilities and connects each client to an open facility such that the sum of the open cost and connection cost is minimized. For the k-UFL problem, Jain and Vazirani [34] employed the primal–dual technique and Lagrangian relaxation to derive a 6-approximation algorithm. They also implicitly demonstrated that any Lagrangian multiplier preserving (LMP) α-approximation algorithm for the UFL problem can be extended to yield a 2α-approximation algorithm for the k-UFL problem. Jain et al. [23] later improved this approximation ratio to 4 by using a scaling method and the conclusion of [66]. Zhang [67] proposed an approximation algorithm with a ratio of $2 + \sqrt{3} + \varepsilon \approx 3.732 + \varepsilon$ based on the local search technique. Charikar and Li [68] developed a 3.25-approximation algorithm using the LP-rounding technique. Kong and Zhang [69] proposed a $(1 + \varepsilon)$-approximation algorithm for the k-UFL problem by employing a new sampling-based technique to estimate the locations of the facilities opened in optimal solutions. Moreover, Jain and Vazirani [34] proposed a more general model, the squared metric k-UFL (SMk-UFL) problem, where the connection cost c_{ij} is defined as the squared metric distance between i and j. They obtained a $(108 + \varepsilon)$-approximation algorithm by combining Lagrangian relaxation with primal–dual techniques. Zhang et al. [70] further proposed an approximation algorithm with a ratio of $22 + \sqrt{505} + \varepsilon \approx 44.472 + \varepsilon$ for the SMk-UFL problem based on the local search technique. Zhang et al. [71] proposed a $(36.342 + \varepsilon)$-approximation algorithm for the SMk-UFL problem within the framework of Lagrangian relaxation. Note that there is a big difference in the ratio between k-UFL and SMk-UFL; this is because the connection costs in the SMk-UFL problem do not satisfy the triangle inequality. Wang et al. [72] extended the problem to the squared metric k-UFL problem with linear penalties (SMk-UFLPLP) and derived an approximation algorithm with a ratio of $22 + \sqrt{505} + \varepsilon \approx 44.472 + \varepsilon$ using the local search technique.

For ease of understanding, we organize the approximation ratios of the algorithms for k-UFL problem (and its variants) and the methods used in Table 3.

Table 3. The related results of k-UFL problem and its variants.

Problem	Ratio	Reference	Methods
k-UFL	6	Jain and Vazirani [34]	Combinatorial method
	4	Jain et al. [23]	Combinatorial method
	$3.732+\varepsilon$	Zhang [67]	Local search
	3.25	Charikar and Li [68]	LP rounding
	$1+\varepsilon$	Kong and Zhang [73]	Combinatorial method
SMk-UFL	$108+\varepsilon$	Jain and Vazirani [34]	Combinatorial method
	$44.472+\varepsilon$	Zhang et al. [70]	Local search
	$36.342+\varepsilon$	Zhang et al. [71]	Lagrangian relaxation
SMk-UFLPLP	$44.472+\varepsilon$	Wang et al. [72]	Local search

The fourth important variant of the facility location problem is the k-level UFL (kL-UFL) problem, also referred to as the multi-level UFL problem, where k represents the number of facility levels. Notably, when $k=1$, the k-level UFL problem is exactly the UFL problem. Formally, in the k-level UFL problem, we are given a set of facilities organized into k hierarchical levels, denoted as $\mathcal{F}=\mathcal{F}_1\cup\mathcal{F}_2\cup\cdots\cup\mathcal{F}_k$, and a set $\mathcal{D}$ of clients. Each client $j\in\mathcal{D}$ has a demand $d_j\geq 0$, and each facility $i\in\mathcal{F}$ has an open cost $f_i\geq 0$. Let $\mathcal{P}=\mathcal{F}_1\times\mathcal{F}_2\times\cdots\times\mathcal{F}_k$. A path $p=(i_1,i_2,\cdots,i_k)\in\mathcal{P}$ is termed open if and only if each facility $i_l(1\leq l\leq k)$ is open. The unit connection cost between client $j\in\mathcal{D}$ and the first-level facility $i_1\in\mathcal{F}_1$ is denoted by c_{i_1j}, and the unit connection between a facility at $(l-1)$ level and a facility at l level is denoted by $c_{i_{l-1}i_l}$, both of which are metric. The unit connection cost of client $j\in\mathcal{D}$ to any path $p=(i_1,i_2,\cdots,i_k)\in\mathcal{P}$ is denoted as $c_{pj}=c_{i_1i_2\cdots i_k,j}=c_{i_1,j}+\sum_{l=1}^{k-1}c_{i_li_{l+1}}$, which is metric in $\mathcal{D}\cup\mathcal{F}_1\cup\mathcal{F}_2\cdots\mathcal{F}_k$. The k-level UFL problem is to open a subset of facilities and connect each client to a path along open facilities, such that the total cost, including the open cost and connection cost, is minimized. For the k-level UFL problem, Meyerson et al. [74] proposed the first approximation algorithm with a ratio of $O(\log n)$, where $n=|\mathcal{D}|$. Bumb and Kern [75] developed a 6-approximation algorithm based on the primal–dual technique. Ageev et al. [76] introduced a path reduction technique that reduces the k-level UFL problem to the UFL problem, achieving a 3.27-approximation algorithm. Aardal and Chudak [77] formulated a linear program with multiple index variables for the k-level UFL problem and derived a 3-approximation algorithm based on the LP-rounding technique. Subsequently, Gabor and van Ommeren [78] introduced a new linear program with polynomial variables for the k-level UFL problem and achieved the same approximation ratio of 3 based on the LP-rounding technique. Byrka et al. [79] proposed a new integer program based on the forest structure characteristics of the optimal solution, yielding an α_k-approximation algorithm, where α_k is monotonically increasing with respect to k and satisfies $\lim_{k\to\infty}\alpha_k=3$. They further applied the randomization to improve the approximation ratio for all $k>2$, obtaining 1.97, 2.09, and 2.19 for k = 3, 4, and 5, respectively. The best ratio for $k=2$ currently is 1.77, achieved by Zhang [80] through a quasi-greedy technique. For the lower bound of the k-level UFL problem, Krishnaswamy and Sviridenko [81] established that no polynomial-time approximation algorithm can achieve performance guarantees better than 1.539 for the 2-level UFL problem and 1.61 for the k-level UFL problem unless $NP=P$. Moreover, Wang et al. [82] investigated the dynamic k-level UFL (DkL-UFL) problem, developing a combinatorial primal–dual approximation algorithm with a ratio of 6. They also designed a 6-approximation algorithm for the DkL-UFL problem with submodular

penalties (DkL-UFLPSP) and another 6-approximation algorithm for the DkL-UFL problem with outliers (DkL-UFLPO).

For ease of understanding, we organize the approximation ratios of the algorithms for k-level UFL problem (and its variants) and the methods used in Table 4.

Table 4. The related results of k-level UFL problem and its variants.

Problem	Ratio	Reference	Methods
k-level UFL	$O(\log n)$ [1]	Meyerson et al. [74]	Combinatorial method
	6	Bumb and Kern [75]	Primal–dual
	3.27	Ageev et al. [76]	Reduction method
	3	Aardal and Chudak [77]	LP rounding
	3	Gabor and van Ommeren [78]	LP rounding
	α_k [2]	Byrka et al. [79]	Combinatorial method
2-level UFL	1.77	Zhang [80]	Quasi-greedy method
DkL-UFL	6	Wang et al. [82]	Combinatorial method
DkL-UFLPSP	6	Wang et al. [82]	Combinatorial method
DkL-UFLPO	6	Wang et al. [82]	Combinatorial method

[1] n is the number of clients, i.e., $n = |\mathcal{D}|$. [2] α_k is monotonically increasing with respect to k and satisfies $\lim_{k\to\infty} \alpha_k = 3$.

The UFL problem serves as a foundational model for optimizing facility placement and client assignment to minimize costs. However, real-world applications often involve resource-constrained facilities, such as storage limits in warehouses, bandwidth thresholds in communication networks, and service capacity ceilings in public infrastructure. These constraints render the traditional UFL model inadequate for practical implementation, necessitating more generalized frameworks. Capacity constraints, a hallmark of resource-limited scenarios, have been extensively studied across combinatorial optimization problems, including the capacitated vertex cover problem [83–86], the capacitated set cover problem [87–89], and the capacitated allocation problem [90]. To enhance the practical relevance of facility location models, capacity constraints have been integrated into the framework, leading to the development of the universal facility location (Uni-FL) problem. Introduced by Mahdian and Pál [91], the Uni-FL problem offers a more flexible and comprehensive approach by incorporating both capacitated (with hard and soft constraints) and uncapacitated cases. Advancing the theoretical understanding and algorithmic solutions for the Uni-FL problem can significantly improve the practicality and efficiency of facility location strategies.

This paper systematically examines the Uni-FL problem and its special cases, including the hard capacitated facility location (HCFL) problem and the soft capacitated facility location (SCFL) problem. Unlike previous works that primarily focus on either uncapacitated or capacitated versions in isolation, this research provides a unified perspective on these models. By surveying existing approximation algorithms and theoretical advancements, it highlights the connections between different problem variants. Furthermore, this work identifies open research questions and proposes future directions, contributing to the ongoing development of more efficient and applicable facility location algorithms. The remainder of this paper is organized as follows. Section 2 introduces the Uni-FL problem with its relevant work. Section 3 and Section 4 discuss two special cases of the Uni-FL problem, the hard capacitated facility location (HCFL) problem and the soft capacitated facility location (SCFL) problem, respectively, along with their relevant work. Finally, Section 5 summarizes our survey and proposes some future research directions in this field.

2. The Universal Facility Location Problem

The universal facility location (Uni-FL) problem is a generalized framework that extends various facility location problems, including the capacitated facility location problem (both hard and soft capacity constraints) and the UFL problem.

Before formally introducing this problem, we first define the concepts of splittable and unsplittable cases of a client's demand [27,92]. Notably, in the hard capacitated facility location problem, the splittable case is commonly referred to as the multiple-source capacitated facility location problem, whereas the unsplittable case is referred to as the single-source capacitated facility location problem [93,94].

