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Insights into New Generalization of q-Legendre-Based Appell Polynomials: Properties and Quasi Monomiality

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Abstract: In this paper, by using the zeroth-order q-Tricomi functions, the theory of three-variable q-Legendre-based Appell polynomials is introduced. These polynomials are studied by means of generating functions, series expansions, and determinant representation. Further, by utilizing the concepts of q-quasi-monomiality, these polynomials are examined as several q-quasi-monomial and operational representations; the q-differential equations for the three-variable q-Legendre-based Appell polynomials were obtained. In addition, we established a new generalization of three-variable q-Legendre-Hermite-Appell polynomials, and we derive series expansion, determinant representation, and q-quasi-monomial and q-differential equations. Some examples are framed to better illustrate the theory of three-variable q-Legendre-based Appell polynomials, and this is characterized by the above properties.

Keywords: q-calculus; q-general polynomials; q-Legendre polynomials; three-variable q-Legendre-Appell polynomials; monomiality principle; explicit form; operational connection; determinant form

MSC: 33E20; 33C45; 33B10; 33E30; 11T23



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1. Introduction and Preliminaries

q-calculus or quantum calculus, an extension of ordinary calculus, was created to investigate q-extensions of mathematical structures since the 18th century [1,2]. One of the most important studied aspects of q-calculus is the q-special functions, which relate to this topic and serve as a connection between physics and mathematics. For mathematical physics, some q-special polynomials and functions have been framed and worked with as the representations of quantum algebra [3]. Moreover, q-special polynomials have been considered, and many of their properties and applications have been given for a long time [4–7]. More recently, Raza and colleagues [8] considered bivariate q-Hermite polynomials and explored several of their properties and applications. In addition, in the last year, bivariate q-Laguerre polynomials have been considered, and many applications, relations, and properties have been provided [9]. For example, in [10], the theory of bivariate q-Legendre polynomials by zeroth-order q-Bessel Tricomi functions was defined, and bivariate q-Legendre polynomials from the context of quasi-monomiality were established. Then, q-integrodifferential equations and the operational representations for these polynomials were provided. Recently, Zayed et al. [11] and Fadel et al. [12] worked on q-Hermite-based Appell polynomials and derived some of their properties; they also established operational identities and q-quasi-monomiality properties. In addition, bivariate

q-Laguerre-Appell polynomials were introduced, and the quasi-monomiality characteristics of these polynomials were analyzed. In [13], q-truncated-Appell polynomials were considered by applying the q-monomiality principle methods, and their quasi-monomial properties and applications were studied and investigated. Several operational identities and quasi-monomial features were given. Additionally, diverse q-differential equations of these polynomials were derived. As applications, and by utilizing the operational identity of the mentioned polynomials, specific presentations regarding several q-truncated-Appell polynomial families were drawn. Lastly, the family of q-Legendre-Sheffer polynomials was introduced using an operational approach, and some of its fundamental properties were developed.

At present, many researchers have worked on q-calculus, and they have derived some results using q-fractional, q-differential equation, q-Mittag-Leffler function, q-Gamma function, and q-Beta function processes; they have proved the Caputo q-fractional boundary value problem with a q-Laplacian operator [14–18].

Let us recall that the respective two q-exponential functions are given by [19,20]

$$e_q(\zeta) = \sum_{v=0}^{\infty} \frac{\zeta^v}{[v]_q!}, \quad 0 < q < 1, \tag{1}$$

and

$$E_q(\zeta) = \sum_{v=0}^{\infty} \frac{q^{\binom{v}{2}} \zeta^v}{[v]_q!}, \quad 0 < q < 1, \tag{2}$$

which satisfy

$$e_q(\zeta) E_q(\eta) = \sum_{v=0}^{\infty} \frac{(\zeta + \eta)_q^v}{[v]_q!} \tag{3}$$

and

$$e_q(\zeta) E_q(-\zeta) = e_q(-\zeta) E_q(\zeta) = 1. \tag{4}$$

A q-analog of the Bessel Tricomi function known as the q-Bessel Tricomi function [10] is defined by

$$C_{v,q}(\zeta) = \sum_{\psi=0}^{\infty} \frac{(-1)^\psi \zeta^\psi}{[\psi]_q! [v + \psi]_q!}, \quad 0 < q < 1 \tag{5}$$

and converges absolutely for all values of ζ. These functions are generated as follows:

$$e_q(\delta) e_q(-\zeta \delta^{-1}) = \sum_{v=0}^{\infty} C_{v,q}(\zeta) \delta^v, \quad \delta \neq 0; \ |\zeta| < \infty. \tag{6}$$

The 0th order q-Bessel Tricomi functions, $C_{0,q}(\zeta)$, are obtained for $n = 0$ in Equation (5) such that

$$C_{0,q}(\zeta) = \sum_{\psi=0}^{\infty} \frac{(-1)^\psi \zeta^\psi}{([\psi]_q!)^2}. \tag{7}$$

The q-derivative operator $\widehat{D}_{q,\zeta}$ operating on the function $f(\zeta)$ is defined by

$$\widehat{D}_{q,\zeta} f(\zeta) = \frac{f(q\zeta) - f(\zeta)}{\zeta(1-q)}, \quad 0 < q < 1, \quad \zeta \neq 0 \tag{8}$$

and particularly

$$\widehat{D}_{q,\zeta} \zeta^m = [m]_q \zeta^{m-1}, \tag{9}$$

$$\widehat{D}_{q,\zeta} e_q(\alpha\zeta) = \alpha e_q(\alpha\zeta), \ \alpha \in \mathbb{C} \tag{10}$$

The kth order q-derivative operator $\widehat{D}_{q,\zeta}^{k}$ with respect to ζ is given by

$$\widehat{D}_{q,\zeta}^{k} e_q(\alpha\zeta) = \alpha^k e_q(\alpha\zeta), \quad k \in \mathbb{N}, \ \alpha \in \mathbb{C}. \tag{11}$$

The q-derivative $\widehat{D}_{q,\zeta}$ on two functions $f(\zeta)$ and $g(\zeta)$ is given by [4,20]

$$\widehat{D}_{q,\zeta}(f(\zeta)g(\zeta)) = f(\zeta)\widehat{D}_{q,\zeta}g(\zeta) + g(q\zeta)\widehat{D}_{q,\zeta}f(\zeta). \tag{12}$$

Particularly, we have

$$\widehat{D}_{q,\zeta}^{-1}\{\zeta^m\} = \frac{\zeta^{m+1}}{[m+1]_q}, \quad m \in \mathbb{N} \sqcup \{0\} \tag{13}$$

and

$$\left(\widehat{D}_{q,\zeta}^{-1}\right)^r\{1\} = \frac{\zeta^r}{[r]_q!}, \quad r \in \mathbb{N} \sqcup \{0\} \tag{14}$$

with $\widehat{D}_{q,\zeta}^{-1}\{1\} = \zeta$.

By using Equations (1) and (13), Equation (7) obtains

$$C_{0,q}(\zeta\delta) = e_q\left(-\widehat{D}_{q,\zeta}^{-1}\delta\right)\{1\}. \tag{15}$$

In their work [8], Raza et al. presented the q-Hermite polynomials, denoted as $H_{v,q}(\zeta,\eta)$, which were defined through the application of the subsequent generating function

$$e_q(\zeta\delta)e_q(\eta\delta^2) = \sum_{v=0}^{\infty} H_{v,q}(\zeta,\eta)\frac{\delta^v}{[v]_q!}, \tag{16}$$

and series definition

$$H_{v,q}(\zeta,\eta) = [v]_q! \sum_{\psi=0}^{\left[\frac{v}{2}\right]} \frac{\eta^\psi \zeta^{v-2\psi}}{[\psi]_q![v-2\psi]_q!}. \tag{17}$$

The operational identity of q-Hermite polynomials $H_{v,q}^{(m)}(\zeta,\eta)$ is given by (see [8])

$$H_{v,q}(\zeta,\eta) = e_q\left(\eta D_{q,\zeta}^2\right)\{\zeta^v\}. \tag{18}$$

Recently, Wani et al. [10] presented the two-variable q-Legendre polynomials, denoted as $S_{v,q}(\zeta,\eta)$, which was defined through the application of the subsequent generating function

$$C_{0,q}(-\zeta\delta^2)e_q(\eta\delta) = \sum_{v=0}^{\infty} S_{v,q}(\zeta,\eta)\frac{\delta^v}{[v]_q!}, \tag{19}$$

or, equivalently,

$$e_q(\widehat{D}_{q,\zeta}^{-1}t^2)e_q(\eta\delta)\{1\} = \sum_{v=0}^{\infty} S_{v,q}(\zeta,\eta)\frac{\delta^v}{[v]_q!}. \tag{20}$$

