

Article

Exploring Climate-Induced Oxygen–Plankton Dynamics Through Proportional–Caputo Fractional Modeling

Mohamed A. Barakat ^{1,2,*} , Areej A. Almoneef ³ , Abd-Allah Hyder ^{4,*}  and Tarek Aboelenen ⁵ 

¹ Department of Computer Science, College of Al Wajh, University of Tabuk, Tabuk 71491, Saudi Arabia

² Department of Mathematics, Faculty of Sciences, Al-Azhar University, Assiut 71524, Egypt

³ Department of Mathematical Sciences, College of Science, Princess Nourah Bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia; aaalmoneef@pnu.edu.sa

⁴ Department of Mathematics, College of Science, King Khalid University, P.O. Box 9004, Abha 61413, Saudi Arabia

⁵ Department of Mathematics, College of Science, Qassim University, Buraydah 51452, Saudi Arabia; t.aboelenen@qu.edu.sa

* Correspondence: m.barakat@ut.edu.sa (M.A.B.); abahahmed@kku.edu.sa (A.-A.H.)

Abstract: In this work, we develop and analyze a novel fractional-order framework to investigate the interactions among oxygen, phytoplankton, and zooplankton under changing climatic conditions. Unlike standard integer-order formulations, our model incorporates a Proportional–Caputo (PC) fractional derivative, allowing the system dynamics to capture non-local influences and memory effects over time. Initially, we rigorously verify that a unique solution exists by suitable fixed-point theorems, demonstrating that the proposed fractional system is both well-defined and robust. We then derive stability criteria to ensure Ulam–Hyers stability (UHS), confirming that small perturbations in initial states lead to bounded variations in long-term behavior. Additionally, we explore extended UHS to assess sensitivity against time-varying parameters. Numerical simulations illustrate the role of fractional-order parameters in shaping oxygen availability and plankton populations, highlighting critical shifts in system trajectories as the order of differentiation approaches unity.

Keywords: fractional calculus; fractional derivative; Proportional–Caputo fractional derivative; fractional-order climate change model; Banach fixed point theorem; Ulam–Hyers stability; predictor-corrector fractional Adams method

MSC: 34A08; 47H10; 86A10; 65L20



Academic Editor: Juan Eduardo Nápoles Valdés

Received: 15 February 2025

Revised: 6 March 2025

Accepted: 9 March 2025

Published: 17 March 2025

Citation: Barakat, M.A.; Almoneef, A.A.; Hyder, A.-A.; Aboelenen, T. Exploring Climate-Induced Oxygen–Plankton Dynamics Through Proportional–Caputo Fractional Modeling. *Mathematics* **2025**, *13*, 980. <https://doi.org/10.3390/math13060980>

Copyright: © 2025 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

1. Introduction

In recent decades, the impacts of climate change have become apparent globally, affecting not only human populations but also microorganisms and animal species across various ecosystems. These organisms are adapting to the evolving climate, especially in coastal and low-lying regions where temperature increases are felt most acutely. Due to industrialization and increased mechanization, the average global temperature has risen approximately by 0.3 °C, with more significant effects observed along coastal zones and at sea surfaces. By the end of the 21st century, climate models suggest that ecosystems may face temperature increases approaching 4 °C, exacerbated by the ongoing loss of forests and natural carbon absorbers. Industrial activities elevate carbon dioxide levels, intensifying the greenhouse effect and further warming the atmosphere, making conditions increasingly unsuitable for numerous species [1–6].

The warming of sea surfaces has also disrupted marine ecosystems and their natural movements. Global temperature rises have triggered northward shifts in species distributions, impacting plankton populations and causing poleward migrations in oceanic climates [7,8]. Consequently, plankton movement patterns are undergoing noticeable changes [9–13]. Environmental scientists and climate experts are closely monitoring these changes, aiming to slow phytoplankton migration over the next century. They predict a shift in these patterns from the Atlantic to Greenland's colder waters, prompting further exploration of plankton species' responses to potential marine biodiversity losses through mathematical modeling. Identifying viable solutions for the survival of these species in increasingly adverse environments remains crucial [14–18].

Fractional derivatives and integrals have become vital tools in mathematical modeling, particularly for complex systems where traditional calculus cannot capture the memory and hereditary properties of processes over time [19–21]. In climate change modeling, fractional calculus provides a framework to describe diffusion processes with increased accuracy, accommodating the anomalous diffusion patterns observed in atmospheric and oceanic data [22]. These models offer a more realistic representation of critical climate variables, such as temperature anomalies and greenhouse gas distributions, that often display non-Gaussian behavior [23]. By incorporating fractional time derivatives, models are able to simulate non-local interactions and persistence effects, which are crucial in understanding long-term climate dynamics, significantly improving forecasting capabilities for extreme weather events, prolonged heatwaves, and seasonal droughts [24]. Recent studies have also highlighted how fractional calculus aids in capturing the energy transfer processes across scales in fluid dynamics, which is essential for simulating ocean currents and heat fluxes—two key factors impacted by climate change [25]. In addition, fractional models have demonstrated advantages in predicting shifts in sea level and salinity profiles, directly linking these aspects to climate-driven changes in thermohaline circulation [26]. For instance, the fractional approach has been applied to track the complex patterns of CO₂ sequestration and methane emissions in various ecosystems, revealing feedback loops that can exacerbate or mitigate warming trends [27]. This holistic approach allows scientists to understand not just isolated events, but also the interconnected impacts across the climate system, enhancing the precision of projections for phenomena such as ice melt and rising ocean acidity levels [28]. The use of fractional calculus is increasingly validated by research, which emphasizes its role in improving model robustness and accuracy over extended timescales by addressing the non-local dynamics and memory effects inherent in climate processes. This method provides crucial insights into adaptive responses and potential mitigation strategies, offering a pathway to design more effective climate policies [29]. The growing application of fractional derivatives in climate science exemplifies the shift towards more sophisticated, comprehensive models that are better equipped to forecast and address the impacts of climate change in the coming decades [30].

This work introduces a new perspective on modeling the interactions among oxygen, phytoplankton, and zooplankton by employing a $\mathbb{P}\mathbb{C}$ fractional derivative, which captures non-local effects and memory mechanisms. The paper rigorously establishes the existence and uniqueness of solutions within this framework, demonstrating that memory factors play a crucial role in ecological stability and population dynamics. A thorough stability analysis is also carried out, highlighting the classical UHS criteria and its extended version, which confirm the model's robustness under small perturbations. The numerical experiments further reinforce the theoretical findings by showcasing diverse dynamical behaviors influenced by the fractional order and proportional parameters.

The paper is structured as follows. In Section 2, an overview of fractional calculus fundamentals is provided, with particular focus on the $\mathbb{P}\mathbb{C}$ operator. Section 3 provides the

climate-driven oxygen–plankton system, describing its components and the underlying assumptions. Section 4 presents the main analytical results on existence and uniqueness using fixed-point methods. Subsequently, Section 5 conducts a thorough stability investigation, covering both UH and extended UH criteria. Section 6 illustrates the model's behavior through numerical experiments, demonstrating how fractional effects shape the long-term outcomes. Finally, Section 7 summarizes the key conclusions.

2. Fundamentals of Fractional Operators

This section outlines the foundational fractional operators applied in our research. These operators are essential for the exploration and analyzing of fractional differential equations.

Consider $\vartheta \in \mathbb{C}$ with $\operatorname{Re}(\vartheta) > 0$, and let $\psi : [\omega, \phi] \rightarrow \mathbb{R}$, where $\omega < \phi$. Based on these assumptions, we now present the subsequent definitions.

Definition 1 ([31]). The left-sided Riemann–Liouville fractional integral of ψ of order ϑ is expressed as follows:

$$({}_{\omega}\mathcal{F}^{\vartheta}\psi)(\zeta) = \frac{1}{\Gamma(\vartheta)} \int_{\omega}^{\zeta} (\zeta - v)^{\vartheta-1} \psi(v) dv, \quad \zeta > \omega. \quad (1)$$

The right-sided Riemann–Liouville fractional integral of ψ with order ϑ is given by

$$(\mathcal{F}_{\phi}^{\vartheta}\psi)(\zeta) = \frac{1}{\Gamma(\vartheta)} \int_{\zeta}^{\phi} (v - \zeta)^{\vartheta-1} \psi(v) dv, \quad \zeta < \phi. \quad (2)$$

Definition 2 ([31]). Assuming $\operatorname{Re}(\vartheta) \geq 0$ and setting $\rho = [\vartheta] + 1$, we define the left-sided Riemann–Liouville fractional derivative of ψ of order ϑ as follows:

$$({}_{\omega}\mathcal{G}^{\vartheta}\psi)(\zeta) = \left(\frac{d}{d\zeta}\right)^{\rho} ({}_{\omega}\mathcal{F}^{\rho-\vartheta}\psi)(\zeta). \quad (3)$$

The right-sided Riemann–Liouville fractional derivative of ψ of order ϑ is given by

$$(\mathcal{G}_{\phi}^{\vartheta}\psi)(\zeta) = \left(-\frac{d}{d\zeta}\right)^{\rho} (\mathcal{F}_{\phi}^{\rho-\vartheta}\psi)(\zeta). \quad (4)$$

Definition 3. If $\rho = [\vartheta] + 1$, the left-sided Caputo fractional derivative of order ϑ for the function ψ is expressed as

$$({}_{\omega}^{\mathcal{C}}\mathcal{G}^{\vartheta}\psi)(\zeta) = ({}_{\omega}\mathcal{F}^{\rho-\vartheta}\psi^{(\rho)})(\zeta). \quad (5)$$

The right-sided Caputo fractional derivative of order ϑ for the function ψ is expressed as

$$({}^{\mathcal{C}}\mathcal{G}_{\phi}^{\vartheta}\psi)(\zeta) = (\mathcal{F}_{\phi}^{\rho-\vartheta}(-1)^{\rho}\psi^{(\rho)})(\zeta). \quad (6)$$

In [32], a new category of generalized fractional operators was introduced. These operators exhibit three notable traits: their associated kernels incorporate an exponential term, the resultant fractional integrals uphold a semigroup property, and they furnish a direct extension of the Riemann–Liouville and Caputo fractional frameworks. Below, we provide the precise definitions of these operators.

