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Existence Results and Gap Functions for Nonsmooth Weak Vector Variational-Hemivariational Inequality Problems on Hadamard Manifolds

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Abstract: In this paper, we consider a class of nonsmooth weak vector variational-hemivariational inequality problems (abbreviated as, WVVHVIP) in the framework of Hadamard manifolds. By employing an analogous to the KKM lemma, we establish the existence of the solutions for WVVHVIP without utilizing any monotonicity assumptions. Moreover, a uniqueness result for the solutions of WVVHVIP is established by using generalized geodesic strong monotonicity assumptions. We formulate Auslender, regularized, and Moreau-Yosida regularized type gap functions for WVVHVIP to establish necessary and sufficient conditions for the existence of the solutions to WVVHVIP. In addition to this, by employing the Auslender, regularized, and Moreau-Yosida regularized type gap functions, we derive the global error bounds for the solution of WVVHVIP under the generalized geodesic strong monotonicity assumptions. Several non-trivial examples are furnished in the Hadamard manifold setting to illustrate the significance of the established results. To the best of our knowledge, this is the first time that the existence results, gap functions, and global error bounds for WVVHVIP have been investigated in the framework of Hadamard manifolds via Clarke subdifferentials.

Keywords: vector variational-hemivariational inequality; gap functions; KKM lemma; Hadamard manifolds; error bounds

MSC: 47H05; 47J20; 49J40; 49J52

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Academic Editors: Michel Chipot and Simeon Reich

Received: 20 December 2024
Revised: 9 February 2025
Accepted: 14 March 2025
Published: 18 March 2025

Citation: Upadhyay, B.B.; Sain, S.; Mishra, P.; Stancu-Minasian, I. Existence Results and Gap Functions for Nonsmooth Weak Vector Variational-Hemivariational Inequality Problems on Hadamard Manifolds. *Mathematics* **2025**, *13*, 995. https://doi.org/10.3390/math13060995

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1. Introduction

In the mathematical optimization theory, the concept of variational inequality problems was initially introduced by Hartman and Stampacchia [1]. Giannessi [2,3] introduced the vector-valued versions of Stampacchia [4] and Minty [5] variational inequality problems in the framework of Euclidean space. Due to several real-life applications of vector variational inequality problems, for instance, in traffic equilibrium problems [6], vector equilibria [7], and vector optimization problems [8], vector variational inequality problems have been extensively studied by several researchers in various frameworks (see, for instance, [9–15] and the references cited therein). Furthermore, the notion of hemivariational inequality problem was first introduced by Panagiotopoulos [16], which is based on the properties of Clarke generalized subdifferential for locally Lipschitz functions.

These problems have various applications in the field of science and engineering, such as structured analysis, nonconvex optimization, mechanics, and contact problems (see, for instance, [17,18]). As a result, hemivariational inequality problems emerged as a significant area of research (see, for instance, [19–21] and the references cited therein).

Variational-hemivariational inequality problems are a fusion of variational and hemivariational inequality problems, which incorporate both convex and nonconvex functions. The initial conceptualization and formulation of variational-hemivariational inequality problems are attributed to the work of Motreanu and Rădulescu [22]. Furthermore, these problems have several applications in mathematical modeling and contact mechanics (see, for instance, [23,24]). In recent years, the theory of variational-hemivariational inequality problems has gained significant attention from several researchers (see, for instance, [17,19,25] and the references cited therein). Tang and Huang [26] have studied existence results for variational-hemivariational inequality problems by employing the KKM theorem in reflexive Banach spaces. Moreover, Migórski et al. [27] have investigated the existence and uniqueness of the solution for variational-hemivariational inequality problems in the setting of reflexive Banach spaces.

It is well known that gap functions play an important role in the study of existence of solutions to variational inequality problems, as one can transform a variational inequality problem into an optimization problem by employing gap functions (see, [28]). Moreover, gap functions are used to derive the error bounds, which provide an upper estimate of the distance between an arbitrary point of the feasible set and the solution set of an optimization problem (see, [29,30]). The notions of gap functions and regularized gap functions for variational inequality problems have been introduced by Auslender [31] and Fukushima [32] in the setting of Euclidean space. Moreover, Yamashita and Fukushima [30] introduced Moreau-Yosida type regularized gap functions for variational inequality problems and further, derived several global error bounds for the solution of the considered problem. Li and He [33] have formulated gap functions and derived the existence results for generalized vector variational inequality problems under the monotonicity assumption on topological vector spaces. Various gap functions for vector variational inequality problems have been studied by Charitha et al. [34] in the Euclidean space setting. Furthermore, Hung et al. [35] have studied gap functions and global error bounds for variational-hemivariational inequality problems in the setting of reflexive Banach spaces. For a detailed exposition regarding gap functions and global error bounds for variational and vector variational inequality problems in various settings, we refer the readers to [29,36–39] and the references cited therein.

Over the last few decades, it has been observed that various real-life optimization problems arising in the field of engineering and science can be effectively formulated in the setting of Riemannian manifolds rather than in the Euclidean space setting, see, for instance, [40,41] and the references cited therein. The extension of optimization concepts from the Euclidean space setting to the framework of Riemannian manifolds is associated with several crucial advantages, such as one can transform nonconvex optimization problems in the setting of Euclidean space into convex optimization problems in the framework of Riemannian manifolds. Moreover, constrained optimization problems in the Euclidean space setting can be reformulated as unconstrained optimization problems in the Riemannian manifold framework (see, for instance, [42,43] and the references cited therein). Udrişte [41] introduced the notions of geodesic convex functions on Hadamard manifolds corresponding to the notions of convex functions in the setting of Euclidean space. For further details related to the extension of optimization techniques from linear spaces to Riemannian manifolds, we refer the readers to [9,13,14,44,45] and the references cited therein.

In the setting of Hadamard manifolds, Németh [46] introduced the concept of variational inequality problems and discussed the existence of solutions for the considered problem. Since then, numerous scholars have investigated variational and vector variational inequality problems in the framework of Hadamard manifolds, see, for instance, [9,13,14,38,47] and the references cited therein. Chen and Huang [10] have studied existence results for vector variational inequality problems by employing the KKM lemma in the framework of Hadamard manifolds. Moreover, Jayswal et al. [44] have studied existence results and gap functions for nonsmooth mixed vector variational inequality problems via Clarke subdifferentials. Existence results for the solution of hemivariational inequality problems have been established by Tang et al. [21] in the Hadamard manifolds setting. Further, the notions of gap functions and global error bounds for generalized mixed variational inequality problems have been studied by Li et al. [48]. Ansari et al. [49] have formulated gap functions and derived global error bounds for nonsmooth variational inequality problems in terms of bifunctions. Moroever, Hung et al. [20] have derived global error bounds for the solution of considered mixed quasi-hemivariational inequality problems in the setting of Hadamard manifolds.

It is significant to note that several authors have studied existence results for the solutions of vector variational inequality problems and hemivariational inequality problems in various frameworks (see, for instance, [10,21,26,35,44]). However, the existence results for the solution of WVVHVIP, which belongs to a broader class of vector variational as well as hemivariational inequality problems, have not been investigated before in the setting of Hadamard manifolds. Moreover, gap functions and global error bounds for variational inequality problems, vector variational inequality problems, and hemivariational inequality problems in the framework of manifolds have been studied by numerous researchers (see, for instance, [20,33–35,44,48,49] and the references cited therein). However, the notions of gap functions and global error bounds for vector variational-hemivariational inequality problems involving nonsmooth locally Lipschitz functions, in particular, WVVHVIP, have not been studied yet in the framework of Hadamard manifolds. Furthermore, it is imperative to note that Hadamard manifolds, in general, are not equipped with a linear structure. Therefore, to cope up with the challenges associated with the nonlinear structure of Hadamard manifolds, several tools from Riemannian geometry will be applied to explore existence results, gap functions, and global error bounds for WVVHVIP via Clarke subdifferentials in the Hadamard manifolds setting.

Inspired by the results established in [20,30,31,34,35], in this paper, we consider a class of nonsmooth weak vector variational-hemivariational inequality problems (WVVHVIP) in the framework of Hadamard manifolds. We introduce the notion of the generalized geodesic strong monotonicity of a set-valued vector field in the Hadamard manifold setting. By employing an analogous to KKM lemma, we establish the existence of the solutions for WVVHVIP without using any monotonicity assumptions. Moreover, we establish the uniqueness of the solution to WVVHVIP by using generalized geodesic strong monotonicity assumptions. We formulate several gap functions, in particular, Auslender type, regularized, and Moreau-Yosida type regularized gap functions for WVVHVIP to establish necessary and sufficient conditions for the existence of the solutions to the considered problem, namely, WVVHVIP. Furthermore, we derive the global error bounds for the solution of WVVHVIP by employing the Auslender type as well as regularized gap functions under the assumptions of generalized geodesic strong monotonicity.

The novelty and contributions of this paper are fourfold: Firstly, the existence results for the solution of WVVHVIP generalize the corresponding results derived by Chen and Huang [10] from nonsmooth weak vector variational inequality problems to nonsmooth weak vector variational-hemivariational inequality problems (WVVHVIP) in the Hadamard

manifold setting. Secondly, the existence results for the solution of WVVHVIP generalize the corresponding results derived by Jayswal et al. [44] from mixed weak vector variational inequality problems to a broader category of vector variational-hemivariational inequality problems. Thirdly, several results related to gap functions and global error bounds for WVVHVIP generalize the corresponding results derived in [30–32] from the Euclidean space setting to the framework of Hadamard manifolds, as well as generalize them from variational inequality problems to WVVHVIP, which belongs to a broader class of vector variational and hemivariational inequality problems. Fourthly, the gap functions and error bounds results investigated in this article generalize various corresponding results derived by Charitha et al. [34] from vector variational inequality problems to WVVHVIP and from the setting of Euclidean space to the framework of Hadamard manifolds. However, existence results, uniqueness results, as well as gap functions and global error bounds for WVVHVIP, have not been investigated before in the setting of Hadamard manifolds. Consequently, the results established in this paper are applicable to study more general classes of vector variational inequality problems and hemivariational inequality problems, as compared to the results existing in the available literature (see, for instance, [21,29,31,35,44] and the references cited therein).

The rest of this article is structured as follows. In Section 2, we recall some basic definitions and mathematical preliminaries related to Hadamard manifolds. By employing an analogous to the KKM lemma, we establish the existence results for the solutions of WVVHVIP in Section 3. Moreover, a uniqueness result for the solution of WVVHVIP is established under assumptions of generalized geodesic strong monotonicity. Furthermore, we formulate various gap functions and regularized gap functions, in particular, Auslender, regularized, and Moreau-Yosida type regularized gap functions for WVVHVIP in Section 4. Subsequently, in Section 5, we derive the global error bounds for the solution of WVVHVIP in terms of the Auslender type and regularized gap functions under generalized geodesic strong monotonicity hypotheses. Finally, in Section 6, we draw conclusions and suggest all possible future research avenues.

2. Notations and Mathematical Preliminaries

The following definitions and fundamental concepts related to Riemannian and Hadamard manifolds are from [41,50,51].

Let $\mathbb{R}^n$ and $\mathbb{R}^n_+$ denote the n—dimensional Euclidean space and the non-negative orthant of $\mathbb{R}^n$, respectively. The notations $\langle \cdot, \cdot \rangle$ and $\operatorname{int} \mathbb{R}^n_+$ are employed to denote the standard inner product in $\mathbb{R}^n$ and interior of $\mathbb{R}^n_+$, respectively. The notation $\emptyset$ is employed to denote an empty set.

Let $\mathbb{H}^n$ be the symbol used to represent an n—dimensional connected Riemannian manifold endowed with the Riemannian metric $\mathcal{G}$. The tangent space at $u \in \mathbb{H}^n$ is denoted by $T_u\mathbb{H}^n$, which is a vector space of dimension n over $\mathbb{R}$. The dual space of $T_u\mathbb{H}^n$ is represented by $T_u^*\mathbb{H}^n$. The tangent bundle $T\mathbb{H}^n$ is a disjoint union of tangent spaces at all points in the Riemannian manifold $\mathbb{H}^n$. That is, $T\mathbb{H}^n = \bigcup_{u \in \mathbb{H}^n} T_u\mathbb{H}^n$. The symbols $\langle \cdot, \cdot \rangle_u$ and $|| \cdot ||_u$ are used to represent the inner product and its associated norm on the tangent space $T_u\mathbb{H}^n$ for every $u \in \mathbb{H}^n$, respectively. Let $\Gamma : [0,1] \to \mathbb{H}^n$ be a piecewise differentiable curve joining u and v in $\mathbb{H}^n$. That is, $\Gamma(0) = u$ and $\Gamma(1) = v$. The length of the curve Γ is denoted by $L(\Gamma)$, and is defined as:

$$L(\Gamma) := \int_0^1 \|\Gamma'(t)\| dt,$$

where $\Gamma'(t) = \frac{d\Gamma(t)}{dt}$, for all $t \in (0,1)$. Moreover, the Riemannian distance between u and v in $\mathbb{H}^n$ is defined as follows:

$$d(u,v) := \inf\{L(\Gamma) : \Gamma \text{ is a piecewise smooth curve joining } u \text{ and } v \text{ in } \mathbb{H}^n\}.$$

A differentiable curve $\Gamma\colon [0,1] \to \mathbb{H}^n$ is said to be a geodesic if its vector field Γ' is parallel to itself, that is, $\nabla_{\Gamma'}\Gamma' = 0$, where ∇ is the unique Levi-Civita connection on $\mathbb{H}^n$. Furthermore, a geodesic curve joining $u, v \in \mathbb{H}^n$ is termed as a minimal geodesic if its length is equal to the Riemannian distance between them. The Riemannian manifold $\mathbb{H}^n$ is called geodesically complete at $u \in \mathbb{H}^n$, if every geodesic Γ emanating from u is defined for every real number (see, [50]). Moreover, $\mathbb{H}^n$ is said to be geodesically complete if it is geodesically complete at every point in $\mathbb{H}^n$. From Hopf-Rinow theorem (see, [50]), every geodesically complete Riemannian manifold is a complete metric space, and there always exists a minimal geodesic between any two points in $\mathbb{H}^n$. A Riemannian manifold is called a Hadamard manifold if it is simply connected, complete, and has non-positive sectional curvature everywhere.

From now onwards, we assume that $\mathbb{H}^n$ is an n-dimensional Hadamard manifold.

For any $u \in \mathbb{H}^n$, the exponential map $\exp_u : T_u\mathbb{H}^n \to \mathbb{H}^n$ is given by $\exp_u(v) = \Gamma_v(1)$, for all $v \in T_u\mathbb{H}^n$, where Γ_v is the unique geodesic such that $\Gamma_v(0) = u$ and $\Gamma_v'(0) = v$. Moreover, the inverse of the exponential map $\exp_u^{-1}\colon \mathbb{H}^n \to T_u\mathbb{H}^n$ satisfies $\exp_u^{-1} u = 0_u$, where 0_u is the zero tangent vector in tangent space $T_u\mathbb{H}^n$. In addition, for any $u, v \in \mathbb{H}^n$, the distance between u and v is $d(u,v) = ||\exp_u^{-1}(v)||_u$. The parallel transport from u to v along a geodesic Γ is a linear isometry $L_\Gamma^{u,v} : T_u\mathbb{H}^n \to T_v\mathbb{H}^n$. The symbol $L_{u,v}$ is used to represent a parallel transport from u to v along a minimal geodesic joining u and v.

A function $\Psi : \mathbb{H}^n \to \mathbb{R}$ is known as a locally Lipschitz function at $u \in \mathbb{H}^n$ with rank K ($K \in \mathbb{R}, K > 0$), if there exists a neighborhood U of u such that

$$|\Psi(v) - \Psi(w)| \leq Kd(v,w), \text{ for all } v, w \in U.$$

A function Ψ is known as locally Lipschitz on $\mathbb{H}^n$ with rank K if it is locally Lipschitz at every $u \in \mathbb{H}^n$ with rank K.

The proof of the following lemma will follow on the lines of the proof of Theorem 6 in [52].

Lemma 1. *Let $\mathcal{I}$ be a finite index set and the functions $\phi_i : \emptyset \neq \mathcal{F} \subseteq \mathbb{H}^n \to \mathbb{R}$ be lower semicontinuous. Then, the function $\Phi : \mathcal{F} \to \mathbb{R}$, defined by*

$$\Phi(u) := \inf_{i \in \mathcal{I}} \phi_i(u),$$

is also lower semicontinuous on $\mathcal{F}$.

In the following definition, we recall the notion of a geodesic convex set from Udrişte [41].

Definition 1. *Let $\mathcal{F} \subseteq \mathbb{H}^n$, and u, w be any two points in $\mathcal{F}$. Then, $\mathcal{F}$ is said to be a geodesic convex set, provided that the following inclusion holds:*

$$\exp_u(t \exp_u^{-1}(w)) \in \mathcal{F}, \text{ for all } t \in [0,1].$$

The notion of the geodesic convex hull of a subset of $\mathbb{H}^n$ is recalled in the following definition (see, for instance, [51,53]).

Definition 2. *Let $\mathcal{F} \subseteq \mathbb{H}^n$ be a nonempty set. The geodesic convex hull of a set $\mathcal{F}$ is the smallest geodesic convex subset of $\mathbb{H}^n$ containing $\mathcal{F}$.*

Remark 1. *The geodesic convex hull of a set $\mathcal{F} \subseteq \mathbb{H}^n$ can be defined as the intersection of all the geodesic convex sets containing $\mathcal{F}$ (see, for instance, [51,53]).*

From now onwards, the geodesic convex hull of a set $\mathcal{F} \subseteq \mathbb{H}^n$ is represented by $\text{conv}(\mathcal{F})$, unless specified otherwise.

The notion of the geodesic convex combination of a finite number of elements in $\mathbb{H}^n$ is recalled in the following definition (see, for instance, [49,53]).

