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Tensorial Maclaurin Approximation Bounds and Structural Properties for Mixed-Norm Orlicz–Zygmund Spaces

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Abstract: This article explores two distinct function spaces: Hilbert spaces and mixed-Orlicz–Zygmund spaces with variable exponents. We first examine the relational properties of Hilbert spaces in a tensorial framework, utilizing self-adjoint operators to derive key results. Additionally, we extend a Maclaurin-type inequality to the tensorial setting using generalized convex mappings and establish various upper bounds. A non-trivial example involving exponential functions is also presented. Next, we introduce a new function space, the mixed-Orlicz–Zygmund space $\left(\ell_{q(\cdot)} \log^{\beta}\left(L^{p(\cdot)}\right)\right)$, which unifies Orlicz–Zygmund spaces of integrability and sequence spaces. We investigate its fundamental properties including separability, compactness, and completeness, demonstrating its significance. This space generalizes the existing structures, reducing to mixed-norm Lebesgue spaces when $\beta = 0$ and to classical Lebesgue spaces when $q = \infty$, $\beta = 0$. Given the limited research on such spaces, our findings contribute valuable insights to the functional analysis.

Keywords: self-adjoint operators; Hilbert spaces; operator convexity; mixed-variable exponent spaces; Zygmund space

MSC: 05A30; 26D10; 26D15

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Academic Editor: Francisco Javier Garcia-Pacheco, Rafael de la Rosa Silva, José María Sánchez Delgado

Received: 9 January 2025
Revised: 25 February 2025
Accepted: 7 March 2025
Published: 10 March 2025

Citation: Afzal, W.; Abbas, M.; Meetei, M.Z.; Bourazza, S. Tensorial Maclaurin Approximation Bounds and Structural Properties for Mixed-Norm Orlicz–Zygmund Spaces. *Mathematics* **2025**, *13*, 917. https://doi.org/10.3390/math13060917

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1. Introduction

Inequalities in function spaces are a cornerstone of functional analysis and the associated disciplines, which give basic constraints relating the functions, integrals, derivatives, and their norms. They are used extensively to study the behavior of functions in a wide variety of mathematical fields. For example, Sobolev and Poincaré inequalities have been used to prove the existence and uniqueness of PDE solutions [1], while Hölder's and Minkowski's inequalities are used to prove operator boundedness [2]. In addition, inequalities are a key tool in energy functional minimization and solving variational problems [3], approximating differential operators' eigenvalues via the Rayleigh quotient [4], and proving the stability of numerical schemes using norm inequalities [5].

The literature encompasses, in depth, a number of integral inequalities, e.g., Hermite–Hadamard, Giaccardi, Jensen, midpoint, Young's, Trapezoid, Simpson, Ostrowski, and so forth. For example, the Hermite–Hadamard inequality has been studied in many directions, e.g., its stochastic counterpart [6], fuzzy-valued mappings [7], center–radius order relations [8], inclusion order [9], and Hilbert spaces [10]. Similarly, other inequalities

have been explored using similar techniques, e.g., the Ostrowski-type inequality, which has been generalized in stochastic environments [11] under various order relations [12,13] and in function spaces [14]. For further information on different inequalities, we refer to [15–19] and the references therein.

The Maclaurin series with an integral remainder term is an effective tool for approximating functions and expressing the approximation error precisely. The integral remainder form is particularly useful in theoretical and numerical approximations when error control is crucial.

1.1. Benefits and Significance of the Maclaurin Series Approximation

- Function Approximation: Replace complex functions with simpler polynomial expressions, which are utilized in calculators and computer software to evaluate functions with high precision.
- Economics and Finance: Investigate physical systems near reference states and use Hamiltonians to solve quantum systems with perturbative relativistic formulas for low velocities.
- Numerical Methods: Use expansions to generate numerical differentiation formulas and analyze truncation mistakes to ensure accuracy.
- Machine Learning: Optimize loss functions and enhance algorithms. Gradient descent inequality employs first- and second-order approximations to assess convergence.
- Physics and Engineering: Approximate potential energy functions at equilibrium points are used to expand state equations or partition functions in systems close to equilibrium.

For more applications in various disciplines, we refer to the following articles [20–22]. This widely recognized equality can be stated as follows.

Theorem 1. *Let* $\zeta : \mathsf{R} \to \mathsf{R}$ *be a function on some set* $\Omega \subseteq \mathsf{R}$ *with* $\mathtt{n}+1$ *continuous derivatives. Then, for any* $\nu \in \Omega$*, we have*

$$\zeta(\nu) = \zeta(0) + \zeta'(0)\nu + \frac{\zeta''(0)}{2!}\nu^2 + \cdots + \frac{\zeta^{(\mathtt{n})}(0)}{\mathtt{n}!}\nu^{\mathtt{n}} + \mathtt{R}_{\mathtt{n}}(\nu), \tag{1}$$

where $\mathtt{R}_{\mathtt{n}}(\nu)$ *is the remainder term after* $\mathtt{n}$ *terms. The remainder term* $\mathtt{R}_{\mathtt{n}}(\nu)$ *can be expressed in integral form as*

$$\mathtt{R}_{\mathtt{n}}(\nu) = \frac{1}{\mathtt{n}!}\int_0^{\nu} (\nu - \mathtt{t})^{\mathtt{n}} \zeta^{(\mathtt{n}+1)}(\mathtt{t})\mathtt{dt}.$$

This formula shows that the error $\mathtt{R}_{\mathtt{n}}(\nu)$ depends on the $(\mathtt{n}+1)$-th derivative of $\zeta(\nu)$ and the behavior of $\zeta(\nu)$ on the interval $[0, \nu]$.

Variable exponent spaces and Hilbert spaces are inherent mathematical structures of varied use across various disciplines. Their joint application strengthens the analysis of complex systems via a more efficient and effective approach. They are, for example, essential for modeling relations in complex systems where evidence is characterized by extreme variability [23]. They enhance approximation in approximation theory through the improved representation of functions via integral operators [24].

Variable exponent spaces are used in partial differential equations (PDEs) with non-traditional growth conditions [25] and the study of nonlinear equations, especially in dynamical systems, where normal techniques break down due to irregularities or complexity [26]. Hilbert spaces, nonetheless, are the bedrock of quantum mechanics, wherein quantum states and observables are constructed within their theory [27]. Hilbert spaces are also extensively employed in signal processing and wireless communications, where linear

operator algebra must be used in signal analysis [28]. Moreover, statistical inference has a lot to do with Gaussian Hilbert space measures [29], through which statisticians can place bounds on approximations and the optimization of linear rules under various constraints. For recent developments in other areas, we refer to [30,31].

The significance of variable exponent spaces and Hilbert spaces has led to numerous original results and well-known inequalities in such spaces. Hilbert spaces and tensor operations have been utilized by Frank Hansen and Huzihiro Araki [32] in proving Jensen's operator-type inequalities. In [33], the authors compared the two spaces side by side and derived numerous striking results related to Hadamard and triangular-type inequalities by utilizing various unified and generalized convex mappings. Prasenjit and Tapas Kumar [34] also worked with frames in the tensor product space of n-Hilbert spaces and generalized earlier results from bases to frames and dual frames in tensor product spaces.

Theorem 2 ([34]). *Assume* $\{\mathtt{m_i}\}_{\mathtt{i}=1}^{\infty}$ *and* $\{\mathtt{n_j}\}_{\mathtt{j}=1}^{\infty}$ *be the class of vectors contained in* $\mathtt{H_1}$ *and* $\mathtt{H_2}$ *Hilbert spaces. The vector product* $\{\mathtt{m_i}\otimes\mathtt{n_j}\}_{\mathtt{i,j}=1}^{\infty}\subseteq \mathtt{H_1}\otimes\mathtt{H_2}$ *is a frame related to* $(\sigma_2\otimes\kappa_2,\cdots,\sigma_{\mathtt{n}}\otimes\kappa_{\mathtt{n}})$ *for* $\mathtt{H_1}\otimes\mathtt{H_2}$ *iff* $\{\mathtt{m_i}\}_{\mathtt{i}=1}^{\infty}$ *is a frame related to* $(\sigma_2,\cdots,\sigma_{\mathtt{n}})$ *for* $\mathtt{H_1}$ *and* $\{\mathtt{n_j}\}_{\mathtt{j}=1}^{\infty}$ *is a frame related to* $(\kappa_2,\cdots,\kappa_{\mathtt{n}})$ *for* $\mathtt{H_2}$. *Since* $\{e_i\}_{\mathtt{i}=1}^{\infty}$ *and* $\{\mathtt{k_j}\}_{\mathtt{j}=1}^{\infty}$ *are dual frames corresponding to* $(\sigma_2,\cdots,\sigma_{\mathtt{n}})$ *and* $(\kappa_2,\cdots,\kappa_{\mathtt{n}})$ *of* $\{\mathtt{m_i}\}_{\mathtt{i}=1}^{\infty}$ *and* $\{\mathtt{n_j}\}_{\mathtt{j}=1}^{\infty}$, *respectively, for all* $p\in\mathtt{H_1}, q\in\mathtt{H_2}$,

$$\mathtt{f}=\sum_{\mathtt{i}=1}^{\infty}\langle \mathtt{f},\mathtt{e_i}\mid\sigma_2,\cdots,\sigma_{\mathtt{n}}\rangle_1\mathtt{m_i},\ \textit{and}\ \mathtt{g}=\sum_{\mathtt{j}=1}^{\infty}\langle \mathtt{g},\mathtt{k_j}\mid\kappa_2,\cdots,\kappa_{\mathtt{n}}\rangle_2\mathtt{n_j}.$$

Then, for all $\mathtt{f}\otimes\mathtt{g}\in\mathtt{H_1}\otimes\mathtt{H_2}$, *we have*

$$\begin{aligned}&\mathtt{f}\otimes\mathtt{g}\\&=\left(\sum_{\mathtt{i}=1}^{\infty}\langle \mathtt{f},\mathtt{e_i}\mid\sigma_2,\cdots,\sigma_{\mathtt{n}}\rangle_1\mathtt{m_i}\right)\otimes\left(\sum_{\mathtt{j}=1}^{\infty}\langle \mathtt{g},\mathtt{k_j}\mid\kappa_2,\cdots,\kappa_{\mathtt{n}}\rangle_2\mathtt{n_j}\right)\\&=\sum_{\mathtt{i,j}=1}^{\infty}\langle \mathtt{f},\mathtt{e_i}\mid\sigma_2,\cdots,\sigma_{\mathtt{n}}\rangle_1\langle \mathtt{g},\mathtt{k_j}\mid\kappa_2,\cdots,\kappa_{\mathtt{n}}\rangle_2(\mathtt{m_i}\otimes\mathtt{n_j}).\end{aligned}$$

Vuk Stojiljkovic [35,36] used classical integral operators and twice-differentiable mappings to the continuous functions on self-adjoint operators in Hilbert space to develop Simpson and Ostrowski-type double inequalities.

Theorem 3 ([36]). *Let* $\mathcal{U}$ *and* $\mathcal{V}$ *be self-adjoint operators, and let* $\mathcal{SP}(\mathcal{U}),\mathcal{SP}(\mathcal{V})\subset\Delta$ *be their associated spectrums. Assuming that* ζ *is a continuous function on* Δ, *we obtain*

$$\begin{aligned}&\int_0^1\zeta((1-\alpha)\mathcal{U}\otimes 1+\alpha 1\otimes\mathcal{V})\mathtt{d}\alpha-\zeta\left(\frac{\mathcal{U}\otimes 1+1\otimes\mathcal{V}}{2}\right)\\=&\frac{(1\otimes\mathcal{V}-\mathcal{U}\otimes 1)^2}{16}\left[\int_0^1\alpha^2\zeta''((1-\alpha)\mathcal{U}\otimes 1+\alpha 1\otimes\mathcal{V})\mathtt{d}\alpha\right.\\&\left.+\int_0^1(\alpha-1)^2\zeta''\left(\left(\frac{1-\alpha}{2}\right)\mathcal{U}\otimes 1+\left(\frac{1+\alpha}{2}\right)1\otimes\mathcal{V}\right)\mathtt{d}\alpha\right].\end{aligned}$$

Dragomir [37] investigated multiple bounds and developed a well-known classical trapezoid-type inequality in the context of tensor Hilbert spaces.

Theorem 4 ([37]). *Let $\mathcal{U}$ and $\mathcal{V}$ be self-adjoint operators, and let $\mathcal{SP}(\mathcal{U}), \mathcal{SP}(\mathcal{V}) \subset \Delta$ be their associated spectrums. Assuming that ζ is a continuous function on Δ, we obtain*

$$(1-\alpha)1 \otimes \zeta(\mathcal{V}) + \alpha\zeta(\mathcal{U}) \otimes 1 - \int_0^1 \zeta((1-u)\mathcal{U} \otimes 1 + u1 \otimes \mathcal{V})du$$
$$= (1 \otimes \mathcal{V} - \mathcal{U} \otimes 1) \int_0^1 (u-\alpha)\zeta'((1-u)\mathcal{U} \otimes 1 + u1 \otimes \mathcal{V})du.$$

for all $\alpha \in [0,1]$.

Khan et al. [38] used fractional integral operators and several forms of extended convex mappings to find boundaries for Simpson and Hermite–Hadamard inequalities.

Theorem 5 ([38]). *Let $\mathcal{U}$ and $\mathcal{V}$ be self-adjoint operators, and let $\mathcal{SP}(\mathcal{U}), \mathcal{SP}(\mathcal{V}) \subset \Delta$ be their associated spectrums. Assuming that ζ is a continuous function on Δ, we obtain*

$$\left\| \frac{\mathcal{V} \otimes 1 + 1 \otimes \mathcal{U}}{2} - \frac{2(\mathcal{U} \otimes 1 - \mathcal{V} \otimes 1)}{(\sqrt{\mathcal{U} \otimes 1} - \sqrt{\mathcal{V} \otimes 1})^2} \int_0^1 \zeta((1-u)\mathcal{V} \otimes 1 + u\mathcal{U} \otimes 1))du \right\|$$
$$\leq \frac{(\sqrt{1 \otimes \mathcal{U}} - \sqrt{\mathcal{V} \otimes 1})^2}{4} \frac{(\|\zeta'(\mathcal{U})\| + \|\zeta'(\mathcal{V})\|)}{2}. \quad (2)$$

For more current results related to the results in this article, see [38–46] and their references.

Variable exponent Lebesgue spaces $\left(L^{p(\cdot)}\right)$ are a modern mathematical analysis field that developed from the conventional Lebesgue space. In 1931, W. Orlicz [47] developed sequence spaces ℓ^{p_n} with variable exponent (p_n) and discussed some of their properties. In the same paper, Orlicz defined the class of measurable functions ζ such that $\int_0^1 |\zeta(x)|^{p(\mu)}dx < \infty$, thereby generalizing the concept of classical Lebesgue spaces of integrability. In this work, they let the exponent $p(\mu)$ depend on the point x in the domain Δ. Their adaptability makes them better suitable for simulating physical processes, in which the space's behavior may vary locally in a nonlinear way. Though $L^{p(\cdot)}$ first emerged in 1931, they were initially analyzed as Banach spaces in [48].

Taking inspiration from these various authors, they constructed and generalized numerous other forms of function spaces and examined their features and boundedness of operators with varied exponents. For instance, in [49], the authors used spaces of variable integrability to study uniform convexity; in [50], they studied some interesting problems related to the fixed point theorem in the varying exponent spaces; in [51], they examined the boundedness of certain operators on grand Herz spaces with variable exponents; in [52], they examined interpolation theorems for variable exponent Lebesgue spaces; and in [53], they examined modular-proximal gradient algorithms in varying exponent spaces. Fernanda and James [54] investigated the algebraic and topological reflexivity features of sequence spaces with varying exponents. Amri et al. [55] employed modular function spaces to examine fixed point theorem with application in the space of variable integrability. Haim Brezis [56] explored different kinds of function spaces and discussed problems related to reflexive spaces, separable spaces, and uniform convexity. Zhenghua Luo and Qingjin Cheng [57] investigated a novel convexity property of convex functions, resulting in a characterization of the class of reflexive Banach spaces. Jain et al. [58] investigated quasi-reflexivity and the sup of linear functions with interesting applications. Enrico Pasqualetto and Tapio Rajala [59] studied first-order Sobolev spaces on reflexive Banach spaces, using relaxation, test plans, and divergence.

Completeness, compactness, and the separability of variable exponent spaces play a very essential role in several areas, such as checking the boundedness of operators, checking

the existence of the solution of a partial differential equation, embedding of spaces, etc. For instance, the authors examined these three properties for Herz spaces in [60], demonstrated separation axioms, representation theorems, and compactness in [61], examined completeness for Moore-closed spaces and centered bases in [62], determined compactness and separability in [63], and demonstrated several pertinent results in weighted Lorentz spaces using the Hardy Operator in [64].

Bandaliyev et al. [65] studied the precompactness and completeness of sets in variable exponent Morrey spaces using bounded metric measure spaces.

Theorem 6 ([65]). *Let* (Ω, ρ, ν) *be a bounded doubling metric measure space, and let* $p(\cdot) \in \mathcal{P}(\Omega, \nu)$ *satisfy* $0 < p_- \leq p_+ < \infty$ *and* $\sup_{\sigma\in\Omega} \frac{\alpha(\sigma)}{p(\sigma)} < \frac{1}{p_+}$. *Assume that a subset* $\mathcal{Q}$ *of* $L^{p(\cdot),\alpha(\cdot)}(\Omega)$ *satisfies the following:*

- $\mathcal{Q}$ *is bounded in* $L^{p(\cdot),\alpha(\cdot)}(\Omega)$;
- *For some* $q \in (0, p_-)$,

$$\lim_{\tau\to 0^+} \sup_{\zeta\in\mathcal{Q}} \left\| \int_{\mathcal{B}(\cdot,\tau)} |\zeta(\cdot) - \zeta(y)|^q \, d\nu(y) \right\|_{L^{\frac{p(\cdot)}{q},\alpha(\cdot)}(\Omega)} = 0.$$

Then, $\mathcal{Q}$ *is totally bounded in* $L^{p(\cdot),\alpha(\cdot)}(\Omega)$.

Orlicz–Zygmund spaces are a class of function spaces that generalize both Orlicz spaces and Zygmund spaces. They enable logarithmic changes to the evolution of Orlicz functions, making them helpful in a variety of applications including nonlinear analysis and partial differential equations. For instance in [66], the authors investigated the Lipschitz regularity for local minimizers of convex variational integrals of the form using Orlicz–Zygmund spaces with variable exponents; in [67], the authors investigated the higher differentiability of minimizers of variational integrals with variable Orlicz–Zygmund spaces; in [68], the authors investigated global gradient estimates for the borderline case of double-phase problems with BMO for Orlicz–Zygmund spaces; and in [69] , the authors investigated regularity for double-phase energy functionals of different power growth.

For further recent results relevant to variable exponent spaces, see [70–76] and the references therein.

1.2. Main Contribution

The article makes two significant contributions. The first is that tensor approximation can be used to extend equality (1) from the lower dimension (classical sense) to a higher dimension since tensor approximation deals with a wide range of complex functions in a very straightforward manner. Second, we can introduce a new norm structure, as Orlicz–Zygmund spaces with variable exponents have been extensively studied for analyzing problems involving variable growth conditions and non-standard behavior in partial differential equations.

The Zygmund space $L^p \log^\beta L(\Omega; R^n)$, for $1 \leqslant p < \infty, \beta \in R (\beta \geqslant 0$ for $p = 1)$ is defined as the Orlicz–Zygmund space $L^\Phi(\Omega; R^n)$ generated by the Young function

$$\Phi(\mu) \simeq \mu^p \log^\beta (e + \mu) \quad \text{for every } \mu \geq \mu_0 \geq 0.$$

Therefore, the Orlicz–Zygmund space $L^p \log^\beta L(\Omega; R^n)$ is defined as

$$L^p \log^\beta L(\Omega; R^n) := \left\{ \zeta : \Omega \to R^n \text{ measurable} : \int_\Omega |\zeta|^p \log^\beta (e + |\zeta|) d\nu < \infty \right\}$$

and it becomes a Banach space with the Luxemburg norm

$$\|\zeta\|_{L\log^{\beta} L(\Omega;R^{n})} := \inf\left\{\alpha > 0 : \zeta_{\Omega}\left|\frac{\zeta}{\alpha}\right|^{p} \log^{\beta}\left(e + \left|\frac{\zeta}{\alpha}\right|\right) d\nu \leq 1\right\}.$$

The main idea we use is a combination of the variable sequence space $(\ell_{q(\cdot)})$ and variable Orlicz–Zygmund space $(L^{p}\log^{\beta} L(\Omega;R^{n}))$ that is merged simultaneously, and we introduce a new type of modular functional with an associated norm that is based on these two spaces. In this space, we define a new norm structure that we call mixed-variable Orlicz–Zygmund space $\left(\ell_{q(\cdot)}\log^{\beta}\left(L^{p(\cdot)}\right)\right)$ and investigate several well-known properties, such as completeness, separability, and precompactness. Since these types of spaces related to developed results are rare in the literature, this paper will play a very important role in the advancement of results related to these in function spaces.

As a main motivation for this study, we were primarily influenced by the works of the following authors [32,35,65,66].

