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On Strongly Regular Graphs and the Friendship Theorem

Igal Sason [1,2]

[1] Andrew & Erna Viterbi Faculty of Electrical and Computer Engineering, Technion—Israel Institute of Technology, Haifa 3200003, Israel; sason@ee.technion.ac.il; Tel.: +972-48294699
[2] Faculty of Mathematics, Technion—Israel Institute of Technology, Haifa 3200003, Israel

Abstract: This paper presents an alternative proof of the celebrated friendship theorem, originally established by Erdős, Rényi, and Sós in 1966. The proof relies on a closed-form expression for the Lovász ϑ-function of strongly regular graphs, recently derived by the author. Additionally, this paper considers some known extensions of the theorem, offering discussions that provide insights into the friendship theorem, one of its extensions, and the proposed proof. Leveraging the closed-form expression for the Lovász ϑ-function of strongly regular graphs, this paper further establishes new necessary conditions for a strongly regular graph to be a spanning or induced subgraph of another strongly regular graph. In the case of induced subgraphs, the analysis also incorporates a property of graph energies. Some of these results are extended to regular graphs and their subgraphs.

Keywords: friendship theorem; strongly regular graph; Lovász ϑ-function; windmill graph; graph invariants; graph energy; spanning subgraph; induced subgraph

MSC: 05C50; 05C60; 05E30

1. Introduction

The friendship theorem in graph theory states that if a finite graph has the property that every pair of distinct vertices shares exactly one common neighbor, then the graph consists of edge-disjoint triangles sharing a single, central vertex. In other words, such a graph is necessarily a friendship graph (also called a windmill graph), where a central vertex is adjacent to all others, and additional edges form edge-disjoint triangles. The theorem's name reflects its intuitive human interpretation, which asserts that if every two individuals in a (finite) group have exactly one mutual friend, then there must be someone who is everybody's friend (often referred to as the "politician").

This theorem was first proved by Erdős, Rényi, and Sós [1], and it found diverse applications in combinatorics and other disciplines. While the friendship theorem itself is a result in graph theory, its structural properties make it useful in designing graph-based codes, network coding, and cryptographic schemes. In network coding, graphs represent the information flow between nodes, and the friendship property ensures that all transmissions go through a single relay node, which models centralized communication structures, such as satellite communication, where every ground station interacts via a single hub. This property is also useful in wireless sensor networks, where multiple sensors communicate through a central processing node. Furthermore, friendship graphs can be linked to block designs and combinatorial structures used to construct error-correcting codes with good covering properties. Beyond these applications, the friendship theorem has a broad range of applications in computer science, biology, social sciences, and economics. The fact that every two vertices share exactly one mutual neighbor makes it a useful model

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Academic Editors: Jonnathan Rodriguez, Luis Medina and Iliya Bouyukliev

Received: 19 February 2025
Revised: 5 March 2025
Accepted: 12 March 2025
Published: 14 March 2025

Citation: Sason, I. On Strongly Regular Graphs and the Friendship Theorem. *Mathematics* **2025**, *13*, 970. https://doi.org/10.3390/math13060970

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for systems where a single hub controls multiple independent interactions, such as in social influence, network protocols, and centralized coordination problems.

The original proof of the theorem involves arguments based on graph connectivity, degree counting, and combinatorial reasoning. The reader is referred to a nice exposition of a proof of the friendship theorem in Chapter 44 of [2], which combines combinatorics and linear algebra (spectral graph theory). Due to its importance and simplicity, the theorem was proved in various ways. For historical insights into these proofs, as well as a novel combinatorial proof and an extension of the theorem, the reader is referred to [3].

The aim of this paper was twofold. First, it provides an alternative proof of this theorem that relies on a recent result by the author [4], in which a closed-form expression for the celebrated Lovász ϑ-function was derived for the structured family of strongly regular graphs. We believe this proof offers a new perspective on the existing collection of proofs of the friendship theorem. This paper further considers some known extensions of the theorem, offering discussions that provide insights into the friendship theorem, one of its extensions from [3], and the proposed proof. The second aim of this paper, building on the closed-form expression for the Lovász ϑ-function of all strongly regular graphs, is to establish new necessary conditions for a strongly regular graph to be a spanning or induced subgraph of another strongly regular graph. For induced subgraphs, the analysis also incorporates a property of graph energies. Some of these results are extended to regular graphs and their subgraphs. In general, determining whether a graph is a spanning or induced subgraph of another given graph is an important and broadly applicable problem across theoretical, algorithmic, and applied areas of graph theory.

This paper is structured as follows: Section 2 introduces the notation and essential background required for the analysis in this paper. Section 3 provides our alternative proof of the friendship theorem, followed by a variation of a known spectral-graph-theoretic proof, and a consideration of the friendship theorem and our alternative proof to gain further insights into the theorem, our proposed proof, and one of the theorem's extensions in [3]. Based on the Lovász ϑ-function of strongly regular graphs, Sections 4 and 5 establish new necessary conditions for one strongly regular graph to be a spanning or induced subgraph of another strongly regular graph. The proposed conditions can be easily checked, their utility is demonstrated, and they are extended to regular graphs and subgraphs, with a consideration of their computational complexity.

2. Preliminaries

This section presents the notation and necessary background, including the definitions and theorems essential for the analysis in this paper.

Let G be a graph with vertex set $\mathsf{V}(\mathsf{G})$ and edge set $\mathsf{E}(\mathsf{G})$. A graph G is called simple if it has no self-loops and no multiple edges between any pair of vertices. Throughout this paper, unless explicitly mentioned, all graphs under consideration are simple, finite, and undirected. The following standard notation and definitions are used.

Definition 1 (Graph complement)**.** *The complement of a graph* G*, denoted by* $\overline{\mathsf{G}}$*, is a graph whose vertex set is* $\mathsf{V}(\mathsf{G})$*, and its edge set is the complement set* $\overline{\mathsf{E}(\mathsf{G})}$*. Every vertex in* $\mathsf{V}(\mathsf{G})$ *is nonadjacent to itself in* G *and* $\overline{\mathsf{G}}$*, so* $\{i,j\} \in \mathsf{E}(\overline{\mathsf{G}})$ *if and only if* $\{i,j\} \notin \mathsf{E}(\mathsf{G})$ *with* $i \neq j$*.*

Definition 2 (Integer-valued graph invariants)**.**

- *Let* $k \in \mathbb{N}$*. A proper k-coloring of a graph* G *is a function* $c\colon \mathsf{V}(\mathsf{G}) \to \{1,2,\ldots,k\}$*, where* $c(v) \neq c(u)$ *for every* $\{u,v\} \in \mathsf{E}(\mathsf{G})$*. The chromatic number of* G*, denoted by* $\chi(\mathsf{G})$*, is the smallest k for which there exists a proper k-coloring of* G*.*

- *A clique in a graph* G *is a subset of vertices* $U \subseteq \mathsf{V}(\mathsf{G})$*, where* $\{u, v\} \in \mathsf{E}(\mathsf{G})$ *for every* $u, v \in U$ *with* $u \neq v$*. The clique number of* G*, denoted by* $\omega(\mathsf{G})$*, is the largest size of a clique in* G*.*
- *An independent set in a graph* G *is a subset of vertices* $U \subseteq \mathsf{V}(\mathsf{G})$*, where* $\{u, v\} \notin \mathsf{E}(\mathsf{G})$ *for every* $u, v \in U$*. The independence number of* G*, denoted by* $\alpha(\mathsf{G})$*, is the largest size of an independent set in* G*. Consequently,* $\alpha(\mathsf{G}) = \omega(\overline{\mathsf{G}})$ *for every graph* G*.*

These integer-valued functions of a graph are invariant under graph isomorphisms, so they are referred to as graph invariants.

We next introduce the Lovász ϑ-function of a graph G, and then consider some of its useful properties. To this end, orthonormal representations of graphs are introduced.

Definition 3 (Orthonormal representations of a graph)**.** *Let* G *be a finite, simple, and undirected graph, and* $d \in \mathbb{N}$*. An orthonormal representation of* G *in the d-dimensional Euclidean space* $\mathbb{R}^d$ *assigns to each vertex* $i \in \mathsf{V}(\mathsf{G})$ *a unit vector* $\mathbf{u}_i \in \mathbb{R}^d$ *such that for every two distinct and nonadjacent vertices in* G*, their assigned vectors are orthogonal.*

Definition 4 (Lovász ϑ-function, [5])**.** *Let* G *be a finite, undirected, and simple graph. The Lovász* ϑ*-function of* G *is defined as*

$$\vartheta(\mathsf{G}) \triangleq \min_{\{\mathbf{u_i}\}, \mathbf{c}} \max_{i \in \mathsf{V}(\mathsf{G})} \frac{1}{\left(\mathbf{c}^{\mathrm{T}} \mathbf{u}_i\right)^2}, \tag{1}$$

where the minimum is taken over all orthonormal representations $\{\mathbf{u}_i : i \in \mathsf{V}(\mathsf{G})\}$ *of* G *and over all unit vectors* $\mathbf{c}$*. The unit vector* $\mathbf{c}$ *is called the* handle *of the orthonormal representation. By the Cauchy–Schwarz inequality,* $\left|\mathbf{c}^{\mathrm{T}} \mathbf{u}_i\right| \leq \|\mathbf{c}\| \, \|\mathbf{u}_i\| = 1$*, so* $\vartheta(\mathsf{G}) \geq 1$*, with equality if and only if* G *is a complete graph.*

The dimension d of the Euclidean space $\mathbb{R}^d$ in the orthonormal representations of G, over which the minimization on the right-hand side of (1) is performed, can be set to the order of G, i.e., $d = |\mathsf{V}(\mathsf{G})|$ (see p. 183 of [6]). Let the following notation be used:

- $\mathbf{A}$ is the $n \times n$ adjacency matrix of G ($n \triangleq |\mathsf{V}(\mathsf{G})|$), where $A_{i,j} = 1$ if and only if $\{i, j\} \in \mathsf{E}(\mathsf{G})$ and $A_{i,j} = 0$ otherwise.
- $\mathbf{J}_n$ is the all-ones $n \times n$ matrix.
- $\mathcal{S}_+^n$ is the set of all $n \times n$ positive semidefinite matrices.
- The eigenvalues of $\mathbf{A}$ are given in decreasing order by

$$\lambda_{\max}(\mathsf{G}) = \lambda_1(\mathsf{G}) \geq \lambda_2(\mathsf{G}) \geq \ldots \geq \lambda_n(\mathsf{G}) = \lambda_{\min}(\mathsf{G}). \tag{2}$$

- The *adjacency spectrum* of G is the multiset of the eigenvalues of $\mathbf{A}$, counted with multiplicities.