Definition 3. *Let $u_1, u_2, \ldots, u_m$ $(m \in \mathbb{N})$ be a finite number of elements chosen arbitrarily from $\mathbb{H}^n$. Then, the geodesic convex combination of $u_1, u_2, \ldots, u_m$ is the geodesic joining u_m to any geodesic convex combination of $u_1, u_2, \ldots, u_{m-1}$, denoted by $\text{comb}_{u_1,u_2,\ldots,u_m}(t_2, t_3, \ldots, t_m)$, and is defined as follows:*

$$\text{comb}_{u_1,u_2,\ldots,u_m}(t_2, t_3, \ldots, t_m) := \exp_{u_m}(t_m \exp_{u_m}^{-1}(\text{comb}_{u_1,u_2,\ldots,u_{m-1}}(t_2, t_3, \ldots, t_{m-1}))),$$

for every $t_i \in [0,1]$ with $i = 2, 3, \ldots, m$.

A relationship between geodesic convex hull, geodesic convex set, and geodesic convex combination is provided in the following lemmas (see, [53]).

Lemma 2. *Let $\emptyset \neq \mathcal{F} \subseteq \mathbb{H}^n$. Then $\mathcal{F}$ is said to be a geodesic convex set if and only if it contains all the geodesic convex combinations of its elements.*

Lemma 3. *Let $\emptyset \neq \mathcal{F} \subseteq \mathbb{H}^n$ be any set. Then, geodesic convex hull of $\mathcal{F}$ consists of all the geodesic convex combinations of elements of $\mathcal{F}$.*

In the following definitions, we recall the notions of generalized directional derivative and generalized subdifferential of a locally Lipschitz function from Bento et al. [54].

Definition 4. *Let $\Psi : \mathbb{H}^n \to \mathbb{R}$ be a locally Lipschitz function on $\mathbb{H}^n$.*

(i) For any $u, w \in \mathbb{H}^n$, the generalized directional derivative of Ψ at u in some direction $v \in T_u\mathbb{H}^n$ is denoted by $\Psi^o(u; v)$, and is defined as follows:

$$\Psi^\circ(u; v) := \limsup_{t \downarrow 0,\, w \to u} \frac{\Psi(\exp_w t(D \exp_u)_{\exp_u^{-1}(w)} v) - \Psi(w)}{t},$$

where $(D \exp_u)_{\exp_u^{-1}(w)} : T_{\exp_u^{-1}(w)}(T_u M) \to T_w M$ is the differential of exponential map at $\exp_u^{-1}(w)$.

(ii) The generalized subdifferential of function Ψ at $u \in \mathbb{H}^n$ is denoted by $\partial_c \Psi(u)$, and is given by:

$$\partial_c \Psi(u) := \{\xi \in T_u\mathbb{H}^n \mid \Psi^o(u; v) \geq \langle \xi, v \rangle, \text{ for all } v \in T_u\mathbb{H}^n\}.$$

In the following lemmas, we discuss some basic properties of the generalized directional derivative and generalized subdifferential for a locally Lipschitz function from Hosseini and Pouryayevali [55].

Lemma 4. *Let $\Psi : \mathbb{H}^n \to \mathbb{R}$ be a locally Lipschitz function on $\mathbb{H}^n$ with rank K. Then, the following conditions hold:*

(i) *The function* $\Psi^\circ(u;\cdot)$ *is finite, positively homogeneous and subadditive on tangent space* $T_u\mathbb{H}^n$*, for every* $u \in \mathbb{H}^n$*. Moreover,* $\Psi^\circ(u;v)$ *satisfies the following condition:*

$$\|\Psi^\circ(u;v)\|_u \leq K\|v\|_u, \text{ for all } v \in T_u\mathbb{H}^n.$$

(ii) $\Psi^\circ(u;v)$ *is upper semicontinuous as a function of* (u,v) *and, as a function of* v *alone, is Lipschitz of rank* K *on* $T_u\mathbb{H}^n$*.*

(iii) $\Psi^\circ(u;-v) = (-\Psi)^\circ(u;v)$*, for every* $u \in \mathbb{H}^n$ *and* $v \in T_u\mathbb{H}^n$*.*

Lemma 5. *Let* $\Psi : \mathbb{H}^n \to \mathbb{R}$ *be a locally Lipschitz function on* $\mathbb{H}^n$ *with rank* K *and* $u \in \mathbb{H}^n$ *be an arbitrary element. Then:*

(i) *The Clarke subdifferential* $\partial_c\Psi(u)$ *is a nonempty, convex, compact subset of* $T_u\mathbb{H}^n$ *and* $\|\zeta\|_u \leq K$*, for every* $\zeta \in \partial_c\Psi(u)$*.*

(ii) *Let* $\{u_i\}_{i=1}^\infty$ *and* $\{\zeta_i\}_{i=1}^\infty$ *be arbitrary sequences in* $\mathbb{H}^n$ *and* $T\mathbb{H}^n$*, respectively, such that* $\zeta_i \in \partial_c\Psi(u_i)$ *for each* i*, and* $\{u_i\}_{i=1}^\infty$ *converges to* u*. Further, we assume that* ζ *is a cluster point of the sequence* $\{L_{u_i,u}\zeta_i\}_{i=1}^\infty$*. Then, we have* $\zeta \in \partial_c\Psi(u)$*.*

The following definition of a geodesic convex function is from Udrişte [41].

Definition 5. *Let* $\mathcal{F} \subseteq \mathbb{H}^n$ *be a geodesic convex set and* $\Psi : \mathcal{F} \to \mathbb{R}$ *be any real-valued function. Then,* Ψ *is said to be a geodesic convex function on* $\mathcal{F}$ *if for any* $u, w \in \mathcal{F}$*, the following condition holds:*

$$\Psi(\exp_u(\tau \exp_u^{-1}(w))) \leq (1-\tau)\Psi(u) + \tau\Psi(w), \text{ for all } \tau \in [0,1].$$

Remark 2. *It is worth noting that* α*-strongly geodesic convex* ($\alpha > 0$) *and strictly geodesic convex functions are geodesic convex functions on a geodesic convex set* $\mathcal{F}$ *(see, [41,56]).*

Now, we furnish an example of a real-valued geodesic convex function defined on the Poincaré half-plane.

Example 1. *Consider the following Poincaré half-plane:*

$$\mathbb{H}^2 := \{u = (u_1, u_2) \in \mathbb{R}^2 \mid u_2 > 0\}.$$

Furthermore, $\mathbb{H}^2$ *is a Hadamard manifold of dimension* 2 *and has constant negative sectional curvature* -1 *(see, [41]). The tangent space at any point* $u \in \mathbb{H}^2$ *is* $T_u\mathbb{H}^2 = \mathbb{R}^2$*.*

The Riemannian metric induces an inner product on $T_u\mathbb{H}^2$*, for all* $u \in \mathbb{H}^2$*. That is,*

$$\langle v, \mu\rangle_u = \langle \mathcal{G}(u)v, \mu\rangle, \text{ for all } v, \mu \in T_u\mathbb{H}^2 = \mathbb{R}^2,$$

where $\mathcal{G}(u) = \begin{bmatrix} \frac{1}{(u_2)^2} & 0 \\ 0 & \frac{1}{(u_2)^2} \end{bmatrix}$.

For any arbitrary element $u \in \mathbb{H}^2$ *and* $v \in T_u\mathbb{H}^2$*, the exponential map* $\exp_u : T_u\mathbb{H}^2 \to \mathbb{H}^2$ *is given by:*

If $v_1 = 0$*,*

$$\exp_u(v) = \left(u_1, u_2 e^{\frac{v_2}{u_2}}\right).$$

If $\nu_1 \neq 0$,

$$\exp_u(\nu) = \left(u_1 + \frac{\nu_2}{\nu_1} + \sqrt{1 + \left(\frac{\nu_2}{\nu_1}\right)^2} \tanh(p_{\nu_1,\nu_2}(1)), u_2 \sqrt{1 + \left(\frac{\nu_2}{\nu_1}\right)^2} \frac{1}{\cosh\left(q_{\nu_1,\nu_2(1))}\right)} \right),$$

where

$$p_{\nu_1,\nu_2}(t) = \begin{cases} t\sqrt{\nu_1^2+\nu_2^2} - \operatorname{arcsinh} \frac{\nu_2}{\nu_1}, & \text{if } \nu_1 > 0, \\ -t\sqrt{\nu_1^2+\nu_2^2} - \operatorname{arcsinh} \frac{\nu_2}{\nu_1}, & \text{if } \nu_1 < 0, \end{cases}$$

$$q_{\nu_1,\nu_2}(t) = \begin{cases} t\frac{\sqrt{\nu_1^2+\nu_2^2}}{u_2} - \operatorname{arcsinh} \frac{\nu_2}{\nu_1}, & \text{if } \nu_1 > 0, \\ -t\frac{\sqrt{\nu_1^2+\nu_2^2}}{u_2} - \operatorname{arcsinh} \frac{\nu_2}{\nu_1}, & \text{if } \nu_1 < 0. \end{cases}$$

The inverse of exponential map $\exp_u^{-1} : \mathbb{H}^2 \to T_u\mathbb{H}^2$ *is given by:*

$$\exp_u^{-1}(w) = \begin{cases} \left(0, u_2 \ln \frac{w_2}{u_2}\right), & \text{if } u_1 = w_1, \\ \frac{u_2}{a}\left(\operatorname{arctanh} \frac{b-u_1}{a} - \operatorname{arctanh} \frac{b-w_1}{a}\right)(u_2, b-u_1), & \text{if } u_1 \neq w_1, \end{cases}$$

where

$$b = \frac{(u_1)^2 + (u_2)^2 - \left((w_1)^2 + (w_2)^2\right)}{2(u_1 - w_1)}, \quad a = \sqrt{(u_1 - b)^2 + (u_2)^2}.$$

Let $\mathcal{F} := \{(u_1, u_2) \in \mathbb{H}^2 : u_1 = 0, \frac{1}{2} \leq u_2\}$ *and* $u, w \in \mathcal{F}$. *Then the unique minimal geodesic joining* u *and* w *is given by:*

$$\exp_u(\tau \exp_u^{-1}(w)) = (0, u_2^{1-\tau} w_2^{\tau}) \in \mathcal{F}, \text{ for all } \tau \in [0,1].$$

This implies that $\mathcal{F}$ *is a geodesic convex set.*

Now, we define a function $\Psi \colon \mathcal{F} \to \mathbb{R}$ *as follows:*

$$\Psi(u) := |\ln u_2 - 1|, \text{ for all } u \in \mathcal{F}.$$

Let $u, w \in \mathcal{F}$ *and* $\tau \in [0,1]$. *Then*

$$\begin{aligned} \Psi(\exp_u(\tau \exp_u^{-1}(w))) &= |\ln\left(u_2^{1-\tau} w_2^{\tau}\right) - 1| \\ &= |\ln(u_2^{1-\tau}) + \ln(w_2^{\tau}) - 1| \\ &= |(1-\tau)(\ln u_2 - 1) + \tau(\ln w_2 - 1)| \\ &\leq (1-\tau)\Psi(u) + \tau\Psi(w), \text{ for all } \tau \in [0,1]. \end{aligned}$$

Therefore, Ψ *is a geodesic convex function on* $\mathcal{F}$.

Let us define a set-valued vector field $\mathcal{A} \colon \mathbb{H}^n \to 2^{T\mathbb{H}^n}$, such that $\mathcal{A}(u) \subseteq T_u\mathbb{H}^n$, for every $u \in \mathbb{H}^n$. The domain of $\mathcal{A}$, denoted by $\mathcal{D}(\mathcal{A})$ is defined as follows:

$$\mathcal{D}(\mathcal{A}) := \{u \in \mathbb{H}^n \mid \mathcal{A}(u) \neq \emptyset\}.$$

The graph of set-valued vector field $\mathcal{A}$, denoted by $\operatorname{Graph}(\mathcal{A})$ is defined as follows:

$$\operatorname{Graph}(\mathcal{A}) := \{(u, \xi) : u \in \mathcal{D}(\mathcal{A}),\ \xi \in \mathcal{A}(u)\}.$$

In the following definition, we introduce the generalized version of geodesic strong monotonicity of a set-valued vector field in the setting of Hadamard manifolds. For further details related to monotone vector fields, we refer the readers to [57,58].

Definition 6. *Let $\mathcal{A}\colon \mathbb{H}^n \to 2^{T\mathbb{H}^n}$ be any set-valued vector field and $\sigma \in \bigcap_{u\in\mathbb{H}^n} T_u\mathbb{H}^n$. Then, $\mathcal{A}$ is said to be geodesic strongly monotone with respect to σ, if there exists a positive constant $m^{\sigma}_{\mathcal{A}}$, such that for every $u, w \in \mathcal{D}(\mathcal{A})$, and for every $\xi \in \mathcal{A}(u), \zeta \in \mathcal{A}(w)$, the following condition holds:*

$$\langle \xi - \sigma, \exp_u^{-1}(w)\rangle_u + \langle \zeta - \sigma, \exp_w^{-1}(u)\rangle_w \leq -m^{\sigma}_{\mathcal{A}} || \exp_w^{-1}(u)||^2_w.$$

Remark 3.

1. *If $\sigma = 0$, then the geodesic strong monotonicity of $\mathcal{A}$ with respect to σ reduces to the strong monotonicity of the vector field $\mathcal{A}$, presented by Li et al. [57].*
2. *If $\sigma = 0$, then geodesic strong monotonicity of set-valued vector field $\mathcal{A}$ with respect to σ reduces to the corresponding definition of strong monotonicity presented by Barani [59].*
3. *If $\mathbb{H}^n$ is a finite-dimensional real Hilbert space and $\sigma = 0$, then $\bigcap_{u\in\mathbb{H}^n} T_u\mathbb{H}^n = \mathbb{H}^n$, $T\mathbb{H}^n = \mathbb{H}^n$, $\exp_u^{-1}(w) = w - u$, for all $w,\ u \in \mathbb{H}^n$. In this case, geodesic strong monotonicity of $\mathcal{A}$ with respect to σ reduces to the corresponding definition of strong monotonicity presented by Tang and Huang [29].*

Now, we provide an example of a geodesic strong monotone vector field with respect to σ in the framework of real symmetric positive definite matrices.

Example 2. *Let $\mathbb{P}^2_+$ and $\mathbb{S}^2$ denote the sets of all real symmetric positive definite matrices and real symmetric matrices of order 2×2, respectively. For any matrix $U \in \mathbb{P}^2_+$, $\mathrm{trace}(U)$ and $\det U$ denote the trace and determinant of matrix U, respectively. Equivalently, $\mathbb{P}^2_+$ can be defined as follows:*

$$\mathbb{P}^2_+ := \left\{ U = \begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix} \mid u_i \in \mathbb{R}, \text{ for all } i = 1,2,3,4,\ u_1,\ u_4 > 0,\ u_3 = u_2,\ \det U > 0 \right\}.$$

Moreover, $\mathbb{P}^2_+$ is a Riemannian manifold endowed with the following Riemannian metric (see, [42])

$$\langle Y, W\rangle_U = \mathrm{trace}(YU^{-1}WU^{-1}), \text{ for all } W,\ Y \in T_U\mathbb{P}^2_+,\ U \in \mathbb{P}^2_+.$$

From [42], it follows that $\mathbb{P}^2_+$ is a Hadamard manifold with tangent space $T_U\mathbb{P}^2_+ = \mathbb{S}^2$, for all $U \in \mathbb{P}^2_+$. Therefore, $\bigcap_{U\in\mathbb{P}^2_+} T_U\mathbb{P}^2_+ = \mathbb{S}^2$, which is a nonempty set.

The exponential map $\exp_U : T_U\mathbb{P}^2_+ \to \mathbb{P}^2_+$ for $U \in \mathbb{P}^2_+$ is defined as follows:

$$\exp_U(W) := U^{\frac{1}{2}} e^{U^{-\frac{1}{2}} W U^{-\frac{1}{2}}} U^{\frac{1}{2}}, \text{ for all } W \in T_U\mathbb{P}^2_+.$$

The inverse of the exponential map $\exp_U^{-1} : \mathbb{P}^2_+ \to T_U\mathbb{P}^2_+$ is defined as follows:

$$\exp_U^{-1}(\mathcal{X}) := U^{\frac{1}{2}} Log(U^{-\frac{1}{2}} \mathcal{X} U^{-\frac{1}{2}}) U^{\frac{1}{2}}, \text{ for all } \mathcal{X} \in \mathbb{P}^2_+,$$

where Log denotes the usual logarithmic function on $\mathbb{P}^2_+$. Let $\Psi : \mathbb{P}^2_+ \to \mathbb{R}$ be a real-valued function. Then, the Riemannian gradient of function $\Psi : \mathbb{P}^2_+ \to \mathbb{R}$ is given as follows (see, for instance, [42]):

$$\mathrm{grad}\,\Psi(U) = U\,\nabla\Psi(U)U,$$

where $\nabla\Psi(U)$ denotes the Euclidean gradient of Ψ at U. Let I be an identity matrix of order 2×2, $\mathcal{F} := \{U \in \mathbb{P}^2_+ \mid U = e^{cI},\ c \in [-1,2]\}$ and let $\mathcal{A}\colon \mathcal{F} \to 2^{T\mathbb{P}^2_+}$ be defined as follows:

$$\mathcal{A}(U) := \left\{ 2(1-t)U \ln\det U + I \mid t \in \left[0, \frac{1}{2}\right] \right\} \subseteq T_U\mathbb{P}^2_+.$$

Let $\sigma = \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix} \in \bigcap_{U \in \mathbb{P}_+^2} T_U \mathbb{P}_+^2 = \mathbb{S}^2$. *Then, one can verify that for every* U, $W \in \mathcal{F}$ *and for every* $\xi \in A(U)$, $\zeta \in A(W)$, *there exists a positive constant* $m_{\mathcal{A}}^{\sigma} = 1$, *such that the following inequality holds:*

$$\left\langle \zeta - \sigma, \exp_W^{-1}(U) \right\rangle_W + \left\langle \xi - \sigma, \exp_U^{-1}(W) \right\rangle_U \leq -m_{\mathcal{A}}^{\sigma} \left\| \exp_W^{-1}(U) \right\|_W^2 .$$

Therefore, $\mathcal{A}$ *is a geodesic strongly monotone vector field with respect to* $\sigma = \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix}$.

The following lemma from Li et al. [60] will be used in the sequel.