The work is organized into six sections. In Section 2, we will recall some basic definitions and arithmetic operations in tensor Hilbert spaces. In Section 3, we construct our major conclusions, including Maclaurin-type approximation, using various types of generalized convex mappings in tensorial Hilbert spaces. In Section 4, we define a few well-known notions and results that we need to support our primary conclusions regarding Orlicz–Zygmund spaces and other related spaces. In Section 5, we demonstrate the separability, completeness, and compactness of newly defined mixed-Orlicz–Zygmund spaces. In Section 6, we discuss our main findings and some future possible work related to these results.

2. Preliminaries

In this section, we recall some fundamental concepts related to extended convex mappings and arithmetic operations in tensor Hilbert spaces. The following articles [32,39] contain further important concepts and discoveries related to this section.

Definition 1 ([33]). *An inner product on a complex linear space $\mathcal{K}$ is a map*

$$(\cdot,\cdot) : \mathcal{K}\times\mathcal{K} \to C$$

such that for all $\xi_1, \xi_2, \xi_3 \in \mathcal{K}$ and $\alpha \in C$, we have

$$\begin{aligned}\langle \xi_1+\xi_2, \xi_3\rangle &= \langle \xi_1, \xi_3\rangle + \langle \xi_2+\xi_3\rangle\\ \langle \alpha\xi_1, \xi_2\rangle &= \alpha\langle \xi_1, \xi_2\rangle\\ \langle \xi_1, \xi_2\rangle &= \overline{\langle \xi_2, \xi_1\rangle}\\ \langle \xi_1, \xi_1\rangle &\geq 0, \quad \langle \xi_1, \xi_1\rangle = 0 \Longleftrightarrow \xi_1 = 0.\end{aligned}$$

A complete inner product space is called a Hilbert space.

Definition 2 ([33]). *Let $\zeta : \mathcal{U}\times\mathcal{V} \to \mathcal{K}$ be a bilinear mapping. The related tensor product of $\mathcal{U}$ with $\mathcal{V}$ on Hilbert space $\mathcal{K}$ is defined as follows:*

- *The Hilbert space $\mathcal{K}$ contains the class of $\zeta(\xi_1,\xi_2)(\xi_1 \in \mathcal{U}, \xi_2 \in \mathcal{V})$ all vectors, and $\mathcal{K}$ is its span;*
- *$(\zeta(\xi_1,\xi_2) \mid \zeta(\xi_3,\xi_4)) = (\xi_1 \mid \xi_2)(\xi_3 \mid \xi_4)$ for $\xi_1,\xi_2 \in \mathcal{U}, \xi_3,\xi_4 \in \mathcal{V}$. If $(\mathcal{K},\zeta)$ is a Cartesian product of $\mathcal{U}$ and $\mathcal{V}$, it is typical to read as $\xi_1\otimes\xi_2$ instead of $\zeta(\xi_1,\xi_2)$. A Cartesian*

product $\mathcal{U} \otimes \mathcal{V}$ *and a function* $(\xi_1, \xi_2) \mapsto \xi_1 \otimes \xi_2$ *of* $\mathcal{U} \times \mathcal{V}$ *into* $\mathcal{U} \otimes \mathcal{V}$*, satisfy the following axioms*

$$\begin{aligned}(\xi_1 + \xi_2) \otimes \xi_2 &= \xi_1 \otimes \xi_2 + \xi_2 \otimes \xi_2\\ (\alpha\xi_1) \otimes \xi_2 &= \alpha(\xi_1 \otimes \xi_2)\\ \xi_1 \otimes (\xi_3 + \xi_4) &= \xi_1 \otimes \xi_3 + \xi_1 \otimes \xi_3\\ \xi_1 \otimes (\alpha\xi_2) &= \alpha(\xi_1 \otimes \xi_2),\end{aligned}$$

where $\alpha \in \mathcal{K}$.

Let $\zeta : \Delta_1 \times \ldots \times \Delta_p \to R$ is taken to be bounded mapping defined in terms of intervals. Assume that $\wp = (\wp_1, \ldots, \wp_p)$ be a p-tuple of the self-adjoint operators associated with $E_1, \ldots, E_p$ Hilbert spaces. Then,

$$\wp_i = \int_{\Delta_i} \alpha_i dE_i(\alpha_i)$$

is the spectrum of possible operators for $i = 1, \ldots, p$; following [32], we define $\wp_i$ as follows:

$$\zeta(\wp_1, \ldots, \wp_p) := \int_{\Delta_1} \ldots \int_{\Delta_p} \zeta(\alpha_1, \ldots, \alpha_p) dE_1(\alpha_1) \otimes \ldots \otimes dE_p(\alpha_p).$$

When the dimensions of Hilbert spaces are finite, many complicated processes can be considerably reduced by integrating processes into finite summations. The authors [77] expand the construction, and it is defined as follows:

$$\zeta(\wp_1, \ldots, \wp_p) = \zeta_1(\wp_1) \otimes \ldots \otimes \zeta_p(\wp_p),$$

where ζ can be split as a product of one-variable mappings $\zeta(a_1, \ldots, a_p) = \zeta_1(a_1) \ldots \zeta_p(a_p)$.

If ζ is sub(super)-multiplicative across the interval Δ, then

$$\zeta(\xi_1\xi_2) \geq (\leq)\zeta(\xi_1)\zeta(\xi_2) \text{ for all } \xi_1\xi_2 \in [0, \infty),$$

and additionally, if ζ is continuous on $[0, \infty)$, one has

$$\zeta(\mathcal{U} \otimes \mathcal{V}) \geq (\leq)\zeta(\mathcal{U}) \otimes \zeta(\mathcal{V}) \text{ for all } \mathcal{U}, \mathcal{V} \geq 0.$$

Consequently, it may be shown that if

$$\mathcal{U} = \int_{[0,\infty)} \xi_1 dE(\xi_1) \text{ and } \mathcal{V} = \int_{[0,\infty)} \xi_2 dF(\xi_2),$$

where $\mathcal{U}$ and $\mathcal{V}$ are spectral resolutions for function ζ on $[0, \infty)$, then

$$\zeta(\mathcal{U} \otimes \mathcal{V}) = \int_{[0,\infty)} \int_{[0,\infty)} \zeta(\xi_1\xi_2) dE(\xi_1) \otimes dF(\xi_2).$$

The geometric property of linear bounded operator $\mathcal{U}, \mathcal{V} > 0$ is represented as follows:

$$\mathcal{U} \#_p \mathcal{V} := \mathcal{U}^{1/2}\left(\mathcal{U}^{-1/2}\mathcal{V}\mathcal{U}^{-1/2}\right)^p \mathcal{U}^{1/2},$$

where $p \in [0, 1]$ and

$$\mathcal{U} \# \mathcal{V} := \mathcal{U}^{1/2}\left(\mathcal{U}^{-1/2}\mathcal{V}\mathcal{U}^{-1/2}\right)^{1/2} \mathcal{U}^{1/2}.$$

By the definitions of # and $\otimes$, we have

$$\mathcal{U}\#\mathcal{V} = \mathcal{V}\#\mathcal{U} \text{ and } (\mathcal{U}\#\mathcal{V}) \otimes (\mathcal{V}\#\mathcal{U}) = (\mathcal{U} \otimes \mathcal{V})\#(\mathcal{V} \otimes \mathcal{U}).$$

Examine the final result that is comparable to the tensorial product that is

$$(\mathcal{U}\beta) \otimes (\mathcal{V}\mathcal{S}) = (\mathcal{U} \otimes \mathcal{V})(\mathcal{U} \otimes \mathcal{S}),$$

that holds $\forall \mathcal{U}, \mathcal{V}, \beta, \mathcal{S} \in \mathtt{B}(\mathtt{H})$. If we take $\beta = \mathcal{U}$ and $\mathcal{S} = \mathcal{V}$, then we obtain

$$\mathcal{U}^2 \otimes \mathcal{V}^2 = (\mathcal{U} \otimes \mathcal{V})^2.$$

Through induction, we have

$$\mathcal{U}^\sigma \otimes \mathcal{V}^\sigma = (\mathcal{U} \otimes \mathcal{V})^\sigma \text{ for any natural number } \sigma \geq 0.$$

Specifically,

$$\mathcal{U}^\sigma \otimes 1 = (\mathcal{U} \otimes 1)^\sigma \text{ and } 1 \otimes \mathcal{V}^\sigma = (1 \otimes \mathcal{V})^\sigma,$$

for all $\sigma \geq 0$. Additionally, we note that the $1 \otimes \mathcal{V}$ and $\mathcal{U} \otimes 1$ are commutative with each other that is

$$(1 \otimes \mathcal{V})(\mathcal{U} \otimes 1) = \mathcal{U} \otimes \mathcal{V} = (\mathcal{U} \otimes 1)(1 \otimes \mathcal{V}).$$

Moreover, for any two natural numbers σ_1, σ_2

$$(\mathcal{U} \otimes 1)^{\sigma_1}(1 \otimes \mathcal{V})^{\sigma_2} = (1 \otimes \mathcal{V})^{\sigma_1}(\mathcal{U} \otimes 1)^{\sigma_2} = \mathcal{U}^{\sigma_2} \otimes \mathcal{V}^{\sigma_1}.$$

Using the functional calculus and tensorial product characteristics for continuous functions of self-adjoint operators the associated tensorial perspective $\mathcal{Q}_{\zeta,\otimes}$, is defined as follows:

$$\begin{aligned}\mathcal{Q}_{\zeta,\otimes}(\mathcal{U}, \mathcal{V}) &= (1 \otimes \mathcal{V})\zeta\Big((\mathcal{U} \otimes 1)(1 \otimes \mathcal{V})^{-1}\Big) = \zeta\Big((\mathcal{U} \otimes 1)(1 \otimes \mathcal{V})^{-1}\Big)(1 \otimes \mathcal{V}) \\ &= \zeta\Big((1 \otimes \mathcal{V})^{-1}(\mathcal{U} \otimes 1)\Big)(1 \otimes \mathcal{V}),\end{aligned}$$

due to the commutativity of $\mathcal{U} \otimes 1$ and $1 \otimes \mathcal{V}$.

Lemma 1 ([33]). *Let $\mathcal{U}$ and $\mathcal{V}$ be self-adjoint operators with $\mathcal{SP}(\mathcal{U}) \subset \Delta_1$ and $\mathcal{SP}(\mathcal{V}) \subset \Delta_2$. If θ, ζ are continuous on Δ_1, then $\Phi, \Im$ are continuous on Δ_2, and φ is continuous on Δ, and subsequently the class of intervals $\zeta(\Delta_1) + \Im(\Delta_2)$ is*

$$\begin{aligned}&(\theta(\mathcal{U}) \otimes 1 + 1 \otimes \Phi(\mathcal{V}))\varphi(\zeta(\mathcal{U}) \otimes 1 + 1 \otimes \Im(\mathcal{V})) \\ &= \int_{\Delta_1}\int_{\Delta_2} (\theta(\sigma) + \Phi(\mathfrak{u}))\varphi(\zeta(\sigma) + \Im(\mathfrak{u}))\mathtt{dE}_{\Delta_1} \otimes \mathtt{dF}_{\Delta_2},\end{aligned} \tag{3}$$

where $\mathcal{U}$ and $\mathcal{V}$ are the spectral resolutions

$$\mathcal{U} = \int_{\Delta_1} \alpha \mathtt{dE}(\alpha) \text{ and } \mathcal{V} = \int_{\Delta_2} \beta \mathtt{dF}(\beta).$$

Definition 3 ([39]). *A mapping $\zeta : \Delta \subseteq \mathsf{R} \to \mathsf{R}$ is stated to be convex (concave) on Δ, if*

$$\zeta(\alpha\xi_1 + (1-\alpha)\xi_2) \leq (\geq)\alpha\zeta(\xi_1) + (1-\alpha)\zeta(\xi_2),$$

valid for all $\xi_1, \xi_2 \in \Delta$ and $\alpha \in [0,1]$.

Definition 4 ([39]). *A mapping $\zeta : \Delta \to \mathsf{R}$ is stated to be convex in a quasi sense if*

$$\zeta((1-\alpha)\xi_1 + \alpha\xi_2) \leq \max\{\zeta(\xi_2), \zeta(\xi_1)\} = \frac{1}{2}(\zeta(\xi_2) + \zeta(\xi_1) + |\zeta(\xi_2) - \zeta(\xi_1)|),$$

for all $\xi_1, \xi_2 \in \Delta$ and $\alpha \in [0,1]$.

Definition 5 ([39]). *A mapping $\zeta : \Delta \subseteq \mathsf{R} \to \mathsf{R}$ is stated to be Godunova convex on Δ, if*

$$\zeta(\alpha\xi_1 + (1-\alpha)\xi_2) \leq \frac{\zeta(\xi_1)}{\alpha} + \frac{\zeta(\xi_2)}{(1-\alpha)},$$

valid for all $\xi_1, \xi_2 \in \Delta$ and $\alpha \in (0,1)$.

The following lemma, whose properties are also utilized in our main findings, is presented in a similar manner using synchronous functions in the work of Dragomir [78].

Lemma 2. *Assume that ζ, ψ are asynchronous and continuous on Δ, while φ, ϑ are continuous and non-negative on Δ. If $\mathcal{U}, \mathcal{V}$ are self-adjoint with spectrums $\mathcal{SP}(\mathcal{U}), \mathcal{SP}(\mathcal{V}) \subset \Delta$, then*

$$\begin{aligned}&[\varphi(\mathcal{U})\zeta(\mathcal{U})\psi(\mathcal{U})] \otimes \vartheta(\mathcal{V}) + \varphi(\mathcal{U}) \otimes [\vartheta(\mathcal{V})\zeta(\mathcal{V})\psi(\mathcal{V})]\\ &\leq [\varphi(\mathcal{U})\zeta(\mathcal{U})] \otimes [\vartheta(\mathcal{V})\psi(\mathcal{V})] + [\varphi(\mathcal{U})\psi(\mathcal{U})] \otimes [\vartheta(\mathcal{V})\zeta(\mathcal{V})],\end{aligned}$$

or, equivalently

$$\begin{aligned}&(\varphi(\mathcal{U}) \otimes \vartheta(\mathcal{V}))[(\zeta(\mathcal{U})\psi(\mathcal{U})) \otimes 1 + 1 \otimes (\zeta(\mathcal{V})\psi(\mathcal{V}))]\\ &\leq (\varphi(\mathcal{U}) \otimes \vartheta(\mathcal{V}))[\zeta(\mathcal{U}) \otimes \psi(\mathcal{V}) + \psi(\mathcal{U}) \otimes \zeta(\mathcal{V})].\end{aligned}$$

Proof. Suppose ζ and ψ are asynchronous functions on Δ, then

$$\zeta(\tau)\psi(\tau) + \zeta(v)\psi(v) \leq \zeta(\tau)\psi(v) + \zeta(v)\psi(\tau),$$

for all $\tau, v \in \Delta$.

To obtain this inequality, we multiply it by $\varphi(\tau)\vartheta(v) \leq 0$, and it follows that

$$\begin{aligned}&\zeta(\tau)\psi(\tau)\varphi(\tau)\vartheta(v) + \varphi(\tau)\zeta(v)\psi(v)\vartheta(v)\\ &\leq \zeta(\tau)\varphi(\tau)\psi(v)\vartheta(v) + \zeta(v)\vartheta(v)\psi(\tau)\varphi(\tau),\end{aligned}$$

for all $\tau, v \in \Delta$.

If we use the double integral, we obtain

$$\begin{aligned}&\int_\Delta \int_\Delta [\zeta(\tau)\psi(\tau)\varphi(\tau)\vartheta(v) + \varphi(\tau)\zeta(v)\psi(v)\vartheta(v)]\mathtt{dE}(\tau) \otimes \mathtt{dF}(v)\\ &\leq \int_\Delta \int_\Delta [\zeta(\tau)\varphi(\tau)\psi(v)\vartheta(v) + \zeta(v)\vartheta(v)\psi(\tau)\varphi(\tau)]\mathtt{dE}(\tau) \otimes \mathtt{dF}(v).\end{aligned}$$

Note that

$$\begin{aligned}&\int_\Delta \int_\Delta [\zeta(\tau)\psi(\tau)\varphi(\tau)\vartheta(v) + \varphi(\tau)\zeta(v)\psi(v)\vartheta(v)]\mathtt{dE}(\tau) \otimes \mathtt{dF}(v)\\ &= \int_\Delta \int_\Delta \zeta(\tau)\psi(\tau)\varphi(\tau)\vartheta(v)\mathtt{dE}(\tau) \otimes \mathtt{dF}(v)\\ &+ \int_\Delta \int_\Delta \varphi(\tau)\zeta(v)\psi(v)\vartheta(v)\mathtt{dE}(\tau) \otimes \mathtt{dF}(v)\\ &= [\varphi(\mathcal{U})\zeta(\mathcal{U})\psi(\mathcal{U})] \otimes \vartheta(\mathcal{V}) + \varphi(\mathcal{U}) \otimes [\vartheta(\mathcal{V})\zeta(\mathcal{V})\psi(\mathcal{V})]\end{aligned}$$

and

$$\begin{aligned}&\int_{\Delta}\int_{\Delta}[\zeta(\tau)\varphi(\tau)\psi(\upsilon)\vartheta(\upsilon)+\zeta(\upsilon)\vartheta(\upsilon)\psi(\tau)\varphi(\tau)]\mathtt{dE}(\tau)\otimes\mathtt{dF}(\upsilon)\\&=\int_{\Delta}\int_{\Delta}\zeta(\tau)\varphi(\tau)\psi(\upsilon)\vartheta(\upsilon)\mathtt{dE}(\tau)\otimes\mathtt{dF}(\upsilon)\\&+\int_{\Delta}\int_{\Delta}\psi(\tau)\varphi(\tau)\zeta(\upsilon)\vartheta(\upsilon)\mathtt{dE}(\tau)\otimes\mathtt{dF}(\upsilon)\\&=[\varphi(\mathcal{U})\zeta(\mathcal{U})]\otimes[\vartheta(\mathcal{V})\psi(\mathcal{V})]+[\varphi(\mathcal{U})\psi(\mathcal{U})]\otimes[\vartheta(\mathcal{V})\zeta(\mathcal{V})].\end{aligned}$$

Now, by applying the tensorial property

$$(\mathtt{PQ})\otimes(\mathtt{RS})=(\mathtt{P}\otimes\mathtt{R})(\mathtt{Q}\otimes\mathtt{u}),$$

for any $\mathtt{P},\mathtt{Q},\mathtt{R},\mathtt{S}\in\mathcal{B}(\mathcal{H})$, we obtain

$$\begin{aligned}&[\varphi(\mathcal{U})\zeta(\mathcal{U})\psi(\mathcal{U})]\otimes\vartheta(\mathcal{V})+\varphi(\mathcal{U})\otimes[\vartheta(\mathcal{V})\zeta(\mathcal{V})\psi(\mathcal{V})]\\&=(\varphi(\mathcal{U})\otimes\vartheta(\mathcal{V}))[(\zeta(\mathcal{U})\psi(\mathcal{U}))\otimes 1]+(\varphi(\mathcal{U})\otimes\vartheta(\mathcal{V}))[1\otimes(\zeta(\mathcal{V})\psi(\mathcal{V}))]\\&=(\varphi(\mathcal{U})\otimes\vartheta(\mathcal{V}))[(\zeta(\mathcal{U})\psi(\mathcal{U}))\otimes 1+1\otimes(\zeta(\mathcal{V})\psi(\mathcal{V}))]\end{aligned}$$

and

$$\begin{aligned}&[\varphi(\mathcal{U})\zeta(\mathcal{U})]\otimes[\vartheta(\mathcal{V})\psi(\mathcal{V})]+[\varphi(\mathcal{U})\psi(\mathcal{U})]\otimes[\vartheta(\mathcal{V})\zeta(\mathcal{V})]\\&=(\varphi(\mathcal{U})\otimes\vartheta(\mathcal{V}))(\zeta(\mathcal{U})\otimes\psi(\mathcal{V}))+(\varphi(\mathcal{U})\otimes\vartheta(\mathcal{V}))(\psi(\mathcal{U})\otimes\zeta(\mathcal{V}))\\&=(\varphi(\mathcal{U})\otimes\vartheta(\mathcal{V}))[\zeta(\mathcal{U})\otimes\psi(\mathcal{V})+\psi(\mathcal{U})\otimes\zeta(\mathcal{V})],\end{aligned}$$

which proves Lemma 2. □

3. The Major Results

These two results play an important role in our further subsequent findings related to Macalurin approximation equality.

