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Nonlinear Almost Contractions of Pant Type Under Binary Relations with an Application to Boundary Value Problems

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Abstract: In this article, we prove certain fixed point findings for a nonlinear almost contraction map of the Pant type in a metric space endowed with a binary relation. The outcomes presented here expand, develop, enhance, and consolidate several existing findings. With a view to arguing for our findings, we furnish a few examples. As an application of our outcomes, we discuss the existence of a unique solution of a boundary value problem of the first order.

Keywords: fixed points; ϕ-contraction maps; boundary value problems

MSC: 54H25; 47H10; 34B15

1. Introduction

The BCP is a key and essential outcome of the metrical fixed point theory. In fact, BCP ensures that a contraction on a complete MS possesses a unique fixed point. This finding also delivers a method for estimating the unique fixed point. There are numerous generalizations of the BCP in the existing literature. One of natural generalizations of contraction is ϕ-contraction. It is originated from ordinary contraction by the replacement of a suitable function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ for the Lipschitzian constant $c \in [0, 1)$. Browder [1], in 1968, essentially investigated the concept of ϕ-contraction, whereby ϕ appeared as an increasing control function having the right continuity requirement, and implemented it for expanding the BCP. Since then, various researchers have further modified the properties of the involved functions ϕ to expand Browder's fixed point finding; see [2–4].

In recent years, a lot of fixed point outcomes have appeared in the context of MS endowed with a partial ordering (e.g., [5,6]). In 2015, Alam and Imdad [7] revealed an amended and accessible formulation of BCP, wherein the authors imposed an arbitrary BR (rather than partial order) on the ambient MS such that the underlying map persists within this BR. Later on, numerous researchers improved and expanded the relation-theoretic contraction principle, e.g., [8–13]. A relatively weaker contraction condition is adopted for this work comparatively weaker than those found in recent research. Indeed the contraction condition is intended to hold only for comparative elements via an underlying BR rather than the whole space. Because of such restricted characteristics, these findings are to be utilized in domains of typical BVP, nonlinear matrix equations, and nonlinear integral equations, wherein the usual fixed point findings are not applicable.

The problem of the existence and uniqueness of solutions of differential or integral equations is among the most common applications of the fixed point theory. Sometimes, for the nonlinear differential or integral equations admit more than one solution. In such cases,

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the classical BCP is not applicable. However, these problems can be solved employing non-unique fixed point theorems. One of the recent non-uniqueness fixed point theorems was proved by Pant [14], which runs as follows.

Theorem 1. *If* $(\mathbf{U}, \varpi)$ *is a complete MS and* $\mathcal{Q} : \mathbf{U} \to \mathbf{U}$ *is a map for which* $\exists\ c \in [0,1)$*, verifying*

$$\varpi(\mathcal{Q}u, \mathcal{Q}v) \leq c \cdot \varpi(u, v), \quad \forall\ u, v \in \mathbf{U} \text{ with } [u \neq \mathcal{Q}(u) \text{ or } v \neq \mathcal{Q}(v)],$$

then $\mathcal{Q}$ *owns a fixed point.*

During the process, Pant [14] also observed that the fixed point set and domain of the underlying map involved in Theorem 1 displayed interesting geometrical, algebraic, and dynamical features besides determining the cardinality of the fixed point set. In this continuation, Pant [15] established a generalization of Theorem 1 under ϕ-contraction.

In 2004, Berinde [16] invented a new extension of BCP via "almost contraction".

Definition 1 ([16])**.** *A self-map* $\mathcal{Q}$ *on an MS* $(\mathbf{U}, \varpi)$ *constitutes an almost contraction if* $\exists$ $c \in (0,1)$ *and* $\ell \in \mathbb{R}^+$*, which enjoy*

$$\varpi(\mathcal{Q}u, \mathcal{Q}v) \leq c \cdot \varpi(u, v) + \ell \cdot \varpi(v, \mathcal{Q}u), \quad \forall\ u, v \in \mathbf{U},$$

that, by the symmetricity of ϖ*, is identical to*

$$\varpi(\mathcal{Q}u, \mathcal{Q}v) \leq c \cdot \varpi(u, v) + \ell \cdot \varpi(u, \mathcal{Q}v), \quad \forall\ u, v \in \mathbf{U}.$$

Theorem 2 ([16])**.** *An almost contraction self-map on a complete MS admits a fixed point.*

To obtain a uniqueness theorem corresponding to Theorem 2, Babu et al. [17] formulated a significant subclass of almost contractions.