The following semidefinite program (SDP) computes $\vartheta(\mathsf{G})$ (by Theorem 4 of [5]):

$$\boxed{\begin{array}{l} \text{maximize} \quad \mathrm{Trace}(\mathbf{B}\,\mathbf{J}_n) \\ \text{subject to} \\ \begin{cases} \mathbf{B} \in \mathcal{S}_+^n, \quad \mathrm{Trace}(\mathbf{B}) = 1, \\ A_{i,j} = 1 \;\Rightarrow\; B_{i,j} = 0, \quad i, j \in [n]. \end{cases} \end{array}} \tag{3}$$

This renders the computational complexity of $\vartheta(\mathsf{G})$ feasible. Specifically, there exist standard algorithms in convex optimization that numerically compute $\vartheta(\mathsf{G})$ for every graph G with a precision of r decimal digits and in polynomial-time in n and r (see Section 11.3 of [6]).

The Lovász ϑ-function of G and its complement graph $\overline{\mathsf{G}}$ satisfy the following [5,6]:

(1) Sandwich theorem:

$$\alpha(\mathsf{G}) \leq \vartheta(\mathsf{G}) \leq \chi(\overline{\mathsf{G}}), \tag{4}$$

$$\omega(\mathsf{G}) \leq \vartheta(\overline{\mathsf{G}}) \leq \chi(\mathsf{G}). \tag{5}$$

(2) Computational complexity: The graph invariants $\alpha(\mathsf{G})$, $\omega(\mathsf{G})$, and $\chi(\mathsf{G})$ are NP-hard problems. However, as mentioned above, the numerical computation of $\vartheta(\mathsf{G})$ is, in general, feasible, so it provides polynomial-time computable bounds on very useful graph invariants that are hard to compute.

(3) Hoffman–Lovász inequality: Let G be d-regular of order n. Then,

$$\vartheta(\mathsf{G}) \leq -\frac{n\,\lambda_n(\mathsf{G})}{d-\lambda_n(\mathsf{G})}, \tag{6}$$

with equality if G is edge-transitive.

Definition 5 (Strongly regular graphs)**.** *Let G be a d-regular graph of order n. The graph G is a strongly regular graph (SRG) if there exist nonnegative integers λ and μ such that*

- *Every pair of adjacent vertices has exactly λ common neighbors;*
- *Every pair of distinct, nonadjacent vertices has exactly μ common neighbors.*

Such a strongly regular graph is said to be a graph in the family $\mathsf{srg}(n,d,\lambda,\mu)$.

It is important to note that these four parameters are interrelated (see Proposition 2 for the known results). Consequently, for some parameter vectors (n,d,λ,μ), the family $\mathsf{srg}(n,d,\lambda,\mu)$ contains no graphs. Furthermore, for parameter vectors (n,d,λ,μ) such that the set $\mathsf{srg}(n,d,\lambda,\mu)$ is nonempty, there may exist several nonisomorphic strongly regular graphs within it. For example, according to [7], there exist 167 nonisomorphic strongly regular graphs in the family $\mathsf{srg}(64,18,2,6)$. The reader is referred to [7–9] for the enumeration of strongly regular graphs with given parameter vectors.

Proposition 1 ([9])**.** *A graph G is strongly regular if and only if its complement $\overline{\mathsf{G}}$ is so. Furthermore, if G is a strongly regular graph in the family $\mathsf{srg}(n,d,\lambda,\mu)$, then $\overline{\mathsf{G}}$ is a strongly regular graph in the family $\mathsf{srg}(n,n-d-1,n-2d+\mu-2,n-2d+\lambda)$.*

Theorem 1 (Bounds on the Lovász function of regular graphs, [4])**.** *Let G be a d-regular graph of order n, which is a noncomplete and nonempty graph. Then, the following bounds hold for the Lovász ϑ-function of G and its complement $\overline{\mathsf{G}}$:*

(1)

$$\frac{n-d+\lambda_2(\mathsf{G})}{1+\lambda_2(\mathsf{G})} \leq \vartheta(\mathsf{G}) \leq -\frac{n\lambda_n(\mathsf{G})}{d-\lambda_n(\mathsf{G})}. \tag{7}$$

- *Equality holds in the leftmost inequality of (7) if $\overline{\mathsf{G}}$ is both vertex-transitive and edge-transitive, or if G is a strongly regular graph;*
- *Equality holds in the rightmost inequality of (7) if G is edge-transitive, or if G is a strongly regular graph.*

(2)

$$1-\frac{d}{\lambda_n(\mathsf{G})} \leq \vartheta(\overline{\mathsf{G}}) \leq \frac{n\big(1+\lambda_2(\mathsf{G})\big)}{n-d+\lambda_2(\mathsf{G})}. \tag{8}$$

- *Equality holds in the leftmost inequality of (8) if* G *is both vertex-transitive and edge-transitive, or if* G *is a strongly regular graph;*
- *Equality holds in the rightmost inequality of (8) if* $\overline{\mathsf{G}}$ *is edge-transitive, or if* G *is a strongly regular graph.*

As a common sufficient condition, note that all the inequalities in (7) and (8) hold with equality if G is a strongly regular graph. The following result provides a closed-form expression of the Lovász ϑ-function of all strongly regular graphs.

Theorem 2 (The Lovász ϑ-function of strongly regular graphs, [4])**.** *Let* G *be a strongly regular graph in the family* $\mathsf{srg}(n, d, \lambda, \mu)$*. Then,*

$$\vartheta(\mathsf{G}) = \frac{n\,(t+\mu-\lambda)}{2d+t+\mu-\lambda}, \tag{9}$$

$$\vartheta(\overline{\mathsf{G}}) = 1 + \frac{2d}{t+\mu-\lambda}, \tag{10}$$

where

$$t \triangleq \sqrt{(\mu-\lambda)^2 + 4(d-\mu)}. \tag{11}$$

Remark 1. *In light of Theorem 2, all strongly regular graphs in a family* $\mathsf{srg}(n, d, \lambda, \mu)$ *are not only cospectral [9], but they also share the same value of the Lovász* ϑ*-function.*

Corollary 1 ([4])**.** *Let* G *be a strongly regular graph on* n *vertices. Then,*

$$\vartheta(\mathsf{G})\,\vartheta(\overline{\mathsf{G}}) = n. \tag{12}$$

Remark 2. *Equality* (12) *is known to hold for all vertex-transitive graphs (see Theorem 8 of [5]). Corollary 1 shows that it also holds for all strongly regular graphs. It should be noted that strongly regular graphs may not necessarily be vertex-transitive (see, e.g., [10]).*

The closed-form expression in Theorem 2 for the Lovász ϑ-function of strongly regular graphs eliminates the need for a numerical solution to the SDP in (3) while also providing an explicit analytical expression for this function. This facilitates the derivation of analytical bounds on key graph invariants, which are generally NP-hard to compute. These bounds are expressed in terms of the parameters (n, d, λ, μ) characterizing strongly regular graphs, as it is next shown in Corollary 2.

Corollary 2 (Bounds on graph invariants of strongly regular graphs, [4])**.** *Let* G *be a strongly regular graph in the family* $\mathsf{srg}(n, d, \lambda, \mu)$*. Then,*

$$\alpha(\mathsf{G}) \leq \left\lfloor \frac{n\,(t+\mu-\lambda)}{2d+t+\mu-\lambda} \right\rfloor \tag{13}$$

$$\omega(\mathsf{G}) \leq 1 + \left\lfloor \frac{2d}{t+\mu-\lambda} \right\rfloor, \tag{14}$$

$$\chi(\mathsf{G}) \geq 1 + \left\lceil \frac{2d}{t+\mu-\lambda} \right\rceil, \tag{15}$$

$$\chi(\overline{\mathsf{G}}) \geq \left\lceil \frac{n\,(t+\mu-\lambda)}{2d+t+\mu-\lambda} \right\rceil, \tag{16}$$

where t *is given in (11).*

Proof. The bounds in (13)–(16) follow from the combination of the sandwich theorem in (4) and (5) with Theorem 2. □

Example 1 (Bounds on graph invariants of strongly regular graphs)**.** *The tightness of the bounds in Corollary 2 is exemplified for four strongly regular graphs as follows:*

(1) Petersen graph: Let G_1 *be the Petersen graph, the unique strongly regular graph in the family* $\mathsf{srg}(10,3,0,1)$ *(see Section 10.3 of [9]). For* G_1*, the upper bounds on its independence and clique numbers in (13) and (14), respectively, as well as the lower bound on its chromatic number in (15) are tight:*

$$\alpha(\mathsf{G}_1) = 4, \quad \omega(\mathsf{G}_1) = 2, \quad \chi(\mathsf{G}_1) = 3. \tag{17}$$

(2) Schläfli, Shrikhande, and Hall–Janko graphs:

- *The Schläfli graph is (up to an isomorphism) the unique strongly regular graph in the family* $\mathsf{srg}(27,16,10,8)$ *(see Section 10.7 of [9]).*
- *The Shrikhande graph is one of two nonisomorphic strongly regular graphs in the family* $\mathsf{srg}(16,6,2,2)$ *(see Section 10.6 of [9]).*
- *The Hall–Janko graph is the unique strongly regular graph in the family* $\mathsf{srg}(100,36,14,12)$ *(see Section 10.32 of [9]).*

Let G_2*,* G_3*, and* G_4 *denote these graphs, respectively. The lower bounds on the chromatic numbers of these graphs, as given in (15), are all tight:*

$$\chi(\mathsf{G}_2) = 9, \quad \chi(\mathsf{G}_3) = 4, \quad \chi(\mathsf{G}_4) = 10. \tag{18}$$

(3) Shrikhande graph (G_3*):*

- *The upper bound on its independence number in (13) is tight:* $\alpha(\mathsf{G}_3) = 4$*.*
- *The upper bound on its clique number in (14) is, however, not tight, as the upper bound is 4, while the actual clique number is* $\omega(\mathsf{G}_3) = 3$*.*

3. The Friendship Theorem: An Alternative Proof and Further Extensions

This section is structured as follows: It presents the renowned friendship theorem, originally established in [1] (see Section 3.1), and it provides a new alternative proof that relies on the Lovász ϑ-function of strongly regular graphs (see Theorem 2 and Section 3.2). This section further provides a variation of a proof from Chapter 44 of [2] that relies on the spectral properties of strongly regular graphs (see Section 3.3). It finally considers further generalizations, providing discussions in order to obtain further insights into the theorem, our proposed proof, and one of the theorem's extensions in [3], which are also supported by numerical results for (small and large) strongly regular graphs (see Section 3.4).

3.1. The Friendship Theorem

Theorem 3 (Friendship Theorem, [1])**.** *Let* G *be a finite graph in which any two distinct vertices have a single common neighbor. Then,* G *consists of edge-disjoint triangles sharing a single vertex that is adjacent to every other vertex.*

A human interpretation of Theorem 3 is well known. Assume that there is a party with n people, where every pair of individuals has precisely one common friend at the party. Theorem 3 asserts that one of these people is a friend of everyone. Indeed, construct a graph whose vertices represent the n people, and every two vertices are adjacent if and only if they represent two friends. The claim then follows from Theorem 3.

Remark 3 (On Theorem 3)**.** *The windmill graph (see Figure 1) has the desired property of the friendship theorem and it turns out to be the only graph with that property. Notably, the friendship theorem does not hold for infinite graphs. Indeed, for an inductive construction of a counterexample, one may start with a five-cycle* C_5 *and repeatedly add a common neighbor for every pair of vertices that does not yet have one. This process results in a countably infinite friendship graph without a vertex adjacent to all other vertices.*

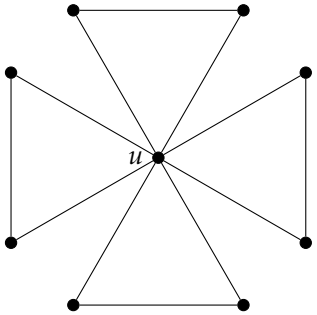


Figure 1. Windmill graph (also known as a friendship graph) consisting of multiple triangles sharing a common central vertex u.

3.2. A New Alternative Proof of the Friendship Theorem Relying on the Lovász ϑ-Function

This subsection presents an alternative proof of Theorem 3 by utilizing the Lovász ϑ-function of strongly regular graphs (and their complements) in Theorem 2. Recall that by Proposition 1, the complements of strongly regular graphs are strongly regular as well.

Proof. Suppose the assertion is false and let G be a counterexample—a finite graph in which any two distinct vertices have a single common neighbor, yet no vertex in G is adjacent to all other vertices. A contradiction is obtained as follows:

The first step shows that the graph G is regular, as proved in Chapter 44 of [2]. We provide a variation of this regularity proof, and then take a different approach that relies on graph invariants, such as $\omega(\mathsf{G})$, $\chi(\mathsf{G})$, and $\vartheta(\overline{\mathsf{G}})$.

- To assert the regularity of G, it is first proved that nonadjacent vertices in G have equal degrees, i.e., $d(u) = d(v)$ if $\{u, v\} \notin \mathsf{E}(\mathsf{G})$.
- The given hypothesis yields that G is a connected graph (having a diameter of at most 2). Let $\{u, v\} \notin \mathsf{E}(\mathsf{G})$, and let $\mathcal{N}(u)$ and $\mathcal{N}(v)$ denote, respectively, the sets of neighbors of the nonadjacent vertices u and v.
- Let $f\colon \mathcal{N}(u) \to \mathcal{N}(v)$ be the injective function where every $x \in \mathcal{N}(u)$ is mapped to the unique $y \in \mathcal{N}(x) \cap \mathcal{N}(v)$. Indeed, if $z \in \mathcal{N}(u) \setminus \{x\}$ satisfies $f(z) = y$, then x and z share two common neighbors (namely, y and u), which contradicts the assumption of the theorem.
- Since $f\colon \mathcal{N}(u) \to \mathcal{N}(v)$ is injective, it follows that $|\mathcal{N}(u)| \leq |\mathcal{N}(v)|$.
- By symmetry, swapping u and v (as nonadjacent vertices of the undirected graph G) also yields $|\mathcal{N}(v)| \leq |\mathcal{N}(v)|$, so we obtain $d(u) = |\mathcal{N}(u)| = |\mathcal{N}(v)| = d(v)$ for all vertices $u, v \in \mathsf{V}(\mathsf{G})$ such that $\{u, v\} \notin \mathsf{E}(\mathsf{G})$.
- To complete the proof that G is regular, let u and v be fixed, nonadjacent vertices in G. Consequently, $d(u) = d(v)$. By the assumption of the theorem, except for one vertex, all other vertices are either nonadjacent to u or v. Hence, except for this vertex, all these vertices must have identical degrees by what we already proved.
- Finally, by our further assumption (later leading to a contradiction), since there is no vertex in G that is adjacent to all other vertices, the single vertex that is adjacent to u and v also has a nonneighbor in G, so it also should have an identical degree to all the degrees of the other vertices (by what is proved in the previous item). Consequently, G is a regular graph.

From this point, our proof proceeds differently.

- Let G be a k-regular graph on n vertices. By the theorem's assumption, every two distinct vertices have exactly one common neighbor, so G is either a connected, strongly regular graph in the family $\mathsf{srg}(n,k,1,1)$ if $k \geq 3$, or $\mathsf{G} = \mathsf{K}_3$ if $k = 2$, or $\mathsf{G} = \mathsf{K}_1$ if $k = 0$ (recall that complete graphs are excluded from the family of strongly regular graphs).
- First, $k = 0$ and $k = 2$ are excluded since complete graphs do not contradict our assumption of having a vertex not adjacent to all other vertices. Hence, let $k \geq 3$.
- Every two adjacent vertices in G share a common neighbor, so G contains a triangle. Moreover, G is C_4-free since every two vertices have exactly one common neighbor, so it must also be K_4-free. Hence, $\omega(\mathsf{G}) = 3$.
- We next show that $\chi(\mathsf{G}) = 3$. First, $\chi(\mathsf{G}) \geq \omega(\mathsf{G}) = 3$. We also need to show that $\chi(\mathsf{G}) \leq 3$, which means that three colors suffice to color all the vertices of G in a way that no two adjacent vertices are assigned the same color. This can be performed recursively by observing that each edge belongs to exactly one triangle (otherwise, the endpoints of that edge would have more than one common neighbor, which is not allowed). Moreover, each newly colored vertex always completes a properly colored triangle, ensuring that the coloring remains valid at every step of the recursion without requiring a fourth color.
- By the sandwich theorem, $\omega(\mathsf{G}) \leq \vartheta(\overline{\mathsf{G}}) \leq \chi(\mathsf{G})$, so $\omega(\mathsf{G}) = \chi(\mathsf{G}) = 3$ implies that $\vartheta(\overline{\mathsf{G}}) = 3$.
- By Theorem 2, where G is a strongly regular graph in the family $\mathsf{srg}(n,k,1,1)$, we also obtain

$$\vartheta(\overline{\mathsf{G}}) = 1 + \frac{k}{\sqrt{k-1}}. \tag{19}$$

This leads to a contradiction since for all $k \geq 3$,

$$\begin{aligned} &(k-2)^2 > 0, \\ \Leftrightarrow\ &k^2 > 4(k-1), \\ \Leftrightarrow\ &1 + \frac{k}{\sqrt{k-1}} > 3, \end{aligned} \tag{20}$$

which completes the proof of the theorem by contradiction. Note that every edge in G lies on a triangle (by the theorem's assumption), and G consists of edge-disjoint triangles since it is C_4-free.

□

3.3. Another Proof of the Friendship Theorem Relying on the Spectral Properties of Strongly Regular Graphs

This subsection presents an alternative proof of the friendship theorem that relies on the adjacency spectrum of strongly regular graphs. This second proof forms a variation of the proof provided in Chapter 44 of [2].

From the point in Section 3.2 where we obtain, by contradiction, that G is a strongly regular graph in the family $\mathsf{srg}(n,k,1,1)$ with $k \geq 3$, it is possible to obtain a contradiction in an alternative way that relies on the following known result for strongly regular graphs.

Proposition 2 (Feasible parameters of strongly regular graphs)**.** *Let* G *be a strongly regular graph in the family* $\mathsf{srg}(n,d,\lambda,\mu)$*. Then, the following holds:*

(1) $(n-d-1)\,\mu = d\,(d-\lambda-1)$.