The q-dilatation operator, T_ζ, operating on any complex variable function, $f(\zeta)$, in the following manner is [11,12,21]

$$T_\zeta^\psi\{f(\zeta)\} = f(q^\psi\zeta), \quad \psi \in \mathbb{R}, \tag{21}$$

which satisfies the property

$$T_\zeta^{-1}T_\zeta f(\zeta) = f(\zeta). \tag{22}$$

The q-derivative operator $\widehat{D}_{q,\delta}$ operating on the function $e_q(\zeta\delta^m)$ is given as [9]

$$\widehat{D}_{q,\delta}\{e_q(\zeta\delta^m)\} = \zeta\delta^{m-1}T_{(\zeta;m)}e_q(\zeta\delta^m), \tag{23}$$

where

$$T_{(\zeta;m)} = \frac{1-q^mT_\zeta^m}{1-qT_\zeta} = 1+qT_\zeta+\cdots+q^{m-1}T_\zeta^{m-1}.$$

The q-derivative operator $\widehat{D}_{q,\delta}$ operating on the function $C_{0,q}(-\zeta\delta^m)$ is given as [4]

$$\widehat{D}_{q,\delta}\{C_{0,q}(-\zeta\delta^m)\} = \widehat{D}_{q,\zeta}^{-1}\delta^{m-1}T_{(\widehat{D}_{q,\zeta}^{-1};m)}C_{0,q}(-\zeta\delta^m), \tag{24}$$

where

$$T_{(\widehat{D}_{q,\zeta}^{-1};m)} = \frac{1-q^mT_{\widehat{D}_{q,\zeta}^{-1}}^m}{1-qT_{\widehat{D}_{q,\zeta}^{-1}}} = 1+qT_{\widehat{D}_{q,\zeta}^{-1}}+\cdots+q^{m-1}T_{\widehat{D}_{q,\zeta}^{-1}}^{m-1}.$$

Recently, Alam et al. [22] introduced the two-variable q-general polynomials, $p_{n,q}(\zeta,\eta)$, in the following manner:

$$e_q(\zeta\delta)\phi_q(\eta,\delta) = \sum_{v=0}^{\infty}p_{v,q}(\zeta,\eta)\frac{\delta^v}{[v]_q!}, \quad (p_{0,q}(\zeta,\eta)=1), \tag{25}$$

where $\phi_q(\eta,t)$ has (at least the formal) series expansion

$$\phi_q(\eta,\delta) = \sum_{v=0}^{\infty}\phi_{v,q}(\eta)\frac{\delta^v}{[v]_q!}, \quad (\phi_{0,q}(\eta)\neq 0). \tag{26}$$

The q-Appell polynomials, $\mathcal{A}_{v,q}(\zeta)$, are elucidated by Al Salam, as delineated in the formula presented in [22,23] (see also [24]),

$$\mathcal{A}_q(\delta)\, e_q(\zeta\delta) = \sum_{v=0}^{\infty}\mathcal{A}_{v,q}(\zeta)\frac{\delta^v}{[v]_q!}, \tag{27}$$

$$\mathcal{A}_q(\delta) = \sum_{v=0}^{\infty}\mathcal{A}_{v,q}\frac{\delta^v}{[v]_q!}, \qquad \mathcal{A}_q(\delta)\neq 0, \qquad A_{0,q}=1. \tag{28}$$

The following demonstrates an appropriate option for certain individuals within the class of q-Appell polynomials.

The elements of the q-Appell polynomials class $\mathcal{A}_{v,q}(\zeta)$ generate the corresponding q-numbers, $\mathcal{A}_{v,q}$, when $\zeta=0$. Table 1 illustrates the initial occurrences of three specific q-numbers: q-Bernoulli numbers, $\mathcal{B}_{v,q}$ [25,26], q-Euler numbers, $\mathcal{E}_{v,q}$ [25,26] and q-Genocchi numbers, $\mathcal{G}_{v,q}$ [25,26] (Table 2).

Table 1. Some q-Appell polynomial families.

S. No.	q-Appell Polynomials	Generating Function	$\mathcal{A}_q(t)$
I.	**The q-Bernoulli Polynomials [25,26]**	$\frac{\delta}{e_q(\delta)-1}e_q(\zeta\delta)=\sum_{v=0}^{\infty}\mathcal{B}_{v,q}(\zeta)\frac{\delta^v}{[v]_q!}$	$\mathcal{A}_q(\delta)=\frac{\delta}{e_q(\delta)-1}$
II.	**The q-Euler Polynomials [25,26]**	$\frac{[2]_q}{e_q(\delta)+1}e_q(\zeta\delta)=\sum_{v=0}^{\infty}\mathcal{E}_{v,q}(\zeta)\frac{\delta^v}{[v]_q!}$	$\mathcal{A}_q(\delta)=\frac{[2]_q}{e_q(\delta)+1}$
III.	**The q-Genocchi Polynomials [25,26]**	$\frac{[2]_q\delta}{e_q(\delta)+1}e_q(\zeta\delta)=\sum_{v=0}^{\infty}\mathcal{G}_{v,q}(\zeta)\frac{\delta^v}{[v]_q!}$	$\mathcal{A}_q(\delta)=\frac{[2]_q\delta}{e_q(\delta)+1}$

Table 2. The first five q-numbers for $\mathcal{B}_{v,q}$, $\mathcal{E}_{v,q}$, and $\mathcal{G}_{v,q}$.

n	0	1	2	3	4
$\mathcal{B}_{v,q}$ [25,26]	1	$-(1+q)^{-1}$	$q^2([3]_q!)^{-1}$	$(1-q)q^3([2]_q)^{-1}([4]_q)^{-1}$	$q^4(1-q^2-2q^3-q^4+q^6)([2]_q^2![3]_q[5]_q)^{-1}$
$\mathcal{E}_{v,q}$ [25,26]	1	$-\frac{1}{2}$	$\frac{1}{4}(-1+q)$	$\frac{1}{8}(-1+2q+q^2-q^3)$	$\frac{1}{16}(q-1)[3]_q!(q^2-4q+1)$
$\mathcal{G}_{v,q}$ [25,26]	0	$\frac{2-q}{1+q}$	$\frac{q(q-5)}{(1+q)^2}$	$-\frac{3q^2(q-5)}{(1+q)^3}-\frac{3q(2-q)}{(1+q)^2}-\frac{q}{(1+q)})$	$\frac{-3q}{1+q}\left(\frac{3q^3+10q^2-28q+7}{(1+q)^3}\right)$

The monomiality principle stands as a fundamental tool for investigating specific special polynomials and functions, as well as their properties. Originating from Steffensen's work [27] in the early 19th century, this concept underwent significant refinement and expansion through Dattoli's contributions in 1996. Recent academic research has utilized the monomiality principle to explore innovative hybrid special polynomial sequences and families [28–30]. Of particular note, Cao et al. [9] extended the application of the monomiality principle to q-special polynomials, opening avenues for the creation of novel q-special polynomial families and shedding light on the quasi-monomiality of certain existing q-special polynomials. This expansion provides a theoretical foundation for interpreting q-special polynomials as unique solutions to generalized q-partial differential and q-integro differential equations. Through the application of q-operation-specific techniques, researchers can derive additional classes of q-generating functions and various generalizations of q-special functions. Among these techniques, the q-operational process demonstrates greater alignment with traditional mathematical approaches and implementations employed in the resolution of q-differential equations.