Lemma 3. *Suppose that $\mathcal{U}$ and $\mathcal{V}$ are self-adjoint operators such that $\mathcal{SP}(\mathcal{U})\subset\Delta_1$ and $\mathcal{SP}(\mathcal{V})\subset\Delta_2$. If ζ,φ are continuous on Δ_1,ψ,ϑ are continuous on Δ_2 and ξ,η are continuous on interval $\mathcal{K}$, subsequently the sum of $\zeta(\Delta_1)+\psi(\Delta_2),\varphi(\Delta_1)+\vartheta(\Delta_2)\subset\mathcal{K}$ is*

$$\begin{aligned}&\xi(\zeta(\mathcal{U})\otimes 1+1\otimes\psi(\mathcal{V}))\eta(\varphi(\mathcal{U})\otimes 1+1\otimes\vartheta(\mathcal{V}))\\&=\int_{\Delta_1}\int_{\Delta_2}\xi(\zeta(\tau)+\psi(\upsilon))\eta(\varphi(\tau)+\vartheta(\upsilon))\mathtt{dE}_{\upsilon}\otimes\mathtt{dF}_{\tau}.\end{aligned}$$

Proof. Let $\mathtt{a},\mathtt{b},\mathtt{c}$ and $\mathtt{d}$ positive-continuous functions such that $\zeta(\tau)=\ln\mathtt{a}(\tau)$, $\varphi(\tau)=\ln\mathtt{c}(\tau)$ for $\tau\in\Delta_1$ and $\psi(\upsilon)=\ln\mathtt{b}(\mathbb{E}),\vartheta(\upsilon)=\ln\mathtt{d}(\mathbb{E})$ for $\upsilon\in\Delta_2$. Then,

$$\begin{aligned}&\int_{\Delta_1}\int_{\Delta_2}\xi(\zeta(\tau)+\psi(\upsilon))\eta(\varphi(\tau)+\vartheta(\upsilon))\mathtt{dE}_{\upsilon}\otimes\mathtt{dF}_{\tau}\\&=\int_{\Delta_1}\int_{\Delta_2}\xi(\ln\mathtt{a}(\tau)+\ln\mathtt{b}(\mathbb{E}))\eta(\ln\mathtt{c}(\tau)+\ln\mathtt{d}(\mathbb{E}))\mathtt{dE}_{\upsilon}\otimes\mathtt{dF}_{\tau}\\&=\int_{\Delta_1}\int_{\Delta_2}(\xi\circ\ln)(\mathtt{a}(\tau)\mathtt{b}(\mathbb{E}))(\eta\circ\ln)(\mathtt{c}(\tau)\mathtt{d}(\mathbb{E}))\mathtt{dE}_{\upsilon}\otimes\mathtt{dF}_{\tau}.\end{aligned}$$

By applying Lemma 2 to the functions $(\xi\circ\ln)$ and $(\eta\circ\ln)$, we obtain

$$\begin{aligned}&\int_{\Delta_1}\int_{\Delta_2}(\xi\circ\ln)(\mathtt{a}(\tau)\mathtt{b}(\mathbb{E}))(\eta\circ\ln)(\mathtt{c}(\tau)\mathtt{d}(\mathbb{E}))\mathtt{dE}_{\upsilon}\otimes\mathtt{dF}_{\tau}\\&=(\xi\circ\ln)(\mathtt{a}(\mathcal{U})\otimes\mathtt{b}(\mathcal{V}))(\eta\circ\ln)(\mathtt{c}(\mathcal{U})\otimes\mathtt{d}(\mathcal{V}))\\&=\xi[\ln(\mathtt{a}(\mathcal{U})\otimes\mathtt{b}(\mathcal{V}))]\eta[\ln(\mathtt{c}(\mathcal{U})\otimes\mathtt{d}(\mathcal{V}))].\end{aligned}$$

Now, observe that, by the commutativity of the operators $\mathtt{a}(\mathcal{U}) \otimes 1$ and $1 \otimes \mathtt{b}(\mathcal{V})$,

$$\begin{aligned}\ln(\mathtt{a}(\mathcal{U}) \otimes \mathtt{b}(\mathcal{V})) &= \ln[(\mathtt{a}(\mathcal{U}) \otimes 1)(1 \otimes \mathtt{b}(\mathcal{V}))] \\ &= \ln(\mathtt{a}(\mathcal{U}) \otimes 1) + \ln(1 \otimes \mathtt{b}(\mathcal{V})) \\ &= [\ln \mathtt{a}(\mathcal{U})] \otimes 1 + 1 \otimes \ln \mathtt{b}(\mathcal{V}) \\ &= \zeta(\mathcal{U}) \otimes 1 + 1 \otimes \psi(\mathcal{V})\end{aligned}$$

and, similarly

$$\ln(\mathtt{c}(\mathcal{U}) \otimes \mathtt{d}(\mathcal{V})) = \varphi(\mathcal{U}) \otimes 1 + 1 \otimes \vartheta(\mathcal{V}).$$

Hence, we obtain the desired representation for Lemma 3. □

Theorem 7. *If ζ is a Godunova–Levin mapping defined over the positive half-line, then $\mathcal{Q}_{\zeta,\otimes}$ is also Godunova–Levin in pairs of positive definite operators. If $\mathcal{U} \geq \mathcal{G} > 0$ and $\mathcal{V} \geq \mathcal{H} > 0$, then also*

$$\mathcal{Q}_{\zeta,\otimes}(\mathcal{U}, \mathcal{V}) \geq \mathcal{Q}_{\zeta,\otimes}(\mathcal{G}, \mathcal{H}).$$

Proof. Suppose that ζ is a operator Godunova–Levin function in the positive half-line. Given the commutative nature of $\mathcal{U} \otimes 1$ and $1 \otimes 0$, we have

$$\mathcal{Q}_{\zeta,\otimes}(\mathcal{U}, \mathcal{V}) = (1 \otimes \mathcal{V})\zeta\Big((\mathcal{U} \otimes 1)(1 \otimes \mathcal{V})^{-1}\Big) = \mathcal{Q}_{\zeta}(\mathcal{U} \otimes 1, 1 \otimes \mathcal{V})$$

for $\mathcal{U}, \mathcal{V} > 0$. If $\mathcal{U}, \mathcal{V}, \mathcal{G}, \mathcal{H} > 0$ and $\alpha \in (0,1)$, then we have

$$\begin{aligned}&\mathcal{Q}_{\zeta}, \otimes((1-\alpha)(\mathcal{U}, \mathcal{V}) + \alpha(\mathcal{G}, \mathcal{H})) \\ &= \mathcal{Q}_{\zeta}, \otimes(((1-\alpha)\mathcal{U} + \alpha\mathcal{G}, (1-\alpha)\mathcal{V} + \alpha\mathcal{H})) \\ &= \mathcal{Q}_{\zeta}(((1-\alpha)\mathcal{U} + \alpha\mathcal{G}) \otimes 1, 1 \otimes ((1-\alpha)\mathcal{V} + \alpha\mathcal{H})) \\ &= \mathcal{Q}_{\zeta}((1-\alpha)\mathcal{U} \otimes 1 + \alpha\mathcal{G} \otimes 1, (1-\alpha)1 \otimes \mathcal{V} + \alpha 1 \otimes \mathcal{H}) \\ &= \mathcal{Q}_{\zeta}((1-\alpha)(\mathcal{U} \otimes 1, 1 \otimes \mathcal{V}) + \alpha(\mathcal{G} \otimes 1, 1 \otimes \mathcal{H})) \\ &\leq \frac{\mathcal{Q}_{\zeta}(\mathcal{U} \otimes 1, 1 \otimes \mathcal{V})}{(1-\alpha)} + \frac{\mathcal{Q}_{\zeta}(\mathcal{G} \otimes 1, 1 \otimes \mathcal{H})}{\alpha} \\ &= \frac{\mathcal{Q}_{\zeta}, \otimes(\mathcal{U}, \mathcal{V})}{(1-\alpha)} + \frac{\mathcal{Q}_{\zeta}, \otimes(\mathcal{G}, \mathcal{H})}{\alpha},\end{aligned}$$

and it demonstrates that $\mathcal{Q}_{\zeta,\otimes}$ is a Godunova–Levin operator in pairs of positive definite operators. If $\mathcal{U} \geq \mathcal{G} > 0$ and $\mathcal{V} \geq \mathcal{H} > 0$, then $\mathcal{U} \otimes 1 \geq \mathcal{G} \otimes 1 > 0$ and $1 \otimes 0 \geq 1 \otimes \mathcal{H} > 0$, so we have

$$\mathcal{Q}_{\zeta}(\mathcal{U} \otimes 1, 1 \otimes 0) \leq \mathcal{Q}_{\zeta}(\mathcal{G} \otimes 1, 1 \otimes \mathcal{H}).$$

□

Tensorial Maclaurin Approximation Using Different Types of Generalize Convex Mappings

Lemma 4. *Assume that ζ is of class $\mathcal{C}^{\mathtt{n}+1}$ on the open interval Δ. Let $\mathcal{U}$ and $\mathcal{V}$ be self-adjoint operators with respective spectrums, $\mathcal{SP}(\mathcal{U}) \subset \Delta$ and $\mathcal{SP}(\mathcal{V}) \subset \Delta$, then one has*

$$\begin{aligned}\zeta(\mathcal{U}) \otimes 1 = &\sum_{\mathtt{k}=0}^{\mathtt{n}} \frac{1}{\mathtt{k}!}(\mathcal{U} \otimes 1 - 1 \otimes 0)^{\mathtt{k}}\Big(1 \otimes \zeta^{(\mathtt{k})}(0)\Big) \\ &+ \frac{1}{\mathtt{n}!}(\mathcal{U} \otimes 1 - 1 \otimes 0)^{\mathtt{n}+1} \\ &\times \int_0^1 (1-\mathtt{u})^{\mathtt{n}} \zeta^{(\mathtt{n}+1)}((1-\mathtt{u})1 \otimes 0 + \mathtt{u}\mathcal{U} \otimes 1)\mathtt{du} \qquad (4)\end{aligned}$$

Proof. Using Maclaurin's representation with the integral remainder, we can express the following two identities:

$$\zeta(\mu) = \sum_{k=0}^{n} \frac{1}{k!}\zeta^{(k)}(\mathcal{U})(\mu - 0)^k + \frac{1}{n!}\int_0^{\mu} \zeta^{(n+1)}(\nu)(\mu - \nu)^n d\nu$$

for $\mu \in \Delta$.

Using the substitution $\nu = (1-u)\sigma_1 + u\sigma_2, u \in [0,1]$ and for any continous function φ with distinct values σ_1, σ_2 in that interval, one has

$$\int_{\sigma_1}^{\sigma_2} \varphi(\nu)d\nu = (\sigma_2 - \sigma_1)\int_0^1 \varphi((1-u)\sigma_1 + u\sigma_2)du.$$

Therefore,

$$\begin{aligned}
&\int_0^{\mu} \zeta^{(n+1)}(\nu)(\mu - \nu)^n d\nu \\
&= (\mu - 0)\int_0^1 \zeta^{(n+1)}((1-u)0 + ux)(x - (1-u)0 - ux)^n du \\
&= (\mu - 0)^{n+1}\int_0^1 \zeta^{(n+1)}((1-u)0 + ux)(1-u)^n du
\end{aligned}$$

and it follows that

$$\begin{aligned}
\zeta(\tau) = &\sum_{k=0}^{n} \frac{1}{k!}\zeta^{(k)}(0)(\tau - 0)^k \\
&+ \frac{1}{n!}(\tau - 0)^{n+1}\int_0^1 \zeta^{(n+1)}((1-u)0 + u\tau)(1-u)^n du
\end{aligned}$$

for all $\tau, v \in \Delta$. If $\mathcal{U}$ and $\mathcal{V}$ have the spectral resolutions

$$\mathcal{U} = \int_{\Delta} \tau dE(\tau) \text{ and } \mathcal{V} = \int_{\Delta} v dF(v),$$

and after that, we obtain the integral applying $\int_{\Delta}\int_{\Delta}$ over $dF_\tau \otimes dF_v$, that is

$$\begin{aligned}
&\int_{\Delta}\int_{\Delta} \zeta(\tau) dF_\tau \otimes dF_v \\
&= \int_{\Delta}\int_{\Delta} \left(\sum_{k=0}^{n} \frac{1}{k!}\zeta^{(k)}(0)(\tau - 0)^k\right) dF_\tau \otimes dF_v \\
&+ \frac{1}{n!}\int_{\Delta}\int_{\Delta} (\tau - 0)^{n+1}\left(\int_0^1 \zeta^{(n+1)}((1-u)0 + u\tau)(1-u)^n du\right) dF_\tau \otimes dF_v.
\end{aligned}$$

We have

$$\int_{\Delta}\int_{\Delta} \zeta(\tau) dF_\tau \otimes dF_v = \zeta(\mathcal{U}) \otimes 1$$

and, by Lemma 3, we have

$$\begin{aligned}
&\int_{\Delta}\int_{\Delta} \left(\sum_{k=0}^{n} \frac{1}{k!}\zeta^{(k)}(0)(\tau - 0)^k\right) dF_\tau \otimes dF_v \\
&= \sum_{k=0}^{n} \frac{1}{k!}\int_{\Delta}\int_{\Delta} \left((\tau - 0)^k \zeta^{(k)}(0)\right) dF_\tau \otimes dF_v \\
&= \sum_{k=0}^{n} \frac{1}{k!}(\mathcal{U} \otimes 1 - 1 \otimes 0)^k \left(1 \otimes \zeta^{(k)}(0)\right).
\end{aligned}$$

Using Fubini's theorem, we obtain

$$\begin{aligned}
&\int_{\Delta}\int_{\Delta}(\tau-0)^{n+1}\left(\int_0^1 \zeta^{(n+1)}((1-u)0+u\tau)(1-u)^n du\right)dF_\tau\otimes dF_v\\
&=\int_0^1(1-u)^n\left(\int_{\Delta}\int_{\Delta}(\tau-0)^{n+1}\left(\zeta^{(n+1)}((1-u)0+u\tau)\right)dF_\tau\otimes dF_v\right)du\\
&=(\mathcal{U}\otimes 1-1\otimes 0)^{n+1}\int_0^1(1-u)^n\left(\zeta^{(n+1)}((1-u)1\otimes 0+u\mathcal{U}\otimes 1)\right)du,
\end{aligned}$$

and so we obtain the required outcome. □

Theorem 8. *Assume that ζ is of class $\mathcal{C}^{2\wp+1}$ on the open interval Δ such that $\alpha_{2\wp+1}\le\zeta^{(2\wp+1)}\le\beta_{2\wp+1}$ for some constants $\alpha_{2\wp+1},\beta_{2\wp+1}$. Let $\mathcal{U}$ and $\mathcal{V}$ be self-adjoint operators with respective spectrums, $\mathcal{SP}(\mathcal{U})\subset\Delta$ and $\mathcal{SP}(\mathcal{V})\subset\Delta$, then one has*

$$\begin{aligned}
&\frac{1}{(2\wp+1)!}(\mathcal{U}\otimes 1-1\otimes 0)^{2\wp+1}\alpha_{2\wp+1}\\
&\le\zeta(\mathcal{U})\otimes 1-\sum_{k=0}^{2\wp}\frac{1}{k!}(\mathcal{U}\otimes 1-1\otimes 0)^k\left(1\otimes\zeta^{(k)}(0)\right)\\
&\le\frac{1}{(2\wp+1)!}(\mathcal{U}\otimes 1-1\otimes 0)^{2\wp+1}\beta_{2\wp+1}.
\end{aligned}\tag{5}$$

Proof. For $n=2\wp$ with $\wp\ge 1$ in (4), we have

$$\begin{aligned}
&\zeta(\mathcal{U})\otimes 1\\
&=\sum_{k=0}^{2\wp}\frac{1}{k!}(\mathcal{U}\otimes 1-1\otimes 0)^k\left(1\otimes\zeta^{(k)}(0)\right)\\
&+\frac{1}{(2\wp)!}(\mathcal{U}\otimes 1-1\otimes 0)^{2\wp+1}\\
&\times\int_0^1(1-u)^{2\wp}\zeta^{(2\wp+1)}((1-u)1\otimes 0+u\mathcal{U}\otimes 1)du.
\end{aligned}\tag{6}$$

Based on the boundedness assumption, we have that

$$\alpha_{2\wp+1}\le\zeta^{(2\wp+1)}((1-u)0+ut)\le\beta_{2\wp+1},\tag{7}$$

for all $t\in\Delta$ and $u\in[0,1]$.

Multiplying the inequality (7) by $(t-0)^{2\wp+1}\ge 0$ with $t\in\Delta$ yields

$$\alpha_{2\wp+1}(t-0)^{2\wp+1}\le(t-0)^{2\wp+1}\zeta^{(2\wp+1)}((1-u)0+ut)\le\beta_{2\wp+1}(t-0)^{2\wp+1}.\tag{8}$$

By taking the integral $\int_{\Delta}\int_{\Delta}$ over $dE_t\otimes dF_s$ in (8), we obtain

$$\begin{aligned}
&\alpha_{2\wp+1}\int_{\Delta}\int_{\Delta}(t-0)^{2\wp+1}dE_t\otimes dF_s\\
&\le\int_{\Delta}\int_{\Delta}(t-0)^{2\wp+1}\zeta^{(2\wp+1)}((1-u)0+ut)dE_t\otimes dF_s\\
&\le\beta_{2\wp+1}\int_{\Delta}\int_{\Delta}(t-0)^{2\wp+1}dE_t\otimes dF_s.
\end{aligned}$$

Using Lemma 4, we derive

$$\begin{aligned}&\alpha_{2\wp+1}(\mathcal{U}\otimes 1-1\otimes 0)^{2\wp+1}\\&\leq(\mathcal{U}\otimes 1-1\otimes 0)^{2\wp+1}\zeta^{(2\wp+1)}((1-\mathtt{u})1\otimes 0+\mathtt{u}\mathcal{U}\otimes 1)\\&\leq\beta_{2\wp+1}(\mathcal{U}\otimes 1-1\otimes 0)^{2\wp+1}.\end{aligned}$$

In addition, if we multiply this inequality by $(1-\mathtt{u})^{2\wp+1}\geq 0$, $\mathtt{u}\in[0,1]$ and integrate, we obtain

$$\begin{aligned}&\alpha_{2\wp+1}\int_0^1(1-\mathtt{u})^{2\wp+1}\mathtt{du}(\mathcal{U}\otimes 1-1\otimes 0)^{2\wp+1}\\&\leq\int_0^1(1-\mathtt{u})^{2\wp+1}(\mathcal{U}\otimes 1-1\otimes 0)^{2\wp+1}\zeta^{(2\wp+1)}((1-\mathtt{u})1\otimes 0+\mathtt{u}\mathcal{U}\otimes 1)\mathtt{du}\\&\leq\beta_{2\wp+1}\int_0^1(1-\mathtt{u})^{2\wp+1}\mathtt{du}(\mathcal{U}\otimes 1-1\otimes 0)^{2\wp+1},\end{aligned}$$

which gives that

$$\begin{aligned}&\frac{\alpha_{2\wp+1}}{2\wp+1}(\mathcal{U}\otimes 1-1\otimes 0)^{2\wp+1}\\&\leq(\mathcal{U}\otimes 1-1\otimes 0)^{2\wp+1}\int_0^1(1-\mathtt{u})^{2\wp+1}\zeta^{(2\wp+1)}((1-\mathtt{u})1\otimes 0+\mathtt{u}\mathcal{U}\otimes 1)\mathtt{du}\\&\leq\frac{\beta_{2\wp+1}}{2\wp+1}(\mathcal{U}\otimes 1-1\otimes 0)^{2\wp+1},\end{aligned}$$

namely

$$\begin{aligned}&\frac{\alpha_{2\wp+1}}{(2\wp+1)!}(\mathcal{U}\otimes 1-1\otimes 0)^{2\wp+1}\\&\leq\frac{1}{(2\wp)!}(\mathcal{U}\otimes 1-1\otimes 0)^{2\wp+1}\times\int_0^1(1-\mathtt{u})^{2\wp+1}\zeta^{(2\wp+1)}((1-\mathtt{u})1\otimes 0+\mathtt{u}\mathcal{U}\otimes 1)\mathtt{du}\\&\leq\frac{\beta_{2\wp+1}}{(2\wp+1)!}(\mathcal{U}\otimes 1-1\otimes 0)^{2\wp+1}.\end{aligned}$$

By utilizing identity (6), we derive the desired inequality. □

Theorem 9. *Assume that ζ is of class $\mathcal{C}^{2\wp+1}$ on the open interval Δ such that $\zeta^{2\wp+1}$ is a convex function on Δ. Let $\mathcal{U}$ and $\mathcal{V}$ be self-adjoint operators with respective spectrums, $\mathcal{SP}(\mathcal{U})\subset\Delta$ and $\mathcal{SP}(\mathcal{V})\subset\Delta$, then one has*

$$\begin{aligned}&\zeta(\mathcal{U})\otimes 1-\sum_{\mathtt{k}=0}^{2\wp}\frac{1}{\mathtt{k}!}(\mathcal{U}\otimes 1-1\otimes 0)^{\mathtt{k}}\Big(1\otimes\zeta^{(\mathtt{k})}(0)\Big)\\&\leq\frac{1}{(2\wp+1)!}(\mathcal{U}\otimes 1-1\otimes 0)^{2\wp+1}\left[\frac{2\wp+1\Big(1\otimes\zeta^{(2\wp+1)}(0)\Big)+\zeta^{(2\wp+1)}(\mathcal{U})\otimes 1}{2\wp+2}\right].\end{aligned}\tag{9}$$

Proof. Since $\zeta^{(2\wp+1)}$ is convex on Δ, then

$$\zeta^{(2\wp+1)}((1-\mathtt{u})0+\mathtt{ut})\leq(1-\mathtt{u})\zeta^{(2\wp+1)}(0)+\mathtt{u}\zeta^{(2\wp+1)}(\mathtt{t}),\tag{10}$$

for all $\mathtt{t}\in\Delta$ and $\mathtt{u}\in[0,1]$.