(2) $n-1-\frac{2d+(n-1)(\lambda-\mu)}{\sqrt{(\lambda-\mu)^2+4(d-\mu)}}$ *is a nonnegative even integer.*

(3) $6|(nd\lambda)$.

Proof. This holds by relying on some basic properties of strongly regular graphs (see Chapter 21 of [11]) as follows.

- Item 1 is a combinatorial equality for strongly regular graphs. It is obtained by letting the vertices in G lie in three levels, where an arbitrarily selected vertex is at the root in Level 0, its d neighbors lie in Level 1, and all the others lie in Level 2. The equality in Condition 1 then follows from a double-counting argument of the number of edges between Levels 1 and 2.
- Item 2 holds by the integrality and nonnegativity of the multiplicities $m_{1,2}$ of the second-largest and least eigenvalues of the adjacency matrix, respectively, which satisfy

$$0 \leq 2m_{1,2} = n-1 \mp \frac{2d+(n-1)(\lambda-\mu)}{\sqrt{(\lambda-\mu)^2+4(d-\mu)}}. \tag{21}$$

- Item 3 holds since the number of triangles in G is given by $\frac{1}{6}nd\lambda$. Indeed, every vertex has d neighbors, every pair of adjacent vertices shares exactly λ common neighbors (forming a triangle), and each triangle is counted six times due to the six possible permutations of its three vertices.

□

- By applying Item 1 in Proposition 2 to G with the parameters $d=k$ and $\lambda=\mu=1$, we obtain $n=k^2-k+1$. This does not lead to a contradiction yet since summing over all the degrees of the neighbors of an arbitrary vertex u in G gives k^2. Then, by the assumption of the theorem that every two vertices have exactly one common neighbor, it follows that the above summation counts each vertex in G exactly once, except for vertex u, which is counted k times. Hence, indeed $n=k^2-k+1$.
- By Item 2 in Proposition 2 with $d=k$ and $\lambda=\mu=1$, it follows that $\frac{k}{\sqrt{k-1}} \in \mathbb{N}$. Consequently, $(k-1)|k^2$. Since $k^2=(k-1)(k+1)+1$, it follows that $(k-1)|1$, so $k=2$. If $k=2$, the only graph that satisfies the condition of Theorem 3 is $\mathsf{G}=\mathsf{K}_3$, which also satisfies the assertion of the theorem. Hence, this argument contradicts the assumption in the proof since it leads to the conclusion that G is a strongly regular graph in the family $\mathsf{srg}(n,k,1,1)$ with $k \geq 3$, which is subsequently rejected (as $(k-1) \nmid 1$).

3.4. Further Discussions and Insights on Extensions of the Friendship Theorem

This subsection considers further generalizations and provides discussions on the friendship theorem and its proofs. The following discussions are presented in Remarks 4–6 and are supplemented by examples that examine strongly regular graphs of different orders, both small and large. In particular, Remark 6 builds on our proof technique, which is based on the Lovász ϑ-function of strongly regular graphs, to offer insights into the extended friendship theorem studied in [3]. It also establishes additional results, which are further illustrated through Examples 2–4.

Remark 4. *By the friendship theorem (Theorem 3), every finite graph in which each pair of distinct vertices shares exactly one common neighbor must be isomorphic to a windmill graph (see Figure 1). Consequently, its order n must be an odd integer, and the size of the graph is given by* $|\mathsf{E}(\mathsf{G})| = \frac{3}{2}(n-1)$. *By definition, a graph* G *satisfying the assumption of Theorem 3 is* C_4*-free.*

A combinatorial proof, based on double counting, asserts that the size of a C_4*-free graph* G *on* n *vertices satisfies (see pp. 200–201 in [2])*

$$|\,\mathsf{E}(\mathsf{G})| \leq \left\lfloor \tfrac{1}{4}n(1+\sqrt{4n-3}) \right\rfloor. \tag{22}$$

Furthermore, it was shown that this upper bound on the size of G *is nearly tight in the sense that if* p *is a prime number, and* $n = p^2 + p + 1$*, then a specific construction of a* C_4*-free graph on* n *vertices has a size that is given by (see pp. 201–202 in [2])*

$$|\,\mathsf{E}(\mathsf{G})| = \tfrac{1}{4}(n-1)(1+\sqrt{4n-3}). \tag{23}$$

The significant difference between the size of a windmill graph on n *vertices, scaling linearly as* $\frac{3}{2}n$*, and the largest size of a general* C_4*-free graph on* n *vertices, scaling super-linearly as* $\frac{1}{2}n^{\frac{3}{2}}$*, stems from an additional structural constraint on* G*. Specifically, beyond being a* C_4*-free graph,* G *must satisfy a stronger condition that uniquely determines it (up to an isomorphism) as a windmill graph.*

Remark 5. *The friendship theorem was recently generalized in [12] to directed graphs (digraphs), allowing for asymmetry in liking relationships. The digraph version of the friendship theorem (see Theorem 1.1 of [12]) states that if a finite, simple, directed graph has the property that every pair of vertices has exactly one common out-neighbor, then it must be either of the following:*

(1) A k*-regular digraph—where each vertex has out-degree* k *and in-degree* k*—on* $n = k^2 - k + 1$ *vertices with* $k \geq 2$*;*

(2) A so-called "fancy wheel digraph," which consists of a disjoint union of directed cycles with an additional vertex that has arcs to and from every vertex on these cycles.

Notably, the order n *of the* k*-diregular digraph matches that of the undirected graph* G *in the second proof (Section 3.3), in which it was established—prior to reaching the contradiction—that* G *is a* k*-regular graph of order* $n = k^2 - k + 2$*.*

Remark 6. *As a possible extension of the friendship theorem, Proposition 1 of [3] states that if* G *is a finite, simple, and undirected graph such that every pair of vertices has exactly* ℓ *common neighbors for some fixed* $\ell \geq 2$*, then* G *is a regular graph. Consequently,* G *is a strongly regular graph in the family* $\mathsf{srg}(n, d, \ell, \ell)$*. Furthermore, by Item 1 of Proposition 2 (here), it follows by substituting* $\lambda = \mu = \ell$ *that the parameters* n*,* d*, and* ℓ *are related by the equation* $(n-1)\ell = d(d-1)$*. As an easy generalization of (19) in our proof of the friendship theorem, it follows from Theorem 2 that*

$$\vartheta(\overline{\mathsf{G}}) = 1 + \frac{d}{\sqrt{d-\ell}}. \tag{24}$$

Similar to our proof of the friendship theorem, which is based on the sandwich theorem stating that $\omega(\mathsf{G}) \leq \vartheta(\overline{\mathsf{G}}) \leq \chi(\mathsf{G})$*, it follows that the clique and chromatic numbers of* G *satisfy, respectively,*

$$\omega(\mathsf{G}) \leq 1 + \left\lfloor \frac{d}{\sqrt{d-\ell}} \right\rfloor, \tag{25}$$

$$\chi(\mathsf{G}) \geq 1 + \left\lceil \frac{d}{\sqrt{d-\ell}} \right\rceil. \tag{26}$$

Furthermore, by Corollary 1, it follows from (24) that

$$\vartheta(\mathsf{G}) = \frac{n\sqrt{d-\ell}}{d+\sqrt{d-\ell}}. \tag{27}$$

Similarly, by the sandwich theorem, the independence number of G *and the chromatic number of its complement* $\overline{\mathsf{G}}$ *satisfy the following bounds:*

$$\alpha(\mathsf{G}) \leq \left\lfloor \frac{n\sqrt{d-\ell}}{d+\sqrt{d-\ell}} \right\rfloor, \tag{28}$$

$$\chi(\overline{\mathsf{G}}) \geq \left\lceil \frac{n\sqrt{d-\ell}}{d+\sqrt{d-\ell}} \right\rceil. \tag{29}$$

Hence, our proposed proof of the friendship theorem that relies on the Lovász ϑ*-function of strongly regular graphs, combined with Theorem 2 and Corollary 1 (here), and Proposition 1 of [3], lead to bounds on graph invariants of* G *and* $\overline{\mathsf{G}}$ *with respect to the extended version of the friendship theorem of [3] for an arbitrary* $\ell \geq 2$.

The next three examples, preceded by definitions and notation, refer to Remark 6.

Definition 6 (Cartesian product of graphs)**.** *Let* G *and* H *be graphs. The Cartesian product, denoted by* $\mathsf{G} \square \mathsf{H}$*, is a graph with a vertex set that is given by* $\mathsf{V}(\mathsf{G} \square \mathsf{H}) = \mathsf{V}(\mathsf{G}) \times \mathsf{V}(\mathsf{H})$ *(i.e., it is equal to the Cartesian product of the vertex sets of* G *and* H*), and its edge set is characterized by the property that distinct vertices* (g,h) *and* (g',h')*, where* $g,g' \in \mathsf{V}(\mathsf{G})$ *and* $h,h' \in \mathsf{V}(\mathsf{H})$*, are adjacent in* $\mathsf{G} \square \mathsf{H}$ *if and only if one of the following conditions holds:*
(1) $g = g'$ *and* $\{h,h'\} \in \mathsf{E}(\mathsf{H})$*, or (2)* $\{g,g'\} \in \mathsf{E}(\mathsf{G})$ *and* $h = h'$.

Definition 7 (Line graph)**.** *Let* $\mathsf{G} = (\mathsf{V},\mathsf{E})$ *be a graph. The line graph of* G*, denoted by* $\mathsf{L}(\mathsf{G})$*, is a graph whose vertices are the edges of* G*, and two vertices are adjacent in the line graph* $\mathsf{L}(\mathsf{G})$ *if the corresponding edges are incident in* G.