In the context of a q-polynomial set, $\tau_{v,q}(\zeta)(v \in \mathbb{N}, \zeta \in \mathbb{C})$, two q-operators are defined: $\widehat{M}_q$ and $\widehat{P}_q$. These operators, known as the q-multiplicative and q-derivative operators, respectively, are implemented as described in [9]:

$$\widehat{M}_q\{\tau_{v,q}(\zeta)\} = \tau_{v+1,q}(\zeta), \tag{29}$$

and

$$\widehat{P}_q\{\tau_{v,q}(\zeta)\} = [v]_q\tau_{v-1,q}(\zeta). \tag{30}$$

The operators $\widehat{M}_q$ and $\widehat{P}_q$ satisfy the following commutation relation, which plays a fundamental role in the structure of q-deformed algebraic systems and their associated representations.

$$[\widehat{M}_q, \widehat{P}_q] = \widehat{P}_q\widehat{M}_q - \widehat{M}_q\widehat{P}_q. \tag{31}$$

An analysis of the $\widehat{M}_q$ and $\widehat{P}_q$ operators facilitates the determination of polynomial, $\tau_{v,q}(\zeta)$, properties. When $\widehat{M}_q$ and $\widehat{P}_q$ demonstrate differential realization, the polynomials, $p_{v,q}(\zeta)$, conform to a particular differential equation:

$$\widehat{M}_q\widehat{P}_q\{\tau_{v,q}(\zeta)\} = [v]_q\tau_{v,q}(\zeta), \tag{32}$$

and

$$\widehat{P}_q\widehat{M}_q\{\tau_{v,q}(\zeta)\} = [v+1]_q\tau_{v,q}(\zeta). \tag{33}$$

In light of (29) and (30), it follows that

$$[\widehat{M}_q, \widehat{P}_q] = [v+1]_q - [v]_q. \tag{34}$$

From (29), we have

$$\widehat{M}_q{}^r\{\tau_{v,q}\} = \tau_{v+r,q}(\zeta). \tag{35}$$

In particular, we have

$$\tau_{v,q}(\zeta) = \widehat{M_q}^{\,v}\{\tau_{0,q}\} = \widehat{M_q}^{\,v}\{1\}, \tag{36}$$

where $\tau_{0,q}(\zeta) = 1$ serves as the q-counterpart of the polynomial $\tau_{v,q}(\zeta)$. Moreover, the generating function for $\tau_{v,q}(\zeta)$ is expressed as

$$e_q(\widehat{M_q}\delta)\{1\} = \sum_{v=0}^{\infty} \tau_{v,q}(\zeta)\frac{\delta^v}{[v]_q!}. \tag{37}$$

The q-multiplicative and q-derivative operators associated with the q-Appell polynomials, $A_{v,q}$, are given by [24]

$$\widehat{M}_{qA} = \eta + \frac{A_q'(\widehat{D}_{q,\eta})}{A_q(\widehat{D}_{q,\eta})}T_\eta, \tag{38}$$

or, alternatively,

$$\widehat{M}_{qA} = \eta\frac{A_q(q\widehat{D}_{q,\eta})}{A_q(\widehat{D}_{q,\eta})} + \frac{A_q'(\widehat{D}_{q,\eta})}{A_q(\widehat{D}_{q,\eta})}, \tag{39}$$

and

$$\widehat{P}_{qA} = \widehat{D}_{q,\eta}, \tag{40}$$

respectively.

The structure of this article is organized as follows: Section 2 introduces a new generalization of q-Legendre polynomials with three variables, emphasizing their connections with well-known q-polynomials; we also introduce a new generalization of q-Legendre-based Appell polynomials with three variables and derive their series representation, determinant form, and q-recurrence relations. Furthermore, we focus on the monomiality principle, exploring its role in the structural properties of these polynomials. Section 3 presents various applications of the new generalization of q-Legendre-based Appell polynomials with three variables. In Section 4, we analyze some examples of these polynomials through definitions of special polynomials. Finally, the article concludes with a summary of key findings and potential directions for future research.

2. New Generalization of q-Legendre-Based Appell Polynomials

This section of our research paper introduces a new generalization of three-variable q-Legendre-based Appell polynomials, denoted as ${}_pS\mathbb{A}_{v,q}(\zeta,\eta,\gamma)$. We elucidate their series definition, q-quasi-monomiality characteristics, operational identities, and associated q-differential equations. Our analysis begins with the formulation of the new generalization of three-variable q-Legendre polynomials, referred to as 3VqLeP ${}_pS_{v,q}(\zeta,\eta,\gamma)$.

By utilizing relations (7) and (25), we introduce the new generalization of three-variable q-Legendre polynomials, ${}_pS_{v,q}(\zeta,\eta,\gamma)$, in the following form:

$$e_q(\zeta\delta)\phi_q(\eta,\delta)C_{0,q}(-\gamma\delta^2) = \sum_{v=0}^{\infty} {}_pS_{v,q}(\zeta,\eta,\gamma)\frac{\delta^v}{[v]_q!}, \quad (p_{0,q}(\zeta,\eta) = 1), \tag{41}$$

where $\phi_q(\eta,\delta)$ has (at least the formal) series expansion

$$\phi_q(\eta,\delta) = \sum_{v=0}^{\infty} \phi_{v,q}(\eta)\frac{\delta^v}{[v]_q!}, \quad (\phi_{0,q}(\eta) \neq 0). \tag{42}$$

Upon simplification of the left-hand side of Equation (41) via the application of Equations (26) and (28), the series definitions of the generalization of three-variable q-Legendre polynomials 3VqLeP ${}_pS_{v,q}(\zeta,\eta,\gamma)$ are the following:

$$ {}_pS_{v,q}(\zeta,\eta,\gamma) = \sum_{\psi=0}^{v} \binom{v}{\psi}_q \phi_{\psi,q}(\eta) S_{v-\psi,q}(\zeta,\gamma). \tag{43} $$

We hereby establish the following concerning quasi-monomial identities of three-variable q-Legendre polynomials, 3VqLeP ${}_pS_{n,q}(\zeta,\eta,\gamma)$.

Theorem 1. *The new generalization of three-variable q-Legendre polynomials 3VqLeP* ${}_pS_{v,q}(\zeta,\eta,\gamma)$ *demonstrates quasi-monomial characteristics when subjected to the aforementioned q-multiplicative and q-derivative operators:*

$$ \widehat{M}_{3VqgLeP} = \zeta + \widehat{D}_{q,\gamma}^{-1}\widehat{D}_{q,\zeta}(1+qT_\gamma) + \frac{\phi_q'(\eta,\widehat{D}_{q,\zeta})}{\phi_q(\eta,\widehat{D}_{q,\zeta})} T_\zeta T_\gamma, \tag{44} $$

or, alternatively,

$$ \widehat{M}_{3VqgLP} = \zeta T_\gamma + \widehat{D}_{q,\gamma}^{-1}\widehat{D}_{q,\zeta}(1+qT_\gamma) + \frac{\phi_q'(\eta,\widehat{D}_{q,\zeta})}{\phi_q(\eta,\widehat{D}_{q,\zeta})} T_\zeta T_\gamma, \tag{45} $$

and

$$ \widehat{P}_{3VqgLeP} = \widehat{D}_{q,\zeta}, \tag{46} $$

respectively, where $T_{q,\zeta}$ and $T_{q,\gamma}$ denote the q-dilatation operators given by Equation (21).

Proof. By taking the q-derivative on both sides of Equation (41), partially with respect to δ and by using Equation (12), we have

$$ \sum_{v=1}^{\infty} {}_pS_{v,q}(\zeta,\eta,\gamma)\widehat{D}_{q,\delta}\frac{\delta^v}{[v]_q!} = e_q(q\zeta\delta)C_{0,q}(-q\gamma\delta^2)\widehat{D}_{q,\delta}\phi_q(\eta,\delta) + \phi_q(\eta,\delta)\widehat{D}_{q,\delta}e_q(\zeta\delta)C_{0,q}(-\gamma\delta^2), \tag{47} $$

By using Equation (12) by taking $f_q(\delta) = \phi_q(\eta,\delta)$ and $g_q(\delta) = e_q(\zeta\delta)C_{0,q}(-\gamma\delta^2)$ and then simplifying the resultant equation by using Equations (9), (21), and (29) in the right-hand side, we have

$$ \left(\zeta + \widehat{D}_{q,\gamma}^{-1}\delta(1+qT_\gamma) + \frac{\phi_q'(\eta,\delta)}{\phi_q(\eta,\delta)}T_\zeta T_\gamma\right)\left\{e_q(\zeta\delta)\phi_q(\eta,\delta)C_{0,q}(-\gamma\delta^2)\right\} = \sum_{v=0}^{\infty} {}_pS_{v+1,q}(\zeta,\eta,\gamma)\frac{\delta^v}{[v]_q!}. \tag{48} $$

since

$$ \widehat{D}_{q,\zeta}\left\{e_q(\zeta\delta)\phi_q(\eta,\delta)C_{0,q}(-\gamma\delta^2)\right\} = \delta e_q(\zeta\delta)\phi_q(\eta,\delta)C_{0,q}(-\gamma\delta^2), \tag{49} $$

and $\frac{\phi_q'(\eta,\delta)}{\phi_q(\eta,\delta)}$ has q-power series expansion in δ as $\phi_q(\eta,\delta)$-invertible series of δ.