Lemma 6. *Let* u, v, w, *and* z *be arbitrary elements of* $\mathbb{H}^n$ *such that* $d(u,z) = d(v,z) = \frac{d(u,v)}{2}$. *Then, one has*

$$d^2(w,z) \leq \frac{1}{2} d^2(w,u) + \frac{1}{2} d^2(w,v) - \frac{1}{4} d^2(u,v).$$

Remark 4. *In view of Lemma 6, we have (see, for instance, [48])*

$$\frac{1}{2} d^2(u,v) \leq d^2(v,z) + d^2(u,z), \text{ for all } u, v, z \in \mathbb{H}^n. \tag{1}$$

The following lemma will be employed in the sequel (see, for instance, [48]).

Lemma 7. *Let the set-valued vector field* $\mathcal{A} : \mathbb{H}^n \rightrightarrows T\mathbb{H}^n$ *and the function* $h : Graph(\mathcal{A}) \to \mathbb{R} \bigcup \{+\infty\}$ *be given. Moreover, if we assume that* h *and* $\mathcal{A}$ *are upper semicontinuous and the values of* $\mathcal{A}$ *are compact, then the function* $f : M \to \mathbb{R} \bigcup \{+\infty\}$ *defined by*

$$f(u) := \sup_{v \in \mathcal{A}(u)} h(u,v)$$

is upper semicontinuous.

Now, we recall the definition of the KKM map and an analogous to the KKM lemma from Zhou and Huang [61]. For further details, we refer to [62].

Definition 7. *Let* $\emptyset \neq \mathcal{F} \subseteq \mathbb{H}^n$ *be a closed geodesic convex set and* $\{u_1, u_2, \ldots, u_k\} \subseteq \mathcal{F}$ *be any finite set. Further, assume that* $\mathcal{G} : \mathcal{F} \to 2^{\mathcal{F}}$ *is a set-valued map. Then,* $\mathcal{G}$ *is known as a KKM map if the following inclusion holds:*

$$\text{conv}(\{u_1, u_2, \ldots, u_k\}) \subseteq \bigcup_{i=1}^{k} \mathcal{G}(u_i).$$

Lemma 8. *Let* $\emptyset \neq \mathcal{F} \subseteq \mathbb{H}^n$ *be a closed geodesic convex set and* $\mathcal{G} : \mathcal{F} \to 2^{\mathcal{F}}$ *be a KKM map. Further, we assume that the following conditions hold:*

(i) $\mathcal{G}(u)$ *is a closed set for every* $u \in \mathcal{F}$.
(ii) There exists $u_0 \in \mathcal{F}$, *such that* $\mathcal{G}(u_0)$ *is a compact set.*

Then, we have

$$\bigcap_{u \in \mathcal{F}} \mathcal{G}(u) \neq \emptyset.$$

3. Existence and Uniqueness Results for WVVHVIP

In this section, by employing an analogous to the KKM lemma, we derive the existence of the solutions for WVVHVIP via Clarke subdifferentials. Moreover, the uniqueness of the solution of WVVHVIP is established under generalized geodesic strong monotonicity assumptions.

In the rest of the paper, we assume that the following conditions hold:

(B1) Let $\mathcal{F}$ be a nonempty, closed, and geodesic convex subset of $\mathbb{H}^n$. Moreover, we assume that $\sigma \in \bigcap_{u\in\mathbb{H}^n} T_u\mathbb{H}^n$.

(B2) Let $\mathcal{I} := \{1, \ldots, l\}$ be an index set and $h_i \colon \mathcal{F} \to \mathbb{R}$, $\mathcal{P}_i \colon \mathbb{H}^n \to \mathbb{R}$ be locally Lipschitz functions on $\mathcal{F}$ for every $i \in \mathcal{I}$.

(B3) Let for every $i \in \mathcal{I}$, $\Psi_i(\cdot,\cdot) \colon \mathcal{F} \times \mathcal{F} \to \mathbb{R}$ be a continuous function.

Now, we consider a nonsmooth weak vector variational-hemivariational inequality problem in the framework of Hadamard manifolds as follows:

Nonsmooth Weak Vector Variational-Hemivariational Inequality Problem (WVVHVIP): Find $u^* \in \mathcal{F}$, such that there exist $\xi_i^* \in \partial_c h_i(u^*)$ $(i \in \mathcal{I})$, satisfying:

$$\begin{aligned}\Big(&\Big\langle \xi_1^* - \sigma, \exp_{u^*}^{-1}(w)\Big\rangle_{u^*} + \Psi_1(u^*, w) - \Psi_1(u^*, u^*) + \mathcal{P}_1^\circ(u^*; \exp_{u^*}^{-1}(w)), \ldots,\\ &\Big\langle \xi_l^* - \sigma, \exp_{u^*}^{-1}(w)\Big\rangle_{u^*} + \Psi_l(u^*, w) - \Psi_l(u^*, u^*) + \mathcal{P}_l^\circ(u^*; \exp_{u^*}^{-1}(w))\Big) \notin -\operatorname{int}\mathbb{R}^l_+,\end{aligned} \tag{2}$$

for all $w \in \mathcal{F}$.

Remark 5.

1. *It is worthwhile to note that if for every $i \in \mathcal{I}$, $\mathcal{P}_i$ is a constant function, then $\mathcal{P}_i^\circ(u;\cdot) = 0$, for all $u \in \mathbb{H}^n$. Therefore, if for every $i \in \mathcal{I}$, $\mathcal{P}_i$ is a constant function, $\sigma = 0$, and $\Psi_i(u,w) = \Psi_i(w)$, for all u, $w \in \mathbb{H}^n$, then WVVHVIP reduces to the mixed weak vector variational inequality problem of the form: Find $u^* \in \mathcal{F}$ and $\xi_i^* \in \partial_c h_i(u^*)$ such that*

$$\begin{aligned}\Big(&\langle \xi_1^*, \exp_{u^*}^{-1}(w)\rangle_{u^*} + \Psi_1(w) - \Psi_1(u^*), \ldots,\\ &\langle \xi_l^*, \exp_{u^*}^{-1}(w)\rangle_{u^*} + \Psi_l(w) - \Psi_l(u^*)\Big) \notin -\operatorname{int}\mathbb{R}^l_+, \text{ for all } w \in \mathcal{F},\end{aligned}$$

   as considered by Jayswal et al. [44].

2. *If for every $i \in \mathcal{I}$, $\mathcal{P}_i$ is a constant function, $\sigma = 0$, $\Psi_i(u,w) = 0$, for all $u, w \in \mathcal{F}$, for all $i \in \mathcal{I}$, then WVVHVIP reduces to weak Stampacchia vector variational inequality problem (WSVVIP) as discussed in [9,10].*

3. *If $\mathcal{I} = \{1\}$, $\sigma = 0$, $\Psi_1(u,w) = 0$, for every u, $w \in \mathcal{F}$ and $\partial_c h_1(u) = \mathcal{A}(u)$, where $\mathcal{A}$ is a single-valued vector field, then WVVHVIP reduces to a hemivariational inequality problem (HVIP($\mathcal{A}, \mathcal{P}_1, \mathcal{F}$)) of the form: Find $u^* \in \mathcal{F}$ such that*

$$\langle \mathcal{A}(u^*), \exp_{u^*}^{-1}(w)\rangle_{u^*} + \mathcal{P}_1^\circ(u^*; \exp_{u^*}^{-1}(w)) \geq 0, \text{ for all } w \in \mathcal{F},$$

   as considered by Tang et al. [21].

4. *If $\mathcal{I} = \{1\}$, $\sigma = 0$, $\Psi_1(u,w) = 0$, for every u, $w \in \mathcal{F}$, $\partial_c h_1(u) = \mathcal{A}(u)$ is a single-valued vector field foe every $u \in \mathcal{F}$, and $\mathcal{P}_1$ is a constant function, then $\mathcal{P}_1^\circ(u;\cdot) = 0$, for all $u \in \mathbb{H}^n$. In this case, WVVHVIP reduces to a variational inequality problem of the form: Find $u^* \in \mathcal{F}$ such that*

$$\langle \mathcal{A}(u^*), \exp_{u^*}^{-1}(w)\rangle_{u^*} \geq 0, \text{ for all } w \in \mathcal{F},$$

   as introduced by Németh [46].

5. *If $\mathbb{H}^n = \mathbb{R}^n$, $\sigma = 0$, $\Psi_i(u,w) = 0$, for all u, $w \in \mathcal{F}$, $\partial_c h_i(u)$ $(i \in \mathcal{I})$ is a single-valued vector field for every $u \in \mathcal{F}$, and $\mathcal{P}_i^\circ(u;\cdot) = 0$, for all $u \in \mathcal{F}$ $(i \in \mathcal{I})$, then WVVHVIP*

reduces to the weak Stampacchia vector variational inequality problem (SVVI)^w of the form: Find $u^* \in \mathcal{F}$ such that

$$(\langle \partial_c h_1(u^*), w - u^* \rangle, \ldots, \langle \partial_c h_l(u^*), w - u^* \rangle) \notin -\operatorname{int} \mathbb{R}^l_+, \text{ for all } w \in \mathcal{F},$$

as considered by Charitha et al. [34].

6. *If $\mathbb{H}^n = \mathbb{R}^n$, $\sigma = 0$, $I = \{1\}$, $\Psi_1(u, w) = 0$, for all $u, w \in \mathcal{F}$ and $\mathcal{P}_1$ is a constant function, then $\mathcal{P}_1^\circ(u; \cdot) = 0$, for all $u \in \mathbb{H}^n$, $\exp_u^{-1}(w) = w - u$, for all $w, u \in \mathbb{H}^n$. Moreover, if $\partial_c h_1(u)$ is a single-valued vector field for every $u \in \mathcal{F}$, then WVVHVIP reduces to the variational inequality problem of the form: Find $u^* \in \mathcal{F}$ such that*

$$\langle \partial_c h_1(u^*), w - u^* \rangle \geq 0, \text{ for all } w \in \mathcal{F},$$

as considered by Yamashita and Fukushima [30].

From now onwards, we assume that $\mathcal{F}$ is a nonempty, compact, and geodesic convex subset of $\mathbb{H}^n$, unless specified otherwise.

In the following theorem, we establish the existence of the solutions to WVVHVIP without relying on the monotonicity assumption on $\partial_c h_i$ $(i \in \mathcal{I})$.

Theorem 1. *Let for every $i \in \mathcal{I}$, $\Psi_i(u, \cdot) \colon \mathcal{F} \to \mathbb{R}$ be a geodesic convex function on $\mathcal{F}$ for every $u \in \mathcal{F}$. Moreover, we suppose that for every $u \in \mathcal{F}$ and for every $i \in \mathcal{I}$, the following set*

$$B_i := \Big\{ w \in \mathcal{F} \mid \forall\, \xi_i \in \partial_c h_i(u) : \langle \xi_i - \sigma, \exp_u^{-1}(w) \rangle_u + \Psi_i(u, w) - \Psi_i(u, u) + \mathcal{P}_i^\circ(u; \exp_u^{-1}(w)) < 0 \Big\},$$

is a geodesic convex set. Then, WVVHVIP has a solution in $\mathcal{F}$.

Proof. Let $u \in \mathcal{F}$ be an arbitrary element. A set-valued map $\mathcal{G} \colon \mathcal{F} \to 2^{\mathcal{F}}$ is defined as follows:

$$\begin{aligned} \mathcal{G}(u) := \{ w \in \mathcal{F} \mid \exists \xi_i \in \partial_c h_i(w)\ (i \in \mathcal{I}) : \\ \Big(\langle \xi_1 - \sigma, \exp_w^{-1}(u) \rangle_w + \Psi_1(w, u) - \Psi_1(w, w) + \mathcal{P}_1^\circ(w; \exp_w^{-1}(u)), \ldots, \\ \langle \xi_l - \sigma, \exp_w^{-1}(u) \rangle_w + \Psi_l(w, u) - \Psi_l(w, w) + \mathcal{P}_l^\circ(w; \exp_w^{-1}(u)) \Big) \notin -\operatorname{int} \mathbb{R}^l_+ \}. \end{aligned}$$

Notably, $\mathcal{G}(u)$ is a nonempty set for every $u \in \mathcal{F}$.

To prove the existence of a solution for WVVHVIP, we divide the proof into two parts:

(i) In this part, we prove that $\mathcal{G}$ is a KKM map. On the contrary, we suppose that there exists a finite set $\{w_1, w_2, \ldots, w_k\} \subseteq \mathcal{F}$ such that for some $\tilde{u} \in \operatorname{conv}(\{w_1, w_2, \ldots, w_k\})$ we have

$$\tilde{u} \notin \bigcup_{j=1}^{k} \mathcal{G}(w_j).$$

This implies that for every $j = 1, \ldots, k$, and for every $\xi_i \in \partial_c h_i(\tilde{u})$ $(i \in \mathcal{I})$, the following inclusion relation holds:

$$\begin{aligned} \Big(\langle \xi_1 - \sigma, \exp_{\tilde{u}}^{-1}(w_j) \rangle_{\tilde{u}} + \Psi_1(\tilde{u}, w_j) - \Psi_1(\tilde{u}, \tilde{u}) + \mathcal{P}_1^\circ(\tilde{u}; \exp_{\tilde{u}}^{-1}(w_j)), \ldots, \\ \langle \xi_l - \sigma, \exp_{\tilde{u}}^{-1}(w_j) \rangle_{\tilde{u}} + \Psi_l(\tilde{u}, w_j) - \Psi_l(\tilde{u}, \tilde{u}) + \mathcal{P}_l^\circ(\tilde{u}; \exp_{\tilde{u}}^{-1}(w_j)) \Big) \in -\operatorname{int} \mathbb{R}^l_+. \end{aligned}$$

Hence, for every $i \in \mathcal{I}$, $j \in \{1, 2, \ldots, k\}$, and for every $\xi_i \in \partial_c h_i(\tilde{u})$ $(i \in \mathcal{I})$, we have

$$\langle \xi_i - \sigma, \exp_{\tilde{u}}^{-1}(w_j) \rangle_{\tilde{u}} + \Psi_i(\tilde{u}, w_j) - \Psi_i(\tilde{u}, \tilde{u}) + \mathcal{P}_i^\circ(\tilde{u}; \exp_{\tilde{u}}^{-1}(w_j)) < 0. \tag{3}$$

Let us consider the following set, which is defined as follows:

$$\mathcal{H} := \{w \in \mathcal{F} \mid \forall\, \xi_i \in \partial_c h_i(\tilde{u})\ (i \in \mathcal{I}) : \langle \xi_i - \sigma, \exp_{\tilde{u}}^{-1}(w)\rangle_{\tilde{u}} + \Psi_i(\tilde{u}, w) - \Psi_i(\tilde{u}, \tilde{u}) + \mathcal{P}_i^{\circ}(\tilde{u}; \exp_{\tilde{u}}^{-1}(w)) < 0\}.$$

Notably, $\mathcal{H}$ is a nonempty subset of $\mathcal{F}$. From the given hypotheses, $\mathcal{H}$ being an intersection of geodesic convex sets is a geodesic convex set. Therefore, $\tilde{u} \in \mathcal{H}$, which implies that

$$0 > \langle \xi_i - \sigma, \exp_{\tilde{u}}^{-1}(\tilde{u})\rangle_{\tilde{u}} + \Psi_i(\tilde{u}, \tilde{u}) - \Psi_i(\tilde{u}, \tilde{u}) + \mathcal{P}_i^{\circ}(\tilde{u}; \exp_{\tilde{u}}^{-1}(\tilde{u})) = 0,$$

for every $\xi_i \in \partial_c h_i(\tilde{u})$ $(i \in \mathcal{I})$, which is a contradiction. Therefore, $\mathcal{G}$ is a KKM map.

(ii) In this part, we show that $\mathcal{G}(u)$ is a closed set-valued map for every $u \in \mathcal{F}$.
Let $w \in \mathcal{F}$ and $\{w_m\}_{m=1}^{\infty} \subseteq \mathcal{G}(u)$ such that $w_m \to w$ as $m \to \infty$. Then there exist $\xi_i^m \in \partial_c h_i(w_m)$ $(i \in \mathcal{I})$, satisfying:

$$\begin{aligned}\Big(&\langle \xi_1^m - \sigma, \exp_{w_m}^{-1}(u)\rangle_{w_m} + \Psi_1(w_m, u) - \Psi_1(w_m, w_m) + \mathcal{P}_1^{\circ}(w_m; \exp_{w_m}^{-1}(u)), \ldots,\\ &\langle \xi_l^m - \sigma, \exp_{w_m}^{-1}(u)\rangle_{w_m} + \Psi_l(w_m, u) - \Psi_l(w_m, w_m)\\ &\qquad + \mathcal{P}_l^{\circ}(w_m; \exp_{w_m}^{-1}(u))\Big) \notin -\operatorname{int}\mathbb{R}_+^l.\end{aligned} \tag{4}$$

Employing continuity of Ψ_i $(i \in \mathcal{I})$, upper semicontinuity of $\mathcal{P}_i^{\circ}$ $(i \in \mathcal{I})$ and Lemma 5, there exist $\xi_i \in \partial_c h_i(w)$ $(i \in \mathcal{I})$ such that

$$\begin{aligned}\Big(&\langle \xi_1 - \sigma, \exp_{w}^{-1}(u)\rangle_{w} + \Psi_1(w, u) - \Psi_1(w, w) + \mathcal{P}_1^{\circ}(w; \exp_{w}^{-1}(u)), \ldots,\\ &\langle \xi_l - \sigma, \exp_{w}^{-1}(u)\rangle_{w} + \Psi_l(w, u) - \Psi_l(w, w) + \mathcal{P}_l^{\circ}(w; \exp_{w}^{-1}(u))\Big) \notin -\operatorname{int}\mathbb{R}_+^l.\end{aligned} \tag{5}$$

Therefore, $w \in \mathcal{G}(u)$.