Definition 7. *When a client's demand can be served by multiple facilities, it is referred to as the splittable case; otherwise, it is referred to as the unsplittable case.*

In the Uni-FL problem, we are given a set $\mathcal{F}$ of facilities and a set $\mathcal{D}$ of clients. Each client $j \in \mathcal{D}$ has a demand $d_j \geq 0$, and the unit connection cost that client $j \in \mathcal{D}$ incurs to connect to facility $i \in \mathcal{F}$ is denoted as c_{ij}, which is metric. The Uni-FL problem is to allocate a certain capacity u_i to each facility $i \in \mathcal{F}$ and connect all demands to the facilities subject to the constraint that the total demands served by any facility $i \in \mathcal{F}$ cannot exceed its allocated capacity u_i. Let $f_i(u_i)$ represent the cost of allocating u_i units of capacity at facility i, where $f_i : R_{\geq 0} \to R_{\geq 0} \cup \{+\infty\}$ satisfies

(a) **Non-decreasing.** $f_i(x) \geq f_i(y)$ for every $x \geq y \geq 0$.
(b) **Normalization.** $f_i(0) = 0$.
(c) **Left continuous.** $\lim_{x \to u^-} f_i(x) = f_i(u)$, for every $u > 0$.

The *facility cost* is defined as the total cost of allocating capacity at all facilities, given by $\sum_{i:i\in\mathcal{F}} f_i(u_i)$. The *connection cost* is defined as the total cost of connecting all client demands to facilities. The Uni-FL problem is to minimize the sum of the facility cost and connection cost.

By introducing variable x_{ij} to indicate whether client j is served by facility i or not, we can obtain the integer linear program of the unsplittable Uni-FL problem as follows.

$$
\begin{aligned}
\textit{Uni-FL}: \min \quad & \sum_{i\in\mathcal{F}} f_i(u_i) + \sum_{i\in\mathcal{F}} \sum_{j\in\mathcal{D}} d_j c_{ij} x_{ij} \\
\text{s.t.} \quad & \sum_{i\in\mathcal{F}} x_{ij} \geq 1, \quad \forall j \in \mathcal{D}, && (2)\\
& \sum_{j\in\mathcal{D}} d_j x_{ij} \leq u_i, \quad \forall i \in \mathcal{F}, && (3)\\
& x_{ij} \in \{0,1\}, \quad \forall i \in \mathcal{F}, j \in \mathcal{D}, && (4)
\end{aligned}
$$

where constraint (2) means each client's demand must be satisfied; constraint (3) means the total demands served by facility i cannot exceed its capacity u_i.

In the splittable case, we define variable x_{ij} as the proportion of client j's demand assigned to facility i, which is in [0, 1]. Consequently, in the splittable case, we only need to replace constraint (4) to the following constraint in the *Uni-FL* program.

$$x_{ij} \in [0,1], \quad \forall i \in \mathcal{F}, j \in \mathcal{D}.$$

Few results are available in the literature for the unsplittable Uni-FL problem. In the remainder of this section, we focus on the splittable Uni-FL problem.

For the non-metric splittable Uni-FL problem, Li and Khuller [95] proposed a greedy algorithm with an approximation ratio of $\ln n + 1$, where $n = |\mathcal{D}|$.

For the metric splittable Uni-FL problem, Mahdian and Pál [91] first introduced this problem and developed an approximation algorithm with a ratio of $\sqrt{15}+4+\varepsilon \approx 7.873+\varepsilon$ based on the local search technique. Vygen [96] improved this result, achieving an approximation ratio of $\frac{\sqrt{41}+7}{2}+\varepsilon \approx 6.702+\varepsilon$ using the same technique while extending the pivoting operation to a more general structure. Garg et al. [97] introduced extended operations in the local search technique and proposed an analytical framework to achieve an approximation ratio of $3+2\sqrt{2}+\varepsilon \approx 5.828+\varepsilon$, but one of their operations is unlikely to be polynomially computable. Angel et al. [98] proposed a polynomially computable operation called open–close, closely following the analysis of [97,99], and achieved an approximation algorithm with a ratio of $3+2\sqrt{2}+\varepsilon \approx 5.828+\varepsilon$, which is the current best result.

For ease of understanding, we organize the approximation ratios of the algorithms for splittable Uni-FL problem and the methods used in Table 5.

Table 5. The related results of splittable Uni-FL problem.

Problem	Ratio	Reference	Methods
Non-metric space	$\ln n+1$ [1]	Li and Khuller [95]	Greedy method
Metric space	$7.873+\varepsilon$	Mahdian and Pál [91]	Local search
	$6.702+\varepsilon$	Vygen [96]	Local search
	$5.828+\varepsilon$	Garg et al. [97]	Local search
	$5.828+\varepsilon$	Angel et al. [98]	Local search

[1] n is the number of clients, i.e., $n=|\mathcal{D}|$.

Open problem 1. *Find a polynomial-time algorithm with an approximation ratio less than* 5.828 *for the splittable Uni-FL problem.*

2.1. The Prize-Collecting Uni-FL Problem

The prize-collecting Uni-FL (PC-Uni-FL) problem is a variant of the Uni-FL problem and extends the prize-collecting UFL problem. A penalty function $h: 2^{\mathcal{D}} \to \mathbb{R}_+$ is defined over the set of clients. Based on the Uni-FL problem, for any client subset $S \subseteq \mathcal{D}$, the penalty function $h(S)$ satisfies the following conditions: (1) $h(\emptyset)=0$; (2) $h(\cdot)$ is nondecreasing; (3) $h(\cdot)$ is either submodular or modular. The prize-collecting Uni-FL problem is to allocate a certain capacity u_i to each facility $i \in \mathcal{F}$ and connect the demands to the facilities, where the demands are defined as the sum of the demands of non-penalized clients, subject to the constraint that the total demands served by any facility $i \in \mathcal{F}$ cannot exceed its allocated capacity u_i, with the objective of minimizing the total cost, including the facility cost, connection cost, and penalty cost.

By introducing variable x_{ij} to indicate the proportion of client j's demand assigned to facility i, and variable z_S to indicate whether client set S is penalized or not, we can obtain the mixed-integer linear program of the splittable prize-collecting Uni-FL problem.

$$
\begin{aligned}
\textit{PC-Uni-FL}: \min \quad & \sum_{i\in\mathcal{F}} f_i(u_i) + \sum_{i\in\mathcal{F}}\sum_{j\in\mathcal{D}} d_j c_{ij} x_{ij} + \sum_{S\subseteq\mathcal{D}} h(S) z_S \\
\text{s.t.} \quad & \sum_{i\in\mathcal{F}} x_{ij} + \sum_{S\subseteq\mathcal{D}: j\in S} z_S \geq 1, \quad \forall j \in \mathcal{D}, \\
& \sum_{j\in\mathcal{D}} d_j x_{ij} \leq u_i, \quad \forall i \in \mathcal{F}, \\
& x_{ij} \in [0,1], \quad \forall i \in \mathcal{F}, j \in \mathcal{D}, \\
& z_S \in \{0,1\}, \quad \forall S \subseteq \mathcal{D},
\end{aligned}
\tag{5}
$$

where constraint (5) means, for any client $j \in \mathcal{D}$, either its demand is satisfied, or it is penalized.

When the penalty function $h(\cdot)$ is submodular, few results are available in the literature. When $h(\cdot)$ is modular, the problem is referred to as the Uni-FL problem with linear penalties. This problem was first proposed by Xu et al. [100], who proposed an approximation algorithm with a ratio of $\sqrt{15}+4+\varepsilon \approx 7.873+\varepsilon$ based on the local search technique. Subsequently, Xu et al. [101] further improved the approximation ratio to $3+2\sqrt{2}+\varepsilon \approx 5.828+\varepsilon$ by the local search technique, which is currently the best result.

For ease of understanding, we organize the approximation ratios of the algorithms for splittable prize-collecting Uni-FL problem when $h(\cdot)$ is modular and the methods used in Table 6.

Table 6. The related results of splittable prize-collecting Uni-FL problem when $h(\cdot)$ is modular.

Ratio	Reference	Methods
$7.873+\varepsilon$	Xu et al. [100]	Local search
$5.828+\varepsilon$	Xu et al. [101]	Local search

Open problem 2. *Design a constant approximation algorithm for the splittable prize-collecting Uni-FL problem when $h(\cdot)$ is submodular.*

2.2. The Robust Uni-FL Problem

The robust Uni-FL (R-Uni-FL) problem represents another variant of the Uni-FL problem with a robust set, and extends the robust UFL problem. This problem is commonly referred to as the Uni-FL problem with outliers and, in some problems, is also known as the partial problem [102–104]. Unlike the prize-collecting Uni-FL problem, the robust Uni-FL problem stipulates that at most L clients may remain unserved (without incurring any penalty cost), corresponding to scenarios where the number of penalized clients is explicitly limited (as discussed in [105,106]).

Based on the Uni-FL problem, we are given an integer $L \geq 0$. The robust Uni-FL problem is to find an outlier set, which is a client subset with a cardinality not exceeding L, allocate a certain capacity u_i to each facility $i \in \mathcal{F}$, and connect the total demands of clients (excluding those in the outlier set) to the facilities subject to the constraint that the total demands served by any facility $i \in \mathcal{F}$ cannot exceed its allocated capacity u_i, with the objective of minimizing the sum of the facility cost and connection cost.