Therefore, according to (41) and (48), we have

$$ \left(\zeta + \widehat{D}_{q,\gamma}^{-1}\widehat{D}_{q,\zeta}(1+qT_\gamma) + \frac{\phi_q'(\eta,\widehat{D}_{q,\zeta})}{\phi_q(\eta,\widehat{D}_{q,\zeta})}T_\zeta T_\gamma\right)\left\{\sum_{v=0}^{\infty} {}_pS_{v,q}(\zeta,\eta,\gamma)\frac{\delta^v}{[v]_q!}\right\} = \sum_{v=0}^{\infty} {}_pS_{v+1,q}(\zeta,\eta,\gamma)\frac{\delta^v}{[v]_q!}. \tag{50} $$

Comparing the coefficients of the like powers of δ on both sides gives

$$ \left(\zeta + \widehat{D}_{q,\gamma}^{-1}\widehat{D}_{q,\zeta}(1+qT_\gamma) + \frac{\phi_q'(\eta,\widehat{D}_{q,\zeta})}{\phi_q(\eta,\widehat{D}_{q,\zeta})}T_\zeta T_\gamma\right)\left\{{}_pS_{v,q}(\zeta,\eta,\gamma)\right\} = {}_pS_{v+1,q}(\zeta,\eta,\gamma). \tag{51} $$

Thus, in view of the monomiality principle in Equation (29), the above equation yields assertion (44) of Theorem 1. By choosing $f_q(\delta)$ and $g_q(\delta)$ appropriately among the functions in the l.h.s. of Equation (41) and then differentiating with respect to δ by using Equation (12) while following the same procedure involved in the proof of assertion (44), we obtain the assertions (45) for these alternate forms of the q-multiplicative operator for 3VqGLegP. Again, by using the identity (49) in (41), we have

$$\widehat{D}_{q,\zeta}\left\{\sum_{v=0}^{\infty} {}_pS_{v,q}(\zeta,\eta,\gamma)\frac{\delta^v}{[v]_q!}\right\} = \sum_{v=1}^{\infty} {}_pS_{v-1,q}(\zeta,\eta,\gamma)\frac{\delta^v}{[v-1]_q!}. \tag{52}$$

By equating the coefficients of like powers of δ on both sides of (52), we find

$$\widehat{D}_{q,\zeta}\{{}_pS_{v,q}(\zeta,\eta,\gamma)\} = [v]_q\,{}_pS_{v-1,q}(\zeta,\eta,\gamma), \quad (v \geqslant 1)$$

which, in view of the monomiality principle in Equation (30), yields assertion (46) of Theorem 1. □

Theorem 2. *The following q-differential equations for three-variable generalized q-Legendre polynomials* 3$VqGLeP$ ${}_pS_{n,q}(\zeta,\eta,\gamma)$ *are*

$$\left(\zeta\widehat{D}_{q,\zeta} + \widehat{D}_{q,\gamma}^{-1}\widehat{D}_{q,\zeta}^2(1+qT_\gamma) + \frac{\phi_q'(\eta,\widehat{D}_{q,\zeta})}{\phi_q(\eta,\widehat{D}_{q,\zeta})}\widehat{D}_{q,\zeta}T_\zeta T_\gamma - [v]_q\right){}_pS_{v,q}(\zeta,\eta,\gamma) = 0, \tag{53}$$

or, alternatively,

$$\left(\zeta\widehat{D}_{q,\zeta}T_\gamma + \widehat{D}_{q,\gamma}^{-1}\widehat{D}_{q,\zeta}^2(1+qT_\gamma) + \frac{\phi_q'(\eta,\widehat{D}_{q,\zeta})}{\phi_q(\eta,\widehat{D}_{q,\zeta})}\widehat{D}_{q,\zeta}T_\zeta T_\gamma - [v]_q\right){}_pS_{v,q}(\zeta,\eta,\gamma) = 0. \tag{54}$$

Proof. In view of Equations (44), (45), and (46) in (32), we obtain the assertions (53) and (54) of Theorem 2. □

Remark 1. *Since* $p_{0,q}(\zeta,\eta) = 1$, *in view of the monomiality principle in Equation (35), we, therefore, have*

$${}_pS_{v,q}(\zeta,\eta,\gamma) = \left(\zeta + \widehat{D}_{q,\gamma}^{-1}\widehat{D}_{q,\zeta}(1+qT_\gamma) + \frac{\phi_q'(\eta,\widehat{D}_{q,\zeta})}{\phi_q(\eta,\widehat{D}_{q,\zeta})}T_\zeta T_\gamma\right)^v\{1\} \quad (p_{0,q}(\zeta,\eta) = 1).$$

Moreover, in view of Equations (35), (41), and (44), we have

$$e_q\left(\widehat{M}_{3VqgP}\right)\{1\} = e_q(\zeta\delta)\phi_q(\eta,\delta)C_{0,q}(-\gamma\delta^2) = \sum_{v=0}^{\infty} {}_pS_{v,q}(\zeta,\eta,\gamma)\frac{\delta^v}{[v]_q!}. \tag{55}$$

Now, we proceed to introduce the new generalization of three-variable q-Legendre-based Appell polynomials (3VqLebAP). We derive the generating functions for the new generalization of three-variable q-Legendre-based Appell polynomials by means of the exponential-generating function of q-Appell polynomials. Thus, by replacing ζ in the left-hand side of (29) using the q-multiplicative operator ${}_pS_{v,q}(\zeta,\eta,\gamma)$ given by (44) (denoting the new generalization of three-variable q-Legendre-based Appell polynomials ${}_pS\mathbb{A}_{v,q}(\zeta,\eta,\gamma)$), we obtain

$$A_q(\delta)e_q\left(\widehat{M}_{3VqgP}\right)\{1\} = \sum_{v=0}^{\infty} {}_pS\mathbb{A}_{v,q}(\zeta,\eta,\gamma)\frac{\delta^v}{[v]_q!}, \tag{56}$$

for which, upon using Equation (44), we obtain the following two equivalent forms of ${}_pS\mathbb{A}_{v,q}(\zeta,\eta,\gamma)$:

$$A_q(\delta)e_q\left(\zeta + \widehat{D}_{q,\gamma}^{-1}\widehat{D}_{q,\zeta}(1+qT_\gamma) + \frac{\phi_q'(\eta,\widehat{D}_{q,\zeta})}{\phi_q(\eta,\widehat{D}_{q,\zeta})}T_\zeta T_\gamma\right)\{1\} = \sum_{v=0}^{\infty} {}_pS\mathbb{A}_{v,q}(\zeta,\eta,\gamma)\frac{\delta^v}{[v]_q!}. \tag{57}$$

By using relation (55) in the left-hand side of Equation (56), the generating function for the new generalization of three-variable q-Legendre-based Appell polynomials ${}_pS\mathbb{A}_{v,q}(\zeta,\eta,\gamma)$ is in the following form:

$$A_q(\delta)e_q(\zeta\delta)\phi_q(\eta,\delta)C_{0,q}(-\gamma\delta^2) = \sum_{v=0}^{\infty} {}_pS\mathbb{A}_{v,q}(\zeta,\eta,\gamma)\frac{\delta^v}{[v]_q!}. \tag{58}$$

Theorem 3. *The three-variable generalized q-Legendre-based Appell polynomials ${}_pS\mathbb{A}_{v,q}(\zeta,\eta,\gamma)$ are defined by the series:*

$${}_pS\mathbb{A}_{v,q}(\zeta,\eta,\gamma) = \sum_{\psi=0}^{v}\binom{v}{\psi}_q A_{\psi,q}\,{}_pS_{v-\psi,q}(\zeta,\eta,\gamma), \tag{59}$$

with $A_{\psi,q}$ given by Equation (28).

Proof. In view of Equations (41) and (58), we can write

$$\sum_{v=0}^{\infty} {}_pS\mathbb{A}_{v,q}(\zeta,\eta,\gamma)\frac{\delta^v}{[v]_q!} = A_q(\delta)\sum_{v=0}^{\infty} {}_pS_{v,q}(\zeta,\eta,\gamma)\frac{\delta^v}{[v]_q!}. \tag{60}$$

Now, by using the Expansion (28) of $A_q(\delta)$ in the l.h.s. of Equation (60), simplifying and then equating the coefficients of the like powers of δ on both sides of the resultant equation, we obtain assertion (59). □

By using a similar approach given in [31,32], and in view of Equation (58), the following determinant form for ${}_pS\mathbb{A}_{v,q}(\zeta,\eta,\gamma)$ is obtained.