Moreover, $\mathcal{G}(u) \subseteq \mathcal{F}$ and $\mathcal{F}$ is a compact set implies that $\mathcal{G}(u)$ is a bounded set for every $u \in \mathcal{F}$. In view of (ii), $\mathcal{G}(u)$ is a closed set for every $u \in \mathcal{F}$. Therefore, $\mathcal{G}$ is a KKM map such that $\mathcal{G}(u)$ is a compact set for every $u \in \mathcal{F}$. From Lemma 8, there exists $\tilde{u} \in \mathcal{F}$ such that $\tilde{u} \in \bigcap\limits_{u \in \mathcal{F}} \mathcal{G}(u)$. Therefore, there exist $\xi_i \in \partial_c h_i(\tilde{u})$ $(i \in \mathcal{I})$ such that

$$\begin{aligned}\Big(&\langle \xi_1 - \sigma, \exp_{\tilde{u}}^{-1}(u)\rangle_{\tilde{u}} + \Psi_1(\tilde{u}, u) - \Psi_1(\tilde{u}, \tilde{u}) + \mathcal{P}_1^{\circ}(\tilde{u}; \exp_{\tilde{u}}^{-1}(u)), \ldots,\\ &\langle \xi_l - \sigma, \exp_{\tilde{u}}^{-1}(u)\rangle_{\tilde{u}} + \Psi_l(\tilde{u}, u) - \Psi_l(\tilde{u}, \tilde{u}) + \mathcal{P}_l^{\circ}(\tilde{u}; \exp_{\tilde{u}}^{-1}(u))\Big) \notin -\operatorname{int}\mathbb{R}_+^l,\end{aligned} \tag{6}$$

for every $u \in \mathcal{F}$. Hence, $\tilde{u} \in \mathcal{F}$ is a solution of WVVHVIP. This completes the proof. □

Remark 6. *Theorem 1 generalizes Theorem 3.5 derived by Jayswal et al. [44] from mixed weak vector variational inequality problem to weak vector variational-hemivariational inequality problem, namely WVVHVIP. Moreover, Theorem 1 is applicable to a broader class of weak vector variational inequality problems and hemivariational inequality problems as the proof of Theorem 1 has not utilized the monotonicity assumption as employed in prior works (see, for instance, [35,44]).*

In the following example, we illustrate the significance of Theorem 1.

Example 3. *Let us consider the Hadamard manifold $\mathbb{P}_+^2$ as considered in Example 2.*

Let I be an identity matrix of order 2×2 and $\mathcal{F} := \{U \in \mathbb{P}_+^2 \mid U = e^{c_1 I}, -1 \leq c_1 \leq 1\}$ be a nonempty, compact, and geodesic convex subset of $\mathbb{P}_+^2$. Moreover, let $h_1, h_2 : \mathcal{F} \to \mathbb{R}$ be defined as follows:

$$h_1(U) := |\ln \det U| \quad \text{and} \quad h_2(U) := \ln u_1 + \ln u_4,$$

where $U = \begin{bmatrix} u_1 & u_2 \\ u_2 & u_4 \end{bmatrix} \in \mathcal{F}$. The Clarke subdifferentials of h_1 and h_2 are given by:

$$\partial_c h_1(U) = \begin{cases} U, & \det U > 1, \\ (2t-1)I & \det U = 1,\ t \in [0,1], \\ -U, & \det U < 1, \end{cases} \text{ and } \partial_c h_2(U) = \{U\}.$$

Define $\Psi_1, \Psi_2 : \mathcal{F} \times \mathcal{F} \to \mathbb{R}$ as follows:

$$\Psi_1(U, W) := 2 \ln \det UW \text{ and } \Psi_2(U, W) := \frac{\det U}{2} + \ln \det W.$$

It can be verified that, $\Psi_i(U, \cdot)$ $(i = 1, 2)$ are geodesic convex functions on $\mathcal{F}$ for every $U \in \mathcal{F}$.

The functions $\mathcal{P}_1, \mathcal{P}_2 : \mathbb{P}_+^2 \to \mathbb{R}$ are given by:

$$\mathcal{P}_1(U) := |\det U - 1| \text{ and } \mathcal{P}_2(U) := \det U.$$

The Clarke generalized directional derivatives of $\mathcal{P}_1$ and $\mathcal{P}_2$ are given by:

$$\mathcal{P}_1^\circ(U; \exp_U^{-1}(W)) = \begin{cases} \det U(\ln \det W - \ln \det U), & \det U > 1, \\ |\ln \det W - \ln \det U|, & \det U = 1, \\ \det U(-\ln \det W + \ln \det U), & \det U < 1. \end{cases}$$

$$\mathcal{P}_2^\circ(U; \exp_U^{-1}(W)) = \{\det U(\ln \det W - \ln \det U)\}.$$

Consider the following nonsmooth weak vector variational-hemivariational inequality problem (WVVHVIP1): Find $U \in \mathcal{F}$ such that there exist $\xi_i^* \in \partial_c h_i(U)$ $(i = 1, 2)$, satisfying:

$$\Big(\langle \xi_1^* - \sigma, \exp_U^{-1}(W)\rangle_U + \Psi_1(U, W) - \Psi_1(U, U) + \mathcal{P}_1^\circ(U; \exp_U^{-1}(W)),$$
$$\langle \xi_2^* - \sigma, \exp_U^{-1}(W)\rangle_U + \Psi_2(U, W) - \Psi_2(U, U) + \mathcal{P}_2^\circ(U; \exp_U^{-1}(W))\Big) \notin -\operatorname{int} \mathbb{R}_+^2,$$

for any $W \in \mathcal{F}$ and $\sigma = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

It can be verified that for every $U \in \mathcal{F}$ and for every $i \in \mathcal{I} = \{1, 2\}$, the following set

$$B_i := \Big\{W \in \mathcal{F} \mid \forall\, \xi_i \in \partial_c h_i(U) :\ \langle \xi_i - \sigma, \exp_U^{-1}(W)\rangle_u + \Psi_i(U, W) - \Psi_i(U, U) + \mathcal{P}_i^\circ(U; \exp_U^{-1}(W)) < 0\Big\},$$

is a geodesic convex set. Moreover, $\Psi_i(U, \cdot)$ $(i \in \mathcal{I})$ is a geodesic convex function for every $U \in \mathcal{F}$. Therefore, all the hypotheses of Theorem 1 are satisfied, which concludes that there exists $\hat{U} \in \mathcal{F}$, such that $\hat{U}$ is a solution of WVVHVIP1. Furthermore, one can verify that $U = e^{-I}$ is a solution of WVVHVIP1.

In the following theorem, we establish the uniqueness of the solution to WVVHVIP under generalized geodesic strong monotonicity assumptions.

Theorem 2. *Let for every $i \in \mathcal{I}$, $\Psi_i(u,\cdot) : \mathcal{F} \to \mathbb{R}$ be a geodesic convex function on $\mathcal{F}$ for every $u \in \mathcal{F}$. Further, we assume that the following conditions hold:*

(i) *The set-valued vector field $\bigcup_{i\in\mathcal{I}} \partial_c h_i$ is geodesic strongly monotone with respect to σ having a positive constant m_h^σ.*

(ii) *There exists a constant $m_\Psi > 0$ such that*

$$\Psi_i(u_1,w_2) - \Psi_i(u_1,w_1) + \Psi_p(u_2,w_1) - \Psi_p(u_2,w_2) \le m_\Psi \| \exp_{u_2}^{-1}(u_1)\|_{u_2} \| \exp_{w_2}^{-1}(w_1)\|_{w_2},$$

for every $u_1, u_2, w_1, w_2 \in \mathcal{F}$ and $i, p \in \mathcal{I}$.

(iii) *There exists a constant $m_\mathcal{P} > 0$ such that*

$$\mathcal{P}_i^\circ(w_1; \exp_{w_1}^{-1}(w_2)) + \mathcal{P}_p^\circ(w_2; \exp_{w_2}^{-1}(w_1)) \le m_\mathcal{P} \| \exp_{w_2}^{-1}(w_1)\|_{w_2}^2,$$

for all $w_1, w_2 \in \mathcal{F}$ and $i, p \in \mathcal{I}$.

Furthermore, we assume that $m_h^\sigma > m_\Psi + m_\mathcal{P}$. Then WVVHVIP has a unique solution.

Proof. In view of Theorem 1, there exists at least one solution of WVVHVIP. On the contrary, we suppose that u_1^* and u_2^* in $\mathcal{F}$ are two distinct solutions of WVVHVIP. Therefore, there exist $k, j \in \mathcal{I}$, $\xi_k^* \in \partial_c h_k(u_1^*)$, and $\zeta_j^* \in \partial_c h_j(u_2^*)$ such that the following inequalities hold:

$$\langle \xi_k^* - \sigma, \exp_{u_1^*}^{-1}(u_2^*)\rangle_{u_1^*} + \Psi_k(u_1^*, u_2^*) - \Psi_k(u_1^*, u_1^*) + \mathcal{P}_k^\circ(u_1^*; \exp_{u_1^*}^{-1}(u_2^*)) \ge 0, \tag{7}$$

$$\langle \zeta_j^* - \sigma, \exp_{u_2^*}^{-1}(u_1^*)\rangle_{u_2^*} + \Psi_j(u_2^*, u_1^*) - \Psi_j(u_2^*, u_2^*) + \mathcal{P}_j^\circ(u_2^*; \exp_{u_2^*}^{-1}(u_1^*)) \ge 0. \tag{8}$$

On adding (7) and (8) we get

$$\begin{aligned}\Psi_k(u_1^*, u_2^*) - \Psi_k(u_1^*, u_1^*) + \Psi_j(u_2^*, u_1^*) - \Psi_j(u_2^*, u_2^*) + \mathcal{P}_k^\circ(u_1^*; \exp_{u_1^*}^{-1}(u_2^*)) + \mathcal{P}_j^\circ(u_2^*; \exp_{u_2^*}^{-1}(u_1^*))\\ \ge -\langle \xi_k^* - \sigma, \exp_{u_1^*}^{-1}(u_2^*)\rangle_{u_1^*} - \langle \zeta_j^* - \sigma, \exp_{u_2^*}^{-1}(u_1^*)\rangle_{u_2^*}.\end{aligned}$$

Therefore, from the given hypotheses (i)-(iii), we obtain the following inequality

$$(m_h^\sigma - m_\mathcal{P} - m_\Psi) \| \exp_{u_1^*}^{-1}(u_2^*)\|_{u_1^*}^2 \le 0. \tag{9}$$

In view of the given hypotheses, we have

$$(m_h^\sigma - m_\mathcal{P} - m_\Psi) > 0.$$

Then from (9), it follows that $u_1^* = u_2^*$. This completes the proof. □

Example 4. *Let $\mathbb{H}^2$ be the same manifold as considered in Example 1.*

Let $\mathcal{F} := \left\{u = (u_1, u_2) \in \mathbb{H}^2 \mid u_1 = 0, -\frac{1}{8} \le \ln u_2 \le \frac{1}{8}\right\}$ be a nonempty, compact and geodesic convex subset of $\mathbb{H}^2$. Further, let $h_1, h_2 \colon \mathcal{F} \to \mathbb{R}$ be defined as follows:

$$h_1(u) := |u_1| + 4(\ln u_2)^2 + 1,\ h_2(u) := 4(\ln u_2)^2 + e.$$

The Clarke subdifferentials of h_1 and h_2 at $\tilde{u} = \left(0, e^{\frac{1}{8}}\right)$ are given as follows:

$$\partial_c h_1(\hat{u}) = \left\{\left(0, e^{\frac{1}{8}}\right)\right\},\ \partial_c h_2(\hat{u}) = \left\{\left(0, e^{\frac{1}{8}}\right)\right\}.$$

Let us define $\Psi_1,\ \Psi_2 \colon \mathcal{F} \times \mathcal{F} \to \mathbb{R}$ *as follows:*

$$\Psi_1(u,w) := \ 2\ln u_2 \ln w_2,\ \Psi_2(u,w) := \ln u_2 \ln w_2^2 + \frac{1}{e}.$$

Let $\mathcal{P}_1, \mathcal{P}_2 : \mathbb{H}^2 \to \mathbb{R}$ *be two real-valued functions defined as follows:*

$$\mathcal{P}_1(u) := \frac{1}{2}(\ln u_2)^2,\ \mathcal{P}_2(u) := \begin{cases} \frac{1}{2}(\ln u_2)^2, & u_1 = 0, \\ |u_2 - 1|, & u_1 \neq 0. \end{cases}$$

Then, the generalized directional derivatives of $\mathcal{P}_1$ *and* $\mathcal{P}_2$ *at* $u \in \mathcal{F}$ *in the direction* $\exp_u^{-1}(w)$ *for any* $w \in \mathcal{F}$ *are given as:*

$$\mathcal{P}_1^\circ(u; \exp_u^{-1}(w)) = \ln u_2 \ln\left(\frac{w_2}{u_2}\right),\ \mathcal{P}_2^\circ(u; \exp_u^{-1}(w)) = \ln u_2 \ln\left(\frac{w_2}{u_2}\right).$$

Now, we consider the following nonsmooth weak vector variational-hemivariational inequality problem for $\sigma = (1,2)$*:*

(WVVHVIP2): Find $\tilde{u} \in \mathcal{F}$*, such that there exist* $\xi_i \in \partial_c h_i(\tilde{u})$ *for every* $i \in \mathcal{I} = \{1,2\}$*, satisfying:*

$$\Big(\langle \xi_1 - \sigma, \exp_{\tilde{u}}^{-1}(w)\rangle_{\tilde{u}} + \Psi_1(\tilde{u},w) - \Psi_1(\tilde{u},\tilde{u}) + \mathcal{P}_1^\circ(\tilde{u}; \exp_{\tilde{u}}^{-1}(w)),$$
$$\langle \xi_2 - \sigma, \exp_{\tilde{u}}^{-1}(w)\rangle_{\tilde{u}} + \Psi_2(\tilde{u},w) - \Psi_2(\tilde{u},\tilde{u}) + \mathcal{P}_2^\circ(\tilde{u}; \exp_{\tilde{u}}^{-1}(w))\Big) \notin -\operatorname{int}\mathbb{R}^2_+,$$

for every $w \in \mathcal{F}$*.*

It can be verified that $\bigcup_{i\in\{1,2\}} \partial_c h_i$ *is geodesic strongly monotone with respect to* σ *with a positive constant* $m_h^\sigma = 8$*. Moreover, all the hypotheses (ii)–(iii) of Theorem 2 are satisfied with* $m_\Psi = 2$ *and* $m_\mathcal{P} = \frac{1}{7}$*. Therefore, there exists a unique solution* $\tilde{u} = (0, e^{\frac{1}{8}})$ *of WVVHVIP2.*

4. Gap Functions for WVVHVIP

In this section, we formulate various gap functions, in particular, Auslender, regularized, and Moreau-Yosida type regularized gap functions for WVVHVIP. These gap functions are employed to establish the necessary and sufficient conditions for the existence of the solutions to WVVHVIP.

In the following definition, we provide the definition of a gap function for WVVHVIP.

Definition 8. *A function* $\Phi \colon \mathcal{F} \to \mathbb{R}$ *is called a gap function for WVVHVIP, if it satisfies the following conditions:*

(i) $\Phi(u) \geq 0$*, for all* $u \in \mathcal{F}$*.*

(ii) $\tilde{u} \in \mathcal{F}$ *is a solution of WVVHVIP if and only if* $\Phi(\tilde{u}) = 0$*.*

Now, we consider function $\Pi \colon \mathcal{F} \to \mathbb{R}$, which is defined as follows:

$$\Pi(u) := \inf_{\xi \in B(u)} \sup_{w\in\mathcal{F}} \min_{i\in\mathcal{I}} \Big\{ \langle L_{u,w}\xi_i - L_{u,w}\sigma, \exp_w^{-1}(u)\rangle_w + \Psi_i(u,u) - \Psi_i(u,w) - \mathcal{P}_i^\circ(u; \exp_u^{-1}(w)) \Big\}, \tag{10}$$

where $u \in \mathcal{F},\ B(u) := \partial_c h_1(u) \times \partial_c h_2(u) \times \ldots \times \partial_c h_l(u)$, and $\xi_i \in \partial_c h_i(u)\ (i \in \mathcal{I})$.

For $u \in \mathcal{F}$ and $\xi \in B(u)$, we let

$$M(u,\xi) := \sup_{w\in\mathcal{F}} \min_{i\in\mathcal{I}} \Big\{ \langle L_{u,w}\xi_i - L_{u,w}\sigma, \exp_w^{-1}(u)\rangle_w + \Psi_i(u,u) - \Psi_i(u,w) - \mathcal{P}_i^\circ(u; \exp_u^{-1}(w)) \Big\}.$$

In the following theorem, we prove that the function Π defined in (10) is a gap function for WVVHVIP.