Now, if we multiply the above inequality by $(\mathtt{t}-0)^{2\wp+1}\geq 0$ with $\mathtt{t}\in\Delta$, then we have

$$(\mathtt{t}-0)^{2\wp+1}\zeta^{(2\wp+1)}((1-\mathtt{u})0+\mathtt{ut})$$

$$\leq (1-\mathtt{u})(\mathtt{t}-0)^{2\wp+1}\zeta^{(2\wp+1)}(0)+\mathtt{u}(\mathtt{t}-0)^{2\wp+1}\zeta^{(2\wp+1)}(\mathtt{t}), \tag{11}$$

for all $\mathtt{t}\in\Delta$ and $\mathtt{u}\in[0,1]$.

By applying the integral $\int_\Delta\int_\Delta$ over $\mathtt{dE_t}\otimes\mathtt{dF_s}$ in (11), we obtain

$$\begin{aligned}&\int_\Delta\int_\Delta(\mathtt{t}-0)^{2\wp+1}\zeta^{(2\wp+1)}((1-\mathtt{u})0+\mathtt{ut})\mathtt{dE_t}\otimes\mathtt{dF_s}\\&\leq\int_\Delta\int_\Delta\Big[(1-\mathtt{u})(\mathtt{t}-0)^{2\wp+1}\zeta^{(2\wp+1)}(0)+\mathtt{u}(\mathtt{t}-0)^{2\wp+1}\zeta^{(2\wp+1)}(\mathtt{t})\Big]\mathtt{dE_t}\otimes\mathtt{dF_s},\end{aligned} \tag{12}$$

for all $\mathtt{u}\in[0,1]$. Using Lemma 4, we conclude that

$$\begin{aligned}&\int_\Delta\int_\Delta(\mathtt{t}-0)^{2\wp+1}\zeta^{(2\wp+1)}((1-\mathtt{u})0+\mathtt{ut})\mathtt{dE_t}\otimes\mathtt{dF_s}\\&=(\mathcal{U}\otimes 1-1\otimes 0)^{2\wp+1}\zeta^{(2\wp+1)}((1-\mathtt{u})1\otimes 0+\mathtt{u}\mathcal{U}\otimes 1)\end{aligned}$$

and

$$\begin{aligned}&\int_\Delta\int_\Delta\Big[(1-\mathtt{u})(\mathtt{t}-0)^{2\wp+1}\zeta^{(2\wp+1)}(0)+\mathtt{u}(\mathtt{t}-0)^{2\wp+1}\zeta^{(2\wp+1)}(\mathtt{t})\Big]\mathtt{dE_t}\otimes\mathtt{dF_s}\\&=(1-\mathtt{u})\int_\Delta\int_\Delta(\mathtt{t}-0)^{2\wp+1}\zeta^{(2\wp+1)}(0)\mathtt{dE_t}\otimes\mathtt{dF_s}\\&+\mathtt{u}\int_\Delta\int_\Delta(\mathtt{t}-0)^{2\wp+1}\zeta^{(2\wp+1)}(\mathtt{t})\mathtt{dE_t}\otimes\mathtt{dF_s}\\&=(1-\mathtt{u})(\mathcal{U}\otimes 1-1\otimes 0)^{2\wp+1}\Big(1\otimes\zeta^{(2\wp+1)}(0)\Big)\\&+\mathtt{u}(\mathcal{U}\otimes 1-1\otimes 0)^{2\wp+1}\Big(\zeta^{(2\wp+1)}(\mathcal{U})\otimes 1\Big)\end{aligned}$$

and by (12), we obtain

$$\begin{aligned}&(\mathcal{U}\otimes 1-1\otimes 0)^{2\wp+1}\zeta^{(2\wp+1)}((1-\mathtt{u})1\otimes 0+\mathtt{u}\mathcal{U}\otimes 1)\\&\leq(1-\mathtt{u})(\mathcal{U}\otimes 1-1\otimes 0)^{2\wp+1}\Big(1\otimes\zeta^{(2\wp+1)}(0)\Big)\\&+\mathtt{u}(\mathcal{U}\otimes 1-1\otimes 0)^{2\wp+1}\Big(\zeta^{(2\wp+1)}(\mathcal{U})\otimes 1\Big),\end{aligned}$$

for all $\mathtt{u}\in[0,1]$. If we multiply the above inequality by $(1-\mathtt{u})^{2\wp}$ and integrate, then we have

$$\begin{aligned}&(\mathcal{U}\otimes 1-1\otimes 0)^{2\wp+1}\int_0^1(1-\mathtt{u})^{2\wp}\zeta^{(2\wp+1)}((1-\mathtt{u})1\otimes 0+\mathtt{u}\mathcal{U}\otimes 1)\mathtt{du}\\&\leq\int_0^1(1-\mathtt{u})^{2\wp+1}\mathtt{du}(\mathcal{U}\otimes 1-1\otimes 0)^{2\wp+1}\Big(1\otimes\zeta^{(2\wp+1)}(0)\Big)\\&+\int_0^1\mathtt{u}(1-\mathtt{u})^{2\wp}\mathtt{du}(\mathcal{U}\otimes 1-1\otimes 0)^{2\wp+1}\Big(\zeta^{(2\wp+1)}(\mathcal{U})\otimes 1\Big)\\&=\frac{1}{2\wp+2}(\mathcal{U}\otimes 1-1\otimes 0)^{2\wp+1}\Big(1\otimes\zeta^{(2\wp+1)}(0)\Big)\\&+\frac{1}{2\wp(2\wp+2)}(\mathcal{U}\otimes 1-1\otimes 0)^{2\wp+1}\Big(\zeta^{(2\wp+1)}(\mathcal{U})\otimes 1\Big)\\&=\frac{1}{2\wp+2}(\mathcal{U}\otimes 1-1\otimes 0)^{2\wp+1}\left[1\otimes\zeta^{(2\wp+1)}0)+\frac{1}{2\wp+1}\zeta^{(2\wp+1)}(\mathcal{U})\otimes 1\right]\\&=\frac{1}{2\wp+1}(\mathcal{U}\otimes 1-1\otimes 0)^{2\wp+1}\left[\frac{2\wp 1\otimes\zeta^{(2\wp+1)}0)+\zeta^{(2\wp+1)}(\mathcal{U})\otimes 1}{2\wp+2}\right].\end{aligned}$$

If we multiply by $\frac{1}{(2\wp)!}$, we obtain

$$\begin{aligned}
&\frac{1}{(2\wp)!}(\mathcal{U}\otimes 1-1\otimes 0)^{2\wp+1}\\
&\times\int_0^1(1-\mathtt{u})^{2\wp}\zeta^{(2\wp+1)}((1-\mathtt{u})1\otimes 0+\mathtt{u}\mathcal{U}\otimes 1)\mathtt{du}\\
&\leq\frac{1}{(2\wp+1)!}(\mathcal{U}\otimes 1-1\otimes 0)^{2\wp+1}\left[\frac{(2\wp+1)1\otimes\zeta^{(2\wp+1)}0)+\zeta^{(2\wp+1)}(\mathcal{U})\otimes 1}{2\wp+2}\right],
\end{aligned}$$

and this concludes the proof. □

Theorem 10. *Assume that ζ is of class $\mathcal{C}^{2\wp+1}$ on the open interval Δ such that $\zeta^{2\wp+1}$ is a quasi-convex function on Δ. Let $\mathcal{U}$ and $\mathcal{V}$ be self-adjoint operators with respective spectrums, $\mathcal{SP}(\mathcal{U})\subset\Delta$ and $\mathcal{SP}(\mathcal{V})\subset\Delta$, then one has*

$$\begin{aligned}
&\zeta(\mathcal{U})\otimes 1-\sum_{\mathtt{k}=0}^{2\wp}\frac{1}{\mathtt{k}!}(\mathcal{U}\otimes 1-1\otimes 0)^{\mathtt{k}}\Big(1\otimes\zeta^{(\mathtt{k})}(0)\Big)\\
&\leq\frac{1}{(2\wp+1)!}(\mathcal{U}\otimes 1-1\otimes 0)^{2\wp+1}\\
&\times\Big[\zeta^{(2\wp+1)}(\mathcal{U})\otimes 1+1\otimes\zeta^{(2\wp+1)}(0)+\Big|\zeta^{(2\wp+1)}(\mathcal{U})\otimes 1-1\otimes\zeta^{(2\wp+1)}(0)\Big|\Big].
\end{aligned}$$

Proof. Since $\zeta^{(2\wp+1)}$ is quasi-convex on Δ, then

$$\zeta^{(2\wp+1)}((1-\mathtt{u})0+\mathtt{ut})\leq\frac{1}{2}\Big(\zeta^{(2\wp+1)}(\mathtt{t})+\zeta^{(2\wp+1)}(0)+\Big|\zeta^{(2\wp+1)}(\mathtt{t})-\zeta^{(2\wp+1)}(0)\Big|\Big),$$

for all $\mathtt{t}\in\Delta$ and $\mathtt{u}\in[0,1]$. If we multiply by $(\mathtt{t}-0)^{2\wp+1}$, we obtain

$$\begin{aligned}
&(\mathtt{t}-0)^{2\wp+1}\zeta^{(2\wp+1)}((1-\mathtt{u})0+\mathtt{ut})\\
&\leq\frac{1}{2}(\mathtt{t}-0)^{2\wp+1}\Big(\zeta^{(2\wp+1)}(\mathtt{t})+\zeta^{(2\wp+1)}(0)+\Big|\zeta^{(2\wp+1)}(\mathtt{t})-\zeta^{(2\wp+1)}(0)\Big|\Big),
\end{aligned}\tag{13}$$

for all $\mathtt{t}\in\Delta$ and $\mathtt{u}\in[0,1]$. By taking the integral $\int_\Delta\int_\Delta$ over $\mathtt{dE_t}\otimes\mathtt{dF_s}$ in (13), we obtain

$$\begin{aligned}
&\int_\Delta\int_\Delta(\mathtt{t}-0)^{2\wp+1}\zeta^{(2\wp+1)}((1-\mathtt{u})0+\mathtt{ut})\mathtt{dE_t}\otimes\mathtt{dF_s}\\
&\leq\frac{1}{2}\int_\Delta\int_\Delta(\mathtt{t}-0)^{2\wp+1}\Big(\zeta^{(2\wp+1)}(\mathtt{t})+\zeta^{(2\wp+1)}(0)+\Big|\zeta^{(2\wp+1)}(\mathtt{t})-\zeta^{(2\wp+1)}(0)\Big|\Big)\\
&\times\mathtt{dE_t}\otimes\mathtt{dF_s}.
\end{aligned}\tag{14}$$

Since, by Lemma 4,

$$\begin{aligned}
&\int_\Delta\int_\Delta(\mathtt{t}-0)^{2\wp+1}\zeta^{(2\wp+1)}((1-\mathtt{u})0+\mathtt{ut})\mathtt{dE_t}\otimes\mathtt{dF_s}\\
&=(\mathcal{U}\otimes 1-1\otimes 0)^{2\wp+1}\zeta^{(2\wp+1)}((1-\mathtt{u})1\otimes 0+\mathtt{u}\mathcal{U}\otimes 1)
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\Delta}\int_{\Delta}(\mathtt{t}-0)^{2\wp+1}\Big(\zeta^{(2\wp+1)}(\mathtt{t})+\zeta^{(2\wp+1)}(0)+\Big|\zeta^{(2\wp+1)}(\mathtt{t})-\zeta^{(2\wp+1)}(0)\Big|\Big)\mathtt{dE_t}\otimes\mathtt{dF_s}\\
&=\int_{\Delta}\int_{\Delta}(\mathtt{t}-0)^{2\wp+1}\zeta^{(2\wp+1)}(\mathtt{t})\mathtt{dE_t}\otimes\mathtt{dF_s}+\int_{\Delta}\int_{\Delta}(\mathtt{t}-0)^{2\wp+1}\zeta^{(2\wp+1)}(0)\mathtt{dE_t}\otimes\mathtt{dF_s}\\
&+\int_{\Delta}\int_{\Delta}(\mathtt{t}-0)^{2\wp+1}\Big|\zeta^{(2\wp+1)}(\mathtt{t})-\zeta^{(2\wp+1)}(0)\Big|\mathtt{dE_t}\otimes\mathtt{dF_s}\\
&=(\mathcal{U}\otimes 1-1\otimes 0)^{2\wp+1}\zeta^{(2\wp+1)}(\mathcal{U})\otimes 1+(\mathcal{U}\otimes 1-1\otimes 0)^{2\wp+1}\Big(1\otimes\zeta^{(2\wp+1)}(0)\Big)\\
&+(\mathcal{U}\otimes 1-1\otimes 0)^{2\wp+1}\Big|\zeta^{(2\wp+1)}(\mathcal{U})\otimes 1-1\otimes\zeta^{(2\wp+1)}(0)\Big|\\
&=(\mathcal{U}\otimes 1-1\otimes 0)^{2\wp+1}\\
&\times\Big[\zeta^{(2\wp+1)}(\mathcal{U})\otimes 1+1\otimes\zeta^{(2\wp+1)}(0)+\Big|\zeta^{(2\wp+1)}(\mathcal{U})\otimes 1-1\otimes\zeta^{(2\wp+1)}(0)\Big|\Big],
\end{aligned}$$

then by (14), we obtain

$$\begin{aligned}
&(\mathcal{U}\otimes 1-1\otimes 0)^{2\wp+1}\zeta^{(2\wp+1)}((1-\mathtt{u})1\otimes 0+\mathtt{u}\mathcal{U}\otimes 1)\\
&\leq(\mathcal{U}\otimes 1-1\otimes 0)^{2\wp+1}\\
&\times\Big[\zeta^{(2\wp+1)}(\mathcal{U})\otimes 1+1\otimes\zeta^{(2\wp+1)}(0)+\Big|\zeta^{(2\wp+1)}(\mathcal{U})\otimes 1-1\otimes\zeta^{(2\wp+1)}(0)\Big|\Big],
\end{aligned}$$

for $\mathtt{u}\in[0,1]$. If we multiply the above inequality by $(1-\mathtt{u})^{2\wp}$ and integrate, then we obtain

$$\begin{aligned}
&(\mathcal{U}\otimes 1-1\otimes 0)^{2\wp+1}\int_0^1(1-\mathtt{u})^{2\wp}\zeta^{(2\wp+1)}((1-\mathtt{u})1\otimes 0+\mathtt{u}\mathcal{U}\otimes 1)\mathtt{du}\\
&\leq\frac{1}{2\wp+1}(\mathcal{U}\otimes 1-1\otimes 0)^{2\wp+1}\\
&\times\Big[\zeta^{(2\wp+1)}(\mathcal{U})\otimes 1+1\otimes\zeta^{(2\wp+1)}(0)+\Big|\zeta^{(2\wp+1)}(\mathcal{U})\otimes 1-1\otimes\zeta^{(2\wp+1)}(0)\Big|\Big],
\end{aligned}$$

and the proof is now complete. □

Example 1. *Under the identical hypothesis of Theorem 8, and if* $\zeta(\mu)=\exp(\mu)$, *then by (6), we obtain*

$$\begin{aligned}
\exp(\mathcal{U})\otimes 1=&\sum_{\mathtt{k}=0}^{2\wp-1}\frac{1}{\mathtt{k}!}(\mathcal{U}\otimes 1-1\otimes 0)^{\mathtt{k}}(1\otimes\exp(0))\\
&+\frac{1}{(2\wp-1)!}(\mathcal{U}\otimes 1-1\otimes 0)^{2\wp}\\
&\times\int_0^1(1-\mathtt{u})^{2\wp-1}\exp((1-\mathtt{u})1\otimes 0+\mathtt{u}\mathcal{U}\otimes 1)\mathtt{du},
\end{aligned}\tag{15}$$

for any self-adjoint operators $\mathcal{U},\mathcal{V}$. *If* $\alpha_{2\wp+1}\leq\mathcal{U},\mathcal{V}\leq\beta_{2\wp+1}$, *then by (5), we obtain*

$$\begin{aligned}
&\frac{1}{(2\wp)!}\exp(\alpha_{2\wp+1})(\mathcal{U}\otimes 1-1\otimes 0)^{2\wp}\\
&\leq\exp(\mathcal{U})\otimes 1-\sum_{\mathtt{k}=0}^{2\wp-1}\frac{1}{\mathtt{k}!}(\mathcal{U}\otimes 1-1\otimes 0)^{\mathtt{k}}(1\otimes\exp(0))\\
&\leq\frac{1}{(2\wp)!}\exp(\beta_{2\wp+1})(\mathcal{U}\otimes 1-1\otimes 0)^{2\wp}
\end{aligned}\tag{16}$$

From (9), we obtain

$$\exp(\mathcal{U}) \otimes 1 - \sum_{k=0}^{2\wp-1} \frac{1}{k!}(\mathcal{U} \otimes 1 - 1 \otimes 0)^k (1 \otimes \exp(0))$$
$$\leq \frac{1}{(2\wp)!}(\mathcal{U} \otimes 1 - 1 \otimes 0)^{2\wp} \left[\frac{2\wp(1 \otimes \exp(0)) + \exp(\mathcal{U}) \otimes 1}{2\wp + 1}\right]. \tag{17}$$

for any self-adjoint operators $\mathcal{U}, \mathcal{V}$. If $w, t > 0$ and if we take in (15), $\mathcal{U} = \ln w, \mathcal{U} = \ln t$, then we have

$$w \otimes 1 = \sum_{k=0}^{2\wp-1} \frac{1}{k!}(\ln w \otimes 1 - 1 \otimes \ln t)^k (1 \otimes t)$$
$$+ \frac{1}{(2\wp-1)!}(\ln w \otimes 1 - 1 \otimes \ln t)^{2\wp}$$
$$\times \int_0^1 (1-u)^{2\wp-1} \exp((1-u)1 \otimes \ln t + u \ln w \otimes 1) du.$$

and for $\wp = 1$, we have

$$w \otimes 1 = (1 \otimes t) + (\ln w \otimes 1 - 1 \otimes \ln t)(1 \otimes t)$$
$$+ (\ln w \otimes 1 - 1 \otimes \ln t)^2$$
$$\times \int_0^1 (1-u) \exp((1-u)1 \otimes \ln t + u \ln w \otimes 1) du.$$

If $0 < \psi \leq w, t \leq \Psi$, then by (16), we obtain

$$\frac{1}{(2\wp)!}\psi(\ln w \otimes 1 - 1 \otimes \ln t)^{2\wp}$$
$$\leq w \otimes 1 - \sum_{k=0}^{2\wp-1} \frac{1}{k!}(\ln w \otimes 1 - 1 \otimes \ln t)^k (1 \otimes t)$$
$$\leq \frac{1}{(2\wp)!}\Psi(\ln w \otimes 1 - 1 \otimes \ln t)^{2\wp}$$

and for $\wp = 1$, we have

$$\frac{1}{2}\psi(\ln w \otimes 1 - 1 \otimes \ln t)^2 \leq w \otimes 1 - 1 \otimes t - (\ln w \otimes 1 - 1 \otimes \ln t)(1 \otimes t)$$
$$\leq \frac{1}{2}\Psi(\ln w \otimes 1 - 1 \otimes \ln t)^2. \tag{18}$$

From (17), we obtain

$$w \otimes 1 - \sum_{k=0}^{2\wp-1} \frac{1}{k!}(\ln w \otimes 1 - 1 \otimes \ln t)^k (1 \otimes t)$$
$$\leq \frac{1}{(2\wp)!}(\ln w \otimes 1 - 1 \otimes \ln t)^{2\wp} \left[\frac{2\wp(1 \otimes t) + w \otimes 1}{2\wp + 1}\right]$$

and for $\wp = 1$,

$$w \otimes 1 - 1 \otimes t - (\ln w \otimes 1 - 1 \otimes \ln t)(1 \otimes t)$$
$$\leq \frac{1}{2}(\ln w \otimes 1 - 1 \otimes \ln t)^2 \left[\frac{2(1 \otimes t) + w \otimes 1}{3}\right],$$

for all $w, t > 0$. *Using the commutativity of* $w \otimes 1$ *and* $1 \otimes t$, *we may deduce from (18) that*

$$\begin{aligned}&\frac{1}{2}\psi(\ln w \otimes 1 - 1 \otimes \ln t)^2\left(1 \otimes t^{-1}\right)\\&\leq 1 \otimes \ln t - 1 - \ln w \otimes 1 + w \otimes t^{-1}\\&\leq \frac{1}{2}\Psi(\ln w \otimes 1 - 1 \otimes \ln t)^2\left(1 \otimes t^{-1}\right),\end{aligned}$$

provided that $0 < \psi \leq w, t \leq \Psi$.

4. Preliminary Results

In this section, we will recall some fundamental definitions and conclusions related to mixed norm function spaces with varying exponents. Reference [79] contains the majority of the properties linked to function spaces, whereas mixed Lebesgue spaces are covered in the following articles [39,80,81].

4.1. Semi-Modular Spaces

The variable Lebesgue spaces are categorized as semi-modular spaces, which is a generalization of a normed space. For this purpose, we first review certain modular space definitions and results.