Theorem 4. *The determinant representation of the new generalization of three-variable q-Legendre-based Appell polynomials ${}_pS\mathbb{A}_{v,q}(\zeta,\eta,\gamma)$ of degree v is*

$${}_pS\mathbb{A}_{0,q}(\zeta,\eta,\gamma) = \frac{1}{\beta_{0,q}},$$

$${}_pS\mathbb{A}_{v,q}(\zeta,\eta,\gamma) = \frac{(-1)^v}{(\beta_{0,q})^{v+1}}\begin{vmatrix} 1 & {}_pS_{1,q}(\zeta,\eta,\gamma) & {}_pS_{2,q}(\zeta,\eta,\gamma) & \cdots & {}_pS_{v-1,q}(\zeta,\eta,\gamma) & {}_pS_{v,q}(\zeta,\eta,\gamma) \\ \beta_{0,q} & \beta_{1,q} & \beta_{2,q} & \cdots & \beta_{v-1,q} & \beta_{v,q} \\ 0 & \beta_{0,q} & \binom{2}{1}_q\beta_{1,q} & \cdots & \binom{v-1}{1}_q\beta_{v-2,q} & \binom{v}{1}_q\beta_{v-1,q} \\ 0 & 0 & \beta_{0,q} & \cdots & \binom{v-1}{1}_q\beta_{v-3,q} & \binom{v}{2}_q\beta_{v-2,q} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \\ 0 & 0 & 0 & \cdots & \beta_{0,q} & \binom{v}{v-1}_q\beta_{1,q} \end{vmatrix}, \tag{61}$$

$$\beta_{v,q} = -\frac{1}{A_{0,q}}\left(\sum_{\psi=1}^{v}\binom{v}{\psi}_q A_{\psi,q}\beta_{v-\psi,q}\right), \quad n = 0,1,2,\ldots,$$

where $\beta_{0,q} \neq 0$, $\beta_{0,q} = \frac{1}{A_{0,q}}$ *and* ${}_pS_{v,q}(\zeta,\eta,\gamma)$, $v = 0,1,2,\ldots$, *represent the new generalization of three-variable q-Legendre polynomials defined by Equation (41).*

Theorem 5. *The new generalization of three-variable q-Legendre-based Appell polynomials 3VqLeP* ${}_pS\mathbb{A}_{v,q}(\zeta,\eta,\gamma)$ *demonstrates quasi-monomial characteristics when subjected to the aforementioned q-multiplicative and q-derivative operators:*

$$\widehat{M}_{3VqLeAP} = \left(\zeta + \widehat{D}_{q,\gamma}^{-1}\widehat{D}_{q,\zeta}(1+qT_\gamma) + \frac{\phi_q'(\eta,\widehat{D}_{q,\zeta})}{\phi_q(\eta,\widehat{D}_{q,\zeta})}T_\zeta T_\gamma\right)\frac{A_q(q\widehat{D}_{q,\zeta})}{A_q(\widehat{D}_{q,\zeta})} + \frac{A_q'(\widehat{D}_{q,\zeta})}{A_q(\widehat{D}_{q,\zeta})}, \quad (62)$$

or, alternatively,

$$\widehat{M}_{3VqLeAP} = \left(\zeta T_\gamma + \widehat{D}_{q,\gamma}^{-1}\widehat{D}_{q,\zeta}(1+qT_\gamma) + \frac{\phi_q'(\eta,\widehat{D}_{q,\zeta})}{\phi_q(\eta,\widehat{D}_{q,\zeta})}T_\zeta T_\gamma\right)\frac{A_q(q\widehat{D}_{q,\zeta})}{A_q(\widehat{D}_{q,\zeta})} + \frac{A_q'(\widehat{D}_{q,\zeta})}{A_q(\widehat{D}_{q,\zeta})}, \quad (63)$$

and

$$\widehat{P}_{3VqLeAP} = \widehat{D}_{q,\zeta}, \quad (64)$$

respectively.

Proof. By taking the q-derivative on both sides of Equation (58) with respect to δ by using Equation (12), we obtain

$$\sum_{v=1}^{\infty} {}_pS\mathbb{A}_{v,q}(\zeta,\eta,\gamma)\widehat{D}_{q,\delta}\frac{\delta^v}{[v]_q!}$$

$$= A_q(q\delta)e_q(q\zeta\delta)\widehat{D}_{q,\delta}\Big(C_{0,q}(-\gamma\delta^2)\phi_q(\eta,\delta)\Big) + C_{0,q}(-\gamma\delta^2)\phi_q(\eta,\delta)\widehat{D}_{q,\delta}\big(e_q(\zeta\delta)A_q(\delta)\big), \quad (65)$$

for which, upon using Equation (12) by taking $f_q(\delta) = C_{0,q}(-\gamma\delta^2)\phi_q(\eta,\delta)$ and $g(\delta) = e_q(\zeta\delta)A_q(\delta)$ and then simplifying the resultant equation by using Equations (12), (21) and (29) in the left-hand side, we have

$$\left(\left(\zeta + \widehat{D}_{q,\gamma}^{-1}t(1+qT_\gamma) + \frac{\phi_q'(\eta,\delta)}{\phi_q(\eta,\delta)}T_\zeta T_\gamma\right)\frac{A_q(q\delta)}{A_q(\delta)} + \frac{A_q'(\delta)}{A_q(\delta)}\right)A_q(\delta)e_q(\zeta\delta)\phi_q(\eta,\delta)C_{0,q}(-\gamma\delta^2)$$

$$= \sum_{v=1}^{\infty} {}_pS\mathbb{A}_{v,q}(\zeta,\eta,\gamma)\frac{\delta^{v-1}}{[v-1]_q!}. \quad (66)$$

Since

$$\widehat{D}_{q,\eta}A_q(\delta)e_q(\zeta\delta)\phi_q(\eta,\delta)C_{0,q}(-\gamma\delta^2) = \delta A_q(\delta)e_q(\zeta\delta)\phi_q(\eta,\delta)C_{0,q}(-\gamma\delta^2). \quad (67)$$

Let $A_q(\delta)$ and $\phi_q(\eta,\delta)$ be an invertible series of δ, and let $\frac{A_q'(\delta)}{A_q(\delta)}$ and $\frac{\phi_q'(\eta,\delta)}{\phi_q(\eta,\delta)}$ have q-power series expansion in δ.

Therefore, according to Equation (66), we obtain

$$\left(\left(\zeta + \widehat{D}_{q,\gamma}^{-1}\widehat{D}_{q,\zeta}(1+qT_\gamma) + \frac{\phi_q'(\eta,\widehat{D}_{q,\zeta})}{\phi_q(\eta,\widehat{D}_{q,\zeta})}T_\zeta T_\gamma\right)\frac{A_q(q\widehat{D}_{q,\zeta})}{A_q(\widehat{D}_{q,\zeta})} + \frac{A_q'(\widehat{D}_{q,\zeta})}{A_q(\widehat{D}_{q,\zeta})}\right)A_q(\delta)e_q(\zeta\delta)\phi_q(\eta,\delta)C_{0,q}(-\gamma\delta^2)$$

$$= \sum_{v=1}^{\infty} {}_pS\mathbb{A}_{v,q}(\zeta,\eta,\gamma)\frac{\delta^{v-1}}{[v-1]_q!},$$

for which, upon using (58), gives

$$\left(\left(\zeta + \widehat{D}_{q,\gamma}^{-1}\widehat{D}_{q,\zeta}(1+qT_\gamma) + \frac{\phi_q'(\eta,\widehat{D}_{q,\zeta})}{\phi_q(\eta,\widehat{D}_{q,\zeta})}T_\zeta T_\gamma\right)\frac{A_q(q\widehat{D}_{q,\zeta})}{A_q(\widehat{D}_{q,\zeta})} + \frac{A_q'(\widehat{D}_{q,\zeta})}{A_q(\widehat{D}_{q,\zeta})}\right)\sum_{v=0}^{\infty} {}_pS\mathbb{A}_{v,q}(\zeta,\eta,\gamma)\frac{\delta^v}{[v]_q!}$$

$$= \sum_{v=1}^{\infty} {}_pS\mathbb{A}_{v,q}(\zeta,\eta,\gamma)\frac{\delta^{v-1}}{[v-1]_q!}. \tag{68}$$