Definition 2 ([17])**.** *A self-map* $\mathcal{Q}$ *on an MS* $(\mathbf{U}, \varpi)$ *constitutes a strict almost contraction if* $\exists$ $c \in (0,1)$ *and* $\ell \in \mathbb{R}^+$ *which enjoy*

$$\varpi(\mathcal{Q}u, \mathcal{Q}v) \leq c \cdot \varpi(u, v) + \ell \cdot \min\{\varpi(u, \mathcal{Q}u), \varpi(v, \mathcal{Q}v), \varpi(u, \mathcal{Q}v), \varpi(v, \mathcal{Q}u)\}, \qquad \forall\ u, v \in \mathbf{U}.$$

Every strict almost contraction is obviously an almost contraction, but the reverse is not true as showcased by Example 2.6 [17]. Berinde and Păcurar [18] investigated that an almost contraction remains continuous on a fixed point set. In the same continuation, Berinde [19] presented certain fixed point outcomes under almost ϕ-contractions. Turinici [20] and Alfuraidan et al. [21] initiated nonlinear variants of almost contraction maps and utilized the same to extend the findings of Berinde [16].

Theorem 3 ([17])**.** *A strict almost contraction self-map on a complete MS offers a unique fixed point.*

The collection of the increasing functions $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\sum_{n=1}^{\infty} \phi^n(t) < \infty, \quad \forall\ t \in \mathbb{R}^+ \backslash \{0\}$ is denoted by Φ (c.f. Bianchini and Grandolfi [4]).

Remark 1 ([22])**.** *Each* $\phi \in \Phi$ *verifies*

(i) $\phi(t) < t$, $\forall\ t \in \mathbb{R}^+ \backslash \{0\}$;

(ii) $\lim_{r \to 0^+} \phi(r) = 0 = \phi(0)$.

Turinici [20] and Alfuraidan et al. [21], motivated by Berinde [16], suggested the class of functions $\theta : \mathbb{R}^+ \to \mathbb{R}^+$ with $\lim_{r \to 0^+} \theta(r) = 0 = \theta(0)$. In the sequel, this class is symbolized by Θ.

In the current investigation, we combine the four contractivity conditions that were previously mentioned, viz., ϕ-contraction, almost contraction, Pant contraction, and relational contraction, and employ it to convey the existence and uniqueness of fixed points in the framework of a relational MS. To indicate the profitability of our findings, some instances are constructed. We infer certain current results in the process, particularly owing to Bianchini and Grandolfi [4], Pant [15], Khan [10], Algehyne et al. [11], Aljawi and Uddin [12], Algehyne et al. [13], Babu et al. [17], Berinde [19], Turinici [20], and others that are similar. To explain our findings, we furnish a few examples. To visually demonstrate our outcomes, we carry out the unique solution of a BVP, confirming a few more predictions.

2. Preliminaries

A BR $\mathbf{R}$ on a set $\mathbf{U}$ is a subset of $\mathbf{U}^2$. In the definitions below, $\mathbf{U}$ is a set, ϖ is a metric on $\mathbf{U}$, $\mathcal{Q} : \mathbf{U} \to \mathbf{U}$ is a map, and $\mathbf{R}$ is a BR on $\mathbf{U}$.