Theorem 3. *Let for every $i \in \mathcal{I}$ and $u \in \mathcal{F}$, $\Psi_i(u,\cdot)\colon \mathcal{F} \to \mathbb{R}$ be a geodesic convex function on $\mathcal{F}$. Then the function Π is a gap function for WVVHVIP.*

Proof. On the contrary, we assume that there exists $u \in \mathcal{F}$ such that

$$\Pi(u) < 0.$$

In view of the fact that $B(u)$ is a compact set and $M(u,\cdot)$ is a continuous function for every $u \in \mathcal{F}$, there exists $\xi_i^* \in \partial_c h_i(u)$ $(i \in \mathcal{I})$, satisfying:

$$\sup_{w\in\mathcal{F}} \min_{i\in\mathcal{I}} \{ \langle L_{u,w}\xi_i^* - L_{u,w}\sigma, \exp_w^{-1}(u)\rangle_w + \Psi_i(u,u) - \Psi_i(u,w) - \mathcal{P}_i^\circ(u; \exp_u^{-1}(w)) \} < 0.$$

It follows that for every $w \in \mathcal{F}$, we have

$$\min_{i\in\mathcal{I}} \{ \langle L_{u,w}\xi_i^* - L_{u,w}\sigma, \exp_w^{-1}(u)\rangle_w + \Psi_i(u,u) - \Psi_i(u,w) - \mathcal{P}_i^\circ(u; \exp_u^{-1}(w)) \} < 0,$$

which is a contradiction for $w = u$. Therefore, we have

$$\Pi(u) \geq 0, \text{ for all } u \in \mathcal{F}. \tag{11}$$

Let $\Pi(\tilde{u}) = 0$ for some $\tilde{u} \in \mathcal{F}$. Since $B(\tilde{u})$ is a compact set and $M(\tilde{u},\cdot)$ is continuous, there exists $\xi_i^* \in \partial_c h_i(\tilde{u})$ for every $i \in \mathcal{I}$ such that the following condition holds:

$$\sup_{w\in\mathcal{F}} \min_{i\in\mathcal{I}} \{ \langle L_{\tilde{u},v}\xi_i^* - L_{\tilde{u},v}\sigma, \exp_w^{-1}(\tilde{u})\rangle_w + \Psi_i(\tilde{u},\tilde{u}) - \Psi_i(\tilde{u},w) - \mathcal{P}_i^\circ(\tilde{u}; \exp_{\tilde{u}}^{-1}(w)) \} = 0.$$

It follows that for every $w \in \mathcal{F}$, we have

$$\min_{i\in\mathcal{I}} \Big\{ \langle L_{\tilde{u},w}\xi_i^* - L_{\tilde{u},w}\sigma, \exp_w^{-1}(\tilde{u})\rangle_w + \Psi_i(\tilde{u},\tilde{u}) - \Psi_i(\tilde{u},w) - \mathcal{P}_i^\circ(\tilde{u}; \exp_{\tilde{u}}^{-1}(w)) \Big\} \leq 0.$$

Therefore, for any $w \in \mathcal{F}$ there exists some $p \in \mathcal{I}$ such that the following inequality holds:

$$\langle L_{\tilde{u},w}\xi_p^* - L_{\tilde{u},w}\sigma, \exp_w^{-1}(\tilde{u})\rangle_w + \Psi_p(\tilde{u},\tilde{u}) - \Psi_p(\tilde{u},w) - \mathcal{P}_p^\circ(\tilde{u}; \exp_{\tilde{u}}^{-1}(w)) \leq 0. \tag{12}$$

Applying the parallel transport from w to $\tilde{u}$ in (12), we have

$$\langle \xi_p^* - \sigma, \exp_{\tilde{u}}^{-1}(w)\rangle_{\tilde{u}} + \Psi_p(\tilde{u},w) - \Psi_p(\tilde{u},\tilde{u}) + \mathcal{P}_p^\circ(\tilde{u}; \exp_{\tilde{u}}^{-1}(w)) \geq 0.$$

Therefore, we have

$$\begin{aligned}(&\langle \xi_1^* - \sigma, \exp_{\tilde{u}}^{-1}(w)\rangle_{\tilde{u}} + \Psi_1(\tilde{u},w) - \Psi_1(\tilde{u},\tilde{u}) + \mathcal{P}_1^\circ(\tilde{u}; \exp_{\tilde{u}}^{-1}(w)), \ldots,\\ &\langle \xi_l^* - \sigma, \exp_{\tilde{u}}^{-1}(w)\rangle_{\tilde{u}} + \Psi_l(\tilde{u},w) - \Psi_l(\tilde{u},\tilde{u}) + \mathcal{P}_l^\circ(\tilde{u}; \exp_{\tilde{u}}^{-1}(w))) \notin -\operatorname{int}\mathbb{R}_+^l,\end{aligned}$$

for every $w \in \mathcal{F}$. This implies that $\tilde{u}$ is a solution of WVVHVIP.

For the converse part, we assume that $\tilde{u} \in \mathcal{F}$ is a solution of WVVHVIP. This implies that there exist $\xi_i^* \in \partial_c h_i(\tilde{u})$ $(i \in \mathcal{I})$, satisfying:

$$\Big(\langle \xi_1^* - \sigma, \exp_{\tilde{u}}^{-1}(w)\rangle_{\tilde{u}} + \Psi_1(\tilde{u}, w) - \Psi_1(\tilde{u}, \tilde{u}) + \mathcal{P}_1^\circ(\tilde{u}; \exp_{\tilde{u}}^{-1}(w)), \ldots,$$
$$\langle \xi_l^* - \sigma, \exp_{\tilde{u}}^{-1}(w)\rangle_{\tilde{u}} + \Psi_l(\tilde{u}, w) - \Psi_l(\tilde{u}, \tilde{u}) + \mathcal{P}_l^\circ(\tilde{u}; \exp_{\tilde{u}}^{-1}(w))\Big) \notin -\operatorname{int} \mathbb{R}_+^l,$$

for all $w \in \mathcal{F}$. It follows that for every $w \in \mathcal{F}$ there exists some $p \in \mathcal{I}$ such that

$$\langle \xi_p^* - \sigma, \exp_{\tilde{u}}^{-1}(w)\rangle_{\tilde{u}} + \Psi_p(\tilde{u}, w) - \Psi_p(\tilde{u}, \tilde{u}) + \mathcal{P}_p^\circ(\tilde{u}; \exp_{\tilde{u}}^{-1}(w)) \geq 0. \tag{13}$$

Applying parallel transport from $\tilde{u}$ to w in (13), we infer that

$$\langle L_{\tilde{u},w}\xi_p^* - L_{\tilde{u},w}\sigma, \exp_w^{-1}(\tilde{u})\rangle_w + \Psi_p(\tilde{u}, \tilde{u}) - \Psi_p(\tilde{u}, w) - \mathcal{P}_p^\circ(\tilde{u}; \exp_{\tilde{u}}^{-1}(w)) \leq 0.$$

Hence, for every $w \in \mathcal{F}$, we get the following inequality

$$\min_{i\in\mathcal{I}}\Big\{\langle L_{\tilde{u},w}\xi_i^* - L_{\tilde{u},w}\sigma, \exp_w^{-1}(\tilde{u})\rangle_w + \Psi_i(\tilde{u}, \tilde{u}) - \Psi_i(\tilde{u}, w) - \mathcal{P}_i^\circ(\tilde{u}; \exp_{\tilde{u}}^{-1}(w))\Big\} \leq 0. \tag{14}$$

On taking supremum over $w \in \mathcal{F}$ in (14), we have

$$M(\tilde{u}, \xi^*) = \sup_{w\in\mathcal{F}} \min_{i\in\mathcal{I}}\Big\{\langle L_{\tilde{u},w}\xi_i^* - L_{\tilde{u},w}\sigma, \exp_w^{-1}(\tilde{u})\rangle_w + \Psi_i(\tilde{u}, \tilde{u}) - \Psi_i(\tilde{u}, w) - \mathcal{P}_i^\circ(\tilde{u}; \exp_{\tilde{u}}^{-1}(w))\Big\} \leq 0.$$

Therefore, we have

$$\Pi(\tilde{u}) = \inf_{\xi\in B(\tilde{u})} M(\tilde{u}, \xi) \leq 0. \tag{15}$$

From (11) and (15), it follows that

$$\Pi(\tilde{u}) = 0.$$

This completes the proof. □

Remark 7.

1. *It is worthwhile to note that if for every $i \in \mathcal{I}$, $\mathcal{P}_i$ is a constant function, then $\mathcal{P}_i^\circ(u; \cdot) = 0$, for all $u \in \mathbb{H}^n$. In addition to this, if $\sigma = 0$, and $\Psi_i(u, w) = \Psi_i(w)$, for all $u, w \in \mathbb{H}^n$, $i \in \mathcal{I}$, then Theorem 3 reduces to Theorem 3.3 deduced by Jayswal et al. [44].*
2. *Let $\mathcal{P}_i$ be a constant function for all $i \in \mathcal{I}$. Moreover, if $\mathbb{H}^n = \mathbb{R}^n$, $\sigma = 0$, $\Psi_i(u, w) = 0$ $(i \in \mathcal{I})$, for all $u, w \in \mathcal{F}$, $\partial_c h_i(u)$ $(i \in \mathcal{I})$ is a single-valued vector field for all $u \in \mathcal{F}$, then Theorem 3 reduces to Theorem 4.1 established by Charitha et al. [34].*
3. *If $\mathbb{H}^n = \mathbb{R}^n$, $\sigma = 0$, $\mathcal{I} = \{1\}$, and $\Psi_1(u, w) = 0$, for all $u, w \in \mathcal{F}$, $\mathcal{P}_1$ is a constant function, then $\exp_u^{-1}(w) = w - u$, for all $u, w \in \mathcal{F}$. Moreover, if $\partial_c h_1(u)$ is a single-valued vector field for all $u \in \mathcal{F}$, then Theorem 3 reduces to Lemma 2.1 derived by Yamashita and Fukushima [30].*
4. *It is worth noting that Hadamard manifolds, in general, represent a nonlinear space. For instance, $\exp_u^{-1}(w) \neq -\exp_w^{-1}(u)$ for any $u, w \in \mathbb{H}^n$. However, in the framework of Euclidean space $u - w = -(w - u)$, for all $u, w \in \mathbb{R}^n$. Therefore, the techniques that have been successfully employed in the context of linear spaces cannot be applied to the optimization problems defined on Hadamard manifolds. This limitation underscores the*

significant challenges associated with the development of optimization techniques in the framework of Hadamard manifolds.

Now, we provide a non-trivial example to illustrate the significance of Theorem 3.

Example 5. *Let us consider the sets* $\mathbb{P}_+^2, \mathcal{F}$ *and functions* $h_i, \Psi_i, \mathcal{P}_i$ $(i = 1, 2)$ *as defined in Example 3. Now, we define a function* $\Pi\colon \mathcal{F} \to \mathbb{R}$ *for* $\sigma = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ *as follows:*

$$\Pi(U) := \inf_{\xi \in B(u)} \sup_{w \in \mathcal{F}} \min_{1 \le i \le 2} \Big\{ \langle L_{U,W}\xi_i - L_{U,W}\sigma, \exp_W^{-1}(U)\rangle_W + \Psi_i(U,U) - \Psi_i(U,W) - \mathcal{P}_i^\circ(U; \exp_U^{-1}(W)) \Big\}.$$

Let $U \in \mathcal{F}$ *such that* $1 < \det U \le e^2$. *Then, the function* Π *for* $\sigma = -I$ *is given by:*

$$\begin{aligned}
\Pi(U) &= \inf_{\xi \in B(U)} \sup_{W \in \mathcal{F}} \min_{1 \le i \le 2} \Big\{ \langle L_{U,W}\xi_i - L_{U,W}\sigma, \exp_W^{-1}(U)\rangle_W + \Psi_i(U,U) - \Psi_i(U,W) - \mathcal{P}_i^\circ(U; \exp_U^{-1}(W)) \Big\} \\
&= \sup_{W \in \mathcal{F}} \min \Big\{ \Big\langle L_{U,W}\xi_1 - \sigma, \exp_W^{-1}(U) \Big\rangle_W + \Psi_1(U,U) - \Psi_1(U,W) - \mathcal{P}_1^\circ(U; \exp_U^{-1}(W)), \\
&\qquad \Big\langle L_{U,W}\xi_2 - \sigma, \exp_W^{-1}(U) \Big\rangle_W + \Psi_2(U,U) - \Psi_2(U,W) - \mathcal{P}_2^\circ(U; \exp_U^{-1}(W)) \Big\} \\
&= \sup_{W \in \mathcal{F}} \min \Big\{ -\Big\langle \xi_1 - \sigma, \exp_U^{-1}(W) \Big\rangle_U + \Psi_1(U,U) - \Psi_1(U,W) - \mathcal{P}_1^\circ(U; \exp_U^{-1}(W)), \\
&\qquad -\Big\langle \xi_2 - \sigma, \exp_U^{-1}(W) \Big\rangle_U + \Psi_2(U,U) - \Psi_2(U,W) - \mathcal{P}_2^\circ(U; \exp_U^{-1}(W)) \Big\} \\
&\qquad\qquad (\textit{on applying the parallel transport from } W \textit{ to } U) \\
&= \sup_{W \in \mathcal{F}} \min \Bigg\{ -\Big\langle U + I, \frac{\ln \frac{\det W}{\det U}}{2} U \Big\rangle_U + 2 \ln \frac{\det U}{\det W} + \det U \Big(\ln \frac{\det U}{\det W} \Big), \\
&\qquad\qquad - \Big\langle U + I, \frac{\ln \frac{\det W}{\det U}}{2} U \Big\rangle_U + \ln \frac{\det U}{\det W} + \det U \Big(\ln \frac{\det U}{\det W} \Big) \Bigg\} \\
&= (1 + (\det U)^{-\frac{1}{2}} + 1 + \det U)(\ln \det U + 2) \\
&\ge 0.
\end{aligned}$$

For $U \in \mathcal{F}$ *such that* $\det U = 1$, *the function* Π *is given by:*

$$\begin{aligned}
\Pi(U) &= \inf_{\xi \in B(U)} \sup_{W \in \mathcal{F}} \min_{1 \le i \le 2} \Big\{ \langle L_{U,W}\xi_i - L_{U,W}\sigma, \exp_W^{-1}(U)\rangle_W + \Psi_i(U,U) - \Psi_i(U,W)) - \mathcal{P}_i^\circ(U; \exp_U^{-1} W) \Big\} \\
&= \inf_{\xi \in B(u)} \sup_{W \in \mathcal{F}} \min \Big\{ \Big\langle L_{U,W}\xi_1 - \sigma, \exp_W^{-1}(U) \Big\rangle_W + \Psi_1(U,U) - \Psi_1(U,W), \\
&\qquad -\mathcal{P}_1^\circ(U; \exp_U^{-1}(W)) \\
&\qquad \Big\langle L_{U,W}\xi_2 - \sigma, \exp_W^{-1}(U) \Big\rangle_W + \Psi_2(U,U) - \Psi_2(U,W) - \mathcal{P}_2^\circ(U; \exp_U^{-1}(W)) \Big\} \\
&= \inf_{\xi \in B(u)} \sup_{W \in \mathcal{F}} \min \Big\{ -\Big\langle \xi_1 - \sigma, \exp_U^{-1}(W) \Big\rangle_U + \Psi_1(U,U) - \Psi_1(U,W) \\
&\qquad -\mathcal{P}_1^\circ(U; \exp_U^{-1}(W)) \\
&\qquad -\Big\langle \xi_2 - \sigma, \exp_U^{-1}(W) \Big\rangle_U + \Psi_2(U,U) - \Psi_2(U,W) - \mathcal{P}_2^\circ(U; \exp_U^{-1}(W)) \Big\} \\
&\qquad\qquad (\textit{on applying the parallel transport from } W \textit{ to } U)
\end{aligned}$$

$$= \inf_{t\in[0,1]} \sup_{W\in\mathcal{F}} \min\Big\{ -\Big\langle (2t-1)I + I, \frac{\ln\det W}{2} I \Big\rangle - 2\ln\det W - |\ln\det W|,$$
$$- \Big\langle 2I, \frac{\ln\det W}{2} I \Big\rangle - \ln\det W - \ln\det W \Big\}$$
$$= 2.$$

Let $U \in \mathcal{F}$ such that $\det U \in [e^{-2}, 1)$. Then, we have

$$\Pi(U) = \inf_{\xi\in B(U)} \sup_{W\in\mathcal{F}} \min_{1\le i\le 2} \Big\{ \langle L_{U,W}\xi_i - L_{U,W}\sigma, \exp_W^{-1}(U)\rangle_W + \Psi_i(U,U) - \Psi_i(U,W)$$
$$- \mathcal{P}_i^\circ(U; \exp_U^{-1}(W)) \Big\}$$

$$= \sup_{W\in\mathcal{F}} \min\Big\{ \Big\langle L_{U,W}\xi_1 - \sigma, \exp_W^{-1}(U)\Big\rangle_W + \Psi_1(U,U) - \Psi_1(U,W) - \mathcal{P}_1^\circ(U; \exp_U^{-1}(W)),$$
$$\Big\langle L_{U,W}\xi_2 - \sigma, \exp_W^{-1}(U)\Big\rangle_W + \Psi_2(U,U) - \Psi_2(U,W) - \mathcal{P}_2^\circ(U; \exp_U^{-1}(W)) \Big\}$$
$$= \sup_{W\in\mathcal{F}} \min\Big\{ -\Big\langle \xi_1 - \sigma, \exp_U^{-1}(W)\Big\rangle_U + \Psi_1(U,U) - \Psi_1(U,W) - \mathcal{P}_1^\circ(U; \exp_U^{-1}(W)),$$
$$-\Big\langle \xi_2 - \sigma, \exp_U^{-1}(W)\Big\rangle_U + \Psi_2(U,U) - \Psi_2(U,W) - \mathcal{P}_2^\circ(U; \exp_U^{-1}(W)) \Big\}$$
$$(\textit{on applying the parallel transport from } W \textit{ to } U)$$
$$= \sup_{W\in\mathcal{F}} \min\Big\{ -\Big\langle -U + I, \frac{\ln\frac{\det W}{\det U}}{2} U \Big\rangle + 2\ln\frac{\det U}{\det W} - \det U\Big(\ln\frac{\det U}{\det W}\Big),$$
$$- \Big\langle U + I, \frac{\ln\frac{\det W}{\det U}}{2} U \Big\rangle + \ln\frac{\det U}{\det W} + \det U\Big(\ln\frac{\det U}{\det W}\Big) \Big\}$$
$$= \Big(1 + (\det U)^{-\frac{1}{2}} - \det U\Big)(\ln\det U + 2)$$
$$\ge 0.$$

Notably, $\Pi(U) = 0$ if and only if $U = e^{-I}$. Therefore, Π is a gap function for WVVHVIP1.

Now, we introduce a regularized gap function for WVVHVIP. Let $\gamma > 0$ be any arbitrary parameter and a function $\vartheta^\gamma \colon \mathcal{F} \to \mathbb{R}$ be defined as follows:

$$\vartheta^\gamma(u) := \inf_{\xi\in B(u)} \sup_{w\in\mathcal{F}} \min_{i\in\mathcal{I}} \Big\{ \langle L_{u,w}\xi_i - L_{u,w}\sigma, \exp_w^{-1}(u)\rangle_w + \Psi_i(u,u) - \Psi_i(u,w)$$
$$- \mathcal{P}_i^\circ(u; \exp_u^{-1}(w)) - \frac{1}{2\gamma}\|\exp_w^{-1}(u)\|_w^2 \Big\}, \tag{16}$$

where $u \in \mathcal{F}$, $B(u) := \partial_c h_1(u) \times \partial_c h_2(u) \times \ldots \times \partial_c h_l(u)$, and $\xi_i \in \partial_c h_i(u)$ $(i \in \mathcal{I})$. For any $u \in \mathcal{F}$ and $\xi \in B(u)$, we let

$$M_\gamma(u,\xi) := \sup_{w\in\mathcal{F}} \min_{i\in\mathcal{I}} \Big\{ \langle L_{u,w}\xi_i - L_{u,w}\sigma, \exp_w^{-1}(u)\rangle_w + \Psi_i(u,u) - \Psi_i(u,w)$$
$$- \mathcal{P}_i^\circ(u; \exp_u^{-1}(w)) - \frac{1}{2\gamma}\|\exp_w^{-1}(u)\|_w^2 \Big\}.$$

In the following theorem, we establish that the function ϑ^γ defined in (16) is a gap function for WVVHVIP.