By comparing the coefficients of δ on both sides of Equation (68) and employing (29), we obtain the assertion (62). By choosing $f_q(\delta)$ and $g_q(\delta)$ appropriately among the functions in the l.h.s. of Equation (58) and then differentiating with respect to δ by using Equation (12) and following the same procedure involved in the proof of assertion (62), we obtain the assertion (63); then, (30) and (67) of the resultant equation gives assertion (64). □

Theorem 6. *The following q-differential equation for* ${}_pS\mathbb{A}_{v,q}(\zeta,\eta,\gamma)$ *holds true:*

$$\left(\zeta\widehat{D}_{q,\zeta}\frac{A_q(q\widehat{D}_{q,\zeta})}{A_q(\widehat{D}_{q,\zeta})} + \widehat{D}_{q,\zeta}\frac{\phi_q'(\eta,\widehat{D}_{q,\zeta})}{\phi_q(\eta,\widehat{D}_{q,\zeta})}T_\zeta T_\gamma\frac{A_q(q\widehat{D}_{q,\zeta})}{A_q(\widehat{D}_{q,\zeta})} - \widehat{D}_{q,\gamma}^{-1}\widehat{D}_{q,\zeta}\frac{A_q(q\widehat{D}_{q,\zeta})}{A_q(\widehat{D}_{q,\zeta})}T_{q,\zeta}\right.$$
$$\left. + \frac{A_q'(\widehat{D}_{q,\zeta})}{A_q(\widehat{D}_{q,\zeta})}\widehat{D}_{q,\zeta} - [v]_q\right){}_pS\mathbb{A}_{v,q}(\zeta,\eta,\gamma) = 0, \tag{69}$$

and

$$\left(\zeta\widehat{D}_{q,\zeta}\frac{A_q(q\widehat{D}_{q,\eta})}{A_q(\widehat{D}_{q,\eta})}T_\gamma + \widehat{D}_{q,\zeta}\frac{\phi_q'(\eta,\widehat{D}_{q,\zeta})}{\phi_q(\eta,\widehat{D}_{q,\zeta})}T_\zeta T_\gamma\frac{A_q(q\widehat{D}_{q,\zeta})}{A_q(\widehat{D}_{q,\eta})} - \widehat{D}_{q,\gamma}^{-1}\widehat{D}_{q,\zeta}\frac{A_q(q\widehat{D}_{q,\zeta})}{A_q(\widehat{D}_{q,\zeta})}\right.$$
$$\left. + \frac{A_q'(\widehat{D}_{q,\zeta})}{A_q(\widehat{D}_{q,\zeta})}\widehat{D}_{q,\zeta} - [v]_q\right){}_pS\mathbb{A}_{v,q}(\zeta,\eta,\gamma) = 0, \tag{70}$$

Proof. By using (62), (63), and (64) in (32), we obtain the assertions (69) and (70). □

3. Applications

This study extends the exploration of recently introduced polynomials, focusing on the examination of three-variable q-Legendre polynomials. Specifically, when considering the case where $\phi_q(\eta,\delta) = e_q(\eta\delta^2)$ generates function (58), which results in the reduction of 3VqLeP ${}_pS_{v,q}(\zeta,\eta,\gamma)$ to the q-HP $H_{v,q}(\zeta,\eta)$, it is demonstrated that the q-Legendre-Hermite-Appell polynomials (q-LeHAP) ${}_SH\mathbb{A}_{v,q}(\zeta,\eta,\gamma)$ can be characterized by a specific generating function.

$$A_q(\delta)e_q(\zeta\delta)e_q(\eta\delta^2)C_{0,q}(-\gamma\delta^2) = \sum_{v=0}^{\infty} {}_SH\mathbb{A}_{v,q}(\zeta,\eta,\gamma)\frac{\delta^v}{[v]_q!}. \tag{71}$$

In other words, we have

$${}_SH\mathbb{A}_{v,q}(\zeta,\eta,\gamma) = e_q(\widehat{D}_{q,\eta}^{-1}\widehat{D}_{q,\zeta}^2)\{{}_HA_{v,q}(\zeta,\eta)\}. \tag{72}$$

We note that

$${}_SH\mathbb{A}_{v,q}(\zeta,\eta,\gamma) = \sum_{\psi=0}^{v}\binom{v}{\psi}_q A_{\psi,q}{}_SH_{v-\psi,q}(\zeta,\eta,\gamma). \tag{73}$$

Now, we will demonstrate the determinant form for ${}_SH\mathbb{A}_{v,q}(\zeta,\eta,\gamma)$ by using a similar approach given in [31,32] and using the view of Equation (71) as the following:

Theorem 7. *The determinant representation of three-variable q-Legendre-Hermite-Appell polynomials ${}_{S}H\mathbb{A}_{v,q}(\zeta,\eta,\gamma)$ of degree v is*

$$ {}_{S}H\mathbb{A}_{0,q}(\zeta,\eta,\gamma) = \frac{1}{\beta_{0,q}}, $$

$$ {}_{S}H\mathbb{A}_{v,q}(\zeta,\eta,\gamma) = \frac{(-1)^{v}}{(\beta_{0,q})^{v+1}} \begin{vmatrix} 1 & {}_{S}H_{1,q}(\zeta,\eta,\gamma) & {}_{S}H_{2,q}(\zeta,\eta,\gamma) & \cdots & {}_{S}H_{v-1,q} & {}_{S}H_{v,q} \\ \beta_{0,q} & \beta_{1,q} & \beta_{2,q} & \cdots & \beta_{v-1,q} & \beta_{v,q} \\ 0 & \beta_{0,q} & \binom{2}{1}_{q}\beta_{1,q} & \cdots & \binom{v-1}{1}_{q}\beta_{v-2,q} & \binom{v}{1}_{q}\beta_{v-1,q} \\ 0 & 0 & \beta_{0,q} & \cdots & \binom{v-1}{1}_{q}\beta_{v-3,q} & \binom{v}{2}_{q}\beta_{v-2,q} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \\ 0 & 0 & 0 & \cdots & \beta_{0,q} & \binom{v}{v-1}_{q}\beta_{1,q} \end{vmatrix}, \tag{74} $$

$$ \beta_{v,q} = -\frac{1}{A_{0,q}} \left(\sum_{\psi=1}^{v} \binom{v}{\psi}_{q} A_{\psi,q}\beta_{v-\psi,q} \right), \quad v = 0,1,2,\ldots, $$

where $\beta_{0,q} \neq 0$, $\beta_{0,q} = \frac{1}{A_{0,q}}$ and ${}_{S}H_{v,q}(\zeta,\eta,\gamma)$, $v = 0,1,2,\ldots,$ are the three-variable q-Legendre-Hermite polynomials.

We shall now demonstrate the q-multiplicative and q-derivative operators of ${}_{S}H\mathbb{A}_{v,q}(\zeta,\eta,\gamma)$. The following theorem is presented:

Theorem 8. *The new generalization of three-variable q-Legendre-Hermite Appell polynomials 3VqLeHP ${}_{S}H\mathbb{A}_{v,q}(\zeta,\eta,\gamma)$ demonstrates quasi-monomial characteristics when subjected to the aforementioned q-multiplicative and q-derivative operators:*

$$ \widehat{M}_{3VqLeHAP} = \left(\zeta + \widehat{D}_{q,\gamma}^{-1}\widehat{D}_{q,\zeta}(1+qT_{\gamma})T_{\zeta}T_{\eta} + \eta\widehat{D}_{q,\zeta}T_{(\eta;2)}T_{\zeta} \right) \frac{A_q(q\widehat{D}_{q,\zeta})}{A_q(\widehat{D}_{q,\zeta})} + \frac{A_q'(\widehat{D}_{q,\zeta})}{A_q(\widehat{D}_{q,\zeta})}, \tag{75} $$

or, equivalently

$$ \widehat{M}_{3VqLeHAP} = \left(\zeta T_{\eta} + \widehat{D}_{q,\gamma}^{-1}\widehat{D}_{q,\zeta}(1+qT_{\gamma})T_{\zeta}T_{\eta} + \eta\widehat{D}_{q,\zeta}T_{(\eta;2)} \right) \frac{A_q(q\widehat{D}_{q,\zeta})}{A_q(\widehat{D}_{q,\zeta})} + \frac{A_q'(\widehat{D}_{q,\zeta})}{A_q(\widehat{D}_{q,\zeta})}, \tag{76} $$

and

$$ \widehat{P}_{2VqGHAP} = \widehat{D}_{q,\zeta}, \tag{77} $$

respectively.