Example 2. *The Shrikhande graph and the Cartesian product of two complete graphs on four vertices,* $\mathsf{K}_4 \square \mathsf{K}_4$*, are the two nonisomorphic strongly regular graphs in the family* $\mathsf{srg}(16,6,2,2)$*, where each pair of distinct vertices has exactly two common neighbors. Note that the parameters* $n = 16$*,* $d = 6$*, and* $\ell = 2$ *indeed satisfy the equality* $(n-1)\ell = d(d-1)$*. Let* G_1 *and* G_2 *denote, respectively, these two graphs. By (25), (26), (28), and (29), we obtain*

$$\alpha(\mathsf{G}) \leq 4, \quad \omega(\mathsf{G}) \leq 4, \quad \chi(\mathsf{G}) \geq 4, \quad \chi(\overline{\mathsf{G}}) \geq 4, \tag{30}$$

which hold for G_1 *and* G_2*. A comparison with their exact values (computed by [13]) gives*

$$\alpha(\mathsf{G}_1) = 4, \quad \omega(\mathsf{G}_1) = 3, \quad \chi(\mathsf{G}_1) = 4, \quad \chi(\overline{\mathsf{G}}_1) = 6, \tag{31}$$

$$\alpha(\mathsf{G}_2) = 4, \quad \omega(\mathsf{G}_2) = 4, \quad \chi(\mathsf{G}_2) = 4, \quad \chi(\overline{\mathsf{G}}_2) = 4, \tag{32}$$

so the four bounds on the graph invariants in (30) are tight for $\mathsf{G} = \mathsf{G}_2$.

Example 3. *For* $n \geq 4$*, the line graph of the complete graph on* n *vertices,* $\mathsf{G} = \mathsf{L}(\mathsf{K}_n)$*, is a strongly regular graph in the family* $\mathsf{srg}(\frac{1}{2}n(n-1), 2(n-2), n-2, 4)$*. Consequently, the line graph* $\mathsf{L}(\mathsf{K}_6)$ *is in the family* $\mathsf{srg}(15,8,4,4)$*, so every pair of distinct vertices in this graph has exactly four common neighbors. Using the SageMath software [13] gives* $\alpha(\mathsf{G}) = 3$*,* $\omega(\mathsf{G}) = 5$*,* $\chi(\mathsf{G}) = 5$*, and* $\chi(\overline{\mathsf{G}}) = 4$*. Substituting the graph parameters* $n = 15$*,* $d = 8$*, and* $\ell = 4$ *into (25)–(29) shows that the three bounds in (25), (26), and (28) are tight. However, the bound in (29) is not tight, as it gives* $\chi(\overline{\mathsf{G}}) \geq 3$*, whereas its exact value is 4.*

Example 4 (Complements of symplectic polar graphs). *The symplectic polar graphs are strongly regular graphs belonging to the families* $\mathsf{srg}(v,k,\lambda,\mu)$*, where*

$$v=\frac{q^{2n}-1}{q-1},\quad k=\frac{q(q^{2n-2}-1)}{q-1},\quad \lambda=\frac{q^{2n-2}-1}{q-1}-2,\quad \mu=\frac{q^{2n-2}-1}{q-1}, \tag{33}$$

for all $n\in\mathbb{N}$ *and* $q\geq 2$ *that is a prime power. Consequently,* $\lambda=\mu-2$ *and* $\mu=\frac{k}{q}$*. This symplectic polar graph is denoted by* $\mathsf{G}=\mathsf{Sp}(2n,q)$ *(see Section 2.5 in [9]). The complement graph of a strongly regular graph in the family* $\mathsf{srg}(v,k,\lambda,\mu)$ *is a strongly regular graph in the family* $\mathsf{srg}(v,v-k-1,v-2k+\mu-2,v-2k+\lambda)$ *(see Proposition 1). Substituting v, k, λ, and μ in (33) gives that* $\mathsf{H}=\overline{\mathsf{G}}$ *is a strongly regular graph in the family*

$$\mathsf{srg}\left(\frac{q^{2n}-1}{q-1},q^{2n-1},q^{2n-2}(q-1),q^{2n-2}(q-1)\right). \tag{34}$$

Consequently, the family of the complements of the symplectic polar graphs, $\mathsf{H}=\overline{\mathsf{Sp}(2n,q)}$*, where* $n\in\mathbb{N}$ *and* $q\geq 2$ *is a prime power, forms an infinite family of strongly regular graphs that are characterized by the property that every pair of vertices in such a graph has an identical number of common neighbors, whose value is given by* $\ell=q^{2n-2}(q-1)$*.*

In regard to their graph invariants (as illustrated in Table 1), by Section 2.5.4 of [9],

$$\alpha(\mathsf{H})=\omega(\mathsf{G})=\frac{q^n-1}{q-1}, \tag{35}$$

and, by Theorem 3.29 of [14],

$$\vartheta(\mathsf{H})=\alpha(\mathsf{H})=\frac{q^n-1}{q-1}, \tag{36}$$

$$\vartheta(\overline{\mathsf{H}})=\chi(\mathsf{H})=q^n+1. \tag{37}$$

Consequently, it follows that the bounds in (28) and (29), applied to H *for its independence and chromatic numbers, are tight. Numerical results are presented in Table 1.*

Table 1. (Example 4) Complements of symplectic polar graphs, the families of strongly regular graphs to which they belong (see (34)), the fixed number (ℓ) of common neighbors for each pair of distinct vertices, the values of the Lovász ϑ-function for these graphs and their complements, and their exact matchings with the independence and chromatic numbers of these graphs.

n	q	$\mathsf{H}=\overline{\mathsf{Sp}(2n,q)}$	ℓ	$\vartheta(\mathsf{H})$	$\alpha(\mathsf{H})$	$\vartheta(\overline{\mathsf{H}})$	$\chi(\mathsf{H})$
3	2	$\mathsf{srg}(63,32,16,16)$	16	7	7	9	9
3	3	$\mathsf{srg}(364,243,162,162)$	162	13	13	28	28
3	4	$\mathsf{srg}(1365,1024,768,768)$	768	21	21	65	65
3	5	$\mathsf{srg}(3906,3125,2500,2500)$	2500	31	31	126	126
3	7	$\mathsf{srg}(19608,16807,14406,14406)$	14406	57	57	344	344
4	2	$\mathsf{srg}(255,128,64,64)$	64	15	15	17	17
4	3	$\mathsf{srg}(3280,2187,1458,1458)$	1458	40	40	82	82
4	4	$\mathsf{srg}(21845,16384,12288,12288)$	12288	85	85	257	257
5	2	$\mathsf{srg}(1023,512,256,256)$	256	31	31	33	33
5	3	$\mathsf{srg}(29524,19683,13122,13122)$	13122	121	121	244	244
5	4	$\mathsf{srg}(349525,262144,196608,196608)$	196608	341	341	1025	1025
6	2	$\mathsf{srg}(4095,2048,1024,1024)$	1024	63	63	65	65
6	3	$\mathsf{srg}(265720,177147,118098,118098)$	118098	364	364	730	730
7	2	$\mathsf{srg}(16383,8192,4096,4096)$	4096	127	127	129	129
8	2	$\mathsf{srg}(65535,32768,16384,16384)$	16384	255	255	257	257
9	2	$\mathsf{srg}(262143,131072,65536,65536)$	65536	511	511	513	513

4. Spanning Subgraphs of Strongly Regular Graphs

A spanning subgraph is obtained by removing some edges from the original graph while preserving all its vertices. Let G and H be graphs of the same order and belonging to the strongly regular graph families $\mathsf{srg}(n,d,\lambda,\mu)$ and $\mathsf{srg}(n,d',\lambda',\mu')$, respectively. The specific structure of these graphs may not necessarily be specified, as multiple nonisomorphic strongly regular graphs can exist within a given family [7,9]. We next derive a necessary condition for one of these graphs to be a spanning subgraph of the other that relies on the Lovász ϑ-functions of these graphs and their complements.

We rely on the following simple lemma, which relates the Lovász ϑ-functions of two graphs (not necessarily strongly regular), where one is a spanning subgraph of the other.

Lemma 1. *Let* G *and* H *be undirected and simple graphs with an identical vertex set. If* H *is a spanning subgraph of* G*, then*

$$\vartheta(\overline{\mathsf{G}}) \geq \vartheta(\overline{\mathsf{H}}), \tag{38}$$

$$\vartheta(\mathsf{G}) \leq \vartheta(\mathsf{H}). \tag{39}$$

Proof. Inequality (38) follows from (1) since every orthonormal representation of $\overline{\mathsf{G}}$ is also an orthonormal representation of $\overline{\mathsf{H}}$. This holds since if $\{i,j\} \notin \mathsf{E}(\overline{\mathsf{H}})$ and $i \neq j$, then $\{i,j\} \in \mathsf{E}(\mathsf{H}) \subseteq \mathsf{E}(\mathsf{G})$, and therefore $\{i,j\} \notin \mathsf{E}(\overline{\mathsf{G}})$; hence, if $\mathbf{u}_i^{\mathsf{T}}\mathbf{u}_j = 0$ for all $\{i,j\} \notin \mathsf{E}(\overline{\mathsf{G}})$, then the same also holds for all $\{i,j\} \notin \mathsf{E}(\overline{\mathsf{H}})$. Inequality (38) also follows alternatively from the SDP problem in (3) since every feasible solution that corresponds to $\vartheta(\overline{\mathsf{H}})$ is also a feasible solution that corresponds to $\vartheta(\overline{\mathsf{G}})$. Inequality (39) follows from (38) since, by definition, H is a spanning subgraph of G if and only if $\overline{\mathsf{G}}$ is a spanning subgraph of $\overline{\mathsf{H}}$. □