Theorem 4. *Let $\Psi_i(u,\cdot) : \mathcal{F} \to \mathbb{R}$ $(i \in \mathcal{I})$ be geodesic convex functions on $\mathcal{F}$ for every $u \in \mathcal{F}$. Then, for any $\gamma > 0$, the function ϑ^γ is a gap function for WVVHVIP.*

Proof. On the contrary, we suppose that there exists $u \in \mathcal{F}$ such that

$$\vartheta^{\gamma}(u) < 0.$$

Since $M_{\gamma}(u,\cdot)$ is continuous and $B(u)$ is a compact set for every $u \in \mathcal{F}$, this implies that there exist $\xi_i^* \in \partial_c h_i(u)$ $(i \in \mathcal{I})$ such that

$$\sup_{w\in\mathcal{F}} \min_{i\in\mathcal{I}} \Big\{ \langle L_{u,w}\xi_i^* - L_{u,w}\sigma, \exp_w^{-1}(u)\rangle_w + \Psi_i(u,u) - \Psi_i(u,w) - \mathcal{P}_i^{\circ}(u; \exp_u^{-1}(w)) - \frac{1}{2\gamma}\| \exp_w^{-1}(u)\|_w^2 \Big\} < 0.$$

It follows that for every $w \in \mathcal{F}$, we have

$$\min_{i\in\mathcal{I}} \Big\{ \langle L_{u,w}\xi_i^* - \sigma, \exp_w^{-1}(u)\rangle_w + \Psi_i(u,u) - \Psi_i(u,w) - \mathcal{P}_i^{\circ}(u; \exp_u^{-1}(w)) - \frac{1}{2\gamma}\| \exp_u^{-1}(w)\|_u^2 \Big\} < 0,$$

which is a contradiction for $w = u$. Therefore, we have

$$\vartheta^{\gamma}(u) \geq 0, \text{ for all } u \in \mathcal{F}. \tag{17}$$

Let $\vartheta^{\gamma}(\tilde{u}) = 0$ for some $\tilde{u} \in \mathcal{F}$. In view of the continuity of the function $M_{\gamma}(\tilde{u},\cdot)$ and the compactness of the set $B(\tilde{u})$, there exist $\xi_i^* \in \partial_c h_i(\tilde{u})$ $(i \in \mathcal{I})$ such that

$$\sup_{w\in\mathcal{F}} \min_{i\in\mathcal{I}} \Big\{ \langle L_{\tilde{u},w}\xi_i^* - L_{\tilde{u},w}\sigma, \exp_w^{-1}(\tilde{u})\rangle_w + \Psi_i(\tilde{u},\tilde{u}) - \Psi_i(\tilde{u},w) - \mathcal{P}_i^{\circ}(\tilde{u}; \exp_{\tilde{u}}^{-1}(w)) - \frac{1}{2\gamma}\| \exp_{\tilde{u}}^{-1}(w)\|_{\tilde{u}}^2 \Big\} = 0. \tag{18}$$

That is, for every $w \in \mathcal{F}$, we have

$$\min_{i\in\mathcal{I}} \Big\{ \langle L_{\tilde{u},w}\xi_i^* - L_{\tilde{u},w}\sigma, \exp_w^{-1}(\tilde{u})\rangle_w + \Psi_i(\tilde{u},\tilde{u}) - \Psi_i(\tilde{u},w) - \mathcal{P}_i^{\circ}(\tilde{u}; \exp_{\tilde{u}}^{-1}(w)) - \frac{1}{2\gamma}\| \exp_{\tilde{u}}^{-1}(w)\|_{\tilde{u}}^2 \Big\} \leq 0. \tag{19}$$

In view of the fact that $L_{\tilde{u},w} \exp_{\tilde{u}}^{-1}(w) = -\exp_w^{-1}(\tilde{u})$, for every $w \in \mathcal{F}$ we get from (19)

$$\min_{i\in\mathcal{I}} \Big\{ \langle L_{\tilde{u},w}\xi_i^* - L_{\tilde{u},w}\sigma, -L_{\tilde{u},w} \exp_{\tilde{u}}^{-1}(w)\rangle_w + \Psi_i(\tilde{u},\tilde{u}) - \Psi_i(\tilde{u},w) - \mathcal{P}_i^{\circ}(\tilde{u}; \exp_{\tilde{u}}^{-1}(w)) - \frac{1}{2\gamma}\| \exp_{\tilde{u}}^{-1}(w)\|_{\tilde{u}}^2 \Big\} \leq 0. \tag{20}$$

Now consider an arbitrary but fixed $\tilde{w} \in \mathcal{F}$. For $\tau \in (0,1)$, let $w_\tau = \exp_{\tilde{u}}(\tau \exp_{\tilde{u}}^{-1}(\tilde{w}))$. Since $\mathcal{F}$ is a geodesic convex set, it is evident that $w_\tau \in \mathcal{F}$, for all $\tau \in (0,1)$.

In view of (20) and the fact that $w_\tau \in \mathcal{F}$ $(\tau \in (0,1))$, we have

$$\min_{i\in\mathcal{I}} \Big\{ -\langle L_{\tilde{u},w_\tau}\xi_i^* - L_{\tilde{u},w_\tau}\sigma, L_{\tilde{u},w_\tau} \exp_{\tilde{u}}^{-1}(w_\tau)\rangle_{w_\tau} + \Psi_i(\tilde{u},\tilde{u}) - \Psi_i(\tilde{u},w_\tau) - \mathcal{P}_i^{\circ}(\tilde{u}; \exp_{\tilde{u}}^{-1}(w_\tau)) \Big\} \leq \frac{1}{2\gamma}\| \exp_{\tilde{u}}^{-1}(w_\tau)\|_{\tilde{u}}^2. \tag{21}$$

By employing geodesic convexity assumptions on $\Psi_i(\tilde{u}, \cdot)$ $(i \in \mathcal{I})$, positive homogeneous property of $\mathcal{P}_i^\circ(\tilde{u}; \cdot)$, and $\exp_{\tilde{u}}^{-1}(w_\tau) = \tau \exp_{\tilde{u}}^{-1}(\tilde{w})$ in (21), it follows that

$$\min_{i\in\mathcal{I}}\Big\{-\langle L_{\tilde{u},w_\tau}\xi_i^* - L_{\tilde{u},w_\tau}\sigma, L_{\tilde{u},w_\tau}\exp_{\tilde{u}}^{-1}(\tilde{w})\rangle_{w_\tau} + \Psi_i(\tilde{u},\tilde{u}) - \Psi_i(\tilde{u},\tilde{w}) - \mathcal{P}_i^\circ(\tilde{u}; \exp_{\tilde{u}}^{-1}(\tilde{w}))\Big\} \leq \frac{\tau}{2\gamma}\|\exp_{\tilde{u}}^{-1}(\tilde{w})\|_{\tilde{u}}^2. \tag{22}$$

In light of the fact that parallel transport preserves the inner product, we yield the following from (22):

$$\min_{i\in\mathcal{I}}\Big\{-\langle \xi_i^* - \sigma, \exp_{\tilde{u}}^{-1}(\tilde{w})\rangle_{\tilde{u}} + \Psi_i(\tilde{u},\tilde{u}) - \Psi_i(\tilde{u},\tilde{w}) - \mathcal{P}_i^\circ(\tilde{u}; \exp_{\tilde{u}}^{-1}(\tilde{w}))\Big\} \leq \frac{\tau}{2\gamma}\|\exp_{\tilde{u}}^{-1}(\tilde{w})\|_{\tilde{u}}^2. \tag{23}$$

Letting $\tau \downarrow 0$ in (23), we have

$$\min_{i\in\mathcal{I}}\Big\{-\langle \xi_i^* - \sigma, \exp_{\tilde{u}}^{-1}(\tilde{w})\rangle_{\tilde{u}} + \Psi_i(\tilde{u},\tilde{u}) - \Psi_i(\tilde{u},\tilde{w}) - \mathcal{P}_i^\circ(\tilde{u}; \exp_{\tilde{u}}^{-1}(\tilde{w}))\Big\} \leq 0. \tag{24}$$

This implies that there exists $k \in \mathcal{I}$ depending on $\tilde{w}$ such that

$$\langle \xi_k^* - \sigma, \exp_{\tilde{u}}^{-1}(\tilde{w})\rangle_{\tilde{u}} + \Psi_k(\tilde{u},\tilde{w}) - \Psi_k(\tilde{u},\tilde{u}) + \mathcal{P}_k^\circ(\tilde{u}; \exp_{\tilde{u}}^{-1}(\tilde{w})) \geq 0.$$

By following the similar steps for any $w \in \mathcal{F}$, we have

$$\Big(\langle \xi_1^* - \sigma, \exp_{\tilde{u}}^{-1}(w)\rangle_{\tilde{u}} + \Psi_1(\tilde{u},w) - \Psi_1(\tilde{u},\tilde{u}) + \mathcal{P}_1^\circ(\tilde{u}; \exp_{\tilde{u}}^{-1}(w)), \ldots, \langle \xi_l^* - \sigma, \exp_{\tilde{u}}^{-1}(w)\rangle_{\tilde{u}} + \Psi_l(\tilde{u},w) - \Psi_l(\tilde{u},\tilde{u}) + \mathcal{P}_l^\circ(\tilde{u}; \exp_{\tilde{u}}^{-1}(w))\Big) \notin -\operatorname{int}\mathbb{R}_+^l.$$

Therefore, $\tilde{u}$ is a solution of WVVHVIP.

For the converse part, we assume that there exists a solution $\tilde{u}$ of WVVHVIP in $\mathcal{F}$. This implies that there exist $\xi_i^* \in \partial_c h_i(\tilde{u})$ $(i \in \mathcal{I})$ such that the following condition holds:

$$\Big(\langle \xi_1^* - \sigma, \exp_{\tilde{u}}^{-1}(w)\rangle_{\tilde{u}} + \Psi_1(\tilde{u},w) - \Psi_1(\tilde{u},\tilde{u}) + \mathcal{P}_1^\circ(\tilde{u}; \exp_{\tilde{u}}^{-1}(w)), \ldots, \langle \xi_l^* - \sigma, \exp_{\tilde{u}}^{-1}(w)\rangle_{\tilde{u}} + \Psi_l(\tilde{u},w) - \Psi_l(\tilde{u},\tilde{u}) + \mathcal{P}_l^\circ(\tilde{u}; \exp_{\tilde{u}}^{-1}(w))\Big) \notin -\operatorname{int}\mathbb{R}_+^l,$$

for every $w \in \mathcal{F}$. Equivalently, for every $w \in \mathcal{F}$, there exists some $p \in \mathcal{I}$ such that we have

$$\langle \xi_p^* - \sigma, \exp_{\tilde{u}}^{-1}(w)\rangle_{\tilde{u}} + \Psi_p(\tilde{u},w) - \Psi_p(\tilde{u},\tilde{u}) + \mathcal{P}_p^\circ(\tilde{u}; \exp_{\tilde{u}}^{-1}(w)) + \frac{1}{2\gamma}\|\exp_w^{-1}(\tilde{u})\|_w^2 \geq 0. \tag{25}$$

Applying parallel transport from $\tilde{u}$ to w in (25) we get

$$\langle L_{\tilde{u},w}\xi_p^* - L_{\tilde{u},w}\sigma, \exp_w^{-1}(\tilde{u})\rangle_w + \Psi_p(\tilde{u},\tilde{u}) - \Psi_p(\tilde{u},w) - \mathcal{P}_p^\circ(\tilde{u}; \exp_{\tilde{u}}^{-1}(w)) - \frac{1}{2\gamma}\|\exp_w^{-1}(\tilde{u})\|_w^2 \leq 0. \tag{26}$$

From (26) it follows that

$$\vartheta^{\gamma}(\tilde{u}) = \inf_{\xi \in B(\tilde{u})} \sup_{w \in \mathcal{F}} \min_{i \in \mathcal{I}} \Big\{ \langle L_{\tilde{u},w}\xi_i - L_{\tilde{u},w}\sigma, \exp_w^{-1}(\tilde{u}) \rangle_w + \Psi_i(\tilde{u},\tilde{u}) - \Psi_i(\tilde{u},w) - \mathcal{P}_i^{\circ}(\tilde{u}; \exp_{\tilde{u}}^{-1}(w)) - \frac{1}{2\gamma} \| \exp_w^{-1}(\tilde{u}) \|_w^2 \Big\} \leq 0. \tag{27}$$

From (17) and (27) we have

$$\vartheta^{\gamma}(\tilde{u}) = 0.$$

This completes the proof. □

Remark 8.

1. *If $\mathbb{H}^n = \mathbb{R}^n$, $\Psi_i(u,w) = 0$, for all u, $w \in \mathcal{F}$, $i \in \mathcal{I}$, $\partial_c h_i(u)$ $(i \in \mathcal{I})$ is a single-valued vector field for all $u \in \mathcal{F}$, and $\mathcal{P}_i$ is a constant function, then Theorem 4 reduces to Theorem 4.2 deduced by Charitha et al. [34].*
2. *The results derived in Theorem 4 generalize the corresponding results established in Theorem 3.1 deduced by Fukushima [32] from the Euclidean space to a more general space, namely Hadamard manifolds, as well as generalize them from variational inequality problem to a broader category of vector variational-hemivariational inequality problems, in particular, WVVHVIP.*

In the following example, we illustrate the significance of Theorem 4.

Example 6. *Consider the following Riemannian manifold (see, [47]):*

$$\mathbb{H}^1 = \mathbb{R}_{++} := \{u \in \mathbb{R} \mid u > 0\}.$$

The Riemannian manifold $\mathbb{H}^1$ is endowed with the following Riemannian metric (see, [41]):

$$\mathcal{G}(u) = \frac{1}{u^2}, \text{ for all } u \in \mathbb{H}^1.$$

Further, $\mathbb{H}^1$ is a Hadamard manifold and the tangent space at every $u \in \mathbb{H}^1$ is the set of real numbers. That is, $T_u\mathbb{H}^1 = \mathbb{R}$ (see, [41]). For any $u \in \mathbb{H}^1$ and $v \in T_u\mathbb{H}^1$, the exponential map $\exp_u : T_u\mathbb{H}^1 \to \mathbb{H}^1$ is defined by:

$$\exp_u(v) := ue^{(v/u)}.$$

Moreover, the inverse of the exponential map $\exp_u^{-1} : \mathbb{H}^1 \to T_u\mathbb{H}^1$ is given by:

$$\exp_u^{-1}(w) := u \ln\left(\frac{w}{u}\right), \text{ for all } w \in \mathbb{H}^1.$$

Let $\mathcal{F} := \{u \in \mathbb{H}^1 \mid u = e^{\tau}, \tau \in [-2,2]\}$ be a nonempty, compact, and geodesic convex subset of $\mathbb{H}^1$. Further, let $h_1, h_2 \colon \mathcal{F} \to \mathbb{R}$ be defined as follows:

$$h_1(u) := \begin{cases} \frac{u^2}{4}, & u \geq 2, \\ \frac{u}{2}, & u < 2, \end{cases} \quad \text{and} \quad h_2(u) := \begin{cases} 2u^2, & u \geq 2, \\ 2u + 4, & u < 2. \end{cases}$$

The Clarke subdifferentials of h_1 and h_2 are given by:

$$\partial_c h_1(u) = \begin{cases} \frac{u^3}{2}, & u > 2, \\ [2,4], & u = 2, \\ \frac{u^2}{2}, & u < 2. \end{cases} \quad \text{and} \quad \partial_c h_2(u) = \begin{cases} 4u^3, & u > 2, \\ [8,32], & u = 2, \\ 2u^2, & u < 2, \end{cases}$$

Let the functions $\Psi_1, \Psi_2 \colon \mathcal{F} \times \mathcal{F} \to \mathbb{R}$ be defined as follows:

$$\Psi_1(u,w) := 2\ln uw \quad \text{and} \quad \Psi_2(u,w) := 2u + 2\ln w.$$

It can be verified that the functions $\Psi_i(u,\cdot)$ $(i = 1,2)$ are geodesic convex functions on $\mathcal{F}$.

The functions $\mathcal{P}_1, \mathcal{P}_2 : \mathbb{H}^1 \to \mathbb{R}$ are defined as follows:

$$\mathcal{P}_1(u) := |u-2| \quad \text{and} \quad \mathcal{P}_2(u) := u + 2.$$

The generalized directional derivative of functions $\mathcal{P}_1, \mathcal{P}_2$ are given by:

$$\mathcal{P}_1^\circ(u;v) = \begin{cases} v, & u > 2, \\ |v|, & u = 2, \\ -v, & u < 2, \end{cases} \quad \text{and} \quad \mathcal{P}_2^\circ(u;v) = v, \text{ for all } u \in \mathcal{F},\ v \in T_u\mathbb{H}^1.$$

Let $\gamma = 5$ and $\sigma = -1$. Now, we define a function $\vartheta^\gamma \colon \mathcal{F} \to \mathbb{R}$ as follows:

$$\vartheta^5(\tilde{u}) := \inf_{\xi \in B(\tilde{u})} \sup_{w \in \mathcal{F}} \min_{1 \le i \le 2} \Big\{ \langle L_{\tilde{u},w}\xi_i - L_{\tilde{u},w}\sigma, \exp_w^{-1}(\tilde{u})\rangle_w + \Psi_i(\tilde{u},\tilde{u}) - \Psi_i(\tilde{u},w) - \mathcal{P}_i^\circ(u; \exp_{\tilde{u}}^{-1}(w)) - \frac{1}{2\gamma}\| \exp_w^{-1}(\tilde{u})\|_w^2 \Big\}.$$

It can be verified that ϑ^5 serves as a gap function for the following variational-hemivariational inequality problem (WVVHVIP2): Find $u \in \mathcal{F}$ such that there exist $\xi_i \in \partial_c h_i(\tilde{u})$ $(i \in \mathcal{I} = \{1,2\})$, satisfying:

$$\Big(\Big\langle \xi_1 - \sigma, \exp_{\tilde{u}}^{-1}(w) \Big\rangle_{\tilde{u}} + \Psi_1(\tilde{u},w) - \Psi_1(\tilde{u},\tilde{u}) + \mathcal{P}_1^\circ(u; \exp_{\tilde{u}}^{-1}(w)), \Big\langle \xi_2 - \sigma, \exp_{\tilde{u}}^{-1}(w) \Big\rangle_{\tilde{u}} + \Psi_2(\tilde{u},w) - \Psi_2(\tilde{u},\tilde{u}) + \mathcal{P}_2^\circ(\tilde{u}; \exp_{\tilde{u}}^{-1}(w)) \Big) \notin -\operatorname{int}\mathbb{R}^2_+,$$

for all $w \in \mathcal{F}$.