By introducing variable x_{ij} to indicate the proportion of client j's demand assigned to facility i, and variable r_j to indicate whether client j is an outlier or not, we can obtain the mixed-integer linear program of the splittable robust Uni-FL problem.

$$
\begin{aligned}
R\text{-}Uni\text{-}FL: \min \quad & \sum_{i\in\mathcal{F}} f_i(u_i) + \sum_{i\in\mathcal{F}}\sum_{j\in\mathcal{D}} d_j c_{ij} x_{ij} \\
\text{s.t.} \quad & \sum_{i\in\mathcal{F}} x_{ij} + r_j \geq 1, \quad \forall j \in \mathcal{D}, && (6)\\
& \sum_{j\in\mathcal{D}} d_j x_{ij} \leq u_i, \quad \forall i \in \mathcal{F}, \\
& \sum_{j\in\mathcal{D}} r_j \leq L, && (7)\\
& x_{ij} \in [0,1], \quad \forall i \in \mathcal{F}, j \in \mathcal{D}, \\
& r_j \in \{0,1\}, \quad \forall j \in \mathcal{D},
\end{aligned}
$$

where constraint (6) means, for any client $j \in \mathcal{D}$, either its demand is satisfied, or it is an outlier; constraint (7) means the number of outliers cannot exceed L.

Research on the robust Uni-FL problem remains limited; however, numerous studies have been conducted on specific cases, such as the hard-capacitated case, as detailed in Section 3.

Open problem 3. *Design a constant approximation algorithm for the splittable robust Uni-FL problem.*

2.3. The k-Uni-FL Problem

The k-Uni-FL problem, referred to as the universal k-facility location problem, is an important variant of the Uni-FL problem and extends the k-UFL problem. The primary distinction between the Uni-FL problem and the k-Uni-FL problem lies in the constraint on the number of open facilities. Intuitively, the Uni-FL problem is a special case of the k-Uni-FL problem with $k = \infty$. The k-Uni-FL problem is exactly the k-UFL problem when the cost function $f_i(u_i) = f_i$ for each $u_i > 0$, where $f_i > 0$.

Based on the Uni-FL problem, given an integer $k \geq 0$, the k-Uni-FL problem is to allocate a certain capacity u_i to each facility $i \in \mathcal{F}$ and connect all client demands to the facilities, subject to two constraints: (1) the total demands served by any facility $i \in \mathcal{F}$ must not exceed its allocated capacity u_i, and (2) the total number of facilities allocated is no more than k. The objective of the k-Uni-FL problem is to minimize the sum of facility cost and connection cost.

By introducing variable x_{ij} to indicate the proportion of client j's demand assigned to facility i, we can obtain the mixed-integer linear program of the splittable k-Uni-FL problem.

$$\begin{aligned} k\text{-}Uni\text{-}FL: \min \quad & \sum_{i\in\mathcal{F}} f_i(u_i) + \sum_{i\in\mathcal{F}}\sum_{j\in\mathcal{D}} d_j c_{ij} x_{ij} \\ \text{s.t.} \quad & \sum_{i\in\mathcal{F}} x_{ij} \geq 1, \quad \forall j \in \mathcal{D}, \\ & \sum_{j\in\mathcal{D}} d_j x_{ij} \leq u_i, \quad \forall i \in \mathcal{F}, \\ & \sum_{i\in\mathcal{F}} I[\sum_{j\in\mathcal{D}} x_{ij} > 0] \leq k, \qquad (8) \\ & x_{ij} \in [0,1], \quad \forall i \in \mathcal{F}, j \in \mathcal{D}, \end{aligned}$$

where $I[\sum_{j\in\mathcal{D}} x_{ij} > 0]$ is the indicator function, if $\sum_{j\in\mathcal{D}} x_{ij} > 0$, $I[\sum_{j\in\mathcal{D}} x_{ij} > 0] = 1$; otherwise, $I[\sum_{j\in\mathcal{D}} x_{ij} > 0] = 0$; constraint (8) means that the total number of facilities allocated is no more than k.

Research on the k-Uni-FL problem remains limited; however, numerous studies have been conducted on specific cases, such as the hard-capacitated and soft-capacitated cases, as detailed in Sections 3 and 4.

Open problem 4. *Design a constant approximation algorithm for the splittable k-Uni-FL problem.*

2.4. The k-Level Uni-FL Problem

The k-level Uni-FL (kL-Uni-FL) problem, also referred to as the multi-level Uni-FL problem, is a variant of the Uni-FL problem, where k denotes the number of facility levels. The k-level Uni-FL problem is exactly the k-level UFL problem when the cost function $f_i(u_i) = f_i$ for each $u_i > 0$, where $f_i > 0$.

Based on the Uni-FL problem, the set of facilities is partitioned into k disjoint subsets, denoted as $\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_k$, such that $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \cdots \cup \mathcal{F}_k$. Let $\mathcal{P} = \mathcal{F}_1 \times \mathcal{F}_2 \times \cdots \times \mathcal{F}_k$ represent the set of all possible paths. A path $p = (i_1, i_2, \cdots, i_k) \in \mathcal{P}$ is termed open if and only if the capacity of each facility $i_l (1 \leq l \leq k)$ is allocated. The unit connection cost between client $j \in \mathcal{D}$ and the first-level facility $i_1 \in \mathcal{F}_1$ is denoted by $c_{i_1 j}$, and the unit connection between a facility at the $(l-1)$ level and a facility at the l level is denoted by $c_{i_{l-1} i_l}$, both of which are metric. The unit connection cost of client $j \in \mathcal{D}$ to any path $p = (i_1, i_2, \cdots, i_k) \in \mathcal{P}$ is denoted as $c_{pj} = c_{i_1 i_2 \cdots i_k, j} = c_{i_1, j} + \sum_{l=1}^{k-1} c_{i_l i_{l+1}}$, which is metric in $\mathcal{D} \cup \mathcal{F}_1 \cup \mathcal{F}_2 \cdots \mathcal{F}_k$. The k-level Uni-FL problem is to allocate a certain capacity u_i to each facility $i_l \in \mathcal{F}_l$ and connect the total demands of clients to open paths along these facilities, subject to the constraint that the total demands served by any facility $i \in \mathcal{F}$ does not exceed its allocated capacity u_i, with the objective of minimizing the sum of facility cost and connection cost.

By introducing variable x_{pj} to represent the proportion of client j's demand assigned to the facilities in the path p, the mixed-integer linear program of the splittable k-level Uni-FL problem can be expressed as follows.

$$
\begin{aligned}
kL\text{-}Uni\text{-}FL: \min \quad & \sum_{i \in \mathcal{F}} f_i(u_i) + \sum_{p \in \mathcal{P}} \sum_{j \in \mathcal{D}} c_{pj} d_j x_{pj} \\
\text{s.t.} \quad & \sum_{p \in \mathcal{P}} x_{pj} \geq 1, \quad \forall j \in \mathcal{D}, && (9) \\
& \sum_{p \in \mathcal{P}: i \in p} \sum_{j \in \mathcal{D}} d_j x_{pj} \leq u_i, \quad \forall i \in \mathcal{F}, && (10) \\
& x_{pj} \in [0,1], \quad \forall p \in \mathcal{P}, j \in \mathcal{D},
\end{aligned}
$$

where constraint (9) means each client must be connected to a path; constraint (10) means the total demands served by a facility cannot exceed its capacity.

Research on the k-level Uni-FL problem remains limited; however, numerous studies have been conducted on specific cases, such as the hard-capacitated and soft-capacitated cases, as detailed in Sections 3 and 4.

Open problem 5. *Design a constant approximation algorithm for the splittable k-level Uni-FL problem.*

2.5. The Other Variants of Uni-FL Problem

The Uni-FL problem in the p-th power of metric space (M^{p}UFLP)was introduced by Xu et al. [41]. This formulation generalizes the Uni-FL problem by relaxing the assumption that the unit connection cost c_{ij} is proportional to the distance between the client j and the facility i. Based on the local search technique, Xu et al. [41] proposed a $(3 \cdot 2^p \cdot \lambda + 1)$-approximation algorithm, where λ was the root of the following equation:

$$3 \cdot 2^p \cdot \lambda^2 - 3 \cdot (2^{p-1} + 1) - 1 = 0.$$

When $p = 1$, the algorithm achieves an approximation ratio of $7.873 + \varepsilon$, matching the result of Mahdian and Pál [91]; when $p = 2$, the algorithm achieves an approximation ratio of $11.18 + \varepsilon$.

3. The Hard Capacitated Facility Location Problem

Hard capacity constraints have been extensively studied in the context of the vertex cover problem, as evidenced by [107–111]. Similarly, they are widely investigated in the facility location problem, particularly in the hard capacitated facility location (HCFL) problem, which is a special case of the Uni-FL problem.

In the HCFL problem, each facility $i \in \mathcal{F}$ is associated with an open cost f_i and a hard capacity u_i. An open facility can serve up to u_i units of demand, and each facility $i \in \mathcal{F}$ can be opened at most b_i times. Consequently, when the cost function is defined as Equation (11), the Uni-FL problem is exactly the HCFL problem.

$$f_i(x) = \begin{cases} f_i \cdot \lceil \frac{x}{u_i} \rceil, & \text{if} \quad x \leq b_i u_i, \\ +\infty, & \text{otherwise}, \end{cases} \tag{11}$$

where x is the total service of facility i. Based on the unsplittable Uni-FL problem, we introduce variable y_i to indicate the number of times facility i is opened. The integer linear program of the unsplittable HCFL problem can be obtained as follows.

$$\begin{aligned} HCFL: \min \quad & \sum_{i \in \mathcal{F}} f_i y_i + \sum_{i \in \mathcal{F}} \sum_{j \in \mathcal{D}} c_{ij} d_j x_{ij} \\ \text{s.t.} \quad & \sum_{i \in \mathcal{F}} x_{ij} \geq 1, \quad \forall j \in \mathcal{D}, \\ & \sum_{j \in \mathcal{D}} d_j x_{ij} \leq u_i y_i, \quad \forall i \in \mathcal{F}, \\ & x_{ij} \in \{0,1\}, \quad \forall i \in \mathcal{F}, j \in \mathcal{D}, \\ & y_i \in \{0,1,\cdots,b_i\}, \quad \forall i \in \mathcal{F}. \end{aligned}$$

Typically, it is assumed that $b_i = 1$.

One should note that, in the unsplittable case, the HCFL problem is NP-complete only in determining whether there is a feasible solution, by a straightforward reduction from the bin-packing problem. To analyze the unsplittable HCFL problem, the concept of a bi-criteria approximation factor has been introduced.