Proof. By taking q-derivative on both sides of Equation (71) with respect to δ by using Equation (12), we obtain

$$ \sum_{v=1}^{\infty} {}_{S}H\mathbb{A}_{v,q}(\zeta,\eta,\gamma)\widehat{D}_{q,\delta}\frac{\delta^{v}}{[v]_{q}!} $$

$$ = A_q(q\delta)e_q(q\zeta\delta)\widehat{D}_{q,\delta}\left(e_q(\eta\delta^2)C_{0,q}(-\gamma\delta^2) \right) + e_q(\eta\delta^2)C_{0,q}(-\gamma\delta^2)\widehat{D}_{q,\delta}\left(A_q(\delta)e_q(\zeta\delta) \right), \tag{78} $$

for which, upon using Equation (12) by taking $f(\delta) = e_q(\eta\delta^2)C_{0,q}(-\gamma\delta^2)$ and $g(\delta) = A_q(\delta)e_q(\zeta\delta)$ and then simplifying the resultant equation by using Equation (12), (28) and (29) in the left-hand side, we have

$$ \left(\left(\zeta + \widehat{D}_{q,\gamma}^{-1}\widehat{D}_{q,\zeta}(1+qT_{\gamma})T_{\zeta}T_{\eta} + \eta\widehat{D}_{q,\zeta}T_{(\eta;2)}T_{\zeta} \right) \frac{A_q(q\delta)}{A_q(\delta)} + \frac{A_q'(\delta)}{A_q(\delta)} \right) A_q(\delta)e_q(\zeta\delta)e_q(\eta\delta^2)C_{0,q}(-\gamma\delta^2) $$

$$= \sum_{v=1}^{\infty} {}_{S}H\mathbb{A}_{v,q}(\zeta,\eta,\gamma)\frac{\delta^{v-1}}{[v-1]_q!}. \tag{79}$$

Since

$$\widehat{D}_{q,\zeta}A_q(\delta)e_q(\zeta\delta)e_q(\eta\delta^2)C_{0,q}(-\gamma\delta^2) = \delta A_q(\delta)e_q(\zeta\delta)e_q(\eta\delta^2)C_{0,q}(-\gamma\delta^2), \tag{80}$$

and $\frac{A_q'(\delta)}{A_q(\delta)}$ has q-power series expansion in δ, as $A_q(\delta)$ is an invertible series of δ.

Therefore, by using Equation (71), we obtain

$$\left(\left(\zeta + \widehat{D}_{q,\gamma}^{-1}\widehat{D}_{q,\zeta}(1+qT_\gamma)T_\zeta T_\eta + \eta\widehat{D}_{q,\zeta}T_{(\eta;2)}T_\zeta\right)\frac{A_q(q\widehat{D}_{q,\eta})}{A_q(\widehat{D}_{q,\eta})} + \frac{A_q'(\widehat{D}_{q,\eta})}{A_q(\widehat{D}_{q,\eta})}\right)$$

$$\times \sum_{v=0}^{\infty} {}_{S}H\mathbb{A}_{v,q}(\zeta,\eta,\gamma)\frac{\delta^{v}}{[v]_q!} = \sum_{v=1}^{\infty} {}_{S}H\mathbb{A}_{v-1,q}(\zeta,\eta,\gamma)\frac{\delta^{v-1}}{[v-1]_q!}. \tag{81}$$

Comparing the coefficients of δ on both sides of Equation (81) and then from the view of Equation (29), the resultant equation gives assertion (75).

Again, by using Equation (78) by taking $f_q(\delta) = e_q(\zeta\delta)C_{0,q}(-\gamma\delta^2)$ and $g_q(\delta) = A_q(\delta)e_q(\eta\delta^2)$ and following the same lines of the proof of assertion (75), we obtain assertion (76).

In view of Equations (30) and (80), we obtain assertion (77). □

Theorem 9. *The following q-differential equation for* ${}_{S}L\mathbb{A}_{v,q}(\zeta,\eta,\gamma)$ *holds true:*

$$\left(\zeta\widehat{D}_{q,\zeta}\frac{A_q(q\widehat{D}_{q,\zeta})}{A_q(\widehat{D}_{q,\zeta})} + \widehat{D}_{q,\gamma}^{-1}\widehat{D}_{q,\zeta}^2(1+qT_\gamma)T_\zeta T_\eta\frac{A_q(q\widehat{D}_{q,\zeta})}{A_q(\widehat{D}_{q,\zeta})} + \eta\widehat{D}_{q,\zeta}^2T_{(\eta;2)}\frac{A_q'(\widehat{D}_{q,\eta})}{A_q(\widehat{D}_{q,\zeta})}\widehat{D}_{q,\zeta}T_\zeta\right.$$

$$\left.-[v]_q\right){}_{S}H\mathbb{A}_{v,q}(\zeta,\eta,\gamma) = 0, \tag{82}$$

or, equivalently

$$\left(\zeta\widehat{D}_{q,\zeta}\frac{A_q(q\widehat{D}_{q,\zeta})}{A_q(\widehat{D}_{q,\zeta})}T_\eta + \widehat{D}_{q,\gamma}^{-1}\widehat{D}_{q,\zeta}^2(1+qT_\gamma)T_\zeta T_\eta\frac{A_q(q\widehat{D}_{q,\zeta})}{A_q(\widehat{D}_{q,\zeta})} + \eta\widehat{D}_{q,\zeta}^2T_{(\eta;2)}\frac{A_q'(\widehat{D}_{q,\eta})}{A_q(\widehat{D}_{q,\zeta})}\widehat{D}_{q,\zeta}\right.$$

$$\left.-[v]_q\right){}_{S}H\mathbb{A}_{v,q}(\zeta,\eta,\gamma) = 0, \tag{83}$$

Proof. By using (75), (76), and (77) in (32), we obtain the assertions (82) and (83). □

4. Examples

The significance of the Appell polynomial family extends beyond pure mathematics, finding applications in diverse fields such as quantum mechanics, numerical analysis, and combinatorial mathematics. Their inherent recurrence relations, differential properties, and orthogonality conditions make them powerful tools for solving differential equations, modeling physical systems, and developing approximation techniques. Moreover, the flexibility in selecting $A_q(\delta)$ allows for the construction of novel polynomial sequences tailored to specific problems, further enhancing their utility. In the subsequent sections, we delve deeper into the theoretical foundations and computational aspects of these polynomials, emphasizing their role in advancing mathematical research and practical applications.

In particular, we discuss the generating functions for the q-Bernoulli, q-Euler, and q-Genocchi polynomials (see Table 1).

As $q \to 1$, these polynomials reduce to the classical Bernoulli, Euler, and Genocchi polynomials (see [8,33]). Bernoulli polynomials and numbers, as well as their Euler and Genocchi counterparts, have been extensively utilized in various branches of mathematics,

including number theory, combinatorics, and numerical analysis. Their applications enable mathematicians to solve complex problems and derive essential mathematical formulas.

Beyond their classical applications, these number sequences also play a crucial role in modern mathematical research, particularly in the study of special functions, q-series, and modular forms. Bernoulli numbers, for example, are deeply connected to the Riemann zeta function and have significant implications in the study of the arithmetic properties of functions. Euler numbers, on the other hand, contribute to the analysis of alternating permutations and have applications in topology and coding theory. Likewise, Genocchi numbers are instrumental in combinatorial identities and are frequently used in the study of differential equations and integer partitions. Their widespread utility underscores the deep interconnections between different branches of mathematics, further solidifying their importance in theoretical and applied research.

Thus, the q-polynomials and numbers of Bernoulli, Euler, and Genocchi are fundamental tools in multiple mathematical domains. They facilitate the exploration of mathematical relationships, the derivation of significant formulas, and the analysis of complex structures and patterns.