Remark 7. *Let* G *and* H *be two undirected and simple graphs on* n *vertices, and suppose that we do not know yet whether* H *is a spanning subgraph of* G*. In light of Corollary 1, it follows that if* G *and* H *are either strongly regular or vertex-transitive graphs, then each of the two inequalities (38) and (39) hold if and only if the other inequality holds. In general, by Corollary 2 of [5], we have* $\vartheta(\mathsf{G})\,\vartheta(\overline{\mathsf{G}}) \geq n$ *and* $\vartheta(\mathsf{H})\,\vartheta(\overline{\mathsf{H}}) \geq n$*, so, if* H *is not necessarily a spanning subgraph of* G*, then the satisfiability of one of the inequalities in (38) and (39) does not necessarily imply the satisfiability of the other inequality.*

Before proceeding with the application of Lemma 1 to strongly regular graphs, let us consider the problem by only relying on Definition 5 of strongly regular graphs. Let G and H be strongly regular graphs on n vertices and belonging to the families $\mathsf{srg}(n,d,\lambda,\mu)$ and $\mathsf{srg}(n,d',\lambda',\mu')$, respectively. If H is a spanning subgraph of G, then the following three inequalities hold:

$$d > d', \quad \lambda \geq \lambda', \quad \min\{\lambda,\mu\} \geq \mu'. \tag{40}$$

This follows from the fact that deleting edges can only decrease the fixed degree of the vertices in a regular graph. Moreover, since every pair of adjacent vertices in H are also adjacent in G, the fixed number of common neighbors of any two adjacent vertices in H must be smaller than or equal to the fixed number of common neighbors of the corresponding vertices in G. Furthermore, the fixed number of common neighbors of any two nonadjacent vertices in H, which are either adjacent or nonadjacent vertices in G, cannot be larger than the minimum between the fixed numbers of common neighbors of adjacent or nonadjacent vertices in G.

We obtain a further necessary condition that specifies Lemma 1 for strongly regular graphs.

Corollary 3. *Let* G *and* H *be strongly regular graphs on* n *vertices and belong to the families* $\mathsf{srg}(n, d, \lambda, \mu)$ *and* $\mathsf{srg}(n, d', \lambda', \mu')$*, respectively. Then, a necessary condition for* H *to be a spanning subgraph of* G *is that their parameters satisfy the inequality*

$$\frac{d}{d'} \, \frac{\mu' - \lambda' + \sqrt{(\mu' - \lambda')^2 + 4(d' - \mu')}}{\mu - \lambda + \sqrt{(\mu - \lambda)^2 + 4(d - \mu)}} \geq 1. \tag{41}$$

Proof. The result is obtained by combining Theorem 2 (see (10) and (11)) and Lemma 1 (see (38)), followed by straightforward algebra. □

Example 5. *Let* G *and* H *be strongly regular graphs belonging to the families* $\mathsf{srg}(45, 28, 15, 21)$ *and* $\mathsf{srg}(45, 22, 10, 11)$*, respectively (by [7], there exist strongly regular graphs with these parameters). The inequalities in (40) are satisfied, so the question of whether* H *can be a spanning subgraph of* G *is left open according to these inequalities. However, by Corollary 3,* H *(or any graph isomorphic to* H*) cannot be a spanning subgraph of* G *since the left-hand side of (41) is equal to* $\frac{3\sqrt{5}+1}{11} = 0.7007\ldots < 1$*, so the necessary condition in (41) is violated.*

The following result extends Corollary 3 to a more general setting, allowing for regular spanning subgraphs of a strongly regular graph and, even more broadly, regular spanning subgraphs of regular graphs. This extension relaxes the strong regularity requirement in Corollary 3, replacing it with the milder condition of regularity. This generalization comes, however, at a certain cost: it necessitates knowledge of the second-largest and least eigenvalues of their adjacency spectra, whereas Corollary 3 requires knowing only the parameter vectors that characterize the strongly regular graphs G and H. This distinction arises because nonisomorphic strongly regular graphs in the same family $\mathsf{srg}(n, d, \lambda, \mu)$, for any feasible parameters (n, d, λ, μ), are cospectral, with their adjacency spectra uniquely determined by these parameters. In contrast, nonisomorphic d-regular graphs on n vertices are not necessarily cospectral.

Corollary 4. *Let* G *and* H *be noncomplete and nonempty* d*-regular and* d'*-regular graphs on* n *vertices, with* $d' \leq d$*. Then,* H *is a spanning subgraph of* G *if the following inequality is satisfied:*

$$\frac{n\big(1 + \lambda_2(\mathsf{G})\big)}{n - d + \lambda_2(\mathsf{G})} + \frac{d'}{\lambda_n(\mathsf{H})} \geq 1, \tag{42}$$

where $\lambda_2(\mathsf{G})$ *is the second-largest eigenvalue of the adjacency matrix of* G*, and* $\lambda_n(\mathsf{H})$ *is the smallest eigenvalue of the adjacency matrix of* H*.*

Proof. The necessary condition in (42) holds by combining inequality (39) with the lower bound on $\vartheta(\mathsf{G})$ and the upper bound on $\vartheta(\mathsf{H})$ in the leftmost and rightmost inequalities in (7), respectively. □

Remark 8. *Combining inequality (38) with the lower bound on* $\vartheta(\overline{\mathsf{H}})$ *and the upper bound on* $\vartheta(\overline{\mathsf{G}})$ *in the leftmost and rightmost inequalities in (8), respectively, gives*

$$\frac{n - d + \lambda_2(\mathsf{G})}{1 + \lambda_2(\mathsf{G})} \leq -\frac{n\lambda_n(\mathsf{H})}{d' - \lambda_n(\mathsf{H})}, \tag{43}$$

which is equivalent to (42) (note that for a noncomplete regular graph G*, its second-largest eigenvalue satisfies* $\lambda_2(\mathsf{G}) > -1$*, and* $\lambda_n(\mathsf{H}) < 0$ *for a nonempty graph* H*). Corollary 4 consequently contains a single inequality since the two inequalities in Lemma 1, combined with (7) and (8), yield an identical inequality.*

5. Induced Subgraphs of Strongly Regular Graphs

An induced subgraph is obtained by deleting vertices from the original graph, along with all their incident edges. In analogy with Section 4, which is focused on spanning subgraphs of strongly regular graphs, and an extension of that analysis to regular graphs and spanning subgraphs, this section examines induced subgraphs in a similar manner. Specifically, we first derive the necessary conditions for a graph to be an induced subgraph of a strongly regular graph in the family $\mathsf{srg}(n, d, \lambda, \mu)$. This analysis relies not only on the Lovász ϑ-function for strongly regular graphs but also on their energy.

The graph energy is a graph invariant originally introduced in [15] while exploring its application in chemistry. A comprehensive treatment of graph energy can be found in [16].

Definition 8 (Energy of a graph, [16]). *The energy of a graph* G *on* n *vertices, denoted by* $\mathcal{E}(\mathsf{G})$*, is given by*

$$\mathcal{E}(\mathsf{G}) \triangleq \sum_{i=1}^{n} |\lambda_i(\mathsf{G})|. \tag{44}$$

The following lemma relates the energy and the Lovász ϑ-function of an induced subgraph to those of the original graph. It is general, and it is then applied to strongly regular graphs and extended to regular graphs. It partially relies on the Cauchy interlacing theorem, which is cited as follows:

Theorem 4 (Cauchy interlacing theorem, [17]). *Let* $\lambda_1 \geq \ldots \geq \lambda_n$ *be the eigenvalues of a symmetric matrix* $\mathbf{M}$ *and let* $\mu_1 \geq \ldots \geq \mu_m$ *be the eigenvalues of a principal* $m \times m$ *submatrix of* $\mathbf{M}$ *(i.e., a submatrix that is obtained by deleting the same set of rows and columns from M). Then,* $\lambda_i \geq \mu_i \geq \lambda_{n-m+i}$ *for* $i = 1, \ldots, m$*.*

Lemma 2. *Let* G *and* H *be finite, undirected, and simple graphs. If* H *is an induced subgraph of* G*, then*

$$\vartheta(\mathsf{H}) \leq \vartheta(\mathsf{G}), \tag{45}$$

$$\mathcal{E}(\mathsf{H}) \leq \mathcal{E}(\mathsf{G}). \tag{46}$$

Proof. Inequality (45) holds since the vertex set of an induced subgraph H is a subset of the vertex set of the original graph G, and the adjacency and nonadjacency relations of the unremoved vertices in H are preserved as compared with those in G. The result then follows from (1) since every orthonormal representation of G yields an orthonormal representation of H by taking the subset of the orthonormal vectors $\{\mathbf{u}_i\}$ that correspond to the remaining vertices in H.

Inequality (46) follows from the Cauchy interlacing theorem (Theorem 4) applied to the adjacency matrix of the graph G. Note that the adjacency matrix of the induced subgraph H is obtained from the adjacency matrix of G by deleting the rows and columns that correspond to the deleted vertices from G. □

Remark 9. *Inequality (46) states that the energy of an induced subgraph is at most the energy of the original graph. However, this result does not generally extend to spanning subgraphs (see p. 64 of [16]). A simple counterexample demonstrating that the energy of a spanning subgraph can exceed that of the original graph is given by considering* $\mathsf{G} = \mathsf{C}_4$ *and removing an edge to obtain the path graph* $\mathsf{H} = \mathsf{P}_4$ *(on four vertices) as a spanning subgraph of* G*. The adjacency spectra of* G *and* H *are, respectively, given by* $\{-2, 0, 0, 2\}$ *and* $\{\frac{1}{2}(\sqrt{5}+1), \frac{1}{2}(\sqrt{5}-1), -\frac{1}{2}(\sqrt{5}-1), -\frac{1}{2}(\sqrt{5}+1)\}$*, so it follows that* $\mathcal{E}(\mathsf{G}) = 4 < 2\sqrt{5} = \mathcal{E}(\mathsf{H})$*. On the other hand, a sufficient condition where the energy of a spanning subgraph does not exceed that of the original graph is given in Theorem 4.20*

of [16]. This highlights that the energy of a spanning subgraph can be either greater or smaller than that of the original graph, which is why such a condition is not included in Section 4.