Definition 4 (cf. [32]). Let $\vartheta \in (0, 1]$ and $\mu \in (0, 1]$. The fractional integral of $\mathbb{P}\mathbb{C}$ type of order ϑ applied to a function γ is given by

$${}_{\mathcal{P}\mathcal{C}}\mathcal{F}^{\vartheta,\mu}\gamma(\zeta) = \frac{1}{\mu^{\vartheta}\Gamma(\vartheta)} \int_0^{\zeta} \exp\left(\frac{\mu-1}{\mu}(\zeta-t)\right) (\zeta-t)^{\vartheta-1} \gamma(t) dt.$$

Definition 5 (cf. [32]). Under the same conditions on ϑ and μ , define $\sigma = \lfloor \vartheta \rfloor + 1$. Then the one-sided $\mathbb{P}\mathbb{C}$ fractional derivative of order ϑ is formulated as

$${}_{\mathcal{P}\mathcal{C}}\mathcal{G}^{\vartheta,\mu}\gamma(\xi) = \frac{1}{\mu^{\sigma-\vartheta}\Gamma(\sigma-\vartheta)} \int_0^\xi \exp\left(\frac{\mu-1}{\mu}(\xi-t)\right) (\xi-t)^{\sigma-\vartheta-1} \left(\frac{d^\sigma\gamma(t)}{dt^\sigma}\right) dt.$$

A central result supporting our investigation is found in [32]:

Lemma 1 ([32]). Let $\mathbb{A}\mathbb{C}^\tau[0, T]$ denote the family of absolutely continuous functions $\mathbf{g}$ such that their $(\tau-1)$ -th derivative remains absolutely continuous on $[0, T]$. Suppose $\vartheta \in \mathbb{C}$ satisfies $\text{Re}(\vartheta) > 0$, let $\mu \in (0, 1]$, and set $\tau = \lfloor \text{Re}(\vartheta) \rfloor + 1$. Assume $\mathbf{g} \in L_1[0, T]$. If $({}_{\mathcal{P}\mathcal{C}}\mathcal{F}^{\vartheta,\mu}\mathbf{g})(\xi) \in \mathbb{A}\mathbb{C}^\tau[0, T]$, then

$${}_{\mathcal{P}\mathcal{C}}\mathcal{F}^{\vartheta,\mu} {}_{\mathcal{P}\mathcal{C}}\mathcal{G}^{\vartheta,\mu}\mathbf{g}(\xi) = \mathbf{g}(\xi) - e^{\frac{\mu-1}{\mu}\xi} \sum_{k=1}^{\tau} \frac{\mathcal{G}^{\vartheta-k,\mu}\mathbf{g}(0)}{\Gamma(\vartheta-k+1)\mu^{\vartheta-k}} \xi^{\vartheta-k}. \quad (7)$$

3. Formation of the Fractional Model

A broad spectrum of natural processes is effectively captured by differential equations, particularly using well-known competition frameworks such as Lotka–Volterra or graze-type formulations. As a prime example, the interactions of the marine ecosystem between oxygen, zooplankton, and phytoplankton can be modeled using such equations. In these relationships, phytoplankton produce oxygen through photosynthesis, which also supports zooplankton growth. Both plankton groups rely on oxygen, while various biochemical and sediment-driven processes gradually deplete it. The original system governing this interplay is as follows:

$$\begin{cases} \frac{dQ}{d\tau} = Y\left(1 - \frac{Q(\tau)}{Q(\tau)+\phi_0}\right)M(\tau) - \frac{M(\tau)Q(\tau)}{Q(\tau)+\phi_2} - \frac{\kappa Q(\tau)N(\tau)}{Q(\tau)+\phi_3} - Q(\tau), \\ \frac{dM}{d\tau} = \left(\frac{\theta Q(\tau)}{Q(\tau)+\phi_1} - \gamma M(\tau)\right)M(\tau) - \frac{\psi M(\tau)N(\tau)}{M(\tau)+\lambda} - \omega M(\tau), \\ \frac{dN}{d\tau} = \left(\frac{\kappa Q^2(\tau)}{Q^2(\tau)+\phi_4^2}\right)\frac{M(\tau)N(\tau)}{M(\tau)+\lambda} - \sigma N(\tau). \end{cases} \quad (8)$$

To incorporate memory effects and account for historical data, we use the $\mathbb{P}\mathbb{C}$ fractional derivative. The fractional form of the above equations, with the $\mathbb{P}\mathbb{C}$ derivative of order ϑ , is given by

$$\begin{cases} {}_{\mathcal{P}\mathcal{C}}\mathcal{G}^{\vartheta,\mu}Q(\tau) = Y\left(1 - \frac{Q(\tau)}{Q(\tau)+\phi_0}\right)M(\tau) - \frac{M(\tau)Q(\tau)}{Q(\tau)+\phi_2} - \frac{\kappa Q(\tau)N(\tau)}{Q(\tau)+\phi_3} - Q(\tau), \\ {}_{\mathcal{P}\mathcal{C}}\mathcal{G}^{\vartheta,\mu}M(\tau) = \left(\frac{\theta Q(\tau)}{Q(\tau)+\phi_1} - \gamma M(\tau)\right)M(\tau) - \frac{\psi M(\tau)N(\tau)}{M(\tau)+\lambda} - \omega M(\tau), \\ {}_{\mathcal{P}\mathcal{C}}\mathcal{G}^{\vartheta,\mu}N(\tau) = \left(\frac{\kappa Q^2(\tau)}{Q^2(\tau)+\phi_4^2}\right)\frac{M(\tau)N(\tau)}{M(\tau)+\lambda} - \sigma N(\tau). \end{cases} \quad (9)$$

In this formulation, the fractional order ϑ introduces a degree of flexibility that allows the model to capture more complex behaviors by accounting for non-local dependencies in time. Each parameter retains its biological interpretation as specified in Table 1, with the fractional derivative enhancing the model's realism in ecological applications.

Table 1. Description of parameters.

Parameter	Description	Parameter	Description
Y	Production rate of oxygen in phytoplankton cell	Q	Oxygen concentration
κ	Maximum feeding efficiency of zooplankton	M	Phytoplankton
$\phi_{1,2,3,4}$	Half saturation constants	N	Zooplankton
θ	Growth rate of phytoplankton per capita	λ	Half saturation constant
γ	Intra-specific competition	ψ	Predation rate of phytoplankton
σ	Natural mortality of zooplankton	ω	Natural mortality of phytoplankton

4. Principal Results

In this section, we develop the analytical framework for our fractional climate change model by first introducing the function space and related norms. Consider the space $\mathbb{Y} = C[0, T]$, consisting of all continuous functions on $[0, T]$, with the norm defined as

$$\|\mathbf{g}(\tau)\| = \max_{\tau \in [0, T]} (|Q(\tau)| + |M(\tau)| + |N(\tau)|),$$

where

$$\mathbf{g}(\tau) = \begin{bmatrix} Q(\tau) \\ M(\tau) \\ N(\tau) \end{bmatrix}, \quad \mathbf{g}_0 = \begin{bmatrix} Q_0 \\ M_0 \\ N_0 \end{bmatrix},$$

and the nonlinear operator $\mathcal{M}$ is expressed as

$$\mathcal{M}(\tau, \mathbf{g}(\tau)) = \begin{bmatrix} \mathcal{G}_1(\tau, Q, M, N) \\ \mathcal{G}_2(\tau, Q, M, N) \\ \mathcal{G}_3(\tau, Q, M, N) \end{bmatrix}.$$

Using the PC fractional derivative, the system can be written as

$${}_{\mathcal{P}\mathcal{C}}\mathcal{G}^{\vartheta, \mu} \mathbf{g}(\tau) = \mathcal{M}(\tau, \mathbf{g}(\tau)), \quad \tau \in [0, T], \quad \mathbf{g}(0) = \mathbf{g}_0.$$

Applying Lemma 1, the above system can be reformulated as an integral equation:

$$\mathbf{g}(\tau) = \mathbf{g}_0 + \frac{1}{\mu^{\vartheta} \Gamma(\vartheta)} \int_0^{\tau} (\tau - s)^{\vartheta-1} e^{\frac{\mu-1}{\mu}(\tau-s)} \mathcal{M}(s, \mathbf{g}(s)) ds, \quad \tau \in [0, T]. \quad (10)$$

To guarantee the existence of a solution for this problem, we assume the following conditions:

- (G1) A suitable constant $\delta > 0$ can be identified such that

$$\|\mathcal{M}(\tau, \mathbf{g}_1) - \mathcal{M}(\tau, \mathbf{g}_2)\| \leq \delta \|\mathbf{g}_1 - \mathbf{g}_2\|,$$

for all $\mathbf{g}_1, \mathbf{g}_2 \in \mathbb{Y}$ and $\tau \in [0, T]$. This ensures the Lipschitz continuity of the operator $\mathcal{M}$.