Definition 8. *An (α, β)-bi-criteria approximation algorithm produces a solution of cost at most α times the optimum, while violating the capacities by no more than a factor of β.*

For the metric unsplittable HCFL problem with general case, Shmoys et al. [27] were the first to consider the bi-criteria approximation and proposed a $(9,4)$-factor approximation algorithm based on filtering and rounding techniques.

For the metric unsplittable HCFL problem with uniform capacities case, Bateni and Hajiaghayi [112] developed a bi-criteria $(O(\log n), 1+\varepsilon)$-approximation algorithm for the general metric and a $(1+\varepsilon, 1+\varepsilon)$-approximation algorithm for the tree metric. They also demonstrated that, in the non-metric space, a capacity violation of at least a factor of 1.5 is necessary to achieve any bounded approximation guarantee. Behsaz et al. [113] introduced a framework for bi-criteria approximation algorithms and developed two new approximation algorithms with factors $(9, \frac{3}{2})$ and $(29.315, \frac{4}{3})$. Additionally, they presented a quasi-polynomial-time $(1+\varepsilon, 1+\varepsilon)$-approximation algorithm for the Euclidean metric.

In the splittable case, let variable x_{ij} denote the proportion of client j's demand that is served by facility i, which is in $[0,1]$. Consequently, in the splittable case, we only need to replace the third constraint with the following constraint in the $HCFL$ program.

$$x_{ij} \in [0,1], \quad \forall i \in \mathcal{F}, j \in \mathcal{D}.$$

For the non-metric splittable HCFL problem, Bar-Ilan et al. [114] proved that there is a constant $c < 1$ such that the approximation ratio cannot be allowed to be $c \ln n$ unless $NP = P$.

For the metric splittable HCFL problem with general case, Pál et al. [115] proposed the first constant-factor approximation algorithm using the local search technique, achieving a ratio of $9 + \varepsilon$. They further refined this approach using the scaling technique, improving the approximation ratio to $8.532 + \varepsilon$. Subsequently, Mahdian and Pál [91] enhanced the approximation ratio to $\sqrt{15} + 4 + \varepsilon \approx 7.873 + \varepsilon$, also leveraging the local search technique. Later, Zhang et al. [99] further improved the approximation ratio to $3 + 2\sqrt{2} + \varepsilon \approx 5.828 + \varepsilon$ based on the local search technique. Bansal et al. [116] achieved a 5-approximation algorithm based on a local search algorithm, which is currently the best approximation ratio. All of the aforementioned algorithms rely on the local search technique, as the hard capacity constraints present significant challenges for linear program-based methods (such as the LP-rounding and primal–dual techniques). Aardal et al. [117] conducted a comprehensive study on valid inequalities for the HCFL problem and proposed a generalization that the strength of the obtained formulas remains an open question. Kolliopoulos and Moysoglou [118] demonstrated that many such formulas were insufficient to achieve a constant integrality gap. They further proved that applying the Sherali–Adams hierarchy to the standard LP formula does not reduce the integrality gap. To address these limitations, An et al. [119] proposed a linear program relaxation with a constant integer gap for the HCFL problem. By leveraging multicommodity flows and matching theory, they developed a 288-approximation algorithm using LP rounding, without violating the capacity constraints. Kao [120] refined this approach, presenting an LP-rounding-based approximation algorithm with a ratio of $\frac{(10+\sqrt{67})}{2} \approx 9.093$.

For the metric splittable HCFL problem with uniform capacities case, Korupolu et al. [35] introduced the first constant-factor $(8 + \varepsilon)$-approximation using the local search technique. Chudak and Williamson [121] later refined this technique, improving the approximation ratio to $6 + \varepsilon$. Charikar and Guha [31] applied the scaling technique, achieving an approximation ratio of $3 + 2\sqrt{2} + \varepsilon \approx 5.828 + \varepsilon$. Aggarwal et al. [122] further improved this factor to $3 + \varepsilon$ based on the local search technique, which is currently the best approximation ratio. While most approximation algorithms rely on local search, Grover et al. [123] proposed an LP-rounding-based algorithm that achieves a constant-factor $O(\frac{1}{\varepsilon})$-approximation based on a natural LP program, but violating the capacity of a factor $1 + \varepsilon$.

For the metric splittable HCFL problem with uniform open costs case, Levi et al. [92] proposed a 5-approximation algorithm using the LP-rounding technique. Aardal et al. [124] later developed a combinatorial approximation algorithm with a ratio of $4.562 + \varepsilon$. Subsequently, Kao [125] introduced a two-stage LP-rounding algorithm, further refining the approximation ratio to 4. The current best approximation ratio is $\sqrt{3} + 2 + \varepsilon \approx 3.732 + \varepsilon$, obtained by Dabas et al. [126] using the local search technique. Moreover, Miao and Yuan [127] extended the result of [92] and proposed a $(\frac{R+\sqrt{R^2+8R}}{2} + 3)$-approximation algorithm, where $R = \frac{\max_{i \in \mathcal{F}_{f_i}}}{\min_{i \in \mathcal{F}_{f_i}}}$. When the open cost is uniform, their algorithm achieves a ratio of 5, matching the result of Levi et al. [92].

For ease of understanding, we organize the approximation ratios of the algorithms for the metric HCFL problem and the methods used in Table 7.

Table 7. The related results for the metric HCFL problem.

Problem	Ratio	Reference	Methods
Unsplittable general case	$(9, 4)$	Shmoys et al. [27]	Combinatorial method
Unsplittable uniform capacities case	$(O(\log n), 1+\varepsilon)$	Bateni [112]	Combinatorial method
	$(9, \frac{3}{2})$	Behsaz et al. [113]	Combinatorial method
	$(29.315, \frac{4}{3})$	Behsaz et al. [113]	Combinatorial method
Splittable general case	$8.532+\varepsilon$	Pál et al. [115]	Local search
	$7.873+\varepsilon$	Mahdian and Pál [91]	Local search
	$5.828+\varepsilon$	Zhang et al. [99]	Local search
	5	Banasal et al. [116]	Local search
	288	An et al. [119]	LP rounding
	9.093	Kao [120]	LP rounding
	$\frac{R+\sqrt{R^2+8R}}{2}+3$ [1]	Miao and Yuan [127]	LP rounding
Splittable uniform capacities case	$8+\varepsilon$	Korupolu et al. [35]	Local search
	$6+\varepsilon$	Chudak and Williamson [121]	Local search
	$5.828+\varepsilon$	Charikar and Guha [31]	Combinatorial method
	$3+\varepsilon$	Aggarwal et al. [122]	Local search
	$(O(\frac{1}{\varepsilon}), 1+\varepsilon)$	Grover et al. [123]	LP rounding
Splittable uniform open costs case	5	Levi et al. [92]	LP rounding
	$4.562+\varepsilon$	Aardal et al. [124]	Combinatorial method
	4	Kao [125]	LP rounding
	$3.732+\varepsilon$	Dabas et al. [126]	Local search

[1] R is the ratio of maximum open cost to minimum open cost, i.e., $R = \frac{\max_{i\in\mathcal{F}} f_i}{\min_{i\in\mathcal{F}} f_i}$.

Open problem 6. *Design an LP-based rounding algorithm for the metric splittable HCFL problem with general case with an approximation ratio no more than 9.093.*

3.1. The Prize-Collecting HCFL Problem

The prize-collecting HCFL (PC-HCFL) problem is a variant of the HCFL problem. The prize-collecting Uni-FL is exactly the prize-collecting HCFL problem when the cost function is expressed as Equation (11). By introducing variable x_{ij} to indicate the proportion of client j's demand that is served by facility i, variable y_i to indicate whether facility $i \in \mathcal{F}$ is open or not, and variable z_S to indicate whether client set $S \subseteq \mathcal{D}$ is penalized or not, we can obtain the mixed-integer linear program of the splittable prize-collecting HCFL problem.

$$
\begin{aligned}
PC\text{-}HCFL: \min \quad & \sum_{i\in\mathcal{F}} f_i y_i + \sum_{i\in\mathcal{F}}\sum_{j\in\mathcal{D}} c_{ij} d_j x_{ij} + \sum_{S\subseteq\mathcal{D}} h(S) z_S \\
\text{s.t.} \quad & \sum_{i\in\mathcal{F}} x_{ij} + \sum_{S\subseteq\mathcal{D}: j\in S} z_S \geq 1, \quad \forall j \in \mathcal{D}, \\
& \sum_{j\in\mathcal{D}} d_j x_{ij} \leq u_i y_i, \quad \forall i \in \mathcal{F}, \\
& x_{ij} \in [0,1], \quad \forall i \in \mathcal{F}, j \in \mathcal{D}, \\
& y_i, z_S \in \{0,1\}, \quad \forall i \in \mathcal{F}, S \subseteq \mathcal{D},
\end{aligned}
$$

When the penalty function $h(\cdot)$ is submodular, few results are available in the literature. When $h(\cdot)$ is modular, the problem is referred to as the HCFL problem with linear penalties (HCFLPLP). For the general case, Gupta and Gupta [128] proposed a $(8.532+\varepsilon)$-approximation using the local search technique.

For the uniform capacities case, Gupta and Gupta [128] developed a $(5.83+\varepsilon)$-approximation algorithm. Bansal et al. [129] provided a $(3+\varepsilon)$-approximation algorithm based on the local search technique. Dabas and Gupta [130] utilized the LP-rounding technique to design a polynomial-time algorithm achieving a constant factor of $O(\frac{1}{\varepsilon})$, while

violating the capacities by a factor of at most $1+\varepsilon$. Moreover, Dabas and Gupta [130] investigated a more general model, the uniform capacitated k-facility location problem with penalties (UCk-FLPP), and proposed two constant-approximation algorithms based on the LP-rounding technique. The first algorithm achieves an $O(\frac{1}{\varepsilon})$-approximation, violating the capacities by a factor of at most $1+\varepsilon$ and cardinality by a factor of at most 2. The second algorithm achieves an $O(\frac{1}{\varepsilon^2})$-approximation, violating the capacities by a factor of at most $2+\varepsilon$.