Definition 3 ([7]). *The elements $u, v \in \mathbf{U}$ are termed $\mathbf{R}$-comparative, denoted by $[u, v] \in \mathbf{R}$, if $(u, v) \in \mathbf{R}$ or $(v, u) \in \mathbf{R}$.*

Definition 4 ([23]). *$\mathbf{R}^{-1} := \{(u, v) \in \mathbf{U}^2 : (v, u) \in \mathbf{R}\}$ is referred to as the transpose of $\mathbf{R}$.*

Definition 5 ([23]). *The relation $\mathbf{R}^s := \mathbf{R} \cup \mathbf{R}^{-1}$ is named the symmetric closure of $\mathbf{R}$.*

Proposition 1 ([7]). *$(u, v) \in \mathbf{R}^s \iff [u, v] \in \mathbf{R}$.*

Definition 6 ([7]). *$\mathbf{R}$ is termed $\mathcal{Q}$-closed if $(\mathcal{Q}u, \mathcal{Q}v) \in \mathbf{R}$, whenever $(u, v) \in \mathbf{R}$.*

Proposition 2 ([9]). *If $\mathbf{R}$ is $\mathcal{Q}$-closed, then $\mathbf{R}$ is $\mathcal{Q}^n$-closed.*

Definition 7 ([7]). *A sequence $\{u_n\} \subset \mathbf{U}$ verifying $(u_n, u_{n+1}) \in \mathbf{R}$, $\forall\, n \in \mathbb{N}$ is termed $\mathbf{R}$-preserving.*

Definition 8 ([8]). *$(\mathbf{U}, \varpi)$ is termed $\mathbf{R}$-complete whenever every Cauchy and $\mathbf{R}$-preserving sequence in $\mathbf{U}$ is convergent.*

Definition 9 ([8]). *$\mathcal{Q}$ is termed $\mathbf{R}$-continuous if for any $u \in \mathbf{U}$ and for any $\mathbf{R}$-preserving sequence $\{u_n\} \subset \mathbf{U}$ with $u_n \xrightarrow{\varpi} u$, we have $\mathcal{Q}(u_n) \xrightarrow{\varpi} \mathcal{Q}(u)$.*

Definition 10 ([7]). *$\mathbf{R}$ is named ϖ-self-closed if any convergent and $\mathbf{R}$-preserving sequence in $\mathbf{U}$ contains a subsequence, each term of which is $\mathbf{R}$-comparative with the limit.*

Definition 11 ([22]). *A subset $\mathbf{V} \subseteq \mathbf{U}$ is termed $\mathbf{R}$-directed if for all $v, w \in \mathbf{V}$, $\exists\, u \in \mathbf{U}$ with $(v, u) \in \mathbf{R}$ and $(w, u) \in \mathbf{R}$.*

Using the symmetry of metric ϖ, we propose the following assertions.

Proposition 3. *If $\phi \in \Phi$ and $\theta \in \Theta$, then the following are identical:*

(i) $\varpi(\mathcal{Q}u, \mathcal{Q}v) \leq \phi(\varpi(u, v)) + \min\{\theta(\varpi(v, \mathcal{Q}u)), \theta(\varpi(u, \mathcal{Q}v))\}, \forall (u, v) \in \mathbf{R}$ *with* $[u \neq \mathcal{Q}(u)$ *or* $v \neq \mathcal{Q}(v)]$;

(ii) $\varpi(\mathcal{Q}u, \mathcal{Q}v) \leq \phi(\varpi(u, v)) + \min\{\theta(\varpi(v, \mathcal{Q}u)), \theta(\varpi(u, \mathcal{Q}v))\}, \forall [u, v] \in \mathbf{R}$ *with* $[u \neq \mathcal{Q}(u)$ *or* $v \neq \mathcal{Q}(v)]$.

Proposition 4. *If $\phi \in \Phi$ and $\theta \in \Theta$, then the following are identical:*

(i) $\varpi(\mathcal{Q}u, \mathcal{Q}v) \leq \phi(\varpi(u, v)) + \min\{\theta(\varpi(v, \mathcal{Q}u)), \theta(\varpi(u, \mathcal{Q}v)), \theta(\varpi(u, \mathcal{Q}u)), \theta(\varpi(v, \mathcal{Q}v))\}$, $\forall\ (u, v) \in \mathbf{R}$;

(ii) $\varpi(\mathcal{Q}u, \mathcal{Q}v) \leq \phi(\varpi(u, v)) + \min\{\theta(\varpi(v, \mathcal{Q}u)), \theta(\varpi(u, \mathcal{Q}v)), \theta(\varpi(u, \mathcal{Q}u)), \theta(\varpi(v, \mathcal{Q}v))\}$, $\forall\ [u, v] \in \mathbf{R}$.