- (G2) There is a nondecreasing continuous function $\Xi : [0, \infty) \rightarrow [0, \infty)$ satisfying $\Xi(\lambda z) \leq \lambda \Xi(z)$ for all $\lambda \geq 1$, and $\psi \in \mathbb{A}^{\tau}[0, T]$ such that

$$\|\mathcal{M}(\tau, \mathbf{g}(\tau))\| \leq \psi(\tau) \Xi(\|\mathbf{g}(\tau)\|),$$

for all $\mathbf{g} \in \mathbb{Y}$ and $\tau \in [0, T]$. This condition bounds the growth of the operator $\mathcal{M}$ using the auxiliary functions Ξ and ψ .

- (G3) Suitable constants $\varepsilon > 0$ and $\mathcal{S} > 0$ can be identified such that

$$\frac{\mathcal{S}}{\|\mathbf{g}_0\| + \frac{\sup_{\tau \in [0, T]} \psi(\tau) \Xi(\varepsilon) T^{\vartheta}}{\mu^{\vartheta} \Gamma(\vartheta+1)}} > 1.$$

This ensures that the norm of the solution remains bounded and satisfies the conditions necessary for the application of fixed-point theorems.

Now, we define the operator $\mathcal{T} : \mathbb{Y} \rightarrow \mathbb{Y}$ as

$$\mathcal{T} \mathbf{g}(\tau) = \mathbf{g}_0 + \frac{1}{\mu^{\vartheta} \Gamma(\vartheta)} \int_0^{\tau} (\tau - s)^{\vartheta-1} e^{\frac{\mu-1}{\mu}(\tau-s)} \mathcal{M}(s, \mathbf{g}(s)) ds. \quad (11)$$

Theorem 1. *If conditions (G2) and (G3) hold, then the system (9) admits at least one solution within the interval $[0, T]$.*

Proof. Choose $\varepsilon > 0$ so that $\mathbb{B}_\varepsilon = \{\mathbf{g} \in \mathbb{B} \mid \|\mathbf{g}\| \leq \varepsilon\}$. By invoking (G2), we derive for each $\tau \in [0, T]$ the following estimate:

$$\|\mathcal{T}\mathbf{g}(\tau)\| \leq \|\mathbf{g}_0\| + \frac{1}{\mu^\vartheta \Gamma(\vartheta)} \int_0^\tau (\tau-s)^{\vartheta-1} e^{\frac{\mu-1}{\mu}(\tau-s)} \|\mathcal{M}(s, \mathbf{g}(s))\| ds.$$

Using the growth condition (G2), we have

$$\|\mathcal{M}(s, \mathbf{g}(s))\| \leq \psi(s) \Xi(\|\mathbf{g}(s)\|).$$

Substituting this into the integral, we obtain

$$\|\mathcal{T}\mathbf{g}(\tau)\| \leq \|\mathbf{g}_0\| + \frac{1}{\mu^\vartheta \Gamma(\vartheta)} \int_0^\tau (\tau-s)^{\vartheta-1} e^{\frac{\mu-1}{\mu}(\tau-s)} \psi(s) \Xi(\|\mathbf{g}(s)\|) ds.$$

Let $\|\mathbf{g}(s)\| \leq \varepsilon$ for all $s \in [0, T]$. Then

$$\|\mathcal{T}\mathbf{g}(\tau)\| \leq \|\mathbf{g}_0\| + \frac{1}{\mu^\vartheta \Gamma(\vartheta)} \int_0^\tau (\tau-s)^{\vartheta-1} e^{\frac{\mu-1}{\mu}(\tau-s)} \psi(s) \Xi(\varepsilon) ds.$$

Taking into account $\Xi(\varepsilon)$, this becomes

$$\|\mathcal{T}\mathbf{g}(\tau)\| \leq \|\mathbf{g}_0\| + \frac{\Xi(\varepsilon)}{\mu^\vartheta \Gamma(\vartheta)} \int_0^\tau (\tau-s)^{\vartheta-1} e^{\frac{\mu-1}{\mu}(\tau-s)} \psi(s) ds.$$

Let $C_\psi = \sup_{\tau \in [0, T]} \psi(\tau)$. Then

$$\|\mathcal{T}\mathbf{g}(\tau)\| \leq \|\mathbf{g}_0\| + \frac{\Xi(\varepsilon) C_\psi}{\mu^\vartheta \Gamma(\vartheta)} \int_0^\tau (\tau-s)^{\vartheta-1} e^{\frac{\mu-1}{\mu}(\tau-s)} ds.$$

The integral $\int_0^\tau (\tau-s)^{\vartheta-1} e^{\frac{\mu-1}{\mu}(\tau-s)} ds \leq \frac{T^\vartheta}{\vartheta}$. Substituting this bound, we have

$$\|\mathcal{T}\mathbf{g}(\tau)\| \leq \|\mathbf{g}_0\| + \frac{\Xi(\varepsilon) C_\psi T^\vartheta}{\mu^\vartheta \Gamma(\vartheta + 1)}.$$

This shows that $\mathcal{T}$ meets the boundedness property.

On the other hand, to prove that the operator $\mathcal{T}$ is equicontinuous, consider $\tau_1, \tau_2 \in [0, T]$ with $\tau_1 < \tau_2$. The difference $\|\mathcal{T}\mathbf{g}(\tau_2) - \mathcal{T}\mathbf{g}(\tau_1)\|$ is bounded by

$$\|\mathcal{T}\mathbf{g}(\tau_2) - \mathcal{T}\mathbf{g}(\tau_1)\| \leq \frac{1}{\mu^\vartheta \Gamma(\vartheta)} \int_{\tau_1}^{\tau_2} (\tau-s)^{\vartheta-1} e^{\frac{\mu-1}{\mu}(\tau-s)} \|\mathcal{M}(s, \mathbf{g}(s))\| ds.$$

Using (G2), we substitute $\|\mathcal{M}(s, \mathbf{g}(s))\| \leq \psi(s) \Xi(\|\mathbf{g}(s)\|)$, yielding

$$\|\mathcal{T}\mathbf{g}(\tau_2) - \mathcal{T}\mathbf{g}(\tau_1)\| \leq \frac{\sup_{s \in [0, T]} \psi(s) \Xi(\|\mathbf{g}(s)\|)}{\mu^\vartheta \Gamma(\vartheta)} \int_{\tau_1}^{\tau_2} (\tau-s)^{\vartheta-1} e^{\frac{\mu-1}{\mu}(\tau-s)} ds.$$

The kernel $(\tau-s)^{\vartheta-1} e^{\frac{\mu-1}{\mu}(\tau-s)}$ is smooth and the integral vanishes as $\tau_2 \rightarrow \tau_1$. Thus, the operator $\mathcal{T}$ is equicontinuous.

By the Arzelà-Ascoli theorem, since $\mathcal{T}$ maps bounded sets into equicontinuous and uniformly bounded sets, it is relatively compact. Therefore, $\mathcal{T}$ is completely continuous.

Now, let $\zeta \in \mathbb{Y}$ be a fixed point of $\mathcal{T}$, that is, $\zeta = \mathcal{T}\zeta$. Using (G2), we estimate

$$\|\zeta(\tau)\| \leq \|\mathfrak{g}_0\| + \frac{1}{\mu^\vartheta \Gamma(\vartheta)} \int_0^\tau (\tau-s)^{\vartheta-1} e^{\frac{\mu-1}{\mu}(\tau-s)} \psi(s) \Xi(\|\zeta(s)\|) ds.$$

Let $\sup_{\tau \in [0, T]} \psi(\tau) = C_\psi$. Then

$$\|\zeta(\tau)\| \leq \|\mathfrak{g}_0\| + \frac{C_\psi \Xi(\|\zeta\|)}{\mu^\vartheta \Gamma(\vartheta)} \int_0^\tau (\tau-s)^{\vartheta-1} e^{\frac{\mu-1}{\mu}(\tau-s)} ds.$$

Evaluating the integral gives

$$\|\zeta(\tau)\| \leq \|\mathfrak{g}_0\| + \frac{C_\psi \Xi(\|\zeta\|) T^\vartheta}{\mu^\vartheta \Gamma(\vartheta+1)}.$$

Using (G3), there exists $\varepsilon > 0$ such that

$$\|\zeta\| \leq \|\mathfrak{g}_0\| + \frac{C_\psi \Xi(\varepsilon) T^\vartheta}{\mu^\vartheta \Gamma(\vartheta+1)}.$$

Thus, the set of solutions $\{\zeta : \zeta = \mathcal{T}\zeta\}$ is bounded. By the Leray–Schauder alternative, $\mathcal{T}$ has at least one fixed point. $\square$

The following result demonstrates that the climate change model (9) has a unique solution, established through the application of the fixed-point approach of Banach.

Theorem 2. Suppose the operator $\mathcal{M} : [0, T] \times \mathbb{B} \rightarrow \mathbb{R}^3$ satisfies the Lipschitz condition (G1) for all $\mathfrak{g}_1, \mathfrak{g}_2 \in \mathbb{B}$. Under these assumptions, the $\mathbb{PC}$ fractional integral system (10) admits a unique solution over the interval $[0, T]$ if the following inequality holds:

$$\delta T^\vartheta < \mu^\vartheta \Gamma(\vartheta+1),$$

where $\delta > 0$ is the Lipschitz constant of the operator $\mathcal{M}$, $\vartheta > 0$ is the fractional order, and $\Gamma(\cdot)$ is the Gamma function.