For the uniform open costs case, Lv and Wu [131] proposed a 5.732-approximation algorithm based on the LP-rounding technique. Due to the challenges associated with linear penalties under hard capacity constraints, Miao et al. [132] introduced the hard capacitated uniform facility location problem with soft penalties (HCUFLPSP), where the demand of each client can be fractionally rejected by incurring a penalty cost. They proposed a 5.5122-approximation algorithm using the LP-rounding technique.

For ease of understanding, we organize the approximation ratios of the algorithms for the metric splittable prize-collecting HCFL problem (and its variants) and the methods used in Table 8.

Open problem 7. *Design a constant approximation algorithm for the metric splittable prize-collecting HCFL problem with submodular penalty function.*

Table 8. The related results of the metric splittable prize-collecting HCFL problem and its variants when $h(\cdot)$ is modular.

Problem	Ratio	Reference	Methods
General case	$8.532+\varepsilon$	Gupta and Gupta [128]	Local search
Uniform capacities case	$5.83+\varepsilon$	Gupta and Gupta [128]	Local search
	$3+\varepsilon$	Bansal et al. [129]	Local search
	$(O(\frac{1}{\varepsilon}), 1+\varepsilon)$	Dabas and Gupta [130]	LP rounding
Uniform open costs case	5.732	Lv and Wu [131]	LP rounding
UCk-FLPP	$(O(\frac{1}{\varepsilon}), 1+\varepsilon)$ [1]	Dabas and Gupta [130]	LP rounding
	$(O(\frac{1}{\varepsilon^2}), 2+\varepsilon)$	Dabas and Gupta [130]	LP rounding
HCUFLPSP	5.5122	Miao et al. [132]	LP rounding

[1] Cardinality of facilities violating at most a factor of 2, i.e., use at most $2k$ facilities.

3.2. The Robust HCFL Problem

The robust HCFL (R-HCFL) problem represents another variant of the HCFL problem with a robust set, and extends the HCFL problem by allowing a subset of clients to remain unconnected without incurring any penalty cost, provided that the total number of unconnected clients does not exceed a given threshold $L \geq 0$. In addition, the robust Uni-FL problem is exactly the HCFL problem when the cost function is expressed as the equality (11). By introducing variable x_{ij} to indicate the proportion of client j's demand that is served by facility i, variable y_i to indicate whether facility i is opened or not, and variable r_j to indicate whether client j is an outlier or not, we can obtain the mixed-integer linear program of the splittable robust HCFL problem.

$$
\begin{aligned}
R\text{-}HCFL: \min \quad & \sum_{i\in\mathcal{F}} f_i y_i + \sum_{i\in\mathcal{F}}\sum_{j\in\mathcal{D}} c_{ij} d_j x_{ij} \\
\text{s.t.} \quad & \sum_{i\in\mathcal{F}} x_{ij} + r_j \geq 1, \quad \forall j \in \mathcal{D}, \\
& \sum_{j\in\mathcal{D}} d_j x_{ij} \leq u_i y_i, \quad \forall i \in \mathcal{F}, \\
& \sum_{j\in\mathcal{D}} r_j \leq L, \\
& x_{ij} \in [0,1], \quad \forall i \in \mathcal{F}, j \in \mathcal{D}, \\
& y_i, r_j \in \{0,1\}, \quad \forall i \in \mathcal{F}, j \in \mathcal{D}.
\end{aligned}
$$

Few results are available in the literature for the general case. For the uniform capacities case, Dabas and Gupta [130] utilized the LP-rounding framework to devise a constant-factor $O(\frac{1}{\varepsilon^2})$-approximation algorithm. This algorithm ensures that the maximum capacity violation is bounded by a factor of $1+\varepsilon$ while incurring an outlier loss factor of $1+\varepsilon$.

For the uniform open costs case, Dabas et al. [126] proposed the first constant-factor $(6.372+\varepsilon)$-approximation algorithm, using the local search technique that requires only two operations.

For ease of understanding, we organize the approximation ratios of the algorithms for metric splittable robust HCFL problem and the methods used in Table 9.

Table 9. The related results of metric splittable robust HCFL problem.

Problem	Ratio	Reference	Methods
Uniform capacities case	$(O(\frac{1}{\varepsilon^2}), 1+\varepsilon)$ [1]	Dabas and Gupta [130]	LP rounding
Uniform open costs case	$6.372+\varepsilon$	Dabas et al. [126]	Local search

[1] Cardinality of outliers violating at most a factor of $(1+\varepsilon)$, i.e., at most $(1+\varepsilon)L$ outliers.

Open problem 8. *Design a constant approximation algorithm for the metric splittable robust HCFL problem with the general case.*

3.3. The k-HCFL Problem

The k-HCFL problem, referred to as the hard capacitated k-facility location problem, is an important variant of the HCFL problem. The k-Uni-FL problem is exactly the k-HCFL problem when the cost function is expressed as (11). By introducing variable x_{ij} to indicate the proportion of client j's demand that is served by facility i, and variable y_i to indicate whether facility i is opened or not, we can obtain the mixed-integer linear program of the splittable k-HCFL problem as follows.

$$
\begin{aligned}
k\text{-}HCFL: \min \quad & \sum_{i\in\mathcal{F}} f_i y_i + \sum_{i\in\mathcal{F}}\sum_{j\in\mathcal{D}} c_{ij} d_j x_{ij} \\
\text{s.t.} \quad & \sum_{i\in\mathcal{F}} x_{ij} \geq 1, \quad \forall j \in \mathcal{D}, \\
& \sum_{j\in\mathcal{D}} d_j x_{ij} \leq u_i y_i, \quad \forall i \in \mathcal{F}, \\
& \sum_{i\in\mathcal{F}} y_i \leq k, \\
& x_{ij} \in [0,1], \quad \forall i \in \mathcal{F}, j \in \mathcal{D}, \\
& y_i \in \{0,1\}, \quad \forall i \in \mathcal{F},
\end{aligned}
$$

For the general case of the k-HCFL problem, Mišković and Stanimirović [133] proposed an efficient heuristic algorithm based on the variable domain search to solve this problem. Jiang et al. [134] proposed a $(20 + \frac{5}{n})$-approximation algorithm based on the primal–dual technique algorithm, violating the capacitated constraints by no more than a factor of 25, where $n = |\mathcal{D}|$.

For the uniform capacities case, Byrka et al. [135] developed an $O(\frac{1}{\varepsilon^2})$-approximation algorithm using the dependent rounding technique, which violates the capacity constraints by a factor of at most $2 + \varepsilon$. Grover et al. [123] proposed a constant-factor $O(\frac{1}{\varepsilon})$-approximation algorithm based on LP rounding, violating the capacity by a factor of at most $1 + \varepsilon$ while using at most $(\frac{2}{1+\varepsilon})k$ facilities. Han et al. [136] introduced an approximation algorithm with a ratio of $6(1 + \frac{1}{\gamma}) + \varepsilon$ based on the local search technique, using at most $(3 + \alpha)k$ facilities, where $\gamma \in (\frac{6}{7}, 1)$ and $\alpha > \frac{2+2\gamma}{1-\gamma}$. Kong and Zhang [73] proposed a two-step sampling-based algorithm, demonstrating that the k-HCFL problem can be approximated by a factor of $3 + \varepsilon$ in fixed-parameter tractable time.

For the uniform open costs case, Aardal et al. [124] established that any α-approximation algorithm for the uncapacitated k-median problem ([34,137]) can be extended to achieve a $(1 + 2\alpha)$-approximation algorithm for the k-HCFL problem with uniform open costs, using at most $2k$ facilities. They further obtained a $(7 + \varepsilon)$-approximation guarantee, using at most $2k$ facilities.

For the case with both uniform open costs and uniform capacities, Aardal et al. [124] achieved a $(7 + \varepsilon)$-approximation guarantee, using at most $2k - 1$ facilities.

For ease of understanding, we organize the approximation ratios of the algorithms for the metric splittable k-HCFL problem and the methods used in Table 10.

Table 10. The related results for the metric splittable k-HCFL problem.

Problem	Ratio	Reference	Methods	Facilities used
General case	$((20 + \frac{5}{n}), 25)$ [1]	Jiang et al. [134]	Primal–dual	k
Uniform capacities case	$(O(\frac{1}{\varepsilon^2}), 2 + \varepsilon)$	Byrka et al. [135]	Dependent rounding	k
	$(O(\frac{1}{\varepsilon}), 1 + \varepsilon)$	Grover et al. [123]	LP rounding	$(\frac{2}{1+\varepsilon})k$
	$6(1 + \gamma) + \varepsilon$ [2]	Han et al. [136]	Local search	$(3 + \alpha)k$
	$3 + \varepsilon$	Kong and Zhang [73]	Combinatorial method	k
Uniform open costs case	$7 + \varepsilon$	Aardal et al. [124]	Combinatorial method	$2k$
Uniform capacities and open costs case	$7 + \varepsilon$	Aardal et al. [124]	Combinatorial method	$2k - 1$

[1] n is the number of clients, i.e., $n = |D|$. [2] γ and α are the parameters, where $\gamma \in (\frac{6}{7}, 1)$ and $\alpha > \frac{2+2\gamma}{1-\gamma}$.