In the following lemma, we prove the lower semicontinuity of the gap function ϑ^γ defined in (16).

Lemma 9. *Let for every $i \in \mathcal{I}$, the function $\Psi_i(u,\cdot) : \mathcal{F} \to \mathbb{R}$ be a geodesic convex function on $\mathcal{F}$ for every $u \in \mathcal{F}$. Then, ϑ^γ is a lower semicontinuous function on $\mathcal{F}$.*

Proof. Let $u,\ w \in \mathcal{F}$. Then we define

$$\hat{M}_\gamma((u,\xi),w) := \min_{i \in \mathcal{I}} \Big\{ \langle L_{u,w}\xi_i - L_{u,w}\sigma, \exp_w^{-1}(u)\rangle_w + \Psi_i(u,u) - \Psi_i(u,w) - \mathcal{P}_i^\circ(u; \exp_u^{-1}(w)) - \frac{1}{2\gamma}\| \exp_w^{-1}(u)\|_w^2 \Big\}, \tag{28}$$

where $\xi_i \in \partial_c h_i(u)$ $(i \in \mathcal{I})$.

The functions $(u,w) \mapsto \Psi_i(u,w)$ $(i \in \mathcal{I})$, $(u,\xi_i) \mapsto \langle L_{u,w}\xi_i - L_{u,w}\sigma, \exp_w^{-1}(u)\rangle_w$ $(i \in \mathcal{I})$ are continuous and the function $(u,v) \mapsto -\mathcal{P}_i^\circ(u;v)$ $(v \in T_u\mathbb{H}^n,\ i \in \mathcal{I})$ is lower

semicontinuous. Therefore, in view of Lemma 1, the function $\hat{M}_\gamma$ defined in (28) is a lower semicontinuous function in the argument (u, ξ) for every $w \in \mathcal{F}$.
Then, we have

$$M_\gamma(u, \xi) = \sup_{w \in \mathcal{F}} \hat{M}_\gamma((u, \xi), w), \tag{29}$$

is a lower semicontinuous function. This implies that—$M_\gamma(u, \xi)$ is upper semicontinuous. In view of the fact that $B(\cdot)$ is upper semicontinuous with compact values, we have from Lemma 7

$$\vartheta^\gamma(u) = \inf_{\xi \in B(u)} M_\gamma(u, \xi) = - \sup_{\xi \in B(u)} (-M_\gamma(u, \xi))$$

is a lower semicontinuous function on $\mathcal{F}$. □

Now, we define the Moreau-Yosida regularization of ϑ^γ in the setting of Hadamard manifolds.

Let $\gamma, \zeta > 0$ be two fixed parameters and $\Theta_{\vartheta^\gamma,\zeta} \colon \mathcal{F} \to \mathbb{R}$ be Moreau-Yosida regularization of ϑ^γ, which is defined as follows:

$$\Theta_{\vartheta^\gamma,\zeta}(u) := \inf_{w \in \mathcal{F}} \left\{ \vartheta^\gamma(w) + \zeta \| \exp_w^{-1}(u) \|_w^2 \right\}. \tag{30}$$

In the following theorem, we prove that the function $\Theta_{\vartheta^\gamma,\zeta}$ defined in (30) is a gap function for WVVHVIP.

Theorem 5. *Let for every $i \in \mathcal{I}$ and $u \in \mathcal{F}$, $\Psi_i(u, \cdot) : \mathcal{F} \to \mathbb{R}$ be a geodesic convex function on $\mathcal{F}$. Then, for any $\gamma, \zeta > 0$, $\Theta_{\vartheta^\gamma,\zeta}$ is a gap function for WVVHVIP.*

Proof. From Theorem 4, $\vartheta^\gamma(u) \geq 0$, for all $u \in \mathcal{F}$. Therefore, from (30), we have

$$\Theta_{\vartheta^\gamma,\zeta}(u) \geq 0, \text{ for all } u \in \mathcal{F}. \tag{31}$$

Let $\tilde{u} \in \mathcal{F}$ be an arbitrary solution of WVVHVIP. This implies that $\vartheta^\gamma(\tilde{u}) = 0$. Therefore,

$$\begin{aligned} \Theta_{\vartheta^\gamma,\zeta}(\tilde{u}) &= \inf_{w \in \mathcal{F}} \{ \vartheta^\gamma(w) + \zeta \| \exp_w^{-1}(\tilde{u}) \|_w^2 \} \\ &\leq \vartheta^\gamma(\tilde{u}) + \zeta \| \exp_{\tilde{u}}^{-1}(\tilde{u}) \|_{\tilde{u}}^2 = 0. \end{aligned} \tag{32}$$

From (31) and (32), it follows that

$$\Theta_{\vartheta^\gamma,\zeta}(\tilde{u}) = 0.$$

Conversely, let $\tilde{u} \in \mathcal{F}$ be an element such that $\Theta_{\vartheta^\gamma,\zeta}(\tilde{u}) = 0$. That is,

$$\inf_{w \in \mathcal{F}} \{ \vartheta^\gamma(w) + \zeta \| \exp_w^{-1}(\tilde{u}) \|_w^2 \} = 0.$$

Hence, it follows that there exists a minimizing sequence $\{w_j\}_{j=1}^\infty \subseteq \mathcal{F}$ such that

$$0 \leq \vartheta^\gamma(w_j) + \zeta \| \exp_{w_j}^{-1}(\tilde{u}) \|_{w_j}^2 < \frac{1}{j}. \tag{33}$$

Therefore, there exists a sequence $\{w_j\}_{j=1}^\infty \subseteq \mathcal{F}$ such that $\vartheta^\gamma(w_j) \to 0$ and $\| \exp_{w_j}^{-1}(\tilde{u}) \| \to 0$ as $j \to \infty$. Since, $\vartheta^\gamma(\cdot)$ is lower semicontinuous such that $\vartheta^\gamma(u) \geq 0$, for all $u \in \mathcal{F}$, it follows that

$$0 \leq \vartheta^\gamma(\tilde{u}) \leq \liminf_{j \to \infty} \vartheta^\gamma(w_j) = 0. \tag{34}$$

Therefore, $\vartheta^{\gamma}(\tilde{u}) = 0$, which in turn implies from Theorem 4 that $\tilde{u} \in \mathcal{F}$ is a solution of WVVHVIP. This completes the proof. □

Remark 9. *Theorem 5 generalizes Theorem 2.4 deduced by Yamashita and Fukushima [30] from the Euclidean space setting to the framework of Hadamard manifolds, as well as from variational inequality problems to nonsmooth weak vector variational-hemivariational inequality problem, namely, WVVHVIP.*

5. Global Error Bounds for WVVHVIP

In this section, by employing the Auslender type, regularized, and Moreau-Yosida type regularized gap functions, we derive the global error bounds for the solution of WVVHVIP under generalized geodesic monotonicity assumptions and properties of Clarke subdifferential.

Theorem 6. *Let for every $i \in \mathcal{I}$ and $u \in \mathcal{F}$, $\Psi_i(u, \cdot) : \mathcal{F} \to \mathbb{R}$ be a geodesic convex function on $\mathcal{F}$. Further, we assume that the following conditions hold:*

(i) *The set-valued map $\bigcup_{i\in\mathcal{I}} \partial_c h_i$ is geodesic strongly monotone with respect to σ having a positive constant m_h^{σ}.*

(ii) *There exists a constant $m_{\Psi} > 0$ such that*

$$\Psi_i(u_1, w_2) - \Psi_i(u_1, w_1) + \Psi_p(u_2, w_1) - \Psi_p(u_2, w_2) \leq m_{\Psi} \| \exp_{u_2}^{-1}(u_1)\|_{u_2} \| \exp_{w_2}^{-1}(w_1)\|_{w_2},$$

for every $u_1, u_2, w_1, w_2 \in \mathcal{F}$ and $i, p \in \mathcal{I}$.

(iii) *There exists a constant $m_{\mathcal{P}} > 0$ such that*

$$\mathcal{P}_i^{\circ}(w_1; \exp_{w_1}^{-1}(w_2)) + \mathcal{P}_p^{\circ}(w_2; \exp_{w_2}^{-1}(w_1)) \leq m_{\mathcal{P}} \| \exp_{w_2}^{-1}(w_1)\|_{w_2}^2,$$

for all $w_1, w_2 \in \mathcal{F}$ and $i, p \in \mathcal{I}$.

Further, we assume that $m_h^{\sigma} - m_{\Psi} - m_{\mathcal{P}} > 0$. Then, there exists a unique solution $\tilde{u}$ of WVVHVIP such that the following condition holds:

$$\| \exp_{\tilde{u}}^{-1}(w)\|_{\tilde{u}} \leq \sqrt{\frac{\Pi(w)}{\left(m_h^{\sigma} - m_{\Psi} - m_{\mathcal{P}}\right)}}, \quad \text{for all } w \in \mathcal{F}.$$

Proof. From the given hypotheses and Theorem 2, there exists a unique solution, namely $\tilde{u}$ of WVVHVIP in $\mathcal{F}$. This implies that there exist $\xi_i^* \in \partial_c h_i(\tilde{u})$ $(i \in \mathcal{I})$, satisfying:

$$\begin{aligned} \Big(&\langle \xi_1^* - \sigma, \exp_{\tilde{u}}^{-1}(w)\rangle_{\tilde{u}} + \Psi_1(\tilde{u}, w) - \Psi_1(\tilde{u}, \tilde{u}) + \mathcal{P}_1^{\circ}(\tilde{u}; \exp_{\tilde{u}}^{-1}(w)), \ldots, \\ &\langle \xi_l^* - \sigma, \exp_{\tilde{u}}^{-1}(w)\rangle_{\tilde{u}} + \Psi_l(\tilde{u}, w) - \Psi_l(\tilde{u}, \tilde{u}) + \mathcal{P}_l^{\circ}(\tilde{u}; \exp_{\tilde{u}}^{-1}(w)) \Big) \notin -\operatorname{int} \mathbb{R}_+^l, \end{aligned} \tag{35}$$

for every $w \in \mathcal{F}$. Equivalently, for every $w \in \mathcal{F}$ there exists some $p \in \mathcal{I}$ such that

$$\langle \xi_p^* - \sigma, \exp_{\tilde{u}}^{-1}(w)\rangle_{\tilde{u}} + \Psi_p(\tilde{u}, w) - \Psi_p(\tilde{u}, \tilde{u}) + \mathcal{P}_p^{\circ}(\tilde{u}; \exp_{\tilde{u}}^{-1}(w)) \geq 0. \tag{36}$$

From the definition of $\Pi(w)$ in (10), there exists $\zeta_i \in \partial_c h_i(w)$ $(i \in \mathcal{I})$, satisfying:

$$\begin{aligned}\Pi(w) &= \sup_{u\in\mathcal{F}} \min_{i\in\mathcal{I}} \Big\{ \langle L_{w,u}\zeta_i - L_{w,u}\sigma, \exp_u^{-1}(w)\rangle_u + \Psi_i(w,w) - \Psi_i(w,u) \\ &\qquad - \mathcal{P}_i^\circ(w; \exp_w^{-1}(u)) \Big\} \\ &\geq \min_{i\in\mathcal{I}} \Big\{ \langle L_{w,\tilde{u}}\zeta_i - L_{w,\tilde{u}}\sigma, \exp_{\tilde{u}}^{-1}(w)\rangle_{\tilde{u}} + \Psi_i(w,w) - \Psi_i(w,\tilde{u}) \\ &\qquad - \mathcal{P}_i^\circ(w; \exp_w^{-1}(\tilde{u})) \Big\}.\end{aligned} \tag{37}$$

From the given hypotheses (i)–(iii) and the fact that $\tilde{u}$ is a solution of WVVHVIP, it follows that for every $w \in \mathcal{F}$ there exists $k \in \mathcal{I}$ such that the following inequalities hold:

$$\begin{aligned}&\langle L_{w,\tilde{u}}\zeta_i - L_{w,\tilde{u}}\sigma, \exp_{\tilde{u}}^{-1}(w)\rangle_{\tilde{u}} + \Psi_i(w,w) - \Psi_i(w,\tilde{u}) - \mathcal{P}_i^\circ(w; \exp_w^{-1}(\tilde{u})) \\ &\quad = \Big\langle L_{w,\tilde{u}}\zeta_i - L_{w,\tilde{u}}\sigma, \exp_{\tilde{u}}^{-1}(w)\Big\rangle_{\tilde{u}} - \Big\langle \xi_k^* - \sigma, \exp_{\tilde{u}}^{-1}(w)\Big\rangle_{\tilde{u}} + \Big\langle \xi_k^* - \sigma, \exp_{\tilde{u}}^{-1}(w)\Big\rangle_{\tilde{u}} \\ &\qquad + \Psi_i(w,w) - \Psi_i(w,\tilde{u}) - \mathcal{P}_i^\circ(w; \exp_w^{-1}(\tilde{u})) \\ &\quad \geq \Big\langle \xi_k^* - \sigma, \exp_{\tilde{u}}^{-1}(w)\Big\rangle_{\tilde{u}} + \Psi_k(\tilde{u},w) - \Psi_k(\tilde{u},\tilde{u}) + \mathcal{P}_k^\circ(\tilde{u}; \exp_{\tilde{u}}^{-1}(w)) \\ &\qquad + (m_h^\sigma - m_\Psi - m_\mathcal{P}) \| \exp_{\tilde{u}}^{-1}(w)\|_{\tilde{u}}^2 \\ &\quad \geq (m_h^\sigma - m_\Psi - m_\mathcal{P}) \| \exp_{\tilde{u}}^{-1}(w)\|_{\tilde{u}}^2.\end{aligned} \tag{38}$$

From (36)–(38), the following inequality holds:

$$\Pi(w) \geq (m_h^\sigma - m_\Psi - m_\mathcal{P}) \| \exp_{\tilde{u}}^{-1}(w)\|_{\tilde{u}}^2, \text{ for all } w \in \mathcal{F}.$$

Hence, we complete the proof. □

Theorem 7. *Let $\gamma > 0$ be a fixed parameter such that all the hypotheses in Theorem 6 are satisfied with $m_h^\sigma - m_\Psi - m_\mathcal{P} - \frac{1}{2\gamma} > 0$. Then, there exists a unique solution $\tilde{u}$ of WVVHVIP such that the following inequality holds:*

$$\| \exp_{\tilde{u}}^{-1}(w)\|_{\tilde{u}} \leq \sqrt{\frac{\vartheta^\gamma(w)}{\left(m_h^\sigma - m_\Psi - m_\mathcal{P} - \frac{1}{2\gamma}\right)}}, \quad \text{for all } w \in \mathcal{F}.$$

Proof. From the given hypotheses and Theorem 2, there exists a unique solution, namely $\tilde{u}$ of WVVHVIP in $\mathcal{F}$. This implies that for every $w \in \mathcal{F}$, there exist $\xi_i^* \in \partial_c h_i(\tilde{u})$ $(i \in \mathcal{I})$, satisfying:

$$\begin{aligned}&\Big(\langle \xi_1^* - \sigma, \exp_{\tilde{u}}^{-1}(w)\rangle_{\tilde{u}} + \Psi_1(\tilde{u},w) - \Psi_1(\tilde{u},\tilde{u}) + \mathcal{P}_1^\circ(\tilde{u}; \exp_{\tilde{u}}^{-1}(w)), \ldots, \\ &\quad \langle \xi_l^* - \sigma, \exp_{\tilde{u}}^{-1}(w)\rangle_{\tilde{u}} + \Psi_l(\tilde{u},w) - \Psi_l(\tilde{u},\tilde{u}) + \mathcal{P}_l^\circ(\tilde{u}; \exp_{\tilde{u}}^{-1}(w)) \Big) \notin -\operatorname{int}\mathbb{R}_+^l.\end{aligned} \tag{39}$$

Equivalently, for every $w \in \mathcal{F}$ there exists some $p \in \mathcal{I}$ such that

$$\langle \xi_p^* - \sigma, \exp_{\tilde{u}}^{-1}(w)\rangle_{\tilde{u}} + \Psi_p(\tilde{u},w) - \Psi_p(\tilde{u},\tilde{u}) + \mathcal{P}_p^\circ(\tilde{u}; \exp_{\tilde{u}}^{-1}(w)) \geq 0. \tag{40}$$

From the definition of $\vartheta^{\gamma}(w)$ in (16), there exist $\zeta_i \in \partial_c h_i(w)$ $(i \in \mathcal{I})$, satisfying:

$$\begin{aligned}\vartheta^{\gamma}(w) = \sup_{u\in\mathcal{F}} \min_{i\in\mathcal{I}} \Big\{ &\langle L_{w,u}\zeta_i - L_{w,u}\sigma, \exp_u^{-1}(w)\rangle_u + \Psi_i(w,w) - \Psi_i(w,u) - \mathcal{P}_i^{\circ}(w; \exp_w^{-1}(u)) \\ &- \frac{1}{2\gamma}\|\exp_u^{-1}(w)\|_u^2 \Big\}, \\ \geq \min_{i\in\mathcal{I}} \Big\{ &\langle L_{w,\tilde{u}}\zeta_i - L_{w,\tilde{u}}\sigma, \exp_{\tilde{u}}^{-1}(w)\rangle_{\tilde{u}} + \Psi_i(w,w) - \Psi_i(w,\tilde{u}) - \mathcal{P}_i^{\circ}(w; \exp_w^{-1}(\tilde{u})) \\ &- \frac{1}{2\gamma}\|\exp_{\tilde{u}}^{-1}(w)\|_{\tilde{u}}^2 \Big\}.\end{aligned} \tag{41}$$