Proof. Let $\mathfrak{g} \in [B]_k$, where $k \geq \frac{\mu^\vartheta \Gamma(\vartheta+1) \|\mathfrak{g}_0\| + C_\psi \Xi(k) T^\vartheta}{\mu^\vartheta \Gamma(\vartheta+1) - \delta T^\vartheta}$. Using this assumption and applying the Lipschitz condition (G1), we can estimate

$$\begin{aligned} \|\mathcal{T}\mathfrak{g}(\tau)\| &\leq \|\mathfrak{g}_0\| + \frac{1}{\mu^\vartheta \Gamma(\vartheta)} \int_0^\tau (\tau-s)^{\vartheta-1} e^{\frac{\mu-1}{\mu}(\tau-s)} \|\mathcal{M}(s, \mathfrak{g}(s))\| ds \\ &\leq \|\mathfrak{g}_0\| + \frac{1}{\mu^\vartheta \Gamma(\vartheta)} \int_0^\tau (\tau-s)^{\vartheta-1} e^{\frac{\mu-1}{\mu}(\tau-s)} \left[\|\mathcal{M}(s, \mathfrak{g}(s)) - \mathcal{M}(s, 0)\| + \|\mathcal{M}(s, 0)\| \right] ds \\ &\leq \|\mathfrak{g}_0\| + \frac{(\delta k + C_\psi \Xi(k))}{\mu^\vartheta \Gamma(\vartheta)} \int_0^\tau (\tau-s)^{\vartheta-1} e^{\frac{\mu-1}{\mu}(\tau-s)} ds. \end{aligned}$$

The integral can be evaluated as

$$\int_0^\tau (\tau-s)^{\vartheta-1} e^{\frac{\mu-1}{\mu}(\tau-s)} ds \leq \frac{T^\vartheta}{\vartheta}.$$

Substituting this back, we obtain

$$\|\mathcal{T}\mathfrak{g}(\tau)\| \leq \|\mathfrak{g}_0\| + \frac{[\delta k + C_\psi \Xi(k)] T^\vartheta}{\mu^\vartheta \Gamma(\vartheta+1)} \leq k.$$

Hence, $\mathcal{T}$ maps the ball $[B]_k$ into itself.

Next, let $g_1, g_2 \in [B]_k$. For $\tau \in [0, T]$, we calculate

$$\|\mathcal{T}g_1(\tau) - \mathcal{T}g_2(\tau)\| \leq \frac{1}{\mu^\theta \Gamma(\theta)} \int_0^\tau (\tau - s)^{\theta-1} e^{\frac{\mu-1}{\mu}(\tau-s)} \|\mathcal{M}(s, g_1(s)) - \mathcal{M}(s, g_2(s))\| ds.$$

Using the Lipschitz property (G1), $\|\mathcal{M}(s, g_1(s)) - \mathcal{M}(s, g_2(s))\| \leq \delta \|g_1(s) - g_2(s)\|$, we find

$$\|\mathcal{T}g_1(\tau) - \mathcal{T}g_2(\tau)\| \leq \frac{\delta}{\mu^\theta \Gamma(\theta)} \int_0^\tau (\tau - s)^{\theta-1} e^{\frac{\mu-1}{\mu}(\tau-s)} \|g_1(s) - g_2(s)\| ds.$$

Evaluating the integral, we obtain

$$\|\mathcal{T}g_1 - \mathcal{T}g_2\| \leq \frac{\delta T^\theta}{\mu^\theta \Gamma(\theta + 1)} \|g_1 - g_2\| < \|g_1 - g_2\|.$$

Therefore, $\mathcal{T}$ is a contraction mapping on $[B]_k$. By the Banach fixed-point theorem, $\mathcal{T}$ admits a unique fixed point $g \in [B]_k$. Thus, the system (10) has a unique solution, as required. $\square$

5. Stability Results

Recent studies have explored stability properties and control mechanisms in fractional-order systems with delays, highlighting the importance of tuning fractional orders for dynamic behavior regulation [33]. Inspired by these approaches, our work examines how the Proportional–Caputo fractional operator influences the ecological system's resilience under perturbations. In this section, we derive the necessary conditions for the $\mathbb{PC}$ fractional system (10) to exhibit different types of stability. These include UHS and extended UHS. Before presenting the stability theorems, we introduce some foundational definitions.

Let $h > 0$ be a positive constant, and consider a continuous function $\Psi : [0, T] \rightarrow \mathbb{R}^+$. The stability conditions can be expressed using the following inequalities:

$$\|\mathcal{PC}\mathcal{G}^{\theta, \mu}\zeta(\tau) - \mathcal{M}(\tau, \zeta(\tau))\| \leq h, \quad \forall \tau \in [0, T], \quad (12)$$

$$\|\mathcal{PC}\mathcal{G}^{\theta, \mu}\zeta(\tau) - \mathcal{M}(\tau, \zeta(\tau))\| \leq h\Psi(\tau), \quad \forall \tau \in [0, T], \quad (13)$$

$$\|\mathcal{PC}\mathcal{G}^{\theta, \mu}\zeta(\tau) - \mathcal{M}(\tau, \zeta(\tau))\| \leq \Psi(\tau), \quad \forall \tau \in [0, T]. \quad (14)$$

Definition 6. The system (10) is said to satisfy UHS under condition (20) if there is a solution $\zeta \in \mathbb{B}$ such that for every $h > 0$, there is a constant $C_h > 0$ satisfying

$$\|g(\tau) - \zeta(\tau)\| \leq hC_h, \quad \forall \tau \in [0, T], \quad (15)$$

where $C_h = \max(C_{h_1}, C_{h_2}, C_{h_3}, C_{h_4})$, with C_{h_i} being specific constants related to the problem.

Definition 7. The system (10) is said to satisfy extended UHS under condition (13) if there exists a function $\Psi_h : [0, T] \rightarrow \mathbb{R}^+$, with $\Psi_h(0) = 0$, such that for every solution $\zeta \in \mathbb{B}$, the inequality

$$\|g(\tau) - \zeta(\tau)\| \leq \Psi_h(\tau), \quad \forall \tau \in [0, T], \quad (16)$$

holds, where $\Psi_h(\tau) = \max\{\Psi_{h_1}, \Psi_{h_2}, \Psi_{h_3}, \Psi_{h_4}\}$, and Ψ_{h_i} are problem-dependent functions.

Lemma 2. Let $\vartheta > 0$ and $\mu \in (0, 1]$. The following inequality holds:

$$\|\zeta(\tau) - g_0 - \frac{1}{\mu^\vartheta \Gamma(\vartheta)} \int_0^\tau e^{\frac{\mu-1}{\mu}(\tau-s)} (\tau-s)^{\vartheta-1} \mathcal{M}(s, \zeta(s)) ds\| \leq \frac{\mathfrak{h} T^k}{\mu^\vartheta \Gamma(\vartheta + k)}, \quad (17)$$

provided that $\zeta \in \mathbb{B}$ satisfies the inequality (12).

Proof. Given that ζ satisfies condition (12), there is a $y \in \mathbb{Y}$, depending on ζ , such that

$$\|y(t)\| \leq \mathfrak{h}, \quad y = \max(y_1, y_2, y_3, y_4), \quad \forall t \in [0, T], \quad (18)$$

and

$$\begin{cases} \mathcal{P}_C \mathcal{G}^{\vartheta, \mu} \zeta(t) = \mathcal{M}(t, \zeta(t)) + y(t), & t \in [0, T], \\ \zeta(0) = \zeta_0 \geq 0. \end{cases} \quad (19)$$

Using Lemma 1, the solution $\zeta(t)$ for problem (19) is represented as

$$\begin{aligned} \zeta(t) &= \zeta_0 + \frac{1}{\mu^\vartheta \Gamma(\vartheta)} \int_0^t e^{\frac{\mu-1}{\mu}(t-s)} (t-s)^{\vartheta-1} \mathcal{M}(s, \zeta(s)) ds \\ &\quad + \frac{1}{\mu^\vartheta \Gamma(\vartheta)} \int_0^t e^{\frac{\mu-1}{\mu}(t-s)} (t-s)^{\vartheta-1} y(s) ds. \end{aligned} \quad (20)$$

Substituting Equation (20) into (18), we calculate

$$\begin{aligned} &\left\| \zeta(t) - \zeta_0 - \frac{1}{\mu^\vartheta \Gamma(\vartheta)} \int_0^t e^{\frac{\mu-1}{\mu}(t-s)} (t-s)^{\vartheta-1} \mathcal{M}(s, \zeta(s)) ds \right\| \\ &\leq \frac{1}{\mu^\vartheta \Gamma(\vartheta)} \int_0^t e^{\frac{\mu-1}{\mu}(t-s)} (t-s)^{\vartheta-1} \|y(s)\| ds. \end{aligned}$$

Since $\|y(s)\| \leq \mathfrak{h}$, we simplify the inequality:

$$\begin{aligned} &\|\zeta(t) - \zeta_0 - \frac{1}{\mu^\vartheta \Gamma(\vartheta)} \int_0^t e^{\frac{\mu-1}{\mu}(t-s)} (\ln(t-s))^{\vartheta-1} \mathcal{M}(s, \zeta(s)) ds\| \\ &\leq \frac{\mathfrak{h}}{\mu^\vartheta \Gamma(\vartheta)} \int_0^t e^{\frac{\mu-1}{\mu}(t-s)} (t-s)^{\vartheta-1} ds \leq \frac{\mathfrak{h} T^\vartheta}{\mu^\vartheta \Gamma(\vartheta + 1)}. \end{aligned}$$