Open problem 9. *Design a constant approximation algorithm for the splittable k-HCFL problem with the general case.*

3.4. The k-Level HCFL Problem

The k-level HCFL (kL-HCFL) problem, also referred to as the multi-level HCFL problem, is a variant of the HCFL problem, where k denotes the number of facility levels. The k-level Uni-FL problem is exactly the k-level HCFL problem when the cost function is expressed as Equation (11). By introducing variable x_{pj} to represent the proportion of the client j's demand served by the facilities in the path p, and variable y_i to indicate whether facility i is opened or not, we can obtain the mixed-integer linear program of the splittable k-level HCFL problem as follows.

$$
\begin{aligned}
kL\text{-}HCFL: \min \quad & \sum_{i\in\mathcal{F}} f_i y_i + \sum_{p\in\mathcal{P}} \sum_{j\in\mathcal{D}} c_{pj} d_j x_{pj} \\
\text{s.t.} \quad & \sum_{p\in\mathcal{P}} x_{pj} \geq 1, \quad \forall j \in \mathcal{D}, \\
& \sum_{p\in\mathcal{P}: i\in p} \sum_{j\in\mathcal{D}} d_j x_{pj} \leq u_i y_i, \quad \forall i \in \mathcal{F}, \\
& x_{pj} \in [0,1], \quad \forall p \in \mathcal{P}, j \in \mathcal{D}, \\
& y_i \in \{0,1\} \quad \forall i \in \mathcal{F}.
\end{aligned}
$$

For the splittable k-level HCFL problem, Chen and Wang [138] developed a cost-varied Vogel-based approximation method, namely the dynamic Vogel approximation method, which is a heuristic method, to solve this problem. Du et al. [139] proposed the first combinatorial approximation algorithm with a ratio of $(k+2+\sqrt{k^2+2k+5}+\varepsilon)$.

For ease of understanding, we organize the approximation ratios of the algorithms for splittable k-level HCFL problem and the methods used in Table 11.

Table 11. The related results of splittable k-level HCFL problem.

Ratio	Reference	Methods
– [1]	Chen and Wang [138]	Dynamic Vogel approximation method
$(k+2+\sqrt{k^2+2k+5}+\varepsilon)$	Du et al. [139]	Combinatorial method

[1] This is a heuristic algorithm, so it has no approximation ratio.

Open problem 10. *Design a constant approximation algorithm for the splittable k-level HCFL problem or prove that such an algorithm does not exist.*

3.5. Other Variants of the HCFL Problem

Some other variants of the HCFL problem have also been investigated in the literature. Ageev et al. [140] studied the network HCFL problem, in which clients and facilities were located at the vertices of a given edge-weighted transportation network graph. Based on the dynamic algorithm proposed by Ageev et al. [141] for the UFL problem on path graphs, they improved the multi-complexity to $O(m^2n^2)$, where $m = |\mathcal{F}|$, $n = |\mathcal{D}|$.

For the online hard capacitated facility location problem, Cygan et al. [142] demonstrated that any online algorithm for the UFL problem with arbitrary (e.g., non-uniform) capacities or arbitrary (e.g., non-uniform) open costs has an unbounded worst-case competitive ratio. Consequently, they focused on the case with uniform capacities and uniform open costs, achieving an optimal competitive ratio of $O(\frac{\log n}{\log\log n})$ with no deletions, where n is the length of the sequence. Additionally, they proposed an online algorithm with a competitive ratio of $O(\log m + \log u \log n)$ in the fully dynamic model with deletions, under the uniform capacities assumption, where m is number of points in the input metric and u is the capacity of each facility.

4. The Soft Capacitated Facility Location Problem

The soft capacitated facility location (SCFL) problem is a special case of the HCFL problem. When each facility $i \in \mathcal{F}$ can be opened an unlimited number of times, i.e., $b_i = \sum_{j\in\mathcal{D}} d_j$, the HCFL problem is exactly the SCFL problem. By introducing variable x_{ij} to indicate whether the demand of client j is served by facility i, and variable y_i to indicate the number of times facility i is opened, the integer linear program of the unsplittable SCFL problem can be written as follows.

$$
\begin{aligned}
SCFL: \min \quad & \sum_{i\in\mathcal{F}} f_i y_i + \sum_{i\in\mathcal{F}}\sum_{j\in\mathcal{D}} c_{ij} d_j x_{ij} \\
\text{s.t.} \quad & \sum_{i\in\mathcal{F}} x_{ij} \geq 1, \quad \forall j \in \mathcal{D}, \\
& \sum_{j\in\mathcal{D}} d_j x_{ij} \leq u_i y_i, \quad \forall i \in \mathcal{F}, \\
& x_{ij} \in \{0,1\}, \quad \forall i \in \mathcal{F}, j \in \mathcal{D}, \\
& y_i \in \mathbb{Z}_+ \cup \{0\}, \quad \forall i \in \mathcal{F}.
\end{aligned}
$$

For the non-metric unsplittable SCFL problem, an approximation algorithm with a ratio of $2H(n)$ can be obtained based on the result of Charikar and Guha [31], which states that any α-approximation for the UFL problem yields a 2α-approximation for the SCFL problem, and the $H(n)$-approximation algorithm for the non-metric UFL problem proposed by Hochbaum [26], where $H(n) = \sum_{1\leq i\leq n} \frac{1}{i}$. Subsequently, Alfandari [143] proposed a logarithmic approximation algorithm with a ratio of $(1+\varepsilon)H(n)$ for the SCFL problem using a set-covering greedy heuristic. This approach leveraged subproblem approximations via a FPTAS based on cost scaling and dynamic programming.

For the metric unsplittable SCFL problem, Shmoys et al. [27] first introduced this problem and developed 7.62-approximation algorithm using filtering and rounding techniques. Jain and Vazirani [34] reduced the SCFL problem to the UFL problem and obtained a 4-approximation algorithm for the SCFL problem through the primal–dual algorithm of the UFL problem. Arya et al. [29] later improved the approximation ratio to $2+\sqrt{3}+\varepsilon \approx 3.732+\varepsilon$ based on the local search technique. Charikar and Guha [31] applied an LP-rounding technique to obtain a 3.46-approximation algorithm, concluding that any α-approximation algorithm for the UFL problem yields a 2α-approximation algorithm for the SCFL problem. Based on the techniques of Jain and Vazirani [34], Jain et al. [23] further improved the ratio to 3. Moreover, Jain et al. [66] applied dual-fitting and factor-revealing techniques, demonstrating that any LMP α-approximation algorithm for the UFL problem yields a $(\alpha+1)$-approximation algorithm for the SCFL problem, thereby achieving an approximation ratio of 3. Mahdian et al. [144] showed that any bifactor (r_f, r_c)-approximation algorithm for the UFL problem implies a $(r_f + r_c)$-approximation algorithm for the SCFL problem. They derived a 2.89-approximation algorithm using a $(1.11, 1.78)$-bifactor algorithm for the UFL problem. Geunes et al. [53] proposed a 2.542-approximation algorithm for the SCFL problem. Later, Mahdian et al. [145] combined the reduction technique with the dual-fitting technique, reducing the SCFL problem to the UFL problem and improving the ratio to 2, which is currently the best ratio for the SCFL problem. Moreover, their analysis was shown to be tight, as the linear program relaxation of this problem has an integer gap of 2.

For the metric unsplittable SCFL problem with uniform capacities case, Chudak and Shmoys [146] modified the clustering method in the rounding process and randomly selected open facilities of Shmoys et al. [27], achieving a 3-approximation algorithm.

Similarly, in the splittable case, let variable x_{ij} denote the proportion of client $j \in \mathcal{D}$'s demand served by facility i, which is in $[0, 1]$.

$$
x_{ij} \in [0,1], \quad \forall i \in \mathcal{F}, j \in \mathcal{D}.
$$

For the splittable SCFL problem, the only known result was provided by Shmoys et al. [27], who developed a 5.69-approximation algorithm. Since then, most research efforts have focused on refining approximation algorithms for the unsplittable SCFL problem.

For ease of understanding, we organize the approximation ratios of the algorithms for the unsplittable SCFL problem and the methods used in Table 12.

Table 12. The related results for the unsplittable SCFL problem.

Problem	Ratio	Reference	Methods
General case in non-metric space	$2H(n)$	Using the conclusion of Charikar and Guha [31]	
	$(1+\varepsilon)H(n)$	Alfandari [143]	Using a set covering greedy heuristic
General case in metric space	7.62	shmoys et al. [27]	Combinatorial method
	4	Jain and Vazirani [34]	Combinatorial method
	$3.732+\varepsilon$	Arya et al. [29]	Local search
	3.46	Charikar and Guha [31]	LP rounding
	3	Jain et al. [23]	Combinatorial method
	3	Jain et al. [66]	Combinatorial method
	2.89	Mahdian et al. [144]	Reduction method
	2.542	Geunes et al. [53]	Combinatorial method
	2	Mahdian et al. [145]	Combinatorial method
Uniform capacities case in the metric space	3	Chudak and Shmoys [146]	LP rounding

Open problem 11. *Design a polynomial-time algorithm with an approximation ratio lower than 2 for the unsplittable SCFL problem.*

4.1. The Prize-Collecting SCFL Problem

The prize-collecting SCFL (PC-SCFL) problem is a variant of the SCFL problem. The prize-collecting HCFL is exactly the SCFL problem when $b_i = \sum_{j\in\mathcal{D}} d_j$. By introducing variable x_{ij} to represent whether client $j \in \mathcal{D}$ is connected to facility $i \in \mathcal{F}$ or not, variable y_i to represent the number of times facility $i \in \mathcal{F}$ is opened, and variable z_S to represent whether the client set S is penalized or not, the integer program of the unsplittable prize-collecting SCFL problem can be expressed as follows.

$$
\begin{aligned}
PC\text{-}SCFL: \min \quad & \sum_{i\in\mathcal{F}} f_i y_i + \sum_{i\in\mathcal{F}}\sum_{j\in\mathcal{D}} c_{ij} d_j x_{ij} + \sum_{S\subseteq\mathcal{D}} h(S) z_S \\
\text{s.t.} \quad & \sum_{i\in\mathcal{F}} x_{ij} + \sum_{S\subseteq\mathcal{D}: j\in S} z_S \geq 1, \quad \forall j \in \mathcal{D}, \\
& \sum_{j\in\mathcal{D}} d_j x_{ij} \leq u_i y_i, \quad \forall i \in \mathcal{F}, \\
& x_{ij}, z_S \in \{0,1\}, \quad \forall i \in \mathcal{F}, S \subseteq \mathcal{D}, \\
& y_i \in \mathbb{Z}_+ \cup \{0\}, \quad \forall i \in \mathcal{F}.
\end{aligned}
$$

Research on the prize-collecting SCFL problem remains relatively limited. When the penalty function $h(\cdot)$ is modular, this problem is also referred to as the SCFL problem with linear penalties. Zhu et al. [147] proposed a 2-approximation algorithm for this problem through a reduction technique and demonstrated that the algorithm reaches the integer gap of its standard linear program relaxation. Moreover, Li et al. [148] proposed the lower-bound facility location problem with linear penalties and soft capacity constraints, in which each open facility must serve at least $B \geq 0$ clients. They developed an approximation algorithm with a ratio of $2\frac{1+\alpha}{1-\alpha}\rho$, where ρ is the approximation ratio of the prize-collecting UFL problem when $h(\cdot)$ is modular, and $\alpha \in [\frac{1}{3}, 1)$.