Definition 6. *Let* $\mathcal{M}$ *be a vector space over some field* $\mathcal{K}$ *(real or complex). The mapping* $\rho : \mathcal{M} \longrightarrow [0, \infty]$ *is said to be semi-modular on* $\mathcal{M}$ *if*

- $\rho(0) = 0$;
- $\rho(\alpha\mu) = \rho(\mu)$ *for all* $\mu \in \mathcal{M}$ *and* $\alpha \in \mathcal{K}, |\alpha| = 1$;
- *if* $\rho(\alpha\mu) = 0$ *for all* $\alpha > 0$, *then* $\mu = 0$;
- *function* $[0, \infty) \ni \alpha \mapsto \rho(\alpha\mu)$ *is left-continuous for every* $\mu \in \mathcal{M}$;
- *function* $[0, \infty) \ni \alpha \mapsto \rho(\alpha\mu)$ *is non-decreasing for every* $\mu \in \mathcal{M}$.

If a vector space $\mathcal{M}$ has ρ as a semi-modular on it, then the space

$$\mathcal{M}_\rho := \{\mu \in \mathcal{M} : \exists_{\alpha>0}\rho(\alpha\mu) < \infty\},$$

known as a semi-modular space, is connected to the functional $\|\cdot\|_\rho : \mathcal{K}_\rho \to [0, \infty]$ and is defined as follows:

$$\|\mathbf{s}\|_\rho := \inf\{\alpha > 0 : \rho(\mathbf{s}/\alpha) \leq 1\} \text{ for } \mathbf{s} \in \mathcal{K}_\rho.$$

Proposition 1. *Suppose that on vector space* $\mathcal{M}$, ρ *is semi-modular. Consequently, for each* $\mu \in \mathcal{M}$, *we have*

$$\rho(\mu) \leq 1 \Longleftrightarrow \|\mu\|_\rho \leq 1.$$

Proof. Assume that $\rho(\mu) \leq 1$. Then, $\|\mu\|_\rho \leq 1$ is inferred from the definition of the norm $\|\cdot\|_\rho$. Conversely, if we assume that $\|\mu\|_\rho \leq 1$, then for each $\alpha > 1$ and due to the left continuity of the function, we have

$$\rho(\mu/\alpha) \leq 1.$$

□

4.2. Variable Exponent Spaces

Now let us review the idea of $L^{p(\cdot)}$ space. To do this, we take into consideration that $\Delta \subset \Omega$ is a measurable set and we define the measurable function $p : \Delta \longrightarrow (0, \infty]$, which is normally described as follows:

$$p_+ = \operatorname*{ess\,sup}_{\kappa \in \Delta} p(\kappa), \quad p_- = \operatorname*{ess\,inf}_{\kappa \in \Delta} p(\kappa).$$

We refer to any Lebesgue measurable functions $p : \Delta \longrightarrow (0, \infty)]$ as variable exponents such that $p_- > 0$. For the set Δ, $\mathcal{P}_0(\Delta)$ represents the space of all varying exponents functions. The notation $\mathcal{P}(\Delta)$ marks the pack of all varying exponent functions p such that $p_- \geq 1$.

Consider the vector space $L^0(\Delta)$, which contains all measurable functions over Δ. For $p \in \mathcal{P}_0(\Delta)$, over the space $L^0(\Delta)$, we define the semi-modular as follows:

$$\rho_{p(\cdot)}(\zeta) := \int_\Delta \Psi_{p(\mu)}(|\zeta(\mu)|) d\mu$$

where

$$\Psi_p(\sigma) := \begin{cases} \sigma^p, & \text{if } 0 < p < \infty, \\ 0, & \text{if } \sigma \leq 1, p = \infty, \\ \infty, & \text{if } \sigma > 1, p = \infty \end{cases}$$

for each $\sigma \geq 0$. This modular space is described by the semi-modular $\rho_{p(\cdot)}$, which in turn defines the $L^{p(\cdot)}(\Delta)$ space as follows:

$$\zeta \in L^{p(\cdot)}(\Delta) \Longleftrightarrow \exists_{\alpha > 0} \rho_{p(\cdot)}(\alpha\zeta) = \int_\Delta \varphi_{p(\mu)}(\alpha|\zeta(\mu)|) d\mu < \infty.$$

The following Luxemburg quasi-norm makes the $L^{p(\cdot)}(\Delta)$ space a quasi-Banach space:

$$\|\zeta\|_{L^{p(\cdot)}(\Delta)} := \inf\Big\{\alpha > 0 : \rho_{p(\cdot)}(\zeta/\alpha) \leq 1\Big\} \text{ for } \zeta \in L^{p(\cdot)}(\Delta).$$

It is the particular case of the Musielak–Orlicz space, and whenever $p \in \mathcal{P}(\Delta)$, then $L^{p(\cdot)}(\Delta)$ becomes Banach space. When the exponent is constant, it reduces to the classical Lebesgue space.

Notably, the definition of $L^{p(\cdot)}(\Delta)$ space becomes simpler if $p_+ < \infty$ so that $\zeta \in L^{p(\cdot)}(\Delta)$ if, and only if

$$\int_\Delta |\zeta(\mu)|^{p(\mu)} d\mu < \infty.$$

Certain presumptions on the exponent function p are required for a few claims in this work. Specifically, function $p : \Delta \longrightarrow R$ is locally log-Hölder continuous on Δ, if there exists $c_{\log}(p) > 0$ and for all $\mu, \nu \in \Delta$, we have

$$|p(\mu) - p(\nu)| \leq \frac{c_{\log}(p)}{\log(e + 1/|\mu - \nu|)}.$$

We say that p satisfies the log-Hölder continuous at infinity (or has a log decay at infinity), if there exists $p_\infty \in \Delta$ and a constant $c_{\log} > 0$ and for all $\nu \in \Delta$. We have

$$|p(\mu) - p_\infty| \leq \frac{c_{\log}(p)}{\log(e + |\mu|)}.$$

If both of the aforementioned criteria are met by the function p, we declare that p is globally log-Hölder continuous $\mathcal{C}_{\log}(p)$ on Δ. The corresponding class of functions is defined as follows:

$$\mathcal{P}_0^{\log}(\Delta) := \left\{ p \in \mathcal{P}_0(\Delta) : \frac{1}{p} \in \mathcal{C}_{\log}(p) \right\}.$$

For the semi-modular $\rho_{p(\cdot)}$ and the quasi-norm $\|\cdot\|_{L^{p(\cdot)}(\Delta)}$, we now formulate several well-known and practical statements.

Proposition 2. *Let $p \in \mathcal{P}_0(\Omega)$ with $p_+ < \infty$ and let $\zeta \in L^{p(\cdot)}(\Omega)$. Then,*

$$\min\left\{ \left(\rho_{p(\cdot)}(\zeta)\right)^{\frac{1}{p_-}}, \left(\rho_{p(\cdot)}(\zeta)\right)^{\frac{1}{p_+}} \right\} \leq \|\zeta\|_{L^{p(\cdot)}(\Omega)} \leq \max\left\{ \left(\rho_{p(\cdot)}(\zeta)\right)^{\frac{1}{p_-}}, \left(\rho_{p(\cdot)}(\zeta)\right)^{\frac{1}{p_+}} \right\}.$$

We now recall a generalized Orlicz–Zygmund space with variable integrability.

Definition 7 ([82]). *The space $L^{p(\mu)} \log^{\beta} L$ is defined for each $\beta \in \mathbb{R}$ as*

$$L^{p(\mu)} \log^{\beta} L(\Delta, \Omega) := \left\{ \nu : \Delta \to \Omega : \nu \text{ is measurable and } \int_{\Delta} |\nu|^{p(\mu)} \log^{\beta}(e + |\nu|) d\nu < \infty \right\}$$

and the associated Luxemburg norm is defined as follows:

$$\|\nu\|_{L^{p(\mu)} \log^{\beta} L(\Delta,\Omega)} := \inf\left\{ \lambda > 0 : \int_{\Delta} \left|\frac{\nu}{\lambda}\right|^{p(\mu)} \log^{\beta}\left(e + \frac{|\nu|}{\lambda}\right) d\mu \leqslant 1 \right\},$$

it becomes a Banach space.

4.3. Mixed Norm Function Spaces

Definition 8 ([81]). *Let $p, q \in \mathcal{P}(\mathbb{R}^n)$. The mixed Lebesgue sequence space $\ell_{q(\cdot)}\left(L^{p(\cdot)}\right)$ is defined on sequences of $L^{p(\cdot)}$ as follows:*

$$\rho_{\ell_{q(\cdot)}\left(L^{p(\cdot)}\right)}((\zeta_j)) := \sum_{j=1}^{\infty} \inf\left\{ \alpha_j > 0 : \rho_{p(\cdot)}\left(\frac{\zeta_j}{\alpha_j^{\frac{1}{q(\cdot)}}} \right) \leq 1 \right\},$$

where the corresponding Luxemburg norm, typically defined as

$$\left\|(\zeta_j)\right\|_{\ell^{q(\cdot)}\left(L^{p(\cdot)}\right)} := \inf\left\{ \nu > 0 : \rho_{\ell_{q(\cdot)}}\left(L^{p(\cdot)}\right)((\zeta_j)/\nu) \leq 1 \right\}, (\zeta_j) \in \ell^{q(\cdot)}\left(L^{p(\cdot)}\right).$$

Under certain assumptions [39], the space $\ell^{q(\cdot)}\left(L^{p(\cdot)}\right)$ becomes a normed space.

Definition 9 ([81]). *Consider $p, q, u \in \mathcal{P}(\mathbb{R}^n)$ where $p(\mu) \leq u(\mu)$. The mixed norm Morrey-sequence space $\ell_{q(\cdot)}\left(\mathbf{M}_{p(\cdot),u(\cdot)}\right)$ encompasses all measurable sequences $(\zeta_\sigma)_\kappa \subset \mathcal{K}(\mathbb{R}^n)$ such that*

$$\rho_{\ell_{q(\cdot)}\left(K_{p(\cdot),u(\cdot)}\right)}((\zeta_\sigma)_\kappa) := \sum_{\nu \geq 0} \sup_{\mu \in \mathbb{R}^n, u > 0} \inf\left\{ \sigma_\kappa > 0 : \rho_{p(\cdot)}\left(\frac{u^{\frac{n}{v(\mu)} - \frac{n}{p(\sigma)}} \phi_\nu \chi_{\mathcal{B}(\mu,u)}}{\sigma_\kappa^{\frac{1}{q(\cdot)}}} \right) \leq 1 \right\}.$$

5. Main Results Related to Mixed-Orlicz–Zygmund $\left(\ell_{q(\cdot)} \log^{\beta}\left(L^{p(\cdot)}\right)\right)$ Space

The aim of this section is to develop a new norm structure by merging two different types of function spaces and analyze whether this space is complete, separable, and compact.

First, taking into account Definitions 7 and 8, we propose a new type of function space and its related modular and norm, which we use in our major results.

Definition 10. *Let* $p, q \in \mathcal{P}(R^n)$. *The mixed generalized Orlic Zygmund sequence space* $\ell_{q(\cdot)} \log^{\beta}\left(L^{p(\cdot)}\right)$ *is defined on the sequences* (ζ_j) *of* $L^{p(\cdot)}$ *as follows:*

$$\rho_{\ell q(\cdot) \log^{\beta}\left(L^{p(\cdot)}\right)}((\zeta_j)) := \sum_{j=1}^{\infty} \inf\left\{\alpha_j > 0 : \rho_{p(\cdot)}\left(\frac{\zeta_j}{\alpha_j^{\frac{1}{q(\cdot)}}}\right) \log^{\beta}\left(e + \frac{|\zeta^j|}{\alpha_j^{\frac{1}{q(\cdot)}}}\right) \leq 1\right\}.$$

If we consider the following convention $\alpha^{\frac{1}{\infty}} = 1$, *the norm is defined as follows:*

$$\left\|(\zeta_j)_j\right\|_{\ell q(\cdot) \log^{\beta}\left(L^{p(\cdot)}\right)} := \inf\left\{\nu > 0 \,\middle|\, \rho_{\ell q(\cdot) \log^{\beta}\left(L^{p(\cdot)}\right)}\left(\frac{1}{\nu}(\zeta_j)_j\right) \log^{\beta}\left(e + \frac{|\zeta^j|}{\nu}\right) \leq 1\right\}.$$

If $q^+ < \infty$, *then*

$$\inf\left\{\alpha > 0 \,\middle|\, \rho_{p(\cdot)}\left(\frac{\zeta}{\alpha_j^{\frac{1}{q(\cdot)}}}\right) \log^{\beta}\left(e + \frac{|\zeta|}{\alpha^{\frac{1}{q(\cdot)}}}\right) \leqslant 1\right\} = \left\||\zeta|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}}.$$

5.1. Completeness

First, we demonstrate that this modular space is a quasi-Banach space.

Theorem 11. *If* $p, q \in \mathcal{P}_0$, *then* $\ell_{q(\cdot)} \log^{\beta}\left(L^{p(\cdot)}\right)$ *is a quasi-Banach space.*

Proof. To demonstrate $\ell_{q(\cdot)} \log^{\beta}\left(L^{p(\cdot)}\right)$ is a quasi-Banach space. We first show that it is a quasi-normed space. By Definition 6, we just need to consider the quasi-convexity. Let $\kappa \in \left(0, \min\left\{\frac{p^-}{2}, \frac{q^-}{2}, 1\right\}\right]$ and define $\tilde{p} = \frac{p}{\kappa}$ and $\tilde{q} = \frac{q}{\kappa}$. Then, clearly $\frac{1}{\tilde{p}(\cdot)} + \frac{1}{\tilde{q}(\cdot)} \leqslant 1$. Thus, we have

$$\begin{aligned}
&\|(\eta_\nu)_\nu + (\zeta_\nu)_\nu\|_{\ell_{q(\cdot)} \log^{\beta}\left(L^{p(\cdot)}\right)} = \left\||(\eta_\nu)_\nu + (\zeta_\nu)_\nu|^{\kappa}\right\|^{\frac{1}{\kappa}}_{\ell\tilde{q}(\cdot) \log^{\beta}\left(L^{(\tilde{p}(\cdot)}\right)} \\
&\leqslant \left\|\left(|\eta_\nu|^{\kappa}\right)_\nu + \left(|\zeta_\nu|^{\kappa}\right)_\nu\right\|^{\frac{1}{\kappa}}_{\ell\tilde{q}(\cdot) \log^{\beta}\left(L^{(\tilde{p}(\cdot)}\right)} \\
&\leqslant \left(\left\|\left(|\eta_\nu|^{\kappa}\right)_\nu\right\|_{\ell\tilde{q}(\cdot) \log^{\beta}\left(L^{\tilde{p}(\cdot)}\right)} + \left\|\left(|\zeta_\nu|^{\kappa}\right)_\nu\right\|_{\ell\tilde{q}(\cdot) \log^{\beta}\left(L^{\tilde{p}(\cdot)}\right)}\right)^{\frac{1}{\kappa}} \\
&= \left(\|(\eta_\nu)_\nu\|^{\kappa}_{\ell\tilde{q}(\cdot) \log^{\beta}\left(L^{(\tilde{p}(\cdot)}\right)} + \|(\zeta_\nu)_\nu\|^{\kappa}_{\ell\tilde{q}(\cdot) \log^{\beta}\left(L^{(\tilde{p}(\cdot)}\right)}\right)^{\frac{1}{\kappa}} \\
&\leqslant 2^{\frac{1-\kappa}{\kappa}}\left(\|(\eta_\nu)_\nu\|_{\ell q(\cdot) \log^{\beta}\left(L^{p(\cdot)}\right)} + \|(\zeta_\nu)_\nu\|_{\ell q(\cdot) \log^{\beta}\left(L^{p(\cdot)}\right)}\right),
\end{aligned}$$

and this completes the proof. The next step is to demonstrate the space's completeness. For this purpose, we fix a Cauchy sequence ζ^i in $\ell q^{(\cdot)} \log^{\beta}\left(L^{p(\cdot)}\right)$. Hence, for any $\varepsilon > 0$, there exists $\wp$ such that for all $i, j \geq \wp$, we have

$$\left\|\zeta^i - \zeta^j\right\|_{\ell q(\cdot) \log^{\beta}\left(L^{p(\cdot)}\right)} < \varepsilon$$

By the quasi-norm's definition

$$\rho_{\ell q(\cdot) \log^{\beta}\left(L^{p(\cdot)}\right)}\left(\frac{\zeta^i - \zeta^j}{\varepsilon}\right) \log^{\beta}\left(e + \frac{|\zeta^i - \zeta^j|}{\varepsilon}\right) \leq 1, \quad \text{for} \quad i, j \geq \wp,$$

and hence

$$\sum_{\mu=1}^{\infty} \inf\left\{\sigma_\mu : \rho_{L^{p(\cdot)}}\left(\frac{\zeta_\mu^i-\zeta_\mu^j}{\varepsilon\sigma_\mu^{\frac{1}{q(\cdot)}}}\right)\log^\beta\left(e+\frac{|\zeta_\mu^i-\zeta_\mu^j|}{\varepsilon\sigma_\mu^{\frac{1}{q(\cdot)}}}\right)\le 1\right\}\le 1.$$

In particular, for each $\delta > 1$, we have

$$\rho_{L^{p(\cdot)}}\left(\frac{\zeta_\mu^i-\zeta_\mu^j}{\varepsilon\delta^{\frac{1}{q(\cdot)}}}\right)\log^\beta\left(e+\frac{|\zeta_\mu^i-\zeta_\mu^j|}{\varepsilon\delta^{\frac{1}{q(\cdot)}}}\right)\le 1.$$

Hence, by the definition of the space $L^{p(\cdot)}$, we obtain

$$\frac{1}{\delta^{\frac{1}{q^-}}}\left\|\zeta_\mu^i-\zeta_\mu^j\right\|_{L^{p(\cdot)}}\le\left\|\frac{\zeta_\mu^i-\zeta_\mu^j}{\delta^{\frac{1}{q(\cdot)}}}\right\|_{L^{p(\cdot)}}\le\varepsilon.$$

Thus, passing with $\delta\to 1$, we obtain $\left\|\zeta_\mu^i-\zeta_\mu^j\right\|_{L^{p(\cdot)}}\le\varepsilon$. Thus, we demonstrate that for each μ, ζ_μ^i is a Cauchy sequence in $L^{p(\cdot)}$. Thus, for every μ, there exists $\zeta_\mu\in L^{p(\cdot)}$ such that

$$\zeta_\mu^i\to\zeta_\mu\quad\text{in}\quad L^{p(\cdot)}.$$

Now, we shall prove that

$$\zeta^i\to\zeta\quad\text{in}\quad \ell_{q(\cdot)}\log^\beta\left(L^{p(\cdot)}\right), \tag{19}$$

where $\zeta=(\zeta_\mu)_\mu$. For this purpose, we require the following:

$$\begin{aligned}&\inf\left\{\sigma_\mu : \rho_{L^{p(\cdot)}}\left(\frac{\zeta_\mu^i-\zeta_\mu}{\varepsilon\sigma_\mu^{\frac{1}{q(\cdot)}}}\right)\log^\beta\left(e+\frac{|\zeta_\mu^i-\zeta_\mu^j|}{\varepsilon\sigma_\mu^{\frac{1}{q(\cdot)}}}\right)\le 1\right\}\\&\le\liminf_{j\to\infty}\inf\left\{\sigma_\mu(j) : \rho_{L^{p(\cdot)}}\left(\frac{\zeta_\mu^i-\zeta_\mu^j}{\varepsilon\sigma_\mu(j)^{\frac{1}{q(\cdot)}}}\right)\log^\beta\left(e+\frac{|\zeta_\mu^i-\zeta_\mu^j|}{\varepsilon\sigma_\mu(j)^{\frac{1}{q(\cdot)}}}\right)\le 1\right\}.\end{aligned} \tag{20}$$

In order to prove (20), we introduce the following notation:

$$\mathcal{A}=\liminf_{j\to\infty}\inf\left\{\sigma_\mu(j) : \rho_{L^{p(\cdot)}}\left(\frac{\zeta_\mu^i-\zeta_\mu^j}{\varepsilon\sigma_\mu(j)^{\frac{1}{q(\cdot)}}}\right)\log^\beta\left(e+\frac{|\zeta_\mu^i-\zeta_\mu^j|}{\varepsilon\sigma_\mu(j)^{\frac{1}{q(\cdot)}}}\right)\le 1\right\}.$$

Let j_k be a subsequence such that

$$\mathcal{A}=\lim_{k\to\infty}\inf\left\{\sigma_\mu(j_k) : \rho_{L^{p(\cdot)}}\left(\frac{\zeta_\mu^i-\zeta_\mu^{j_k}}{\varepsilon\sigma_\mu(j_k)^{\frac{1}{q(\cdot)}}}\right)\log^\beta\left(e+\frac{|\zeta_\mu^i-\zeta_\mu^j|}{\varepsilon\sigma_\mu(j_k)^{\frac{1}{q(\cdot)}}}\right)\le 1\right\}.$$

Hence, for each $v>0$ there exists N such that for all $k\ge N$, one has

$$\left\{\sigma_\mu(j_k) : \rho_{L^{p(\cdot)}}\left(\frac{\zeta_\mu^i-\zeta_\mu^{j_k}}{\varepsilon\sigma_\mu(j_k)^{\frac{1}{q(\cdot)}}}\right)\log^\beta\left(e+\frac{|\zeta_\mu^i-\zeta_\mu^j|}{\varepsilon\sigma_\mu(j_k)^{\frac{1}{q(\cdot)}}}\right)\le 1\right\}<\mathcal{A}+v.$$