Hence, inequality (17) is satisfied, completing the proof. $\square$

6. Numerical Simulations

This section develops a numerical approach for solving the climate change model (9) using a predictor-corrector Adams method. The scheme accounts for fractional memory effects and nonlocal properties. An iterative refinement process enhances accuracy through multiple correction steps per time interval. The predictor-corrector fractional Adams method [34,35] for the Proportional–Caputo FDE (10) are, respectively,

$$\begin{aligned} g_n^p &= g(0) + h^\vartheta \sum_{j=1}^{n-1} a_{n-j-1}^\vartheta e^{\frac{\mu-1}{\mu}(\tau_n - \tau_j)} \mathcal{M}(\tau_j, g_j), \\ g_n &= g(0) + h^\vartheta \left(\tilde{b}_n^\vartheta \mathcal{M}(\tau_0, g_0) + \sum_{j=1}^{n-1} b_{n-j}^\vartheta e^{\frac{\mu-1}{\mu}(\tau_n - \tau_j)} \mathcal{M}(\tau_j, g_j) + b_0^\vartheta e^{\frac{\mu-1}{\mu}h} \mathcal{M}(\tau_j, g_n^p) \right). \end{aligned} \quad (21)$$

To enhance the approximation, several corrector iterations, denoted by η , can be performed:

$$\begin{aligned} \mathbf{g}_n^{[0]} &= \mathbf{g}(0) + h^\vartheta \sum_{j=1}^{n-1} a_{n-j-1}^\vartheta e^{\frac{\mu-1}{\mu}(\tau_n-\tau_j)} \mathcal{M}(\tau_j, \mathbf{s}_j), \\ \mathbf{g}_n^{[\eta]} &= \mathbf{g}(0) + h^\vartheta \left(\tilde{b}_n^\vartheta \mathcal{M}(\tau_0, \mathbf{g}_0) + \sum_{j=1}^{n-1} b_{n-j}^\vartheta e^{\frac{\mu-1}{\mu}(\tau_n-\tau_j)} \mathcal{M}(\tau_j, \mathbf{g}_j) + b_0^\vartheta e^{\frac{\mu-1}{\mu}h} \mathcal{M}(\tau_j, \mathbf{g}_n^{[\eta-1]}) \right), \end{aligned} \quad (22)$$

where $\eta = 1, 2, \dots$, and

$$\begin{aligned} a_n^\alpha &= \frac{((n+1)^\alpha - n^\alpha)}{\Gamma(\alpha+1)}, \\ \tilde{b}_n^\alpha &= \frac{((n-1)^{\alpha+1} - n^\alpha(n-\alpha-1))}{\Gamma(\alpha+2)}, \\ b_n^\alpha &= \begin{cases} \frac{1}{\Gamma(\alpha+2)}, & n=0; \\ \frac{((n-1)^{\alpha+1} - 2n^{\alpha+1} + (n+1)^{\alpha+1})}{\Gamma(\alpha+2)}, & n=1, 2, \dots \end{cases} \end{aligned} \quad (23)$$

The parameter values and initial conditions used in the simulations are provided in Table 2.

Table 2. Parameter values and initial conditions.

Symbol	Value	Symbol	Value
Y	1.2	κ	0.8
ϕ_0	0.5	ϕ_1	0.4
ϕ_2	0.3	ϕ_3	0.2
ϕ_4	0.1	θ	0.9
γ	0.6	λ	0.5
ψ	0.7	σ	0.4
ω	0.5	μ	0.95
$M(0)$	0.5	T	10
$Q(0)$	1.0	ϑ	0.8
$N(0)$	0.3		

Plankton densities and oxygen concentrations for various ϑ and μ fractional orders, i.e., $\vartheta = 0.9$, $\vartheta = 0.95$, $\vartheta = 0.99$ and $\mu = 0.95$ in the left column are displayed in Figure 1. The right column illustrates the trajectories of the oxygen–plankton dynamical system within the corresponding three-dimensional phase space. Following a sequence of damped oscillations, the system ultimately stabilizes at steady-state values for $T = 2.12$, $\vartheta = 0.9$, and $\mu = 0.95$. This scenario signifies that the coexistence state functions as a stable focus. The system exhibits periodic oscillations when $\vartheta = 0.95$ and $\mu = 0.95$. In fact, when ϑ shifts from $\vartheta = 0.9$ to $\vartheta = 0.95$, Hopf bifurcation occurs. The plankton and oxygen concentrations exhibit a stable limit cycle, and the system dynamics is periodic for $\vartheta = 0.95$. After a few oscillation sequences, the species densities become extinct and the limit cycle disappears for $\vartheta = 0.99$.

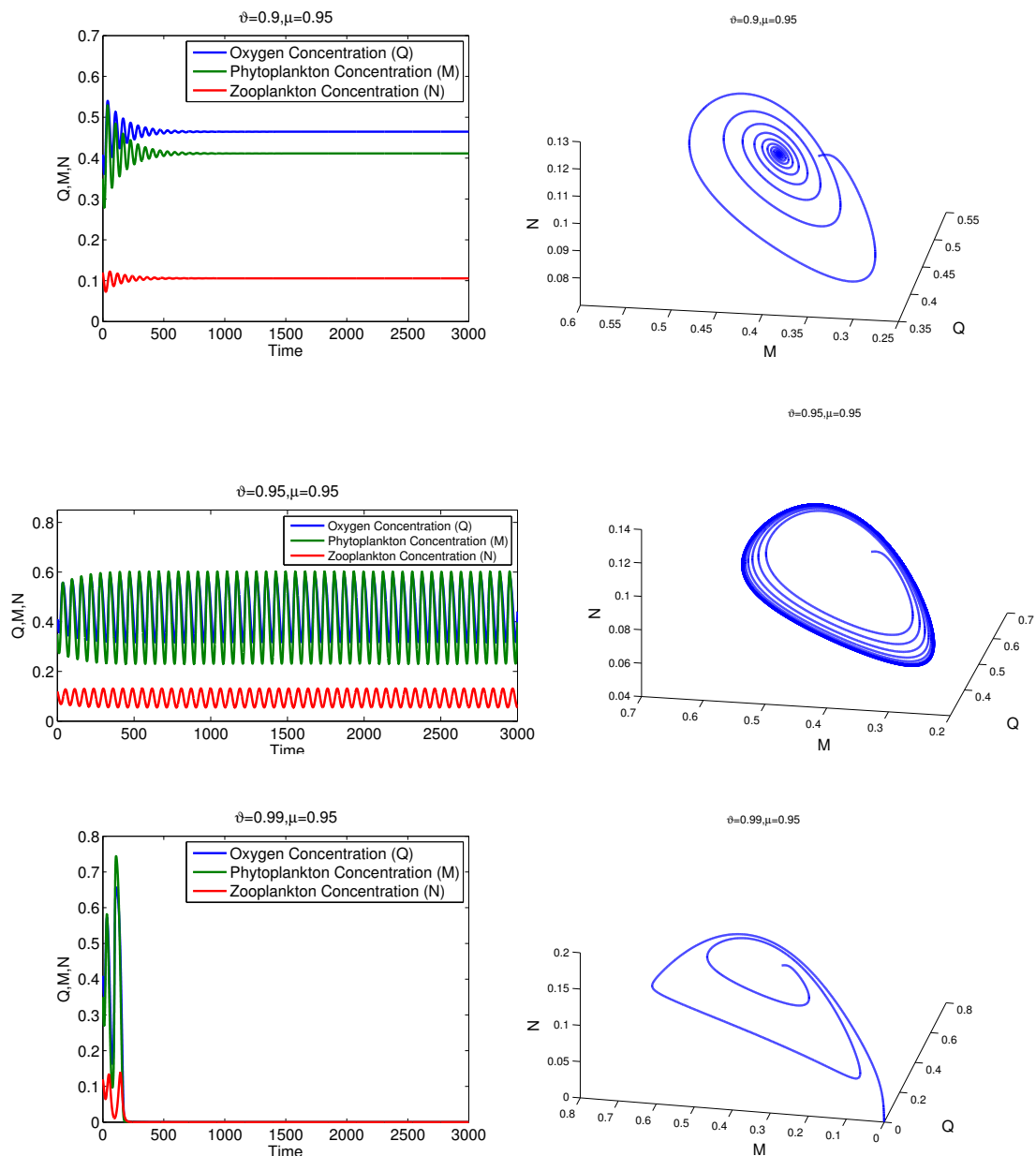


Figure 1. Numerical results obtained for oxygen–phytoplankton–zooplankton densities with corresponding time series solutions at time = 3000 for different ϑ and μ fractional orders, i.e., $\vartheta = 0.9$, $\vartheta = 0.95$, $\vartheta = 0.99$, and $\mu = 0.95$ for $T = 2.12$. Initial conditions are $Q_0 = 0.395$, $M_0 = 0.35$, and $N_0 = 0.12$.

In a theoretical perspective, horizontal movement from a warmer to cooler one can be specified by the following choice of possible $T(t)$, i.e., a linear spatial form:

$$T = T_0, \quad t < t_0, \quad T(t) = T_0 + \omega(t - t_0), \quad t \geq t_0. \quad (24)$$

Here, t_0 represents the initial time when global warming commenced, T_0 denotes the pre-change oxygen production rate, and ω signifies the rate of global warming; for further details on the selection of this function, refer to [36].

Figure 2 illustrates the time evolution of plankton densities and oxygen concentration for $\omega = 2 \times 10^{-5}$ and $T_0 = 2$, along with various fractional orders of ϑ and μ , specifically $\vartheta = 0.95$, $\vartheta = 0.98$, $\vartheta = 0.999$, and $\mu = 0.95$. For $T_0 = 2$ and different fractional orders of ϑ and μ , the coexistence state functions as a stable focus. Consequently, in all three cases, the system initially exhibits damped oscillations before converging to a stable steady state.

However, the system's tendency to increase with respect to ϑ and μ follows a similar pattern. Notably, as ϑ approaches 0.999, the oscillatory behavior becomes more pronounced. In this scenario, the final temperature function T remains below the Hopf bifurcation threshold in the proportional–fractional order models, analogous to its classical counterpart. As previously mentioned, the right column of Figure 2 illustrates the phase plane structure corresponding to these cases.