Open problem 12. *Design a 2-approximation algorithm for the metric unsplittable prize-collecting SCFL problem when $h(\cdot)$ is submodular.*

4.2. The Robust SCFL Problem

The robust SCFL (R-SCFL) problem represents another variant of the SCFL problem with a robust set, and extends the SCFL problem by allowing a subset of clients to remain unconnected without incurring any penalty cost, provided that the total number of unconnected clients does not exceed a given threshold $L \geq 0$. The robust HCFL problem is exactly the robust SCFL problem when $b_i = \sum_{j\in\mathcal{D}} d_j$. By introducing variable x_{ij} to represent whether client $j \in \mathcal{D}$ is connected to facility $i \in \mathcal{F}$ or not, variable y_i to indicate the number of times facility i is opened, and variable r_j to indicate whether client j is an outlier or not, we can obtain the integer linear program of the unsplittable robust SCFL problem.

$$
\begin{aligned}
R\text{-}SCFL: \min \quad & \sum_{i\in\mathcal{F}} f_i y_i + \sum_{i\in\mathcal{F}}\sum_{j\in\mathcal{D}} c_{ij} d_j x_{ij} \\
\text{s.t.} \quad & \sum_{i\in\mathcal{F}} x_{ij} + r_j \geq 1, \quad \forall j \in \mathcal{D}, \\
& \sum_{j\in\mathcal{D}} d_j x_{ij} \leq u_i y_i, \quad \forall i \in \mathcal{F}, \\
& \sum_{j\in\mathcal{D}} r_j \leq L, \\
& x_{ij} \in \{0,1\}, \quad \forall i \in \mathcal{F}, j \in \mathcal{D}, \\
& y_i \in \mathbb{Z}_+ \cup \{0\}, \quad \forall i \in \mathcal{F}, \\
& r_j \in \{0,1\}, \quad \forall j \in \mathcal{D}.
\end{aligned}
$$

There is limited research on this problem.

Open problem 13. *Design a constant approximation algorithm for the metric unsplittable robust SCFL problem.*

4.3. The k-SCFL Problem

The k-SCFL problem, referred to as the soft capacitated k-facility location problem, is an important variant of the SCFL problem, where the number of facilities that can be opened is restricted to at most k. The k-HCFL problem is exactly the k-SCFL problem when $b_i = \sum_{j\in\mathcal{D}} d_j$. By introducing variable x_{ij} to represent whether client j is connected to facility i or not, and variable y_i to indicate the number of times facility i is opened, we can obtain the mixed-integer linear program of the unsplittable k-SCFL problem as follows.

$$
\begin{aligned}
k\text{-}SCFL: \min \quad & \sum_{i\in\mathcal{F}} f_i y_i + \sum_{i\in\mathcal{F}}\sum_{j\in\mathcal{D}} c_{ij} d_j x_{ij} \\
\text{s.t.} \quad & \sum_{i\in\mathcal{F}} x_{ij} \geq 1, \quad \forall j \in \mathcal{D}, \\
& \sum_{j\in\mathcal{D}} d_j x_{ij} \leq u_i y_i, \quad \forall i \in \mathcal{F}, \\
& \sum_{i\in\mathcal{F}} y_i \leq k, \\
& x_{ij} \in \{0,1\}, \quad \forall i \in \mathcal{F}, j \in \mathcal{D}, \\
& y_i \in \mathbb{Z}_+ \cup \{0\}, \quad \forall i \in \mathcal{F}.
\end{aligned}
$$

For the unsplittable k-SCFL problem, Jiang et al. [134] proposed a $(20+\frac{5}{n})$-approximation algorithm based on the primal–dual technique of the SCFL problem and the use of the combinatorial structure of the fractional solution, violating the capacitated constraints by no more than a factor of 25, where $n=|\mathcal{D}|$.

Open problem 14. *Design a constant approximation algorithm for the unsplittable k-SCFL problem.*

4.4. The k-Level SCFL Problem

The k-level SCFL (kL-SCFL) problem, also referred to as the multi-level SCFL problem, is a variant of the SCFL problem. The k-level HCFL problem is exactly the k-level SCFL problem when $b_i=\sum_{j\in\mathcal{D}} d_j$. Let variable y_i represent the number of times facility i is opened and variable x_{pj} represent whether client j is connected to path p; $x_{pj}=1$ if client j is connected to path p; $x_{pj}=0$ otherwise. The integer program of the unsplittable k-level SCFL problem can be expressed as follows.

$$\begin{aligned} kL\text{-}SCFL:\min \quad & \sum_{i\in\mathcal{F}} f_i y_i+\sum_{p\in\mathcal{P}}\sum_{j\in\mathcal{D}} c_{pj}d_j x_{pj} \\ \text{s.t.} \quad & \sum_{p\in\mathcal{P}} x_{pj}\geq 1, \quad \forall j\in\mathcal{D}, \\ & \sum_{p\in\mathcal{P}:i\in p}\sum_{j\in\mathcal{D}} d_j x_{pj}\leq u_i y_i, \quad \forall i\in\mathcal{F}, \\ & x_{pj}\in\{0,1\}, \quad \forall p\in\mathcal{P}, j\in\mathcal{D}, \\ & y_i\in\mathbb{Z}_+\cup\{0\}, \quad \forall i\in\mathcal{F}. \end{aligned}$$

For the unsplittable k-level SCFL problem, Bumb and Kern [75] proposed a 12-approximation algorithm by leveraging the 6-approximation algorithm for the k-level UFL problem. They implicitly demonstrated that any α-approximation algorithm for the k-level UFL problem can be extended to a 2α-approximation algorithm for the k-level SCFL problem. Ageev [149] employed the reduction method combined with the Lagrangian multiplier-preserving technique, achieving a 9-approximation algorithm. Furthermore, based on the findings of Bumb and Kern [75] and the 3-approximation algorithm for the k-level UFL problem proposed by Aardal et al. [77], a 6-approximation algorithm for the k-level SCFL problem can be derived. Wu and Xu [150] introduced a $(\frac{\ln(\frac{1}{\beta})}{1-\beta}, 1+\frac{2}{1-\beta})$-bifactor approximation algorithm for the k-level UFL problem. By setting $\beta=0.2222$, they obtained a 5.5053-approximation algorithm for the k-level SCFL problem. For the special case of $k=2$, Wu et al. [151] developed an approximation algorithm with a ratio of $4+\frac{1}{e-1}+\varepsilon\approx 4.582+\varepsilon$, employing a primal–dual technique by constructing the corresponding uncapacitated case. Zhang [80] proposed a 3-approximation algorithm for the 2-level SCFL problem based on bifactor reduction.

Moreover, Wu et al. [152] introduced the stochastic multi-level facility location problem with soft capacities (SML-FLPSC) and designed a $(\frac{1}{\alpha}+\frac{6}{1-2\alpha})$-approximation algorithm by reducing this problem to the stochastic multi-level facility location (SML-FL) problem, for which they developed a $(\frac{1}{\alpha}, \frac{6}{1-2\alpha})$-bifactor approximation algorithm for the SML-FL problem, where $\alpha\in(0,0.5)$. Setting $\alpha=0.183$ resulted in an approximation ratio of 14.9282.

For ease of understanding, we organize the approximation ratios of the algorithms for metric splittable k-level SCFL problem (and its variants) and the methods used in Table 13.

Table 13. The related results of metric splittable k-level SCFL problem and its variants.

Problem	Ratio	Reference	Methods
k-level SCFL	12	Bumb and Kern [75]	Combinatorial method
	9	Ageev [149]	Combinatorial method
	6	Using the conclusion of [75] and the algorithm of [77]	
	5.5053	Wu et al. [150]	Reduction method
2-level SCFL	$4.582+\varepsilon$	Wu et al. [151]	Primal–dual
	3	Zhang [80]	Combinatorial method
SML-FLPSC	14.9282	Wu et al. [152]	Reduction method

Open problem 15. *Design an approximation algorithm with the ratio less than 5.5053 for unsplittable k-level SCFL problem.*

4.5. Other Variants of the SCFL Problem

Numerous other variants of the SCFL problem have been explored, including the SCFL and cable installation problem, the squared metric SCFL problem, and the online SCFL problem, among others. Below, we provide a brief overview of these variants and their associated results.

The SCFL and cable installation problem is referred to as the soft capacitated facility location and cable installation (SC-FLCI) problem, which is to design a minimum cost network to serve client demands by opening facilities for service provision and installing cables for service shipment, where both facilities and cables are associated with capacity constraints. Chen and Chen [153] first introduced this problem as an extension of the work by Ravi and Sinha [154] on the capacitated-cable facility location problem. They developed a 19.84-approximation algorithm for the SC-FLCI problem for the splittable case and a 28.06-approximation algorithm for the unsplittable case, utilizing an integration of the primal–dual scheme, Lagrangian relaxation, demand clustering, and bifactor approximation.

The squared metric SCFL problem is referred to as the squared metric soft capacitated facility location (SMSCFL) problem, which removes the standard assumption that the unit connection cost is proportional to the distance between a client and its assigned facility. Instead, it considers connection costs that are proportional to the square of the distance measure. Xu et al. [41] proposed an $(11.18+\varepsilon)$-approximation algorithm for the SMSCFL problem based on the local search technique. Subsequently, Han et al. [155] improved the approximation ratio to 10.