We next utilize the spectral properties of strongly regular graphs listed in Theorem 5 to derive a closed-form expression for their energy.

Theorem 5 (The adjacency spectra of strongly regular graphs, Chapter 21 of [11])**.** *The following spectral properties of strongly regular graphs hold:*

(1) A strongly regular graph has at most three distinct eigenvalues.

(2) Let G *be a connected strongly regular graph in the family* $\mathsf{srg}(n,d,\lambda,\mu)$ *(i.e.,* $\mu > 0$*). Then, its adjacency spectrum consists of three distinct eigenvalues, where the largest eigenvalue is given by* $\lambda_1(\mathsf{G}) = d$ *with multiplicity 1, and the other two distinct eigenvalues of its adjacency matrix are given by*

$$p_{1,2} = \tfrac{1}{2}\left(\lambda - \mu \pm \sqrt{(\lambda-\mu)^2 + 4(d-\mu)}\right), \tag{47}$$

with the respective multiplicities

$$m_{1,2} = \tfrac{1}{2}\left(n - 1 \mp \frac{2d + (n-1)(\lambda-\mu)}{\sqrt{(\lambda-\mu)^2 + 4(d-\mu)}}\right). \tag{48}$$

(3) A connected regular graph with exactly three distinct eigenvalues is strongly regular.

(4) Connected strongly regular graphs, for which $2d + (n-1)(\lambda-\mu) \neq 0$*, have integral eigenvalues and their respective multiplicities are distinct.*

(5) A connected regular graph is strongly regular if and only if it has three distinct eigenvalues, where the largest eigenvalue is of multiplicity 1.

(6) A disconnected strongly regular graph is a disjoint union of m *identical complete graphs* K_r*, where* $m \geq 2$ *and* $r \in \mathbb{N}$*. It belongs to the family* $\mathsf{srg}(mr, r-1, r-2, 0)$*, and its adjacency spectrum is* $\{(r-1)^{[m]}, (-1)^{[m(r-1)]}\}$*, where superscripts indicate the multiplicities of the eigenvalues, thus having two distinct eigenvalues.*

A closed-form expression for the energy of strongly regular graphs as a function of a general parameter vector (n,d,λ,μ) does not appear to be explicitly available in the literature (see, e.g., [9,16,18,19]). Such an expression is next introduced for further analysis.

Lemma 3 (The energy of strongly regular graphs)**.** *The energy of a strongly regular graph* G *in the family* $\mathsf{srg}(n,d,\lambda,\mu)$ *is given by*

$$\mathcal{E}(\mathsf{G}) = d\left(1 + \frac{2(n-d) + \lambda - \mu}{\sqrt{(\lambda-\mu)^2 + 4(d-\mu)}}\right). \tag{49}$$

Proof. A connected strongly regular graph G has an adjacency spectrum consisting of exactly three distinct eigenvalues, as determined in Item 2 of Theorem 5. The largest eigenvalue of G is d with multiplicity 1, the second-largest eigenvalue is $\lambda_2(\mathsf{G}) = p_1$ with multiplicity m_1, and the smallest eigenvalue is $\lambda_{\min}(\mathsf{G}) = p_2$ with multiplicity m_2 (see (47) and (48)). Consequently, by (44), the energy of G is given by

$$\mathcal{E}(\mathsf{G}) = d + m_1|p_1| + m_2|p_2|. \tag{50}$$

Substituting (47) and (48) into (50) gives

$$\begin{aligned}
\mathcal{E}(\mathsf{G}) &= d + \tfrac{1}{2}\left(\lambda - \mu + \sqrt{(\lambda-\mu)^2 + 4(d-\mu)}\right) \cdot \tfrac{1}{2}\left(n - 1 - \frac{2d + (n-1)(\lambda-\mu)}{\sqrt{(\lambda-\mu)^2 + 4(d-\mu)}}\right) \\
&\quad + \tfrac{1}{2}\left(\mu - \lambda + \sqrt{(\lambda-\mu)^2 + 4(d-\mu)}\right) \cdot \tfrac{1}{2}\left(n - 1 + \frac{2d + (n-1)(\lambda-\mu)}{\sqrt{(\lambda-\mu)^2 + 4(d-\mu)}}\right) \\
&= d + \tfrac{1}{2}(n-1)\sqrt{(\lambda-\mu)^2 + 4(d-\mu)} + \frac{2d + (n-1)(\lambda-\mu)}{\sqrt{(\lambda-\mu)^2 + 4(d-\mu)}} \cdot \tfrac{1}{2}(\mu - \lambda) \\
&= d + \frac{2(n-1)(d-\mu) + d(\mu-\lambda)}{\sqrt{(\lambda-\mu)^2 + 4(d-\mu)}} \\
&= d + \frac{\big[2(n-1) + \mu - \lambda\big]\, d - 2(n-1)\mu}{\sqrt{(\lambda-\mu)^2 + 4(d-\mu)}} \\
&= d\left(1 + \frac{2(n-d) + \lambda - \mu}{\sqrt{(\lambda-\mu)^2 + 4(d-\mu)}}\right), \qquad (51)
\end{aligned}$$

where the last equality in (51) can be verified by relying on the connection between the four parameters of a strongly regular graph, as given in Item 1 of Proposition 2, followed by straightforward algebra. This establishes (49) for all connected strongly regular graphs.

If G is a disconnected strongly regular graph, then, by Item 6 of Theorem 5, it is a disjoint union of m identical complete graphs K_r, where $m \geq 2$ and $r \in \mathbb{N}$. The adjacency spectrum of G is then given by $\{(r-1)^{[m]}, (-1)^{[m(r-1)]}\}$, so the graph energy is $\mathcal{E}(\mathsf{G}) = 2m(r-1)$. This strongly regular graph G belongs to the family $\mathsf{srg}(mr, r-1, r-2, 0)$, and substituting the parameters $n = mr$, $d = r-1$, $\lambda = r-2$, and $\mu = 0$ into (49) verifies that the latter equality extends to disconnected strongly regular graphs. □

In the context of Lemma 3, it is worth noting the following result on maximal energy graphs, which combines Theorems 5.9 and 5.10 of [16].

Theorem 6 (Maximal energy graphs, [18,19])**.** *Let* G *be a graph on* n *vertices. Then,*

$$\mathcal{E}(\mathsf{G}) \leq \tfrac{1}{2}n(\sqrt{n}+1), \qquad (52)$$

with equality if G *is a strongly regular graph in the family*

$$\mathsf{srg}\Big(n, \tfrac{1}{2}(n+\sqrt{n}), \tfrac{1}{4}(n+2\sqrt{n}), \tfrac{1}{4}(n+2\sqrt{n})\Big). \qquad (53)$$

Furthermore, such strongly regular graphs exist if one of the following conditions hold:

(1) $n = 4^p q^4$ *with* $p, q \in \mathbb{N}$;

(2) $n = 4^{p+1}q^2$ *with* $p \in \mathbb{N}$ *and* $4q-1$ *is a prime power, or* $2q-1$ *is a prime power, or* q *is a square of an integer, or* $q < 167$.

Corollary 5. *Let* H *be an induced subgraph of a strongly regular graph* G *belonging to the family* $\mathsf{srg}(n, d, \lambda, \mu)$*. Then, the following inequalities hold:*

(1)

$$\vartheta(\mathsf{H}) \leq \frac{n\,(t + \mu - \lambda)}{2d + t + \mu - \lambda}, \qquad (54)$$

$$\mathcal{E}(\mathsf{H}) \leq d\left(1 + \frac{2(n-d) + \lambda - \mu}{t}\right), \qquad (55)$$

where $t \triangleq \sqrt{(\lambda-\mu)^2 + 4(d-\mu)}$*, as given in (11).*

(2) *Specifically, if* H *is in the family* $\mathsf{srg}(n', d', \lambda', \mu')$*, where* $n' \leq n$*,* $d' \leq d$*,* $\lambda' \leq \lambda$*, and* $\nu' \leq \nu$*, then*

$$\frac{n'}{n} \cdot \frac{t' + \mu' - \lambda'}{t + \mu - \lambda} \cdot \frac{2d + t + \mu - \lambda}{2d' + t' + \mu' - \lambda'} \leq 1, \tag{56}$$

$$\frac{td'}{t'd} \cdot \frac{t' + 2(n' - d') + \lambda' - \mu'}{t + 2(n - d) + \lambda - \mu} \leq 1, \tag{57}$$

where $t' \triangleq \sqrt{(\lambda' - \mu')^2 + 4(d' - \mu')}$.