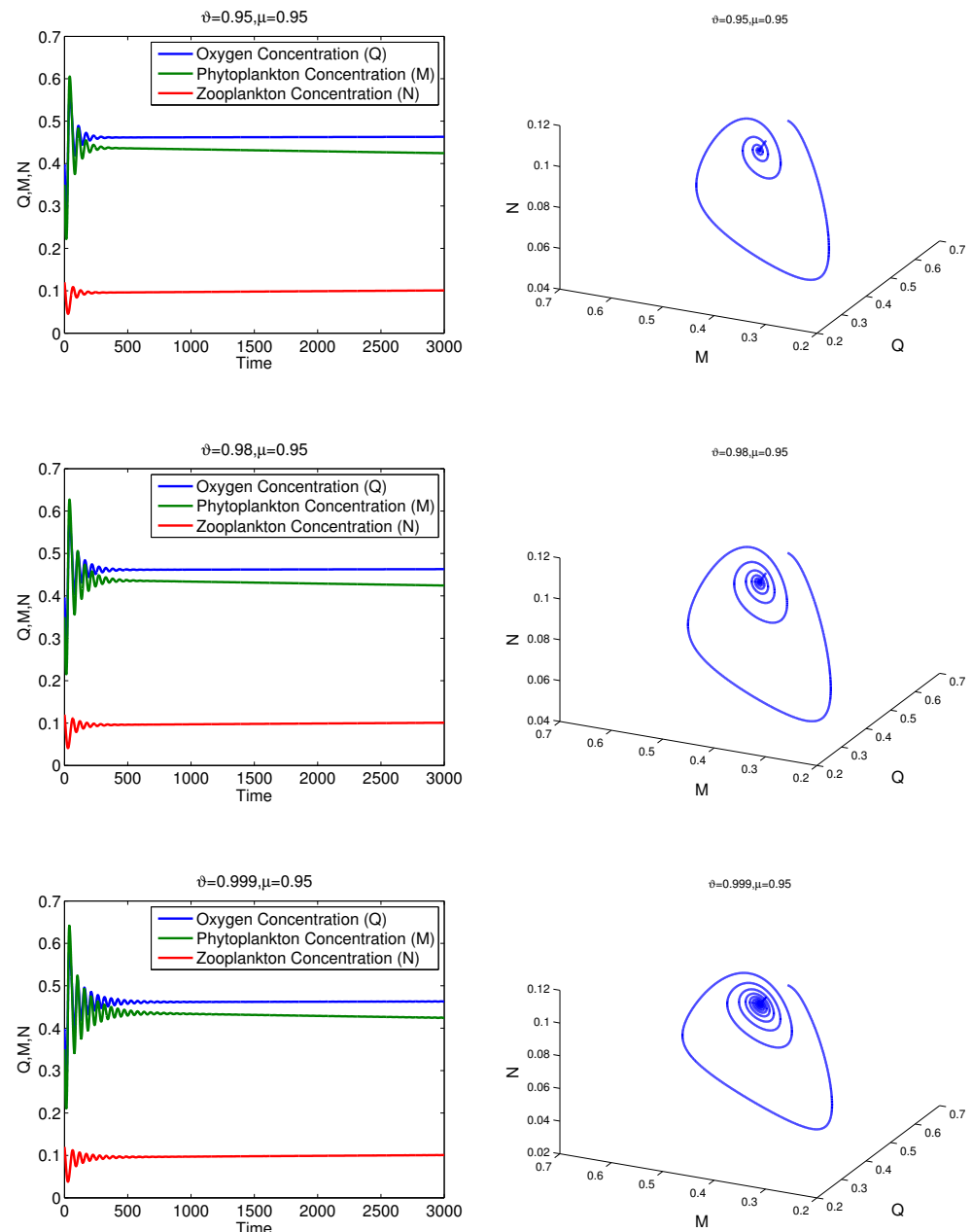


Figure 2. Numerical results obtained for oxygen–phytoplankton–zooplankton densities with corresponding time series solutions at $\omega = 2 \times 10^{-5}$ and time = 3000 for different ϑ and μ fractional orders, i.e., $\vartheta = 0.9$, $\vartheta = 0.95$, $\vartheta = 0.99$ and $\mu = 0.95$ for $T_0 = 2$. Initial conditions are $Q_0 = 0.395$, $M_0 = 0.35$, and $N_0 = 0.12$.

Figure 3 illustrates the oxygen concentration and plankton densities corresponding to different fractional orders of ϑ and μ , specifically $\vartheta = 0.95$, $\vartheta = 0.99$, $\vartheta = 0.999$, and $\mu = 0.95$, for $T_0 = 2.011$ under a warming rate of $\omega = 2 \times 10^{-5}$ over a time period of 3000. The primary distinction between this figure and the previous one lies in the initial temperature. In this case, the initial value of the temperature function is higher than in the

previous scenario. Consequently, as the system reaches the Hopf bifurcation threshold, it transitions into periodic oscillations. Initially, the oscillation amplitude diminishes, but over time, it progressively increases.

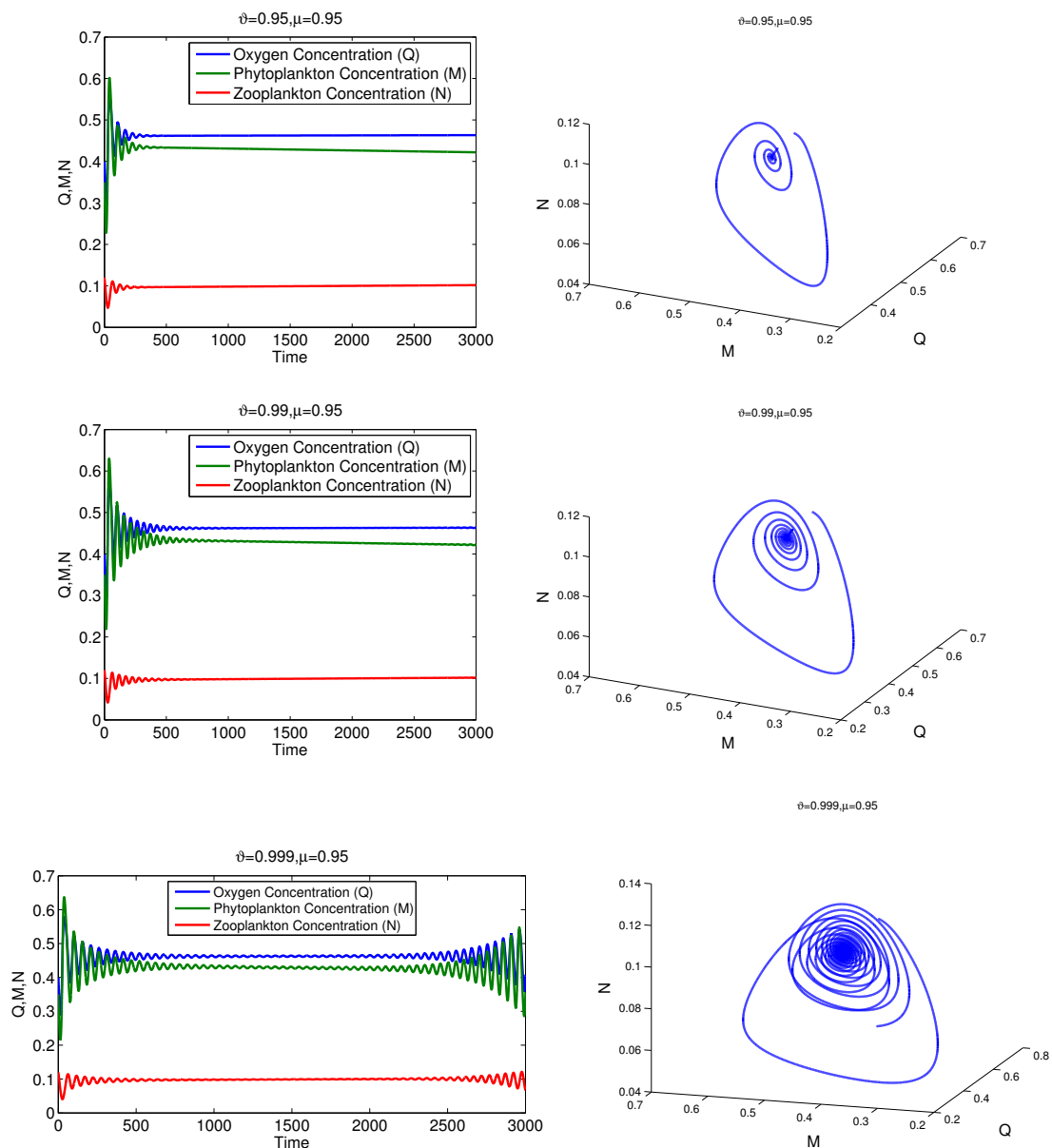


Figure 3. Numerical results obtained for oxygen–phytoplankton–zooplankton densities with corresponding time series solutions at $\omega = 2 \times 10^{-5}$ and time = 3000 for different ϑ and μ fractional orders, i.e., $\vartheta = 0.9$, $\vartheta = 0.95$, $\vartheta = 0.99$ and $\mu = 0.95$ for $T_0 = 2.011$. Initial conditions are $Q_0 = 0.395$, $M_0 = 0.35$, and $N_0 = 0.12$.