The τ-relaxed SCFL problem is a generalization of several SCFL problems, including the SMSCFL problem and the SCFL problem, and was introduced by Han et al. [155]. Inspired by the work of Jain and Vazirani [34], they developed a $(3\tau+1)$-approximation algorithm, where the connection cost satisfied the τ-relaxed triangle inequality, a relaxation of the triangle inequality. This relaxed inequality was originally proposed by Fernandes et al. [156]. The specific definition is as follows.

Definition 9 ([156]). *Given a set of facilities $\mathcal{F}$, a set of clients $\mathcal{D}$, and a parameter τ, the connection cost satisfies the τ-relaxed triangle inequality if, for any $i,i' \in \mathcal{F}$ and $j,j' \in \mathcal{D}$, the following condition is satisfied:*

$$c_{i'j} \leq \tau(c_{i'j'} + c_{ij'} + c_{ij}).$$

The online SCFL problem is referred to as the online soft capacitated facility location (O-SCFL) problem, where there is a set of clients that are unknown in advance and arrive at once. Thang [157] proposed an $O(\log n + \frac{\lambda}{1-\mu})$-competitive algorithm based on the primal–dual technique, where $n = |\mathcal{D}|$, and the quantity $\frac{\lambda}{1-\mu}$ describes the gap between the

facility open cost and the linearity. Han et al. [158] later developed a dual-fitting algorithm achieving an $O(\log n)$-competitive ratio.

In addition to the aforementioned variants, Gabor and van Ommeren [159] studied the facility location problem with subadditive costs. They devised a $2(1+\varepsilon)$-approximation algorithm by being $(1+\varepsilon, 1)$-reduced to the SCFL problem. For a special subclass of subadditive functions, they obtained a 2-approximation algorithm by reducing it to a linear-cost facility location problem. Cai et al. [160] introduced a random soft capacity model, referred to as the two-stage soft capacitated facility location problem, and developed a 3.72-approximation algorithm by simplifying it into the linear-cost version of the two-stage facility location problem and the dynamic facility location problem. Li and Xu [161] proposed the soft capacitated facility location game, designing a 6-approximation cross-monotonic cost-sharing method.

5. Conclusions

In this paper, we conduct a comprehensive investigation into the Uni-FL problem, which extends both the capacitated facility location problem (including the HCFL problem and the SCFL problem) and the UFL problem. The following are our main works: (1) investigate the Uni-FL problem and its special cases, the HCFL problem and the SCFL problem, and give their models and mathematical formulas; (2) summarize existing algorithms and relevant results for the Uni-FL problem and its special cases, organizing their variants alongside the corresponding approximation algorithms, as shown in Figure 2; (3) propose some open questions. The purpose of our research is to provide a valuable reference for researchers and engineers working on practical problems.

Overall, this survey provides a comprehensive overview of the latest algorithms and results for the Uni-FL problem while also highlighting the need for further research in this field. Additionally, we propose several open questions and summarize four future research directions in this field: (1) Improve all existing algorithms. (2) Investigate problems that incorporate multiple constraints simultaneously. For example, consider a problem where the number of penalized clients cannot exceed $k_1 \geq 0$, the number of selected facilities cannot exceed $k_2 \geq 0$, and the objective function includes the penalty cost, thus combining three different problems into a unified framework. (3) Inspired by [102], the concept of semi-splittable can be incorporated into the capacitated facility location problem. The demand is defined as **semi-splittable** if a client's demand can be allocated to multiple copies of one facility, but not to other facilities, as shown in Figure 3. (4) Extend existing models to weighted scenarios, as proposed in [162], rather than limiting the focus solely to cardinality constraints in robust facility location problems.

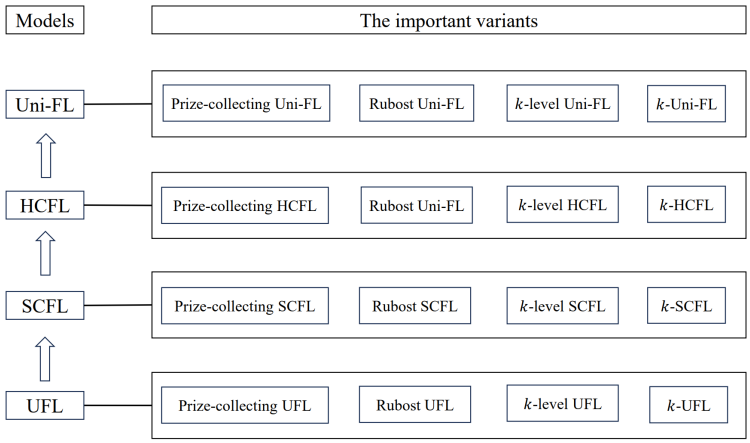


Figure 2. The framework diagram of all models in this paper and their main relationships.

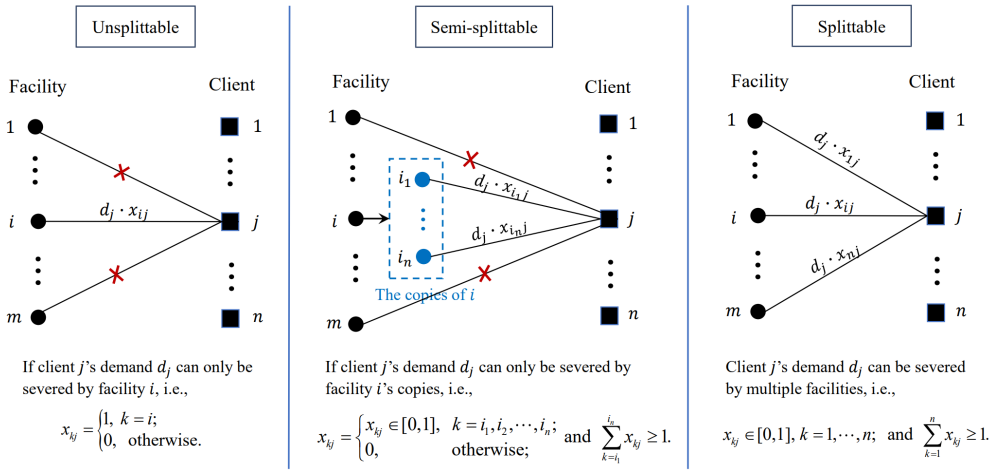


Figure 3. The three splittable cases. The red X indicates that the client cannot be served by that facility, for example, in the unsplittable case, client j cannot be served by any facility other than facility i.

Author Contributions: Conceptualization, H.X. and W.L.; methodology, H.X. and J.Z.; formal analysis, H.X.; investigation, H.X. and W.L.; resources, W.L.; writing—original draft preparation, H.X.; writing—review and editing, H.X., W.L., J.Z. and Z.Z.; visualization, H.X.; supervision, W.L. and Z.Z.; project administration, W.L.; funding acquisition, Z.Z., H.X. and J.Z. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported in part by the Postgraduate Innovation Foundation of Yunnan University [No. TM-23236819] and [No. TM-23236822] and the 16th Postgraduate Innovation Foundation of Yunnan University [No. KC-242410224].

Data Availability Statement: No new data were created or analyzed in this study.

Conflicts of Interest: All authors certify that they have no affiliations with or involvement in any organization or entity with any financial interest or non-financial interest in the subject matter or materials discussed in this manuscript.

Abbreviations

The following abbreviations are used in this manuscript:

UFL	Uncapacitated facility location
Uni-FL	Universal facility location
HCFL	Hard capacitated facility location
SCFL	Soft capacitated facility location
UFLPSP	Uncapacitated facility location problem with submodular penalties
*k*L-UFLPSP	*k*-level uncapacitated facility location problem with submodular penalties
SUFLPSP	Stochastic uncapacitated facility location problem with submodular penalties
UFLPLP	Uncapacitated facility location problem with linear penalties
*k*L-UFLPLP	*k*-level uncapacitated facility location problem with linear penalties
k-UFLPLP	Uncapacitated *k*-facility location problem with linear penalties
OPC-UFL	Online prize-collecting uncapacitated facility location
R-UFL	Robust uncapacitated facility location
PUFLPO	Priority uncapacitated facility location problem with outliers
*k*L-UFLO	*k*-level uncapacitated facility location problem with outliers
*k*L-SMUFLPO	*k*-level squared metric uncapacitated facility location problem with outliers
k-UFL	Uncapacitated *k*-facility location
SM*k*-UFL	Squared metric uncapacitated *k*-facility location
SM*k*-UFLPLP	Squared metric uncapacitated *k*-facility location problem with linear penalties
*k*L-UFL	*k*-level uncapacitated facility location
D*k*L-UFL	Dynamic *k*-level uncapacitated facility location
D*k*L-UFLPSP	Dynamic *k*-level uncapacitated facility location problem with submodular penalties
D*k*L-UFLPO	Dynamic *k*-level uncapacitated facility location problem with outliers
PC-Uni-FL	Prize-collecting universal facility location
R-Uni-FL	Robust universal facility location
k-Uni-FL	Universal k-facility location
*k*L-Uni-FL	*k*-level universal facility location
M^{p}UFLP	Universal facility location problem in the *p*-th power of metric space
PC-HCFL	Prize-collecting hard capacitated facility location
HCFLPLP	Hard capacitated facility location problem with linear penalties
UC*k*-FLPP	Uniform capacitated *k*-facility location problem with penalties
HCUFLPSP	Hard capacitated uniform facility location problem with soft penalties
k-HCFL	Hard capacitated *k*-facility location
*k*L-HCFL	*k*-level hard capacitated facility location
R-HCFL	Robust hard capacitated facility location
PC-SCFL	Prize-collecting soft capacitated facility location
R-SCFL	Robust soft capacitated facility location
k-SCFL	Soft capacitated *k*-facility location
*k*L-SCFL	*k*-level soft capacitated facility location
SML-FLPSC	Stochastic multi-level facility location problem with soft capacities
SML-FL	Stochastic multi-level facility location
SC-FLCI	Soft capacitated facility location and cable installation
SMSCFL	Squared metric soft capacitated facility location
O-SCFL	Online soft capacitated facility location

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