Through the appropriate selection of the function $A_q(\delta)$ in Table 1 in Equation (71), it is possible to establish the following generating functions for the q-Legendre-Hermite-based Bernoulli, ${}_SH\mathbb{B}_{v,q}(\zeta,\eta,\gamma)$, Euler, ${}_SH\mathbb{E}_{v,q}(\zeta,\eta,\gamma)$, and Genocchi, ${}_SH\mathbb{G}_{v,q}(\zeta,\eta,\gamma)$, polynomials:

$$\frac{\delta}{e_q(\delta)-1}C_{0,q}(-\gamma\delta^2)e_q(\zeta\delta)e_q(\eta\delta^2)=\sum_{v=0}^{\infty}{}_SH\mathbb{B}_{v,q}(\zeta,\eta,\gamma)\frac{\delta^v}{[v]_q!}, \tag{84}$$

$$\frac{2}{e_q(\delta)+1}C_{0,q}(-\gamma\delta^2)e_q(\zeta\delta)e_q(\eta\delta^2)=\sum_{v=0}^{\infty}{}_SH\mathbb{E}_{v,q}(\zeta,\eta,\gamma)\frac{\delta^v}{[v]_q!}, \tag{85}$$

and

$$\frac{2\delta}{e_q(\delta)+1}C_{0,q}(-\gamma\delta^2)e_q(\zeta\delta)e_q(\eta\delta^2)=\sum_{v=0}^{\infty}{}_SH\mathbb{G}_{v,q}(\zeta,\eta,\gamma)\frac{\delta^v}{[v]_q!}, \tag{86}$$

Further, in view of expression (73), the polynomials ${}_SH\mathbb{B}_{v,q}(\zeta,\eta,\gamma)$,${}_SH\mathbb{E}_{v,q}(\zeta,\eta,\gamma)$ and ${}_SH\mathbb{G}_{v,q}(\zeta,\eta,\gamma)$ satisfy the following explicit form:

$${}_SH\mathbb{B}_{v,q}(\zeta,\eta,\gamma) = \sum_{\psi=0}^{v}\binom{v}{\psi}_q \mathbb{B}_{\psi,q}\,{}_SH_{v-\psi,q}(\zeta,\eta,\gamma), \tag{87}$$

$${}_SH\mathbb{E}_{v,q}(\zeta,\eta,\gamma) = \sum_{\psi=0}^{v}\binom{v}{\psi}_q \mathbb{E}_{\psi,q}\,{}_SH_{v-\psi,q}(\zeta,\eta,\gamma), \tag{88}$$

and

$${}_SH\mathbb{G}_{v,q}(\zeta,\eta,\gamma) = \sum_{\psi=0}^{v}\binom{v}{\psi}_q \mathbb{G}_{\psi,q}\,{}_SH_{v-\psi,q}(\zeta,\eta,\gamma), \tag{89}$$

where $\mathbb{B}_{\psi,q}$, $\mathbb{E}_{\psi,q}$, $\mathbb{G}_{\psi,q}$ are the q-Bernoulli, q-Euler, and q-Genocchi numbers [3,12,26].

Furthermore, in view of expression (74) and by taking $\beta_{0,q}=1$, $\beta_{i,q}=\frac{1}{[i+1]_q}$ $(i=1,2,\cdots,v)$, $\beta_{0,q}=1$, $\beta_{i,q}=\frac{1}{2}$ $(i=1,2,\cdots,v)$, and $\beta_{0,q}=1$, $\beta_{i,q}=\frac{1}{2[i+1]_q}$ $(i=1,2,\cdots,v)$, respectively, we find that the polynomials ${}_SH\mathbb{B}_{v,q}(\zeta,\eta,\gamma)$, ${}_SH\mathbb{E}_{v,q}(\zeta,\eta,\gamma)$, and ${}_SH\mathbb{G}_{v,q}(\zeta,\eta,\gamma)$ satisfy the following determinant representations:

$$
{}_{S}H\mathbb{B}_{v,q}(\zeta,\eta,\gamma) = (-1)^{v}\begin{vmatrix}
1 & {}_{S}H_{1,q}(\zeta,\eta,\gamma) & {}_{S}H_{2,q}(\zeta,\eta,\gamma) & \cdots & {}_{S}H_{v-1,q}(\zeta,\eta,\gamma) & {}_{S}H_{v,q}(\zeta,\eta,\gamma)\\
1 & \frac{1}{[2]_q} & \frac{1}{[3]_q} & \cdots & \frac{1}{[v]_q} & \frac{1}{[v+1]_q}\\
0 & 1 & \binom{2}{1}_q \frac{1}{[2]_q} & \cdots & \binom{v-1}{1}_q \frac{1}{[v-1]_q} & \binom{v}{1}_q \frac{1}{[v]_q}\\
0 & 0 & 1 & \cdots & \binom{v-1}{2}_q \frac{1}{[v-2]_q} & \binom{v}{2}_q \frac{1}{[v-1]_q}\\
. & . & . & \cdots & . & \\
. & . & . & \cdots & . & \\
0 & 0 & 0 & \cdots & 1 & \binom{v}{v-1}_q \frac{1}{[2]_q}
\end{vmatrix}. \tag{90}
$$

$$
{}_{S}H\mathbb{E}_{v,q}(\zeta,\eta,\gamma) = (-1)^{v}\begin{vmatrix}
1 & {}_{S}H_{1,q}(\zeta,\eta,\gamma) & {}_{S}H_{2,q}(\zeta,\eta,\gamma) & \cdots & {}_{S}H_{v-1,q}(\zeta,\eta,\gamma) & {}_{S}H_{v,q}(\zeta,\eta,\gamma)\\
1 & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2}\\
0 & 1 & \frac{1}{2}\binom{2}{1}_q & \cdots & \frac{1}{2}\binom{v-1}{1}_q & \frac{1}{2}\binom{v}{1}_q\\
0 & 0 & 1 & \cdots & \frac{1}{2}\binom{v-1}{2}_q & \frac{1}{2}\binom{v}{2}_q\\
. & . & . & \cdots & . & \\
. & . & . & \cdots & . & \\
0 & 0 & 0 & \cdots & 1 & \frac{1}{2}\binom{v}{v-1}_q
\end{vmatrix} \tag{91}
$$

and

$$
{}_{S}H\mathbb{G}_{v,q}(\zeta,\eta,\gamma) = (-1)^{n}\begin{vmatrix}
1 & {}_{S}H_{1,q}(\zeta,\eta,\gamma) & {}_{S}H_{2,q}(\zeta,\eta,\gamma) & \cdots & {}_{S}H_{v-1,q}(\zeta,\eta,\gamma) & {}_{S}H_{v,q}(\zeta,\eta,\gamma)\\
1 & \frac{1}{2[2]_q} & \frac{1}{2[3]_q} & \cdots & \frac{1}{2[v]_q} & \frac{1}{2[v+1]_q}\\
0 & 1 & \binom{2}{1}_q \frac{1}{2[2]_q} & \cdots & \binom{v-1}{1}_q \frac{1}{2[v-1]_q} & \binom{v}{1}_q \frac{1}{2[v]_q}\\
0 & 0 & 1 & \cdots & \binom{v-1}{2}_q \frac{1}{2[v-2]_q} & \binom{v}{2}_q \frac{1}{2[v-1]_q}\\
. & . & . & \cdots & . & \\
. & . & . & \cdots & . & \\
0 & 0 & 0 & \cdots & 1 & \binom{v}{v-1}_q \frac{1}{2[2]_q}
\end{vmatrix}. \tag{92}
$$

5. Conclusions

Our interest in the properties of a new generalization of three-variable q-Legendre-based Appell polynomials plays a crucial role in the analysis of charged-beam transport problems in classical mechanics and the computation of quantum-phase-space mechanics. This study has successfully developed and analyzed a new generalization of three-variable q-Legendre-based Appell polynomials by introducing their generating functions, series definitions, q-derivatives, and operational identities. Various properties of these polynomials have been derived and explored. In addition, the same methods and formalism were extended to multi-dimensional and multi-index q-hybrid polynomials, offering valuable insights into their structure. We also employed the quasi-monomial extension technique to explain and implement q-multiplicative and q-derivative operators for the three-variable q-Legendre-based Appell polynomials. Furthermore, several summation formulae for these polynomials were established, contributing to a deeper understanding of their applications. Future research could focus on simplifying and refining the results for multi-dimensional

and multi-index q-polynomials, particularly through the development of more efficient generating functions and operational identities. Investigating the applications of these polynomials in complex systems, including quantum mechanics and statistical mechanics, could provide further avenues for exploration. Additionally, extending the theory of q-derivatives and multiplicative operators to higher-order polynomials and their interrelations would be an interesting direction for future studies.

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