Figure 4 shows the oxygen concentration and plankton densities for each of ϑ and μ fractional orders, i.e., $\vartheta = 0.985$, $\vartheta = 0.99$, and $\vartheta = 0.999$ with $\mu = 0.95$ and for $T_0 = 2.011$ for the warming rate $\omega = 35 \times 10^{-6}$ and for time = 3000. When the system reaches Hopf bifurcation, the magnitude of oscillations grows, while the amplitude of oscillations falls as warming begins. After the system reaches the Hopf bifurcation point, it is observed that an increase in T causes the system to undergo oscillations that grow in size before the oxygen concentration and species densities decrease.

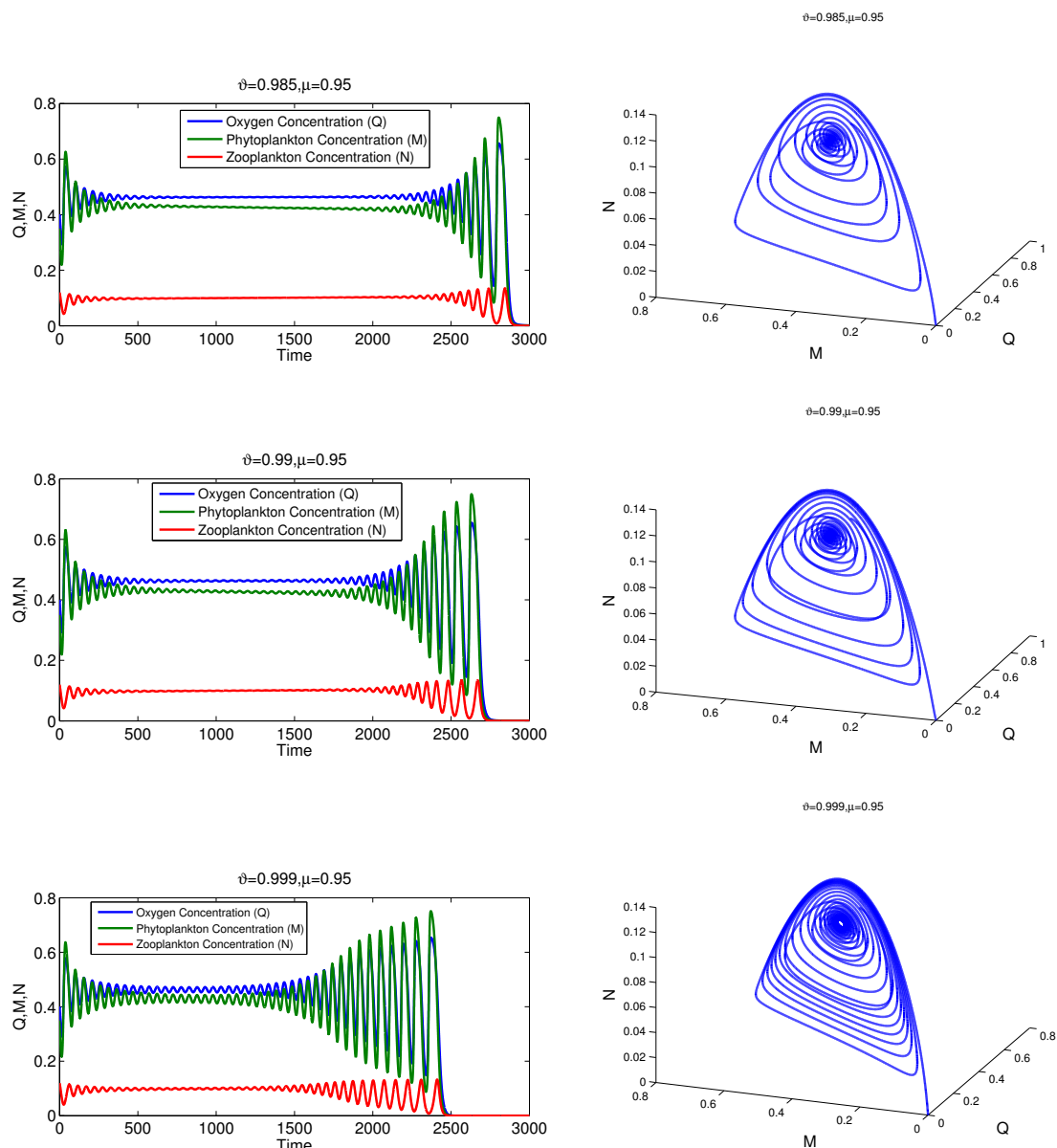


Figure 4. Numerical results obtained for oxygen–phytoplankton–zooplankton densities with corresponding time series solutions at $\omega = 35 \times 10^{-6}$ and time = 3000 for different ϑ and μ fractional orders, i.e., $\vartheta = 0.985$, $\vartheta = 0.99$ and $\vartheta = 0.999$ for $T_0 = 2.011$. Initial conditions are $Q_0 = 0.395$, $M_0 = 0.35$, and $N_0 = 0.12$.

Figure 5 shows several ϑ and μ fractional orders, i.e., $\vartheta = 0.94$ and $\vartheta = 0.999$ with $\mu = 0.95$, such as with oxygen concentration. The figure legend shows the plankton densities vs. time for $T_0 = 2.065$ and $\omega = 1 \times 10^{-4}$. The system experiences oscillations that grow in amplitude before abruptly approaching extinction. Note that the extinction time is furthered by the choice of ϑ and μ ; for smaller values of ϑ and μ , the system persists much longer. Ultimately, the system coincides with its solutions in the classical derivative when ϑ and μ is taken near 1.

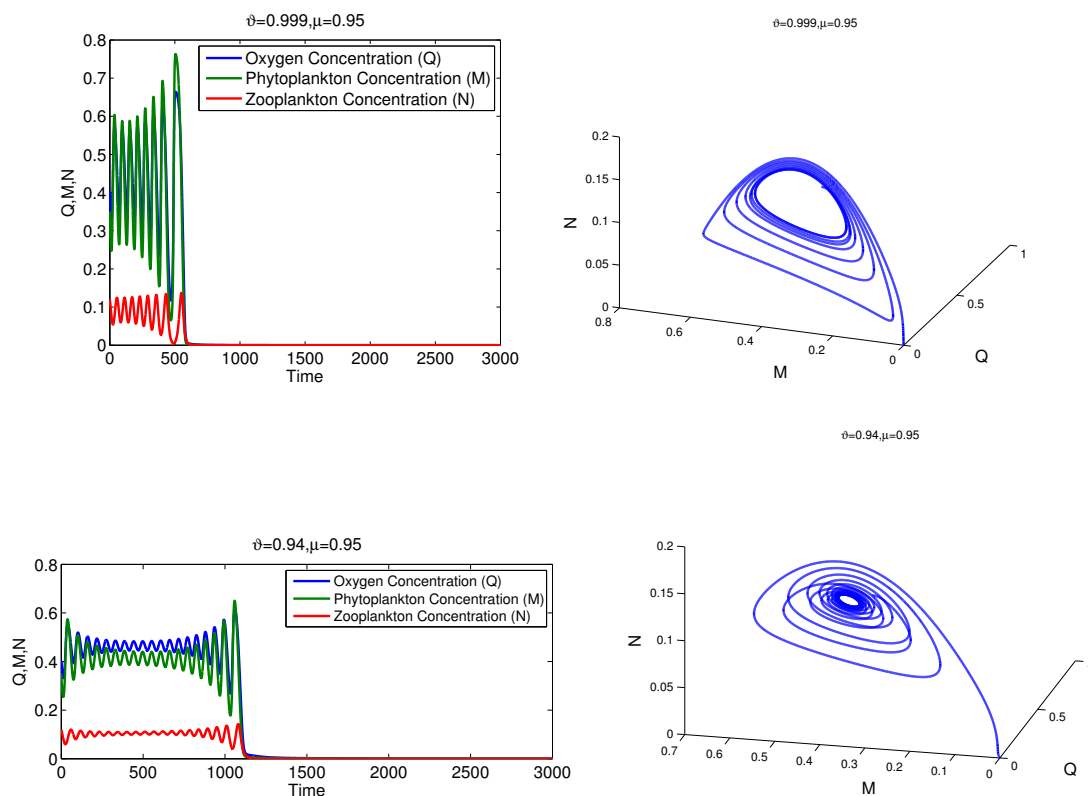


Figure 5. Numerical results obtained for oxygen–phytoplankton–zooplankton densities with corresponding time series solutions at $\omega = 1 \times 10^{-4}$ and time = 3000 for different ϑ and μ fractional orders, i.e., $\vartheta = 0.94$ and $\vartheta = 0.999$ with $\mu = 0.95$ for $T_0 = 2.065$. Initial conditions are $Q_0 = 0.395$, $M_0 = 0.35$, and $N_0 = 0.12$.

7. Conclusions

This paper has presented a fractional-order model of the oxygen–plankton system by employing a $\mathbb{P}\mathbb{C}$ derivative, capturing memory-dependent influences that shape population dynamics. Under this framework, existence and uniqueness results were rigorously established, and both UHS and extended UHS analyses confirmed the model’s robustness against small disturbances in initial conditions or parameter values. Numerical simulations illustrated diverse scenarios where the fractional order and proportional parameters can induce transitions between equilibrium states and long-lasting oscillations. Several special cases further underscore the model’s flexibility. First, setting the fractional order ϑ close to 1 effectively recovers classical behavior, thus serving as a link between standard integer-order systems and memory-inclusive formulations. Second, choosing $\mu = 1$ simplifies the kernel to a more conventional Caputo form, aiding comparison to established fractional models. Finally, considering $\omega = 0$ (no warming) leads to a baseline scenario without externally driven temperature effects, providing insights into how warming rates amplify or dampen population oscillations. These special cases highlight the adaptability of the $\mathbb{P}\mathbb{C}$ model in examining a spectrum of ecological dynamics, with potential applications in other climate-related or biological studies.