Thus,

$$\rho_{L^{p(\cdot)}}\left(\frac{\zeta_\mu^i-\zeta_\mu^{j_k}}{\varepsilon(\mathcal{A}+v)^{\frac{1}{q(\cdot)}}}\right)\log^\beta\left(e+\frac{|\zeta_\mu^i-\zeta_\mu^j|}{\varepsilon(\mathcal{A}+v)^{\frac{1}{q(\cdot)}}}\right)\le 1.$$

Next, we apply the Fatou Lemma and the fundamental characteristics of the L^{∞} norm, and we obtain

$$\rho_{L^{p(\cdot)}}\left(\frac{\zeta_{\mu}^{i}-\zeta_{\mu}}{\varepsilon(\mathcal{A}+v)^{\frac{1}{q(\cdot)}}}\right)\log^{\beta}\left(e+\frac{|\zeta_{\mu}^{i}-\zeta_{\mu}|}{\varepsilon(\mathcal{A}+v)^{\frac{1}{q(\cdot)}}}\right)\leq 1.$$

Hence,

$$\inf\left\{\sigma_{\mu}:\rho_{L^{p(\cdot)}}\left(\frac{\zeta_{\mu}^{i}-\zeta_{\mu}}{\varepsilon\sigma_{\mu}^{\frac{1}{q(\cdot)}}}\right)\log^{\beta}\left(e+\frac{|\zeta_{\mu}^{i}-\zeta_{\mu}|}{\varepsilon\sigma_{\mu}^{\frac{1}{q(\cdot)}}}\right)\leq 1\right\}\leq\mathcal{A}+v.$$

Finally, applying $v\to 0$, we obtain (20). Now, we can prove (19). Since ζ^{i} is a Cauchy sequence in $\ell_{q(\cdot)}\log^{\beta}\left(L^{p(\cdot)}\right)$, for $i,j\geq\wp$ we have

$$\sum_{\mu=1}^{\infty}\inf\left\{\sigma_{\mu}:\rho_{L^{p(\cdot)}}\left(\frac{\zeta_{\mu}^{i}-\zeta_{\mu}^{j}}{\varepsilon\sigma_{\mu}^{\frac{1}{q(\cdot)}}}\right)\log^{\beta}\left(e+\frac{|\zeta_{\mu}^{i}-\zeta_{\mu}^{j}|}{\varepsilon\sigma_{\mu}^{\frac{1}{q(\cdot)}}}\right)\leq 1\right\}\leq 1.$$

Therefore, by (20), for each N and $i\geq\wp$, we have

$$\sum_{\mu=1}^{N}\inf\left\{\sigma_{\mu}:\rho_{L^{p(\cdot)}}\left(\frac{\zeta_{\mu}^{i}-\zeta_{\mu}}{\varepsilon\sigma_{\mu}^{\frac{1}{q(\cdot)}}}\right)\log^{\beta}\left(e+\frac{|\zeta_{\mu}^{i}-\zeta_{\mu}|}{\varepsilon\sigma_{\mu}^{\frac{1}{q(\cdot)}}}\right)\leq 1\right\}\leq 1.$$

Then, passing $N\to\infty$, we obtain

$$\left\|\zeta^{i}-\zeta\right\|_{\ell_{q(\cdot)}\log^{\beta}\left(L^{p(\cdot)}\right)}\leq\varepsilon\quad\text{for}\quad i\geq\wp.$$

□

5.2. Separability

The next result shows that the mixed norm space $\ell_{q(\cdot)}\log^{\beta}\left(L^{p(\cdot)}\right)$ is separable when the exponents satisfy $p^{+},q^{+}<\infty$.

Theorem 12. *If $p,q\in\mathcal{P}_{0}$ and $p^{+},q^{+}<\infty$, then $\ell_{q(\cdot)}\log^{\beta}\left(L^{p(\cdot)}\right)$ is separable.*

Proof. For each $k\in N$, we define

$$\mathcal{V}_{k}=\left\{\zeta\in\ell_{q(\cdot)}\log^{\beta}\left(L^{p(\cdot)}\right);\quad\zeta_{i}=0,i>k\right\}.$$

It is clear that $\mathcal{V}_{k}$ represents a closed subset of $\ell_{q(\cdot)}\log^{\beta}\left(L^{p(\cdot)}\right)$ and

$$\underbrace{L^{p(\cdot)}\times L^{p(\cdot)}\times\ldots\times L^{p(\cdot)}}_{k\text{ times}}\times\{0\}\times\cdots\times\{0\}\times\cdots\subset\mathcal{V}_{k}.$$

We divide the proof into two parts. First, we show that for each $k\in N$, the space $\mathcal{V}_{k}$ is separable. Since $p^{+}<\infty$, we know that $L^{p(\cdot)}$ is separable. So there is a countable subset of $L^{p(\cdot)}$, say, $\mathcal{K}$ such that which $\overline{\mathcal{K}}^{\|\cdot\|_{L^{p(\cdot)}}}=L^{p(\cdot)}$. Set

$$\mathcal{K}_{k}:=\underbrace{\mathcal{K}\times\mathcal{K}\times\ldots\times\mathcal{K}}_{k\text{ times}}\times\{0\}\times\cdots\times\{0\}\times\cdots\subset\mathcal{V}_{k}.$$

Obviously $\mathcal{K}_{\mathtt{k}}$ is countable, so it is enough to show that $\overline{\mathcal{K}}^{\|\cdot\|_{\ell_{\mathtt{q}(\cdot)}\log^{\beta}\left(\mathrm{L}^{\mathtt{p}(\cdot)}\right)}} = \mathcal{V}_{\mathtt{k}}$. Let us fix $\varepsilon > 0$ and $\zeta \in \mathcal{V}_{\mathtt{k}}$. Then, there exists $\eta \in \mathcal{K}_{\mathtt{k}}$ such that for $\nu \in \{1, 2, \cdots, \mathtt{k}\}$, we have

$$\|\zeta_\nu - \eta_\nu\|_{\mathrm{L}^{\mathtt{p}(\cdot)}} \leq \frac{\varepsilon}{\mathtt{k}^{\frac{1}{\mathtt{q}^-}}}.$$

Then, we have $\rho_{\mathtt{p}(\cdot)}\left(\frac{\zeta_\nu - \eta_\nu}{\varepsilon \cdot \left(\frac{1}{k}\right)^{\frac{1}{\mathtt{q}^-}}}\right) \leq 1$, and we obtain

$$\inf\left\{\sigma_\nu; \quad \rho_{\mathtt{p}(\cdot)}\left(\frac{\zeta_\nu - \eta_\nu}{\varepsilon \cdot (\sigma_\nu)^{\frac{1}{\mathtt{q}(\cdot)}}}\right) \leq 1\right\} \leq \frac{1}{\mathtt{k}}$$

Therefore,

$$\begin{aligned}
\rho_{\ell_{\mathtt{q}(\cdot)}\log^{\beta}\mathrm{L}^{\mathtt{p}(\cdot)}}\left(\frac{\zeta - \eta}{\varepsilon}\right) &= \sum_{\nu=1}^{\infty} \inf\left\{\sigma_\nu > 0; \quad \rho_{\mathtt{p}(\cdot)}\left(\frac{\zeta_\nu - \eta_\nu}{\varepsilon\sigma_\nu^{\frac{1}{\mathtt{q}(\cdot)}}}\right)\log^{\beta}\left(e + \frac{|\zeta_\nu - \eta_\nu|}{\varepsilon\sigma_\nu^{\frac{1}{\mathtt{q}(\cdot)}}}\right) \leq 1\right\} \\
&= \sum_{\nu=1}^{\mathtt{k}} \inf\left\{\sigma_\nu > 0; \quad \rho_{\mathtt{p}(\cdot)}\left(\frac{\zeta_\nu - \eta_\nu}{\varepsilon\sigma_\nu^{\frac{1}{\mathtt{q}(\cdot)}}}\right)\log^{\beta}\left(e + \frac{|\zeta_\nu - \eta_\nu|}{\varepsilon\sigma_\nu^{\frac{1}{\mathtt{q}(\cdot)}}}\right) \leq 1\right\} \\
&\leq 1.
\end{aligned}$$

Now, we have show that

$$\overline{\mathrm{p}_{\mathtt{k}=1}^{\infty}\mathcal{V}_{\mathtt{k}}}^{\|\cdot\|_{\ell_{\mathtt{q}(\cdot)}\log^{\beta}\mathrm{L}^{\mathtt{p}(\cdot)}}} = \ell_{\mathtt{q}(\cdot)}\log^{\beta}\mathrm{L}^{\mathtt{p}(\cdot)}.$$

To prove it, we let $\zeta \in \ell_{\mathtt{q}(\cdot)}\log^{\beta}\left(\mathrm{L}^{\mathtt{p}(\cdot)}\right)$, then

$$\sum_{\mathtt{i}=1}^{\infty} \inf\left\{\sigma_{\mathtt{i}}; \quad \rho_{\mathtt{p}(\cdot)}\left(\frac{\zeta}{2\mathcal{M}\sigma_{\mathtt{i}}^{\frac{1}{\mathtt{q}(\cdot)}}}\right)\log^{\beta}\left(e + \frac{|\zeta|}{2\mathcal{M}\sigma_{\mathtt{i}}^{\frac{1}{\mathtt{q}(\cdot)}}}\right) \leq 1\right\} \leq 1,$$

where $\mathcal{M} = \|\zeta\|_{\ell_{\mathtt{q}(\cdot)}\log^{\beta}\mathrm{L}^{\mathtt{p}(\cdot)}}$. Therefore, for every $0 < \varepsilon < 1$, $\exists\, \mathtt{k} \in \mathtt{N}$ such that

$$\sum_{\mathtt{i}=\mathtt{k}+1}^{\infty} \inf\left\{\sigma_{\mathtt{i}}; \quad \rho_{\mathtt{p}(\cdot)}\left(\frac{\zeta}{2\mathcal{M}\sigma_{\mathtt{i}}^{\frac{1}{\mathtt{q}(\cdot)}}}\right)\log^{\beta}\left(e + \frac{|\zeta|}{2\mathcal{M}\sigma_{\mathtt{i}}^{\frac{1}{\mathtt{q}(\cdot)}}}\right) \leq 1\right\} \leq \varepsilon.$$

Thus, we obtain

$$\sum_{\mathtt{i}=\mathtt{k}+1}^{\infty} \inf\left\{\tau_{\mathtt{i}}; \quad \rho_{\mathtt{p}(\cdot)}\left(\frac{\zeta}{2\mathcal{M}(\varepsilon\tau_{\mathtt{i}})^{\frac{1}{\mathtt{q}(\cdot)}}}\right)\log^{\beta}\left(e + \frac{|\zeta|}{2\mathcal{M}(\varepsilon\tau_{\mathtt{i}})^{\frac{1}{\mathtt{q}(\cdot)}}}\right) \leq 1\right\} \leq 1. \tag{21}$$

On the other hand, we have

$$\begin{aligned}
&\rho_{\mathtt{p}(\cdot)}\left(\frac{\zeta}{2\mathcal{M}\varepsilon^{\frac{1}{\mathtt{q}^+}}\tau_{\mathtt{i}}^{\frac{1}{\mathtt{q}(\cdot)}}}\right)\log^{\beta}\left(e + \frac{|\zeta|}{2\mathcal{M}\varepsilon^{\frac{1}{\mathtt{q}^+}}\tau_{\mathtt{i}}^{\frac{1}{\mathtt{q}(\cdot)}}}\right) \\
&\leq \rho_{\mathtt{p}(\cdot)}\left(\frac{\zeta}{2\mathcal{M}(\varepsilon\tau_{\mathtt{i}})^{\frac{1}{\mathtt{q}(\cdot)}}}\right)\log^{\beta}\left(e + \frac{|\zeta|}{2\mathcal{M}(\varepsilon\tau_{\mathtt{i}})^{\frac{1}{\mathtt{q}(\cdot)}}}\right) \leq 1.
\end{aligned} \tag{22}$$

Thanks to (22), we have

$$\left\{\tau_i;\quad \rho_{p(\cdot)}\left(\frac{\zeta}{2\mathcal{M}(\varepsilon\tau_i)^{\frac{1}{q(\cdot)}}}\right)\log^{\beta}\left(e+\frac{|\zeta|}{2\mathcal{M}(\varepsilon\tau_i)^{\frac{1}{q(\cdot)}}}\right)\le 1\right\}$$
$$\subset\left\{\tau_i;\quad \rho_{p(\cdot)}\left(\frac{\zeta}{2\mathcal{M}\varepsilon^{\frac{1}{q^+}}\tau_i^{\frac{1}{q(\cdot)}}}\right)\log^{\beta}\left(e+\frac{|\zeta|}{2\mathcal{M}\varepsilon^{\frac{1}{q^+}}\tau_i^{\frac{1}{q(\cdot)}}}\right)\le 1\right\}. \tag{23}$$

Then, in view of (21), we have

$$\sum_{i=k+1}^{\infty}\inf\left\{\tau_i;\quad \rho_{p(\cdot)}\left(\frac{\zeta}{2\mathcal{M}\varepsilon^{\frac{1}{q^+}}\tau_i^{\frac{1}{q(\cdot)}}}\right)\log^{\beta}\left(e+\frac{|\zeta|}{2\mathcal{M}\varepsilon^{\frac{1}{q^+}}\tau_i^{\frac{1}{q(\cdot)}}}\right)\le 1\right\}$$
$$\le\sum_{i=k+1}^{\infty}\inf\left\{\tau_i;\quad \rho_{p(\cdot)}\left(\frac{\zeta}{2\mathcal{M}(\varepsilon\tau_i)^{\frac{1}{q(\cdot)}}}\right)\log^{\beta}\left(e+\frac{|\zeta|}{2\mathcal{M}(\varepsilon\tau_i)^{\frac{1}{q(\cdot)}}}\right)\le 1\right\}\le 1 \tag{24}$$

Finally, let us define

$$\eta=\begin{cases}\zeta_i & \text{if } i\le k\\ 0 & \text{if } i>k\end{cases}.$$

It is clear that $\zeta\in\mathcal{V}_k$, and by (24), we obtain $\|\zeta-\eta\|_{\ell_{q(\cdot)}\log^{\beta}L^{p(\cdot)}}\le 2\mathcal{M}\varepsilon^{\frac{1}{q^+}}$, which is the desired result. □

Proposition 3. *Let* $p,q\in\mathcal{P}_0(\Omega)$ *and let* $(\zeta_j),(\eta_j)\subset L^{p(\cdot)}(R^n)$. *If* $0\le\zeta_j\le\eta_j$ *for each* $j\in N$, *then*

$$\left\|(\zeta_j)\right\|_{\ell_{q(\cdot)}\log^{\beta}L^{p(\cdot)}}\le\left\|(\eta_j)\right\|_{\ell_{q(\cdot)}\log^{\beta}L^{p(\cdot)}}.$$

It suffices to show that

$$\left\{\sigma>0:\rho_{\ell_{q(\cdot)}(L^{p(\cdot)})}\left(\frac{\eta_j}{\sigma}\right)\log^{\beta}\left(e+\frac{|\eta_j|}{\sigma}\right)\le 1\right\}$$
$$\subset\left\{\sigma>0:\rho_{\ell_{q(\cdot)}(L^{p(\cdot)})}\left(\frac{\zeta_j}{\sigma}\right)\log^{\beta}\left(e+\frac{|\zeta_j|}{\sigma}\right)\le 1\right\}. \tag{25}$$

Suppose that $\sigma>0$ *is such that*

$$\rho_{\ell_{q(\cdot)}\log^{\beta}(L^{p(\cdot)})}\left(\frac{\eta_j}{\sigma}\right)=\sum_{j=1}^{\infty}\inf\left\{\tau_j>0:\rho_{p(\cdot)}\left(\frac{\eta_j}{\sigma\tau_j^{\frac{1}{q(\cdot)}}}\right)\log^{\beta}\left(e+\frac{|\eta_j|}{\sigma\tau_j^{\frac{1}{q(\cdot)}}}\right)\le 1\right\}\le 1.$$

Let us fix $j\in N$ *and let* $\tau_j>0$ *be such that*

$$\rho_{p(\cdot)}\left(\frac{\eta_j}{\sigma\tau_j^{\frac{1}{q(\cdot)}}}\right)\log^{\beta}\left(e+\frac{|\eta_j|}{\sigma\tau_j^{\frac{1}{q(\cdot)}}}\right)\le 1.$$

Due to assumption that $0\le\zeta_j\le\eta_j$, *we obtain*

$$\rho_{p(\cdot)}\left(\frac{\zeta_j}{\sigma\tau_j^{\frac{1}{q(\cdot)}}}\right)\log^{\beta}\left(e+\frac{|\zeta_j|}{\sigma\tau_j^{\frac{1}{q(\cdot)}}}\right)\le\rho_{p(\cdot)}\left(\frac{\eta_j}{\sigma\tau_j^{\frac{1}{q(\cdot)}}}\right)\log^{\beta}\left(e+\frac{|\eta_j|}{\sigma\tau_j^{\frac{1}{q(\cdot)}}}\right)\le 1.$$

Therefore,

$$\left\{\tau_j > 0 : \rho_{p(\cdot)}\left(\frac{\eta_j}{\sigma\tau_j^{\frac{1}{q(\cdot)}}}\right)\log^{\beta}\left(e+\frac{|\eta_j|}{\sigma\tau_j^{\frac{1}{q(\cdot)}}}\right)\le 1\right\}$$
$$\subset\left\{\tau_j > 0 : \rho_{p(\cdot)}\left(\frac{\eta_j}{\sigma\tau_j^{\frac{1}{q(\cdot)}}}\right)\log^{\beta}\left(e+\frac{|\eta_j|}{\sigma\tau_j^{\frac{1}{q(\cdot)}}}\right)\le 1\right\}. \tag{26}$$

Finally, it easily follows that

$$\begin{aligned}\rho_{\ell_{q(\cdot)}\log^{\beta}\left(L^{p(\cdot)}\right)}\left(\frac{\zeta_j}{\sigma}\right) &= \sum_{j=1}^{\infty}\inf\left\{\tau_j > 0 : \rho_{p(\cdot)}\left(\frac{\zeta_j}{\sigma\tau_j^{\frac{1}{q(\cdot)}}}\right)\log^{\beta}\left(e+\frac{|\zeta_j|}{\sigma\tau_j^{\frac{1}{q(\cdot)}}}\right)\le 1\right\}\\ &\le \sum_{j=1}^{\infty}\inf\left\{\tau_j > 0 : \rho_{p(\cdot)}\left(\frac{\eta_j}{\sigma\tau_j^{\frac{1}{q(\cdot)}}}\right)\log^{\beta}\left(e+\frac{|\eta_j|}{\sigma\tau_j^{\frac{1}{q(\cdot)}}}\right)\le 1\right\}\\ &= \rho_{\ell_{q(\cdot)}\log^{\beta}\left(L^{p(\cdot)}\right)}\left(\frac{\eta_j}{\sigma}\right)\log^{\beta}\left(e+\frac{|\eta_j|}{\sigma}\right)\le 1,\end{aligned}$$

and this concludes the proof.

Lemma 5. *Assume that* $p, q \in \mathcal{P}_0(\Omega)$ *with* $(\zeta_i)_{i=1}^{\infty} \in \ell_{q(\cdot)}\log^{\beta}\left(L^{p(\cdot)}\right)$. *Then, for every* $j \in \mathbb{N}$, *we have*

$$\|\zeta_j\|_{L^{p(\cdot)}(\Omega)} \le \|(\zeta_j)\|_{\ell_{q(\cdot)}\log^{\beta}\left(L^{p(\cdot)}\right)}.$$

Proof. Suppose $j \in \mathbb{N}$, and we define the sequence $(\eta_i(j))_{i=1}^{\infty}$ in such a way that $\eta_i(j) := \zeta_j\delta_{ij}$. Then, for each $i \in \mathbb{N}$, we have $|\eta_i(j)| \le |\zeta_j|$, as sequences preserve their order, it follows as

$$\|(\eta_j(j))\|_{\ell_{q(\cdot)}\log^{\beta}\left(L^{p(\cdot)}\right)} \le \|(\zeta_j)\|_{\ell_{q(\cdot)}\log^{\beta}\left(L^{p(\cdot)}\right)}.$$

Finally, by Proposition 3, we obtain

$$\|(\eta_j(j))\|_{\ell_{q(\cdot)}\log^{\beta}\left(L^{p(\cdot)}\right)} = \|\zeta_j\|_{L^{p(\cdot)}(\Omega)},$$

and the proof follows. □

5.3. Precompactness

Theorem 13. *Let* $p \in \mathcal{P}_0^{\log}(\Omega)$, $q \in \mathcal{P}_0(\Omega)$ *and assume that* $p_+, q_+ < \infty$. *Then, the class* $\mathcal{Q} \subset \ell_{q(\cdot)}\log^{\beta}\left(L^{p(\cdot)}\right)$ *is precompact in* $\ell_{q(\cdot)}\log^{\beta}\left(L^{p(\cdot)}\right)$ *if the following assumptions are met:*

- *(a)* $\mathcal{Q}$ *is precompact in* $\ell_{q(\cdot)}\log^{\beta}\left(L^{p(\cdot)}\right)$;
- *(b) For all* $\varepsilon > 0$ *there exists* $\mathcal{K} \in \mathbb{N}$ *such that* $\forall_{\zeta\in\mathcal{Q}}\sum_{r=\mathcal{K}+1}^{\infty}\left\||\zeta_r|^{q(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}(\Omega)} \le \varepsilon$;
- *(c) For all* $r \in \mathbb{N}$ *and* $0 < u < p_-$, *then*

$$\lim_{\sigma\to 0^+}\sup_{\zeta\in\mathcal{Q}}\int_{\Omega}\left(\frac{1}{|\mathcal{B}(\mu,\sigma)|}\int_{B(\mu,\sigma)}|\zeta_r(\mu)-\zeta_r(\nu)|^u d\nu\right)^{\frac{p(\mu)}{u}} d\mu = 0;$$

- *(d) For all* $r \in \mathbb{N}$, $\lim_{R\to\infty}\sup_{\zeta\in\mathcal{Q}}\int_{\Omega\setminus\mathcal{B}(0,R)}|\zeta_r(\mu)|^{p(\mu)}d\mu = 0$.