From the given hypotheses (i)–(iii) and the fact that $\tilde{u}$ is a solution of WVVHVIP, it follows that for every $w \in \mathcal{F}$ there exists $k \in \mathcal{I}$ such that the following inequalities hold:

$$\begin{aligned}&\langle L_{w,\tilde{u}}\zeta_i - L_{w,\tilde{u}}\sigma, \exp_{\tilde{u}}^{-1}(w)\rangle_{\tilde{u}} + \Psi_i(w,w) - \Psi_i(w,\tilde{u}) - \mathcal{P}_i^{\circ}(w; \exp_w^{-1}(\tilde{u})) - \frac{1}{2\gamma}\|\exp_{\tilde{u}}^{-1}(w)\|_{\tilde{u}}^2 \\ &= \left\langle L_{w,\tilde{u}}\zeta_i - L_{w,\tilde{u}}\sigma, \exp_{\tilde{u}}^{-1}(w)\right\rangle_{\tilde{u}} - \left\langle \xi_k^* - \sigma, \exp_{\tilde{u}}^{-1}(w)\right\rangle_{\tilde{u}} + \left\langle \xi_k^* - \sigma, \exp_{\tilde{u}}^{-1}(w)\right\rangle_{\tilde{u}} \\ &\quad + \Psi_i(w,w) - \Psi_i(w,\tilde{u}) - \mathcal{P}_i^{\circ}(w; \exp_w^{-1}(\tilde{u})) - \frac{1}{2\gamma}\|\exp_{\tilde{u}}^{-1}(w)\|_{\tilde{u}}^2 \\ &\geq \left\langle \xi_k^* - \sigma, \exp_{\tilde{u}}^{-1}(w)\right\rangle_{\tilde{u}} + \Psi_k(\tilde{u},w) - \Psi_k(\tilde{u},\tilde{u}) + \mathcal{P}_k^{\circ}(\tilde{u}; \exp_{\tilde{u}}^{-1}(w)) \\ &\quad + \left(m_h^{\sigma} - m_{\Psi} - m_{\mathcal{P}} - \frac{1}{2\gamma}\right)\|\exp_{\tilde{u}}^{-1}(w)\|_{\tilde{u}}^2 \\ &\geq \left(m_h^{\sigma} - m_{\Psi} - m_{\mathcal{P}} - \frac{1}{2\gamma}\right)\|\exp_{\tilde{u}}^{-1}(w)\|_{\tilde{u}}^2.\end{aligned} \tag{42}$$

From (40), (41), and (42), the following inequality holds:

$$\vartheta^{\gamma}(w) \geq \left(m_h^{\sigma} - m_{\Psi} - m_{\mathcal{P}} - \frac{1}{2\gamma}\right)\|\exp_{\tilde{u}}^{-1}(w)\|_{\tilde{u}}^2, \text{ for all } w \in \mathcal{F}.$$

Hence, we complete the proof. □

Remark 10. *Theorem 6 generalizes Lemma 4.1 derived by Yamashita and Fukushima [30] from Euclidean space to an even more general space, namely Hadamard manifolds and from variational inequality problem to a more general problem, in particular, WVVHVIP.*

In the following example, we illustrate the significance of Theorem 6.

Example 7. *Let $\mathbb{H}^2$ be the manifold as considered in Example 1. Moreover, we consider $\mathcal{F}$, h_i $(i = 1,2)$, Ψ_i $(i = 1,2)$, and $\mathcal{P}_i$ $(i = 1,2)$ as defined in Example 4.*

Now, we define a function $\vartheta^{\gamma} : \mathcal{F} \to \mathbb{R}$ corresponding to $\gamma = 5$ as follows:

$$\begin{aligned}\vartheta^5(u) := \sup_{w\in\mathcal{F}} \min_{i\in\{1,2\}} \Big\{ &\left\langle L_{u,w}\xi_i - L_{u,w}\sigma, \exp_w^{-1}(u)\right\rangle_w + \Psi_i(u,u) - \Psi_i(u,w) \\ &- \mathcal{P}_i^{\circ}(u; \exp_u^{-1}(w)) - \frac{1}{10}||\exp_w^{-1} u||_w^2 \Big\}.\end{aligned} \tag{43}$$

On simplifying (43), it follows that

$$\vartheta^5(u) = \left(\left(\frac{-2 + 8u_2 \ln u_2}{u_2}\right) + 3\ln u_2 - \frac{1}{10}\left(\ln u_2 - \frac{1}{8}\right)\right)\left(\ln u_2 - \frac{1}{8}\right), \text{ for all } u \in \mathcal{F}.$$

Moreover, $\vartheta^{\gamma}(\tilde{u}) = 0$ if and only if $\tilde{u} = (0, e^{\frac{1}{8}})$. Therefore, ϑ^{γ} for $\gamma = 5$ is a regularized gap function for WVVHVIP2. Furthermore, $m_h^{\sigma} - m_{\Psi} - m_{\mathcal{P}} - \frac{1}{2\gamma} > 0$ and $\tilde{u} = (0, e^{\frac{1}{8}})$ is the unique

solution of WVVHVIP2. In view of Example 4, all the hypotheses of Theorem 6 are satisfied. Therefore, the following inequality holds:

$$||\exp_{\tilde{u}}^{-1}(w)||_{\tilde{u}} \leq \sqrt{\frac{\vartheta^{\gamma}(w)}{\left(m_h^{\sigma} - m_{\Psi} - m_{\mathcal{P}} - \frac{1}{2\gamma}\right)}}, \text{ for all } w \in \mathcal{F}.$$

Equivalently,

$$\begin{aligned}
&\sqrt{\frac{\vartheta^{\gamma}(w)}{\left(m_h^{\sigma} - m_{\Psi} - m_{\mathcal{P}} - \frac{1}{2\gamma}\right)}} \\
&= \sqrt{5.75\left(\left(\frac{-2 + 8w_2 \ln w_2}{w_2}\right) + 3\ln w_2 - \frac{1}{10}\left(\ln w_2 - \frac{1}{8}\right)\right)\left(\ln w_2 - \frac{1}{8}\right)} \\
&\geq \left|\left(\ln w_2 - \frac{1}{8}\right)\right|, \text{ for all } w = (w_1, w_2) \in \mathcal{F}.
\end{aligned}$$

Theorem 8. *Let $\gamma > 0$ and $\zeta > 0$ be fixed parameters, such that $m_h^{\sigma} - m_{\Psi} - m_{\mathcal{P}} > \frac{1}{2\gamma}$. Furthermore, we assume that all the hypotheses of Theorem 6 are satisfied. Then, there exists a unique solution $\tilde{u}$ of WVVHVIP such that the following inequality holds:*

$$\sqrt{\frac{\Theta_{\vartheta^{\gamma},\zeta}(w)}{\zeta}} \leq \|\exp_{\tilde{u}}^{-1}(w)\|_{\tilde{u}} \leq \sqrt{\frac{2\Theta_{\vartheta^{\gamma},\zeta}(w)}{\min\{m_h^{\sigma} - m_{\Psi} - m_{\mathcal{P}} - \frac{1}{2\gamma}, \zeta\}}}, \text{ for all } w \in \mathcal{F}.$$

Proof. From the given hypotheses and Theorem 2, there exists a unique solution of WVVHVIP in $\mathcal{F}$, namely $\tilde{u}$.

In view of (30), we have

$$\begin{aligned}
\Theta_{\vartheta^{\gamma},\zeta}(w) &= \inf_{\tilde{w}\in\mathcal{F}}\left\{\vartheta^{\gamma}(\tilde{w}) + \zeta\|\exp_{\tilde{w}}^{-1}(w)\|_{\tilde{w}}^2\right\} \\
&\leq \vartheta^{\gamma}(\tilde{u}) + \zeta||\exp_{\tilde{u}}^{-1}(w)||_{\tilde{u}} \\
&= \zeta||\exp_{\tilde{u}}^{-1}(w)||_{\tilde{u}}^2.
\end{aligned}$$

This implies that

$$||\exp_{\tilde{u}}^{-1}(w)||_{\tilde{u}} \geq \sqrt{\frac{\Theta_{\vartheta^{\gamma},\zeta}(w)}{\zeta}}, \text{ for all } w \in \mathcal{F}. \tag{44}$$

Let w be an arbitrary element of $\mathcal{F}$. It follows from Theorem 7 that

$$\begin{aligned}
\Theta_{\vartheta^{\gamma},\zeta}(w) &= \inf_{\tilde{w}\in\mathcal{F}}\left\{\vartheta^{\gamma}(\tilde{w}) + \zeta\|\exp_{\tilde{w}}^{-1}(w)\|_{\tilde{w}}^2\right\} \\
&\geq \inf_{\tilde{w}\in\mathcal{F}}\left\{\left(m_h^{\sigma} - m_{\Psi} - m_{\mathcal{P}} - \frac{1}{2\gamma}\right)\|\exp_{\tilde{w}}^{-1}(\tilde{u})\|_{\tilde{w}}^2 + \zeta\|\exp_{\tilde{w}}^{-1}(w)\|_{\tilde{w}}^2\right\} \\
&\geq \min\left\{m_h^{\sigma} - m_{\Psi} - m_{\mathcal{P}} - \frac{1}{2\gamma}, \zeta\right\}\inf_{\tilde{w}\in\mathcal{F}}\{\|\exp_{\tilde{w}}^{-1}(\tilde{u})\|_{\tilde{w}}^2 + \|\exp_{\tilde{w}}^{-1}(w)\|_{\tilde{w}}^2\} \\
&\geq \frac{1}{2}\min\left\{m_h^{\sigma} - m_{\Psi} - m_{\mathcal{P}} - \frac{1}{2\gamma}, \zeta\right\}\|\exp_{\tilde{u}}^{-1}(w)\|_{\tilde{u}}^2.
\end{aligned} \tag{45}$$

From (44) and (45), the following conclusion holds:

$$\sqrt{\frac{\Theta_{\vartheta^{\gamma},\zeta}(w)}{\zeta}} \leq \|\exp_{\tilde{u}}^{-1}(w)\|_{\tilde{u}} \leq \sqrt{\frac{2\Theta_{\vartheta^{\gamma},\zeta}(w)}{\min\{m_h^{\sigma} - m_{\Psi} - m_{\mathcal{P}} - \frac{1}{2\gamma}, \zeta\}}}, \text{ for all } w \in \mathcal{F}.$$

Hence, we complete the proof. □

Remark 11. *If $\mathbb{H}^n = \mathbb{R}^n$, then $\exp_u^{-1}(w) = w - u$, for all $u, w \in \mathbb{H}^n$. Further, if $\mathcal{I} = \{1\}$, $\mathcal{P}_1$, is a constant function, $\partial_c h_1(u)$ is a single-valued vector field for all $u \in \mathcal{F}$, and $\Psi_1(u, w) = 0$, for all $u, w \in \mathcal{F}$, then Theorem 8 reduces to Theorem 4.3 deduced by Yamashita and Fukushima [30].*

Now, we provide a non-trivial example to demonstrate the significance of Theorem 8.

Example 8. *Consider $\mathbb{H}^2$, $\mathcal{F}$, h_i, Ψ_i, $\mathcal{P}_i$ $(i = 1, 2)$ as defined in Example 7. Then, the function $\Theta_{\vartheta^\gamma,\zeta} : \mathcal{F} \to \mathbb{R}$ corresponding to $\gamma = 5$ and $\zeta = \frac{1}{7}$ is defined as follows:*

$$\begin{aligned}\Theta_{\vartheta^5,\frac{1}{7}}(w) &:= \inf_{\tilde{w}\in\mathcal{F}}\left\{\vartheta^5(\tilde{w}) + \frac{1}{7}\left\|\ln\left(\frac{w_2}{\tilde{w}_2}\right)\right\|_{\tilde{w}_2}^2\right\}\\ &= \inf_{\tilde{w}\in\mathcal{F}}\left\{\left(\left(\frac{-2+8\tilde{w}_2\ln\tilde{w}_2}{\tilde{w}_2}\right) + 3\ln\tilde{w}_2 - \frac{1}{10}\left(\ln\tilde{w}_2 - \frac{1}{8}\right)\right)\left(\ln\tilde{w}_2 - \frac{1}{8}\right)\right.\\ &\qquad\left. + \frac{1}{7}(\ln w_2 - \ln\tilde{w}_2)^2\right\}\\ &= \frac{1}{7}\left(\ln w_2 - \frac{1}{8}\right)^2.\end{aligned}$$

Moreover, $\Theta_{\vartheta^\gamma,\zeta}(\tilde{u}) = 0$ if and only if $\tilde{u} = (0, e^{\frac{1}{8}})$. Therefore, $\Theta_{\vartheta^\gamma,\zeta}$ for $\gamma = 5, \zeta = \frac{1}{7}$ is a Moreau-Yosida regularized gap function for WVVHVIP2.

From Examples 4 and 7, all the hypotheses of Theorem 6 are satisfied, which leads to the following inequality:

$$\sqrt{\frac{\Theta_{\vartheta^\gamma,\zeta}(w)}{\zeta}} \le ||\exp_{\tilde{u}}^{-1}(w)||_{\tilde{u}} \le \sqrt{\frac{2\Theta_{\vartheta^\gamma,\zeta}(w)}{\min\{(m_h^\sigma - m_\Psi - m_\mathcal{P} - \frac{1}{2\gamma}), \zeta\}}}, \text{ for all } w \in \mathcal{F}.$$

Equivalently, for every $w = (w_1, w_2) \in \mathcal{F}$, the following conditions hold:

$$\begin{aligned}\sqrt{\frac{\Theta_{\vartheta^\gamma,\zeta}(w)}{\zeta}} &= \left|\left(\ln u_2 - \frac{1}{8}\right)\right| = \left|\left(\ln u_2 - \frac{1}{8}\right)\right|,\\ \sqrt{\frac{2\Theta_{\vartheta^\gamma,\zeta}(w)}{\min\{(m_h^\sigma - m_\Psi - m_\mathcal{P} - \frac{1}{2\gamma}), \zeta\}}} &= 2\left|\left(\ln u_2 - \frac{1}{8}\right)\right| \ge \left|\left(\ln u_2 - \frac{1}{8}\right)\right|.\end{aligned}$$

6. Conclusions and Future Research Directions

In this paper, we have investigated a class of nonsmooth weak vector variational-hemivariational inequality problems, namely, WVVHVIP in the setting of Hadamard manifolds. By employing an analogous to the KKM lemma, we have established the existence of solutions for WVVHVIP without using any monotonicity assumptions. Moreover, we have established the uniqueness of the solution to WVVHVIP under generalized geodesic strong monotonicity assumptions. We have formulated several gap functions for WVVHVIP, in particular, Auslender, regularized, and Moreau-Yosida type regularized gap functions. Subsequently, we have established necessary and sufficient conditions for the existence of the solutions to the considered vector variational-hemivariational inequality problem. Furthermore, the global error bounds for the solution of WVVHVIP have been derived in terms of the Auslender type, regularized, and Moreau-Yosida type regularized gap functions under generalized geodesic strong monotonicity assumptions of Clarke sub-

differentials. Several non-trivial examples have been provided in the Hadamard manifold setting to demonstrate the significance of the established results.

The results derived in this paper extend and generalize several well-known results existing in the literature. In particular, the results derived in this article generalize the corresponding results established by Chen and Huang [10] from nonsmooth weak vector variational inequality problems to nonsmooth weak vector variational-hemivariational inequality problems in the framework of Hadamard manifolds. Various existence results for the solution of WVVHVIP generalize the corresponding results derived by Jayswal et al. [44] from nonsmooth mixed weak vector variational inequality problems to nonsmooth weak vector variational-hemivariational inequality problems in the Hadamard manifold setting. Furthermore, the results established in the present article generalize the corresponding results derived in [30–32] from the Euclidean space setting to the framework of Hadamard manifolds and from variational inequality problems to WVVHVIP. In addition, several results related to gap functions and global error bounds for WVVHVIP generalize the corresponding results derived by Charitha et al. [34] from vector variational inequality problems to nonsmooth weak vector-variational hemivariational inequality problems and generalize them from the framework of Euclidean space to the setting of Hadamard manifolds.

The results derived in the present article can be applied to various real-life problems in the domains of mechanics and engineering, such as optimal control and frictional contact problems (see, [23,24]). Moreover, it is significant to observe that all the results derived in this paper use the linear property of the inner product in tangent spaces. However, several variational inequalities involve bifunctions (see, for instance, [20,49]), which may not necessarily be linear, then the results derived in the present article may no longer be applicable for nonsmooth vector variational-hemivariational inequalities involving bifunctions. We intend to address this limitation in our future research work.

The results established in the present article suggest various potential avenues for future research work. For instance, investigating gap functions and regularized gap functions for mixed vector quasi-variational-hemivariational inequality problems would be an interesting research problem. Furthermore, in view of the work of Oyewole [63], we would like to develop subgradient extragradient-type algorithms for WVVHVIP in our future course of study.

Author Contributions: Conceptualization, B.B.U. and I.S.-M.; Methodology, B.B.U.; Software, S.S. and P.M.; Validation, B.B.U.; Formal analysis, I.S.-M. and S.S.; Investigation, B.B.U. and S.S.; Resources, B.B.U.; Writing—original draft, P.M.; Writing—review and editing, S.S. All authors have read and agreed to the published version of the manuscript.

Funding: The first author is supported by the Science and Engineering Research Board (SERB), Government of India, under MATRICS scheme via grant number MTR/2022/000925. The second author extends her gratitude to the Ministry of Education, Government of India, for its financial support through the Prime Minister Research Fellowship (PMRF) under grant PMRF ID-2703571.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: The authors affirm that data sharing does not apply to this article since no datasets were generated or analyzed during the current study.

Acknowledgments: The authors would like to thank the anonymous referees for their careful reading of the paper and constructive suggestions that have substantially improved the manuscript.

Conflicts of Interest: The authors declare that there are no actual or potential conflicts of interest in relation to this article.

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