From an ecological perspective, fractional-order models that capture memory effects provide valuable tools for managing marine ecosystems. By accounting for long-term environmental changes, such as temperature shifts and nutrient variations, these models help detect critical thresholds where small disturbances could trigger major changes in plankton populations. This understanding supports proactive conservation actions, such as adjusting nutrient inputs or establishing protected zones, enabling resource managers to anticipate and mitigate ecological risks more effectively.

Author Contributions: Conceptualization, M.A.B., A.A.A., A.-A.H. and T.A.; Methodology, M.A.B., A.A.A., A.-A.H. and T.A.; Formal analysis, M.A.B., A.A.A., A.-A.H. and T.A.; Investigation, M.A.B., A.A.A., A.-A.H. and T.A.; Writing—original draft, M.A.B., A.A.A., A.-A.H. and T.A.; Writing—review & editing, M.A.B., A.A.A., A.-A.H. and T.A. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by King Khalid University, Grant (RGP.2/82/45) and Princess Nourah bint Abdulrahman University, Grant (PNURSP2025R337).

Data Availability Statement: No new data were created or analyzed in this study.

Acknowledgments: The authors extend their appreciation to the Deanship of Research and Graduate Studies at King Khalid University for funding this work through the Research Groups Program under grant (RGP.2/82/45). The authors would like to acknowledge the Princess Nourah bint Abdulrahman University Researchers Supporting Project number (PNURSP2025R337).

Conflicts of Interest: The authors declare no conflict of interest.

References

- Harris, G. *Phytoplankton Ecology: Structure, Function and Fluctuation*; Springer Science and Business Media: Berlin/Heidelberg, Germany, 2012.
- Steele, J.H. *Spatial Pattern in Plankton Communities*; Springer Science and Business Media: Berlin/Heidelberg, Germany, 1978; Volume 3.
- Okubo, A. *Diffusion and Ecological Problems: Mathematical Models*; Biomathematics, Vol. 10; Springer: Berlin/Heidelberg, Germany, 1980.
- Malchow, H.; Petrovskii, S.V.; Venturino, E. *Spatiotemporal Patterns in Ecology and Epidemiology: Theory, Models, and Simulation*; CRC Press: Boca Raton, FL, USA, 2007.
- McCarty, J.P. Ecological Consequences of Recent Climate Change. *Conserv. Biol.* **2001**, *15*, 320–331. [[CrossRef](#)]
- Jonkers, L.; Hillebrand, H.; Kucera, M. Global Change Drives Modern Plankton Communities Away from the Pre-industrial State. *Nature* **2019**, *570*, 372–375. [[CrossRef](#)] [[PubMed](#)]
- Din, A.; Shah, K.; Seadawy, A.; Alrabaiah, H.; Baleanu, D. On a New Conceptual Mathematical Model Dealing with the Current Novel Coronavirus-19 Infectious Disease. *Results Phys.* **2020**, *19*, 103510. [[CrossRef](#)]
- Sabatier, J.A.T.; Agrawal, O.P.; Machado, J.A.T. *Advances in Fractional Calculus*; Springer: Dordrecht, The Netherlands, 2007; Volume 4, p. 9.
- Rayner, N.A.; Parker, D.E.; Horton, E.B.; Foll, C.K.; Alexander L.V.; Rowell, D.P.; Kent, E.C.; Kaplan, A. Global Analyses of Sea Surface Temperature, Sea Ice, and Night Marine Air Temperature Since the Late Nineteenth Century. *J. Geophys. Res. Atmos.* **2003**, *108*. [[CrossRef](#)]
- Poloczanska, E.S.; Brown, C.J.; Sydeman, W.J.; Kiessling, W.; Schoeman, D.S.; Moore, P.J.; Brander, K.; Bruno, J.F.; Buckley, L.B.; Burrows, M.T.; et al. Global Imprint of Climate Change on Marine Life. *Nat. Clim. Change* **2013**, *3*, 919–925. [[CrossRef](#)]
- Houghton, E. *Climate Change 1995: The Science of Climate Change: Contribution of Working Group I to the Second Assessment Report of the Intergovernmental Panel on Climate*; Cambridge University Press: Cambridge, UK, 1996; Volume 2.
- Metz, B.; Davidson, O.; Swart, R.; Pan, J. *Intergovernmental Panel on Climate Change (IPCC) Third Assessment Report. Climate Change 2001: Mitigation*; Cambridge University Press: Cambridge, UK; New York, NY, USA, 2001; p. 398.
- Houghton, J.T.; Ding, Y.; Griggs, D.J. (Eds.) *Climate Change 2001: The Scientific Basis*; The Press Syndicate of the University of Cambridge: Cambridge, UK, 2001.
- Rodenhouse, N.L.; Matthews, S.N.; McFarl, K.P.; Lambert, J.D.; Iverson, L.R.; Prasad, A.; Sillett, T.S.; Holmes, R.T. Potential Effects of Climate Change on Birds of the Northeast. *Mitig. Adapt. Strateg. Glob. Change* **2008**, *13*, 517–540. [[CrossRef](#)]
- Newman, J.A. *Climate Change Biology*; CABI: 2011. Available online: <https://www.cabidigitallibrary.org/doi/book/10.1079/9781845937485.0000> (accessed on 14 February 2025).
- Contessa, G.M.; Guardati, M.; D'Arienzo, M.; Poggi, C.; Sandri, S.; D'Auria, M.C.; Genovese, E.; Cannatà, V. The Impact of Climate Change on Radiological Practices in Italy. *Eur. Phys. J. Plus* **2018**, *133*, 1–12. [[CrossRef](#)]
- Edwards, M.; Johns, D.G.; Licandro, P.; John, A.W.; Stevens, D.P. *Ecological Status Report: Results from the CPR Survey 2006/2007*; Technical Report; SAHFOS Technical Report: Plymouth, UK, 2008; ISSN 1744-0750.
- Worm, B.; Lotze, H.K. Marine Biodiversity and Climate Change. In *Climate Change*; Elsevier: Amsterdam, The Netherlands, 2021; pp. 445–464.

19. Smith, J.; Zhang, W. Fractional Calculus in Climate Modeling: A Review of Anomalous Diffusion Approaches. *J. Clim. Syst.* **2023**, *12*, 201–221.
20. Barakat, M.A.; Soliman, A.H.; Hyder, A.A. Langevin Equations with Generalized Proportional Hadamard–Caputo Fractional Derivative. *Comput. Intell. Neurosci.* **2021**, 2021, 6316477. [[CrossRef](#)]
21. Almoneef, A.; Barakat, M.; Hyder, A.A. Analysis of the Fractional HIV Model under Proportional Hadamard–Caputo Operators. *Fractal Fract.* **2023**, *7*, 220. [[CrossRef](#)]
22. Jones, E.; Li, H. Applications of Fractional Derivatives in Environmental Science and Climate Change. *Environ. Model. Softw.* **2022**, *36*, 145–160.
23. Kumar, A.; Chen, F. Modeling Weather Extremes Using Fractional Derivatives. *Appl. Math. Atmos. Sci.* **2021**, *7*, 87–102.
24. White, L.; Patel, R. Fractional Models for Energy Transfer in Climate Systems. *Renew. Energy Clim. Change J.* **2023**, *4*, 99–113.
25. Wilson, M.; Lee, D. Simulating Ocean Currents with Fractional Differential Equations. *J. Oceanogr.* **2022**, *19*, 212–230.
26. Anderson, R.; Silva, C. Thermohaline Circulation Models with Fractional Diffusion for Climate Projections. *Clim. Fluid Dyn.* **2024**, *13*, 50–66.
27. Brown, L.; Gupta, R. Fractional Models in Carbon Sequestration and Emission Forecasting. *Environ. Sci. Clim.* **2023**, *9*, 98–112.
28. Diaz, M.; Thompson, S. Fractional Dynamics in Polar Ice Melt and Ocean Acidity Studies. *Polar Clim. Res.* **2023**, *6*, 27–43.
29. Johnson, P.; Clark, E. Fractional Calculus in Climate Policy Modeling: Applications and Challenges. *J. Environ. Policy Anal.* **2024**, *5*, 115–132.
30. Adams, P.; García, E. Non-Local Dynamics in Climate Models: The Role of Fractional Derivatives. *Clim. Dyn.* **2022**, *45*, 365–380.
31. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier: Amsterdam, The Netherlands, 2006.
32. Jarad, F.; Abdeljawad, T.; Alzabut, J. Generalized fractional derivatives generated by a class of local proportional derivatives. *Eur. Phys. J. Spec. Top.* **2017**, *226*, 3457–3471. [[CrossRef](#)]
33. Gao, Q.; Cai, J.; Liu, Y.; Shi, L.; Xu, W. Power mapping-based stability analysis and order adjustment control for fractional-order multiple delayed systems. *ISA Trans.* **2023**, *138*, 10–19.
34. Diethelm, K.; Ford, N.J.; Freed, A.D. Detailed error analysis for a fractional Adams method. *Numer. Algorithms* **2004**, *36*, 31–52. [[CrossRef](#)]
35. Diethelm, K.; Ford, N.J.; Freed, A.D. A predictor-corrector approach for the numerical solution of fractional differential equations. *Nonlinear Dyn.* **2002**, *29*, 3–22. [[CrossRef](#)]
36. Sekerci, Y.; Petrovskii, S. Mathematical modelling of plankton–oxygen dynamics under the climate change. *Bull. Math. Biol.* **2015**, *77*, 2325–2353. [[CrossRef](#)]

Disclaimer/Publisher’s Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.