Proof. Initially, we will demonstrate that the conditions (a)-(d) are satisfied. The space $\ell_{q(\cdot)}\log^{\beta}\left(L^{p(\cdot)}\right)$ is a quasi-Banach space; it is sufficient to establish that $\left(\zeta^{k}\right)$ has a convergent subsequence.

For every $r \in N$ sequence, $\left(\zeta_{r}^{k}\right)_{k=1}^{\infty}$ is bounded in $L^{p(\cdot)}(\Omega)$. Given (a), there exists $\alpha \in R$ such that

$$\sup_{k\in N}\left\{k : \rho_{\ell_{q(\cdot)}\log^{\beta}\left(L^{p(\cdot)}\right)}\left(\zeta^{k}\right)\log^{\beta}\left(e+|\zeta^{k}|\right) \leq \alpha\right\},$$

that is

$$\sup_{k\in N}\left\|\zeta^{k}\right\|_{\ell_{q(\cdot)}\log^{\beta}\left(L^{p(\cdot)}\right)} \leq \alpha.$$

Let us fix $r \in N$. Using Lemma 5, we may conclude that

$$\sup_{k\in N}\left\{k : \rho_{L^{p(\cdot)}}\left(\zeta_{r}{}^{k}\right)\log^{\beta}\left(e+|\zeta_{r}{}^{k}|\right) \leq \alpha\right\} \leq \sup_{k\in N}\left\{k : \rho_{\ell_{q(\cdot)}\log^{\beta}\left(L^{p(\cdot)}\right)}\left(\zeta^{k}\right)\log^{\beta}\left(e+|\zeta^{k}|\right) \leq \alpha\right\},$$

and this implies that

$$\sup_{k\in N}\left\|\zeta_{r}^{k}\right\|_{L^{p(\cdot)}(\Omega)} \leq \sup_{k\in N}\left\|\zeta^{k}\right\|_{\ell_{q(\cdot)}\log^{\beta}\left(L^{p(\cdot)}\right)} \leq \alpha,$$

which proves that $\left(\zeta_{r}^{k}\right)_{k=1}^{\infty}$ is bounded in $L^{p(\cdot)}(\Omega)$. Finally, for a constant $r \in N$, we define

$$\mathcal{K}_{r} := \overline{\left\{\zeta_{r}^{k} \in L^{p(\cdot)}(\Omega) : k \in N\right\}}.$$

Since sets $(\mathcal{K}_{r})_{j=1}^{\infty}$ are closed and bounded in $L^{p(\cdot)}(\Delta)$, thus, a strictly increasing sequence $(k_{1}) \subset N$ and a sequence $(\zeta_{r}) \subset L^{p(\cdot)}(\Omega)$ exist by using the Cantor diagonal argument technique for every $r \in N$, that is

$$\zeta_{r}^{k_{1}} \xrightarrow{l\to\infty} \zeta_{r} \text{ in } L^{p(\cdot)}(\Omega).$$

Let $\zeta = (\zeta_{r})$, and we prove

$$\zeta^{k_{1}} \xrightarrow{l\to\infty} \zeta \text{ in } \ell_{q(\cdot)}\log^{\beta}\left(L^{p(\cdot)}\right).$$

We denote $\left(\zeta^{k_{1}}\right)$ by $\left(\zeta^{k}\right)$. Fixing $\varepsilon \in (0,1)$, we demonstrate that

$$\rho_{\ell_{q(\cdot)}\log^{\beta}\left(L^{p(\cdot)}\right)}\left(\frac{\zeta^{k}-\zeta}{\varepsilon}\right)\log^{\beta}\left(e+\frac{|\zeta^{k}-\zeta|}{\varepsilon}\right) \leq 1,$$

for sufficiently large $k \in N$. There exists $\mathcal{K} \in N$ from (b) such that

$$\sum_{r=\mathcal{K}+1}^{\infty}\left\|\left|\zeta_{r}^{k}\right|^{q(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}(\Omega)} \leq \frac{\varepsilon^{q_{+}}}{2^{q_{+}}4\mathcal{L}\left(\log^{\beta}\right)^{q_{+}}} \leq \varepsilon. \tag{27}$$

Subsequently, we can split the modular as follows:

$$\begin{aligned}&\rho_{\ell_{q(\cdot)}\log^{\beta}\left(L^{p(\cdot)}\right)}\left(\frac{\zeta^{k}-\zeta}{\varepsilon}\right)\log^{\beta}\left(e+\frac{|\zeta^{k}-\zeta|}{\varepsilon}\right)\\ &=\sum_{j=1}^{\mathcal{K}}\left\|\left|\frac{\zeta_{r}^{k}-\zeta_{r}}{\varepsilon}\right|^{q(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}(\Omega)}+\sum_{r=\mathcal{K}+1}^{\infty}\left\|\left|\frac{\zeta_{r}^{k}-\zeta_{r}}{\varepsilon}\right|^{q(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}(\Omega)}\end{aligned} \tag{28}$$

Next, by the triangle inequality for $\|\cdot\|_{L^{\frac{p(\cdot)}{q(\cdot)}}(\Omega)}$, Lemma 5, the Fatou lemma for the counting measure, and Equation (27), we have

$$
\begin{aligned}
&\sum_{r=\mathcal{K}+1}^{\infty}\left\|\left|\zeta_r^k-\zeta_r\right|^{q(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}(\Omega)} \le \sum_{r=\mathcal{K}+1}^{\infty}\left\|2^{q_+}\left(\left|\frac{\zeta_r^k}{\varepsilon}\right|^{q(\cdot)}+\left|\frac{\zeta_r}{\varepsilon}\right|^{q(\cdot)}\right)\right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}(\Omega)}\\
&\le 2^{q_+}\mathcal{L}\left(\sum_{r=\mathcal{K}+1}^{\infty}\left\|\left|\frac{\zeta_r^k}{\varepsilon}\right|^{q(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}(\Omega)}+\sum_{r=\mathcal{K}+1}^{\infty}\left\|\left|\frac{\zeta_r}{\varepsilon}\right|^{q(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}(\Omega)}\right)\\
&\le \frac{2^{q_+}\mathcal{L}}{\varepsilon^{q_+}}\left(\sum_{r=\mathcal{K}+1}^{\infty}\left\|\left|\zeta_r^k\right|^{q(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}(\Omega)}+\sum_{r=\mathcal{K}+1}^{\infty}\left\|\left|\zeta_r\right|^{q(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}(\Omega)}\right)\\
&= \frac{2^{q_+}\mathcal{L}}{\varepsilon^{q_+}}\left(\sum_{r=\mathcal{K}+1}^{\infty}\left\|\left|\zeta_r^k\right|^{q(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}(\Omega)}+\sum_{r=\mathcal{K}+1}^{\infty}\inf\left\{\tau_r>0:\rho_{p(\cdot)}\left(\frac{\zeta_r}{\tau_r^{\frac{1}{q(\cdot)}}}\right)\log^{\beta}\left(e+\frac{|\zeta_r|}{\tau_r^{\frac{1}{q(\cdot)}}}\right)\le 1\right\}\right)\\
&\le \frac{2^{q_+}4\mathcal{L}\left(\log^{\beta}\right)^{q_+}}{\varepsilon^{q_+}}\left(\sum_{r=\mathcal{K}+1}^{\infty}\left\|\left|\zeta_r^k\right|^{q(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}(\Omega)}+\sum_{r=\mathcal{K}+1}^{\infty}\liminf_{k\to\infty}\inf\left\{\tau_r(k)>0:\rho_{p(\cdot)}\left(\frac{\zeta_r^k}{\tau_r(k)^{\frac{1}{q(\cdot)}}}\right)\log^{\beta}\left(e+\frac{|\zeta_r^k|}{\tau_r(k)^{\frac{1}{q(\cdot)}}}\right)\le 1\right\}\right)\\
&\le \frac{2^{q_+}4\mathcal{L}\left(\log^{\beta}\right)^{q_+}}{\varepsilon^{q_+}}\left(\sum_{r=\mathcal{K}+1}^{\infty}\left\|\left|\zeta_r^k\right|^{q(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}(\Omega)}+\liminf_{k\to\infty}\sum_{r=\mathcal{K}+1}^{\infty}\inf\left\{\tau_r(k)>0:\rho_{p(\cdot)}\left(\frac{\zeta_r^k}{\tau_r(k)^{\frac{1}{q(\cdot)}}}\right)\log^{\beta}\left(e+\frac{|\zeta_r^k|}{\tau_r(k)^{\frac{1}{q(\cdot)}}}\right)\le 1\right\}\right)\\
&= \frac{2^{q_+}4\mathcal{L}\left(\log^{\beta}\right)^{q_+}}{\varepsilon^{q_+}}\left(\sum_{r=\mathcal{K}+1}^{\infty}\left\|\left|\zeta_r^k\right|^{q(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}(\Omega)}+\liminf_{k\to\infty}\sum_{r=\mathcal{K}+1}^{\infty}\left\|\left|\zeta_r^k\right|^{q(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}(\Omega)}\right)\\
&\le \frac{2^{q_+}4\mathcal{L}\left(\log^{\beta}\right)^{q_+}}{\varepsilon^{q_+}}\cdot\frac{\varepsilon^{q_+}}{2^{q_+}{}^{+2}4\mathcal{L}\left(\log^{\beta}\right)^{q_+}}=\frac{1}{4}\le\varepsilon.
\end{aligned}
$$

Therefore, for each $k\in\mathbb{N}$, we have

$$
\rho_{\ell^{q(\cdot)}\log^{\beta}\left(L^{p(\cdot)}\right)}\left(\frac{\zeta^k-\zeta}{\varepsilon}\right)\log^{\beta}\left(e+\frac{|\zeta^k-\zeta|}{\varepsilon}\right)\le\sum_{j=1}^{\mathcal{K}}\left\|\left|\frac{\zeta_r^k-\zeta_r}{\varepsilon}\right|^{q(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}(\Omega)}+\frac{1}{4}. \tag{29}
$$

Additionally, we assert that for any $r\in\{1,\dots,\mathcal{K}\}$, we also have

$$
\left\|\left|\zeta_r^k-\zeta_r\right|^{q(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}(\Omega)}\xrightarrow{k\to\infty}0.
$$

And for some specific $r\in\{1,\dots,\mathcal{K}\}$, and $\nu\in(0,1)$, we arrive at

$$
\begin{aligned}
&\rho_{\frac{p(\cdot)}{q(\cdot)}}\left(\left|\frac{\zeta_r^k-\zeta_r}{\varepsilon}\right|^{q(\cdot)}\cdot\frac{1}{\nu}\right)\log^{\beta}\left(e+\frac{|\zeta_r^k-\zeta_r|}{\varepsilon\nu}\right)\\
&\le\int_{\Omega}\left|\frac{\zeta_r^k(x)-\zeta_r(x)}{\varepsilon\nu^{\frac{1}{q_-}}}\right|^{p(x)}\log^{\beta}\left(e+\left|\frac{\zeta_r^k(x)-\zeta_r(x)}{\varepsilon\nu^{\frac{1}{q_-}}}\right|\right)dx\\
&=\rho_{p(\cdot)}\left(\frac{\zeta_r^k-\zeta_r}{\varepsilon\nu^{\frac{1}{q_-}}}\right)\log^{\beta}\left(e+\left|\frac{\zeta_r^k(x)-\zeta_r(x)}{\varepsilon\nu^{\frac{1}{q_-}}}\right|\right).
\end{aligned}
\tag{30}
$$

Since $\zeta_r^k\xrightarrow{k\to\infty}\zeta_r$ in $L^{p(\cdot)}(\Omega)$, for sufficiently big $k\in\mathbb{N}$, we have

$$
\begin{aligned}
&\rho_{\frac{p(\cdot)}{q(\cdot)}}\left(\left|\frac{\zeta_r^k-\zeta_r}{\varepsilon}\right|^{q(\cdot)}\cdot\frac{1}{\nu}\right)\log^{\beta}\left(e+\frac{|\zeta_r^k-\zeta_r|}{\varepsilon\nu}\right)\\
&\le\rho_{p(\cdot)}\left(\frac{\zeta_r^k-\zeta_r}{\varepsilon\nu^{\frac{1}{q_-}}}\right)\log^{\beta}\left(e+\left|\frac{\zeta_r^k(x)-\zeta_r(x)}{\varepsilon\nu^{\frac{1}{q_-}}}\right|\right)\le 1,
\end{aligned}
\tag{31}
$$

which proves that

$$
\left\|\left|\frac{\zeta_r^k-\zeta_r}{\varepsilon}\right|^{q(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}(\Omega)}\xrightarrow{k\to\infty}0.
$$

According to Equation (29), for sufficiently large $k \in \mathbb{N}$, we have

$$\rho_{\ell q^{(\cdot)} \log^{\beta}\left(L^{p(\cdot)}\right)}\left(\frac{\zeta^{k}-\zeta}{\varepsilon}\right)\log^{\beta}\left(e+\frac{|\zeta^{k}-\zeta|}{\varepsilon}\right)\leq 1,$$

this indicates that $\zeta^{k} \xrightarrow{k\to\infty} \zeta$ in $\ell q^{(\cdot)} \log^{\beta}\left(L^{p(\cdot)}\right)$ and it proves that the class $\mathcal{Q}$ of measurable functions is bounded. Then, we prove that conditions (a)-(d) are necessary, assuming $\mathcal{Q}$ is totally bounded. Since (a) is evident, we shift our focus to (b). Let us set $\varepsilon \in (0,1)$ and define

$$\kappa := \frac{\varepsilon}{2^{q_{+}+1}\mathcal{L}\left(\log^{\beta}\right)^{\varepsilon}},$$

where the constant $\mathcal{L}$ is once more used for the triangle inequality $\left\|\left|\zeta_{r}^{k}\right|^{q(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}(\Omega)}$. According to our assumptions, there exists $\zeta^{1},\ldots,\zeta^{k} \in \mathcal{Q}$ such that

$$\mathcal{Q} \subset p_{i=1}^{k}\mathcal{B}\left(\zeta^{i},\left(\frac{\varepsilon}{2^{q_{+}+1}\mathcal{L}\left(\log^{\beta}\right)^{\varepsilon}}\right)^{\frac{1}{q_{-}}}\right),$$

where $\mathcal{B}\left(\zeta^{i},\left(\frac{\varepsilon}{2^{q_{+}+1}\mathcal{L}\left(\log^{\beta}\right)^{\varepsilon}}\right)^{\frac{1}{q_{-}}}\right)$ are an open balls in $\ell q^{(\cdot)} \log^{\beta}\left(L^{p(\cdot)}\right)$.

Since $\zeta_{i} \in \ell q^{(\cdot)} \log^{\beta}\left(L^{p(\cdot)}\right)$, $\exists\, \mathcal{K} \in \mathbb{N}$ such that

$$\sum_{r=\mathcal{K}+1}^{\infty}\left\|\left|\zeta_{r}^{i}\right|^{q(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}(\Omega)} < \frac{\varepsilon}{2^{q_{+}+1}\mathcal{L}\left(\log^{\beta}\right)^{\varepsilon}},$$

for $i = 1,\ldots,k$. Let $\zeta \in \mathcal{Q}$, then $\exists\, i \in \{1,\ldots,k\}$, such that

$$\left\|\zeta-\zeta^{i}\right\|_{\ell q^{(\cdot)} \log^{\beta}\left(L^{p(\cdot)}\right)} < \left(\frac{\varepsilon}{2^{q_{+}+1}\mathcal{L}\left(\log^{\beta}\right)^{\varepsilon}}\right)^{\frac{1}{q_{-}}}.$$

If we consider the unit ball property, then it follows that

$$\rho_{\ell q^{(\cdot)} \log^{\beta}\left(L^{p(\cdot)}\right)}\left(\frac{\zeta-\zeta^{i}}{\left(\frac{\varepsilon}{2^{q_{+}+1}\mathcal{L}\left(\log^{\beta}\right)^{\varepsilon}}\right)^{\frac{1}{q_{-}}}}\right)\log^{\beta}\left(e+\left|\frac{\zeta-\zeta^{i}}{\left(\frac{\varepsilon}{2^{q_{+}+1}\mathcal{L}\left(\log^{\beta}\right)^{\varepsilon}}\right)^{\frac{1}{q_{-}}}}\right|\right)\leq 1.$$

Finally, the triangle inequality yields

$$\begin{aligned}
&\sum_{r=\mathcal{K}+1}^{\infty}\left\|\left|\zeta_{r}\right|^{q(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}(\Omega)}\\
&\leq 2^{q_{+}}\mathcal{L}\left(\log^{\beta}\right)^{\varepsilon}\left(\sum_{j=1}^{\infty}\left\|\left|\zeta_{r}-\zeta_{r}^{i}\right|^{q(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}(\Omega)}+\sum_{r=\mathcal{K}+1}^{\infty}\left\|\left|\zeta_{r}^{i}\right|^{q(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}(\Omega)}\right)\\
&\leq 2^{q_{+}}\mathcal{L}\left(\log^{\beta}\right)^{\varepsilon}\left(\frac{\varepsilon}{2^{q_{+}+1}\mathcal{L}\left(\log^{\beta}\right)^{\varepsilon}}\sum_{r=1}^{\infty}\frac{2^{q_{+}+1}\mathcal{L}\left(\log^{\beta}\right)^{\varepsilon}}{\varepsilon}\left\|\left|\zeta_{r}-\zeta_{r}^{i}\right|^{q(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}(\Omega)}+\frac{\varepsilon}{2^{q_{+}+1}\mathcal{L}\left(\log^{\beta}\right)^{\varepsilon}}\right)
\end{aligned}$$

$$\leq 2^{q_+}\mathcal{L}\left(\log^{\beta}\right)^{\varepsilon}\left(\frac{\varepsilon}{2^{q_++1}\mathcal{L}\left(\log^{\beta}\right)^{\varepsilon}}\sum_{r=1}^{\infty}\left\|\left|\frac{\zeta_r-\zeta_r^i}{\kappa^{\frac{1}{q_-}}}\right|^{q(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}(\Omega)}+\frac{\varepsilon}{2^{q_++1}\mathcal{L}\left(\log^{\beta}\right)^{\varepsilon}}\left(\log^{\beta}\right)^{\varepsilon}\right)$$

$$=2^{q_+}\mathcal{L}\left(\log^{\beta}\right)^{\varepsilon}\left(\frac{\varepsilon}{2^{q_++1}\mathcal{L}\left(\log^{\beta}\right)^{\varepsilon}}\rho_{\ell q^{(\cdot)}\log^{\beta}\left(L^{p(\cdot)}\right)}\left(\frac{\zeta-\zeta^i}{\left(\frac{\varepsilon}{2^{q_++1}\mathcal{L}\left(\log^{\beta}\right)^{\varepsilon}}\right)^{\frac{1}{q_-}}}\right)\right.$$

$$\left.\times\log^{\beta}\left(e+\left|\frac{\zeta-\zeta^i}{\left(\frac{\varepsilon}{2^{q_++1}\mathcal{L}\left(\log^{\beta}\right)^{\varepsilon}}\right)^{\frac{1}{q_-}}}\right|\right)+\frac{\varepsilon}{2^{q_++1}\mathcal{L}\left(\log^{\beta}\right)^{\varepsilon}}\left(\log^{\beta}\right)^{\varepsilon}\right)$$

$$\leq 2^{q_++1}\mathcal{L}\left(\log^{\beta}\right)^{\varepsilon}\kappa=\varepsilon.$$

which establishes (b).

To show that conditions (c) and (d) are valid, let us consider $r \in \mathbb{N}$ and analyze Lemma 5, then one has

$$\|\zeta_r\|_{L^{p(\cdot)}(\Omega)} \leq \|\zeta\|_{\ell q^{(\cdot)}\log^{\beta}\left(L^{p(\cdot)}\right)},$$

valid for each $\zeta \in \ell q^{(\cdot)}\log^{\beta}\left(L^{p(\cdot)}\right)$. As a result, the projective map

$$\pi_r : \ell q^{(\cdot)}\log^{\beta}\left(L^{p(\cdot)}\right) \to L^{p(\cdot)}(\Omega),$$

is defined as follows:

$$\pi_r(\zeta) := \zeta_r \quad \text{for } \zeta \in \ell q^{(\cdot)}\log^{\beta}\left(L^{p(\cdot)}\right).$$

As $\ell q^{(\cdot)}\log^{\beta}\left(L^{p(\cdot)}\right)$ is a quasi-Banach space, it is a closed and bounded subset of $\mathcal{Q}$, which is equivalent to its precompactness. Following that, the continuous image of the projective map is defined as

$$\pi_r(\mathcal{Q}) = \{\zeta_r : \zeta \in \mathcal{Q}\},$$

which is precompact in $L^{p(\cdot)}(\Omega)$, and the proof is completed.