Proof. Inequalities (54)–(57) follow from Lemma 2, combined with its specialization to strongly regular graphs according to Theorem 2 and Lemma 3. □

Remark 10. *The computational complexity of the Lovász* ϑ*-function for a graph* G *on* n *vertices, as obtained by numerically solving the SDP problem in (3), is polynomial in* n *and* $\log \frac{1}{\varepsilon}$*, where* ε *denotes the computation precision (see Theorem 11.11 of [6]). The constrained convex optimization problem in (3) involves* $\frac{1}{2}n(n+1)$ *optimization variables—specifically, the entries of the* $n \times n$ *positive semidefinite matrix* $\mathbf{B}$*—and at most quadratically many equality constraints in* n *(scaling linearly in* n *for sparse graphs). Consequently, while the Lovász* ϑ*-function remains computationally feasible for graphs with up to several hundred vertices, it becomes impractical for significantly larger instances. Checking the satisfiability of the necessary conditions for spanning or induced subgraphs in Lemmata 1 and 2 is thus infeasible for large* n*. However, computing only the second-largest or smallest eigenvalue of the adjacency matrix can be efficiently performed using iterative methods, which avoid computing the entire spectrum. These methods are particularly well-suited for large graphs, enabling the verification of the necessary conditions for spanning or induced subgraphs of regular graphs in Corollary 4 and Item 1 of Corollary 5, even for large instances. Furthermore, verifying the necessary conditions in Corollary 3 and Item 2 of Corollary 5 for spanning or induced strongly regular subgraphs of strongly regular graphs is computationally straightforward and imposes no practical limitations, even for exceedingly large strongly regular graphs.*

Example 6 (Cycles as induced subgraphs of the Shrikhande graph)**.** *The Shrikhande graph, denoted by* G*, is a strongly regular graph in the family* $\mathsf{srg}(16, 6, 2, 2)$ *(there are two such nonisomorphic strongly regular graphs, see Section 10.6 of [9]). The graph* G *is thus a strongly regular graph in which every pair of vertices has exactly two common neighbors (by the friendship theorem, it stays in contrast to the non-existence of a regular graph, except for a triangle, in which every pair of vertices has exactly one common neighbor). It can be verified that* G *contains induced cycles of lengths 3, 4, 5, 6, and 8, but no induced cycles of length 7 or greater than 8. Figure 2 illustrates two such induced cycles, with a cycle of length 6 shown on the left plot and a cycle of length 8 on the right plot, with both generated using the SageMath software [13].*

Let $\mathsf{H} = \mathsf{C}_\ell$ *denote a cycle of length* $\ell \geq 3$*. The necessary conditions in (54) and (55) for a graph isomorphic to* C_ℓ *to be an induced subgraph of* G *are given by* $\vartheta(\mathsf{C}_\ell) \leq 4$ *and* $\mathcal{E}(\mathsf{C}_\ell) \leq 36$*, respectively. These follow from the equalities* $\vartheta(\mathsf{G}) = 4$ *by Theorem 2 and* $\mathcal{E}(\mathsf{G}) = 36$ *by Lemma 3. It can be verified that the former condition on the Lovász* ϑ*-function is more restrictive in the studied case than the latter condition on the graph energy, so we focus on the necessary condition* $\vartheta(\mathsf{C}_\ell) \leq 4$*. By [5], for all integers* $\ell \geq 3$*,*

$$\vartheta(\mathsf{C}_\ell) = \begin{cases} \dfrac{\ell}{1 + \sec \frac{\pi}{\ell}}, & \ell \in \{3, 5, 7, \ldots\}, \\ \dfrac{\ell}{2}, & \ell \in \{4, 6, 8, \ldots\}. \end{cases} \tag{58}$$

Consequently, by (58), for all $\ell > 8$*, we obtain* $\vartheta(\mathsf{C}_\ell) \geq \vartheta(\mathsf{C}_9) = 4.361\ldots > 4 = \vartheta(\mathsf{G})$*, verifying that* G *does not contain any induced cycle of length* $\ell > 8$*. Furthermore,* $\vartheta(\mathsf{C}_8) = 4 = \vartheta(\mathsf{G})$ *and* G

contains a cycle of length 8 as an induced subgraph (see the right plot of Figure 2). This provides an example where the necessary condition holds with equality, coinciding with the existence of the longest possible induced cycle. However, since G *does not contain an induced cycle of length 7, despite* $\vartheta(\mathsf{C}_7) = 3.31766\ldots < 4 = \vartheta(\mathsf{G})$*, this serves as a counterexample where the necessary conditions in Item 1 of Corollary 5 fail to exclude the existence of an induced cycle of length 7 in* G*.*

Another counterexample where the necessary conditions in Item 1 of Corollary 5 fail to exclude the existence of an induced cycle of a given length is a connected, triangle-free graph G *with energy of at least 4 (noting that* $\mathcal{E}(\mathsf{C}_3) = 4$*). By definition, such a graph does not contain any triangle as a subgraph, yet the considered necessary conditions do not rule out the absence of an induced triangle in* G*. Among the connected strongly regular graphs, there are seven currently known triangle-free graphs, and all of them are determined by their adjacency spectrum (see, e.g., Section 4.7 of [20]).*

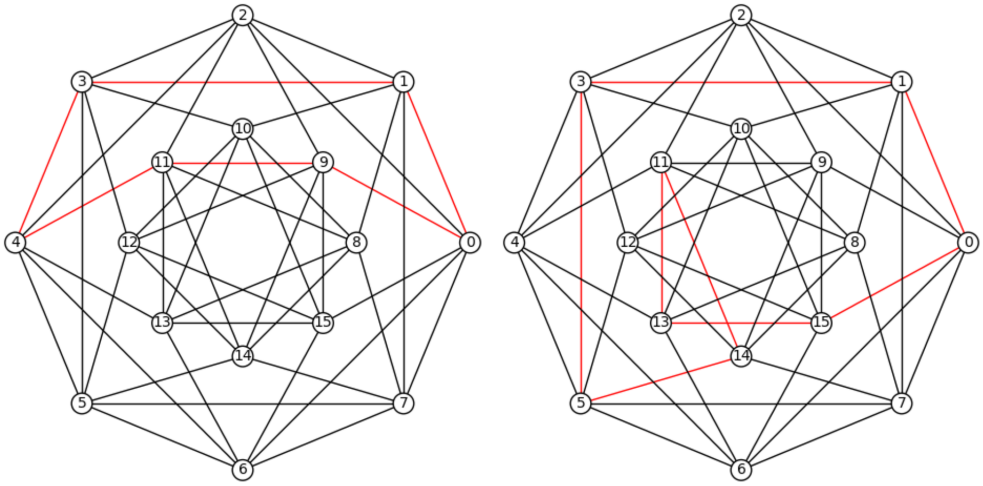


Figure 2. The Shrikhande graph G and two induced cycles of lengths 6 (left plot) and 8 (right plot), with their edges highlighted in red.

Example 7 (Induced subgraphs of triangular graphs)**.** *The family of triangular graphs consists of regular graphs obtained as the line graphs of complete graphs on at least three vertices. Let* T_ℓ *denote the line graph of the complete graph* K_ℓ *for all* $\ell \geq 3$*. This results in a strongly regular graph in the family* $\mathsf{srg}(\frac{1}{2}\ell(\ell-1), 2\ell-4, \ell-2, 4)$ *for all* $\ell \geq 4$ *(see Section 1.1.7 of [9]). For* $\ell \geq 4$ *with* $\ell \neq 8$*, these connected, strongly regular graphs are unique for their respective parameters. Their uniqueness further implies that they are determined by their spectra (following from Theorem 34 of [20]).*

Now, let H *be a strongly regular graph in the family* $\mathsf{srg}(n', d', \lambda', \mu')$ *for some specified (feasible) parameters. We consider whether* H *can appear as an induced subgraph of the triangular graph* T_ℓ *as a function of* $\ell \geq 4$*. Applying Item 2 of Corollary 5, if either inequality (56) or (57) is violated, then no strongly regular graph in the family* $\mathsf{srg}(n', d', \lambda', \mu')$ *(including* H*) can be an induced subgraph of* T_ℓ*. This leads to a general conclusion for those values of* ℓ *where either inequality fails. Applying Item 2 of Corollary 5 to* T_ℓ *with the following parameters:*

$$n = \tfrac{1}{2}\ell(\ell-1), \quad d = 2(\ell-2), \quad \lambda = \ell-2, \quad \mu = 4, \tag{59}$$

yields, by (11),

$$t = \ell - 2, \quad \forall\, \ell \geq 4. \tag{60}$$

With straightforward algebra, the inequalities (56) and (57) simplify to

$$\begin{cases} \dfrac{2n'(t'+\mu'-\lambda')}{2d'+t'+\mu'-\lambda'} \leq \ell, \\ \dfrac{d'\,(t'+2(n'-d')+\lambda'-\mu')}{t'} \leq 2\ell(\ell-3), \end{cases} \tag{61}$$

where $t' \triangleq \sqrt{(\lambda'-\mu')^2+4(d'-\mu')}$. *If, for given parameters* (n', d', λ', μ'), *at least one of these inequalities is violated for a specific* $\ell \geq 4$, *then* H *cannot be an induced subgraph of* T_ℓ.

Example 8 (Continuation of Example 7: the Gewirtz graph)**.** *Consider the Gewirtz graph, denoted by* H, *which is a connected, triangle-free, strongly regular graph in the family* $\mathsf{srg}(56, 10, 0, 2)$ *(see Section 10.20 of [9]). To clarify, since* $\mu = 2 > 0$, *the graph* H *is connected, and* $\lambda = 0$ *confirms that it is triangle-free. Uniqueness in this parameter family ensures it is determined by its spectrum [20]. Specializing (61) for* H, *the first inequality is violated if and only if* $\ell \leq 31$ *(it dominates the second, energy-based inequality in (61)). Hence,* H *cannot be an induced subgraph of* T_ℓ *for all* $\ell \leq 31$. *For* $\ell \leq 11$, *this result is trivial since* $\frac{1}{2}\ell(\ell-1) < 56$, *meaning that the order of* T_ℓ *is too small to contain* H *as a subgraph. However, for* $\ell \geq 32$, *this question remains open, as both inequalities in (61) hold.*

Funding: This research received no external funding.

Data Availability Statement: The original contributions presented in this study are included in this article. Further inquiries can be directed to the corresponding author.

Acknowledgments: The author wishes to acknowledge the two anonymous reviewers for helpful and timely reports.

Conflicts of Interest: The author declares no conflicts of interest.

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