The results of Theorem 13 show that if $p \in \mathcal{P}_0(\Omega)$ instead of $p \in \mathcal{P}_0^{\log}(\Omega)$, assumptions (a)–(d) imply the closedness and boundedness. As a result, the theorem holds true. □

6. Conclusions and Future Remarks

The structure of function spaces and the behavior of operators are clarified by the equalities and inequalities of the Hilbert space, which are essential for functional analysis. The paper contributes in two important ways. The first is the employment of continuous self-adjoint operators on Hilbert spaces, which are essential when working with complex functions on higher dimensions, to extend equality from the classical sense to the tensorial sense using the tensor approximation. Next, the authors introduce a new type of function space, namely mixed-Orlicz–Zygmund space, which combines the Orlicz–Zygmund space of integrability with the sequence space. To prove its validity, we also examine several of its characteristics, including the completeness, compactness, and separability. We believe that this generalized space plays a very essential role in the progress of function spaces and their linked problems, such as the bounds of operators, the regularity of partial differential equations, embedding, etc.

This study advances mathematical inequality theory by investigating inequalities supporting tensor Hilbert spaces and mixed norm function spaces, which is a rare topic

in the literature in terms of produced results. Following these results, we advise readers to attempt to develop the Maclaurin inequality on coordinates using tensor operations and also investigate various other properties of our newly defined mixed spaces, such as checking the boundedness of operators, analysis of differential equations, extending over a complex domain, etc.

Author Contributions: Conceptualization, W.A.; Validation, S.B.; Formal analysis, W.A., M.A. and S.B.; Resources, M.Z.M.; Data curation, S.B.; Writing—original draft, W.A.; Writing—review & editing, M.A. and M.Z.M.; Supervision, M.A.; Funding acquisition, M.Z.M. All authors have read and agreed to the published version of the manuscript.

Funding: APC were supported by the resources of the corresponding author, Mutum Zico Meetei.

Data Availability Statement: Data are contained within the article.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Diening, L.; Lee, M.; Ok, J. Parabolic weighted Sobolev–Poincaré type inequalities. *Nonlinear Anal.* **2022**, *218*, 112772. [CrossRef]
2. Harboure, E.; Salinas, O.; Viviani, B. Boundedness of Operators Related to a Degenerate Schrödinger Semigroup. *Potential Anal.* **2022**, *57*, 401–431. [CrossRef]
3. Poudel, S.; Wang, X.; Lee, S. A Novel Technique for Minimizing Energy Functional Using Neural Networks. *Eng. Appl. Artif. Intell.* **2024**, *133*, 108313. [CrossRef]
4. Duan, S.-Q.; Wang, Q.-W.; Duan, X.-F. On Rayleigh Quotient Iteration for the Dual Quaternion Hermitian Eigenvalue Problem. *Mathematics* **2024**, *12*, 4006. [CrossRef]
5. Lakoba, T.I.; Micula, S. Recent Progress in Studies of Stability of Numerical Schemes. *Symmetry* **2022**, *14*, 2692. [CrossRef]
6. Javed, M.Z.; Awan, M.U.; Ciurdariu, L.; Dragomir, S.S.; Almalki, Y. On Extended Class of Totally Ordered Interval-Valued Convex Stochastic Processes and Applications. *Fractal Fract.* **2024**, *8*, 577. [CrossRef]
7. Costa, T.M.; Román-Flores, H. Some Integral Inequalities for Fuzzy-Interval-Valued Functions. *Inf. Sci.* **2017**, *420*, 110–125. [CrossRef]
8. Liu, W.; Shi, F.; Ye, G.; Zhao, D. The Properties of Harmonically Cr-h-Convex Function and Its Applications. *Mathematics* **2022**, *10*, 2089. [CrossRef]
9. Kara, H.; Budak, H.; Ali, M.A.; Sarikaya, M.Z.; Chu, Y.-M. Weighted Hermite–Hadamard Type Inclusions for Products of Co-Ordinated Convex Interval-Valued Functions. *Adv. Differ. Equ.* **2021**, *2021*, 104. [CrossRef]
10. Balazs, P.; Bellomonte, G.; Hosseinnezhad, H. Frame-Related Sequences in Chains and Scales of Hilbert Spaces. *Axioms* **2022**, *11*, 180. [CrossRef]
11. Afzal, W.; Prosviryakov, E.Y.; El-Deeb, S.M.; Almalki, Y. Some New Estimates of Hermite–Hadamard, Ostrowski and Jensen-Type Inclusions for h-Convex Stochastic Process via Interval-Valued Functions. *Symmetry* **2023,** *15*, 831. [CrossRef]
12. Chalco-Cano, Y.; Flores-Franulič, A.; Román-Flores, H. Ostrowski Type Inequalities for Interval-Valued Functions Using Generalized Hukuhara Derivative. *Comput. Appl. Math.* **2012**, *31*, 457–472.
13. Afzal, W.; Abbas, M.; Macías-Díaz, J.E.; Treanţă, S. Some H-Godunova–Levin Function Inequalities Using Center Radius (Cr) Order Relation. *Fractal Fract.* **2022**, *6*, 518. [CrossRef]
14. Acu, A.M.; Gonska, H.; Raşa, I. Grüss-Type and Ostrowski-Type Inequalities in Approximation Theory. *Ukr. Math. J.* **2011,** *63*, 843–864. [CrossRef]
15. Tariq, M.; Ahmad, H.; Budak, H.; Sahoo, S.K.; Sitthiwirattham, T.; Reunsumrit, J. A Comprehensive Analysis of Hermite-Hadamard Type Inequalities via Generalized Preinvex Functions. *Axioms* **2021**, *10*, 328. [CrossRef]
16. Amer Latif, M. Fejér-Type Inequalities for Harmonically Convex Functions and Related Results. *Symmetry* **2023**, *15*, 1602. [CrossRef]
17. Ahmadini, A.A.H.; Afzal, W.; Abbas, M.; Aly, E.S. Weighted Fejér, Hermite-Hadamard, and Trapezium-Type Inequalities for (h_1, h_2)-Godunova-Levin Preinvex Function with Applications and Two Open Problems.*Mathematics* **2024**, *12*, 382. [CrossRef]
18. Pečarić, J.; Perić, I.; Roqia, G. Exponentially Convex Functions Generated by Wulbert's Inequality and Stolarsky-Type Means. *Math. Comput. Model.* **2012**, *55*, 1849–1857. [CrossRef]
19. A. Khan, Z.; Afzal, W.; Nazeer, W.; K. Asamoah, J.K. Some New Variants of Hermite-Hadamard and Fejér-Type Inequalities for Godunova-Levin Preinvex Class of Interval-Valued Functions. *J. Math.* **2024**, *2024*, 8814585. [CrossRef]

20. Liu, C. The Renormalization Method Based on the Taylor Expansion and Applications for Asymptotic Analysis. *Nonlinear Dyn.* **2017**, *88*, 1099–1124. [CrossRef]
21. Monera, M.G.; Montesinos-Amilibia, A.; Sanabria-Codesal, E. The Taylor Expansion of the Exponential Map and Geometric Applications. *RACSAM* **2014**, *108*, 881–906. [CrossRef]
22. Almalki, Y.; Afzal, W. Some New Estimates of Hermite–Hadamard Inequalities for Harmonical cr-h-Convex Functions via Generalized Fractional Integral Operator on Set-Valued Mappings. *Mathematics* **2023**, *11*, 4041. [CrossRef]
23. Khôi, L.H. Hilbert Spaces of Holomorphic Dirichlet Series and Applications to Convolution Equations. *J. Math. Anal. Appl.* **1997**, *206*, 10–24. [CrossRef]
24. Feng, H.; Hui, S.Y.W.; Shen, R. Approximating Reproducing Kernel Hilbert Space Functions by Bernstein Operators. *Results Math* **2024**, *79*, 234. [CrossRef]
25. Kaltenbach, A.; Růžička, M. Variable Exponent Bochner–Lebesgue Spaces with Symmetric Gradient Structure. *J. Math. Anal. Appl.* **2021**, *503*, 125355. [CrossRef]
26. Bachar, M.; Khamsi, M.A.; Méndez, O. Examining Nonlinear Fredholm Equations in Lebesgue Spaces with Variable Exponents. *Symmetry* **2023**, *15*, 2014. [CrossRef]
27. Mostafazadeh, A. Consistent Treatment of Quantum Systems with a Time-Dependent Hilbert Space. *Entropy* **2024**, *26*, 314. [CrossRef]
28. Ghalyan, N.F.; Ray, A.; Jenkins, W.K. A Concise Tutorial on Functional Analysis for Applications to Signal Processing. *Sci* **2022**, *4*, 40. [CrossRef]
29. Ramer, R. On Nonlinear Transformations of Gaussian Measures. *J. Funct. Anal.* **1974**, *15*, 166–187. [CrossRef]
30. Altwaijry, N.; Feki, K.; Minculete, N. A Generalized Norm on Reproducing Kernel Hilbert Spaces and Its Applications. *Axioms* **2023**, *12*, 645. [CrossRef]
31. Ruiz-Medina, M.D.; Angulo, J.M.; Anh, V.V. Application of Hilbert-0pace Methods to Random Field Modelling and Estimation. *Am. J. Math. Manag. Sci.* **2001**, *21*, 263–282.
32. Araki. H.; Hansen. F. Jensen's operator inequality for functions of several variables. *Proc. Am. Math. Soc.* **2000**, *7*, 2075–2084.
33. Afzal, W.; Abbas, M.; Breaz, D.; Cotîlă, L.-I. Fractional Hermite-Hadamard, Newton-Milne, and Convexity Involving Arithmetic-Geometric Mean-Type Inequalities in Hilbert and Mixed-Norm Morrey Spaces $\ell_{q(\cdot)}\left(M_{p(\cdot),v(\cdot)}\right)$ with Variable Exponents. *Fractal Fract.* **2024**, *8*, 518. [CrossRef]
34. Ghosh, P.; Samanta, T. Introduction of Frame in Tensor Product of N-Hilbert Spaces. *Sahand Commun. Math. Anal.* **2021**, *17*, 16.
35. Stojiljkovic, V. Twice Differentiable Ostrowski Type Tensorial Norm Inequality for Continuous Functions of self-adjoint Operators in Hilbert Spaces. *Eur. J. Pure Appl. Math.* **2023**, *16*, 1421–1433. [CrossRef]
36. Stojiljković, V. Simpson Type Tensorial Norm Inequalities for Continuous Functions of self-adjoint Operators in Hilbert Spaces. *CMI* **2024**, *33*, 105–117. [CrossRef]
37. Silvestru, D. A Trapezoid Type Tensorial Norm Inequality for Continuous Functions of self-adjoint Operators in Hilbert Spaces. *Istanb. J. Math.* **2023**, *1*, 48–56.
38. Khan, Z.A.; Afzal, W.; Abbas, M.; Ro, J.; Aloraini, N.M. A Novel Fractional Approach to Finding the Upper Bounds of Simpson and Hermite-Hadamard-Type Inequalities in Tensorial Hilbert Spaces by Using Differentiable Convex Mappings. *Aims MATH* **2024**, *9*, 35151–35180. [CrossRef]
39. Afzal, W.; Abbas, M.; Alsalami, O.M. Bounds of Different Integral Operators in Tensorial Hilbert and Variable Exponent Function Spaces. *Mathematics* **2024**, *12*, 2464. [CrossRef]
40. Wada, S. On Some Refinement of the Cauchy–Schwarz Inequality. *Linear Algebra Appl.* **2007**, *420*, 433–440. [CrossRef]
41. Gümüş, M.; Hezenci, F.; Budak, H. Some New Approaches to Fractional Euler-Maclaurin-Type Inequalities via Various Function Classes. *Fractal Fract.* **2024**, *8*, 372. [CrossRef]
42. Meftah, B.; Souahi, A.; Merad, M. Some Local Fractional Maclaurin Type Inequalities for Generalized Convex Functions and Their Applications. *Chaos Solitons Fractals* **2022**, *162*, 112504. [CrossRef]
43. Merad, M.; Meftah, B.; Moumen, A.; Bouye, M. Fractional Maclaurin-Type Inequalities for Multiplicatively Convex Functions. *Fractal Fract.* **2023**, *7*, 879. [CrossRef]
44. Stojiljković, V.; Dragomir, S.S. Tensorial Simpson 1/8 Type Inequalities for Convex Functions of self-adjoint Operators in Hilbert Space. *Eur. J. Math. Anal.* **2024**, *4*, 17. [CrossRef]
45. Stojiljkovic, V. Generalized Tensorial Simpson Type Inequalities for Convex Functions of self-adjoint Operators in Hilbert Space. *Maltepe J. Math.* **2024**, *6*, 78–89. [CrossRef]
46. Ghosh, P.; Samanta, T.K. Generalized Fusion Frame in A Tensor Product of Hilbert Space. *J. Indian Math. Soc.* **2022**, *89*, 58–71. [CrossRef]
47. Orlicz, W. Über Konjugierte Exponentenfolgen. *Studia Math.* **1931**, *3*, 200–211. [CrossRef]
48. Kováčik, O.; Rákosník, J. On spaces $L^{p(\mu)}, W^{k,p(\mu)}$. *Czechoslov. Math. J.* **1991**, *41*, 592–618. [CrossRef]

49. Bachar, M.; Mendez, O.; Bounkhel, M. Modular Uniform Convexity of Lebesgue Spaces of Variable Integrability. *Symmetry* **2018**, *10*, 708. [CrossRef]
50. El Amri, A.; Khamsi, M.A. New Modular Fixed-Point Theorem in the Variable Exponent Spaces $\ell_p(\mu)$. *Mathematics* **2022**, *10*, 869. [CrossRef]
51. Sultan, M.; Sultan, B.; Aloqaily, A.; Mlaiki, N.; Sultan, M.; Sultan, B.; Aloqaily, A.; Mlaiki, N. Boundedness of Some Operators on Grand Herz Spaces with Variable Exponent. *Aims MATH* **2023**, *8*, 12964–12985. [CrossRef]
52. Kopaliani, T. Interpolation Theorems for Variable Exponent Lebesgue Spaces. *J. Funct. Anal.* **2009**, *257*, 3541–3551. [CrossRef]
53. Lazzaretti, M.; Calatroni, L.; Estatico, C. Modular-Proximal Gradient Algorithms in Variable Exponent Lebesgue Spaces. *SIAM J. Sci. Comput.* **2022**, *44*, A3463–A3489. [CrossRef]
54. Botelho, F.; Jamison, J. Algebraic and Topological Reflexivity Properties of $\ell_p(\mu)$ Spaces. *J. Math. Anal. Appl.* **2008**, *346*, 141–144. [CrossRef]
55. Amri, A.E.; Khamsi, M.A.; Méndez, O.D. A Fixed Point Theorem in the Lebesgue Spaces of Variable Integrability $L^{p(\cdot)}$. *Symmetry* **2023**, *15*, 1999. [CrossRef]
56. Brezis, H. Weak Topologies. Reflexive Spaces. Separable Spaces. Uniform Convexity. In *Functional Analysis, Sobolev Spaces and Partial Differential Equations*; Brezis, H., Ed.; Springer: New York, NY, USA, 2011; pp. 55–87. ISBN 9780387709147.
57. Luo, Z.; Cheng, Q. Characterization of Reflexivity by Convex Functions. *J. Funct. Spaces* **2016**, *2016*, 1–4. [CrossRef]
58. Jain, P.K.; Arora, K.K.; Sinha, D.P. Quasi Reflexivity and the Sup of Linear Functionals. *Bull. Austral. Math. Soc.* **1997**, *55*, 89–98. [CrossRef]
59. Pasqualetto, E.; Rajala, T. Vector Calculus on Weighted Reflexive Banach Spaces. *Math. Z.* **2024,** *308*, 48. [CrossRef]
60. Samko, S. Variable Exponent Herz Spaces. *Mediterr. J. Math.* **2013**, *10*, 2007–2025. [CrossRef]
61. Rodabaugh, S.E. Separation Axioms: Representation Theorems, Compactness, and Compactifications. In *Mathematics of Fuzzy Sets: Logic, Topology, and Measure Theory*; Höhle, U., Rodabaugh, S.E., Eds.; Springer: Boston, MA, USA, 1999; pp. 481–552, ISBN 9781461550792.
62. Green, J.Wm. Moore-Closed Spaces, Completeness and Centered Bases. *Gen. Topol. Its Appl.* **1974**, *4*, 297–313. [CrossRef]
63. Almira, J. M.; Luther, U. Compactness and Generalized Approximation Spaces. *Numer. Funct. Anal. Optim.* **2002**, *23*, 1–38. [CrossRef]
64. Carro, M.J.; Soria, J. Weighted Lorentz Spaces and the Hardy Operator. *J. Funct. Anal.* **1993**, *112*, 480–494. [CrossRef]
65. Bandaliyev, R.A.; Górka, P.; Guliyev, V.S.; Sawano, Y. Relatively Compact Sets in Variable Exponent Morrey Spaces on Metric Spaces. *Mediterr. J. Math.* **2021**, *18*, 232. [CrossRef]
66. Eleuteri, M.; Passarelli di Napoli, A. Lipschitz Regularity of Minimizers of Variational Integrals with Variable Exponents. *Nonlinear Anal. Real World Appl.* **2023**, *71*, 103815. [CrossRef]
67. Giannetti, F.; Passarelli Di Napoli, A. Higher Differentiability of Minimizers of Variational Integrals with Variable Exponents. *Math. Z.* **2015**, *280*, 873–892. [CrossRef]
68. Byun, S.-S.; Oh, J. Global Gradient Estimates for the Borderline Case of Double Phase Problems with BMO Coefficients in Nonsmooth Domains. *J. Differ. Equ.* **2017**, *263*, 1643–1693. [CrossRef]
69. Vetro, C.; Zeng, S. Regularity and Dirichlet Problem for Double-Phase Energy Functionals of Different Power Growth. *J. Geom. Anal.* **2024**, *34*, 105. [CrossRef]
70. Ferreira, R.; Hästö, P.; Ribeiro, A.M. Characterization of Generalized Orlicz Spaces. *Commun. Contemp. Math.* **2020**, *22*, 1850079. [CrossRef]
71. Sultan, B.; Hussain, A.; Sultan, M. Chracterization of Generalized Campanato Spaces with Variable Exponents via Fractional Integrals. *J. Pseudo-Differ. Oper. Appl.* **2025**, *16*, 22. [CrossRef]
72. Bachar, M. The Variation of Constants Formula in Lebesgue Spaces with Variable Exponents. *Symmetry* **2024**, *16*, 978. [CrossRef]
73. Lukkassen, D.; Persson, L.-E.; Samko, S.; Wall, P. Weighted Hardy-Type Inequalities in Variable Exponent Morrey-Type Spaces. *J. Funct. Spaces Appl.* **2013**, *2013*, 1–11. [CrossRef]
74. Diening, L.; Harjulehto, P.; Hästö, P.; Ruzicka, M. *Lebesgue and Sobolev Spaces with Variable Exponents*; Lecture Notes in Mathematics; Springer: Berlin/Heidelberg, Germany, 2011; Volume 2017, ISBN 9783642183621/9783642183638.
75. Ragusa, M.A.; Tachikawa, A. Regularity of Minimizers for Double Phase Functionals of Borderline Case with Variable Exponents. *Adv. Nonlinear Anal.* **2024**, *13*, 20240017. [CrossRef]
76. Cruz-Uribe, D.; Gogatishvili, A.; Kopaliani, T. On the Embedding between the Variable Lebesgue Space $Lp(\cdot)(\Omega)$ and the Orlicz Space $L(LogL)\alpha(\Omega)$. *J. Math. Anal. Appl.* **2025**, *544*, 129081 [CrossRef]
77. Koranyi, A. On Some Classes of Analytic Functions of Several Variables. *Trans. Am. Math. Soc.* **1961**, *101*, 520. [CrossRef]
78. Dragomir, S. Tensorial and Hadamard Product Inequalities for Synchronous Functions. *Commun. Adv. Math. Sci.* **2023**, *6*, 177–187. [CrossRef]
79. Akın, L. On Innovations of N-Dimensional Integral-Type Inequality on Time Scales. *Adv. Differ. Equ.* **2021**, *2021*, 148. [CrossRef]
80. Zhang, H.; Zhou, J. Mixed-Norm Amalgam Spaces and Their Predual. *Symmetry* **2022**, *14*, 74. [CrossRef]

81. Saibi, K. Variable Besov–Morrey Spaces Associated with Operators. *Mathematics* **2023**, *11*, 2038. [CrossRef]
82. Giannetti, F.; Passarelli Di Napoli, A. Regularity Results for a New Class of Functionals with Non-Standard Growth Conditions. *J. Differ. Equ.* **2013**, *254*, 1280–1305. [CrossRef]

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