3. Main Results

Here, we reveal the fixed point outcomes for a new contraction inequality via two test functions ϕ and θ in the framework of a relational MS.

Theorem 4. *Assume that $(\mathbf{U}, \varpi)$ is an MS assigned with a BR $\mathbf{R}$ and $\mathcal{Q} : \mathbf{U} \to \mathbf{U}$ is a map. Also, we have the following:*

(a) $(\mathbf{U}, \varpi)$ *is an* $\mathbf{R}$*-complete MS.*

(b) $\exists\ u_0 \in \mathbf{U}$ *verifying* $(u_0, \mathcal{Q}u_0) \in \mathbf{R}$.

(c) $\mathbf{R}$ *is* $\mathcal{Q}$*-closed.*

(d) $\mathcal{Q}$ *serves as* $\mathbf{R}$*-continuous or* $\mathbf{R}$ *remains* ϖ*-self-closed.*

(e) $\exists\ \phi \in \Phi$ *and* $\theta \in \Theta$*, verifying*

$$\varpi(\mathcal{Q}u, \mathcal{Q}v) \leq \phi(\varpi(u, v)) + \min\{\theta(\varpi(v, \mathcal{Q}u)), \theta(\varpi(u, \mathcal{Q}v))\},$$
$$\forall\ (u, v) \in \mathbf{R} \text{ with } [u \neq \mathcal{Q}(u) \text{ or } v \neq \mathcal{Q}(v)].$$

Then, $\mathcal{Q}$ admits a fixed point.

Proof. Our proof is accomplished in several steps:
Step–1. Consider the sequence $\{\mathsf{u}_n\} \subset \mathbf{U}$ such that

$$\mathsf{u}_n = \mathcal{Q}^n(\mathsf{u}_0) = \mathcal{Q}(\mathsf{u}_{n-1}),\ \forall\ n \in \mathbb{N}. \tag{1}$$

Step–2. We prove that $\{\mathsf{u}_n\}$ remains an $\mathbf{R}$-preserving sequence. Utilizing (b), the $\mathcal{Q}$-closedness of $\mathbf{R}$, and Proposition 2, we find

$$(\mathcal{Q}^n\mathsf{u}_0, \mathcal{Q}^{n+1}\mathsf{u}_0) \in \mathbf{R}$$

which, upon utilizing (1), becomes

$$(\mathsf{u}_n, \mathsf{u}_{n+1}) \in \mathbf{R}, \quad \forall\ n \in \mathbb{N}. \tag{2}$$

Step–3. Denote $\varpi_n := \varpi(\mathsf{u}_n, \mathsf{u}_{n+1})$. If $\exists\ n_0 \in \mathbb{N}_0$ with $\varpi_{n_0} = 0$, then by (1), it yields $\mathsf{u}_{n_0} = \mathsf{u}_{n_0+1} = \mathcal{Q}(\mathsf{u}_{n_0})$, so $\mathsf{u}_{n_0} \in \mathrm{Fix}(\mathcal{Q})$ and so, it is complete. However, if we conclude $\varpi_n > 0$, $\forall\ n \in \mathbb{N}_0$, we proceed to the next step.
Step–4. We prove that $\{\mathsf{u}_n\}$ is a Cauchy sequence. As for every $n \in \mathbb{N}_0$, it yields $\mathsf{u}_{n-1} \neq \mathsf{u}_n$. Utilizing (e) and (1), it yields

$$\varpi(\mathsf{u}_n, \mathsf{u}_{n+1}) \quad \leq \quad \phi(\varpi(\mathsf{u}_{n-1}, \mathsf{u}_n)) + \theta(\varpi(\mathsf{u}_n, \mathcal{Q}\mathsf{u}_{n-1})) = \phi(\varpi(\mathsf{u}_{n-1}, \mathsf{u}_n)) + \theta(0),$$

i.e.,

$$\varpi_n \leq \phi(\varpi_{n-1}),\ \forall\ n \in \mathbb{N},$$

which, by the increasingness of ϕ, becomes

$$\varpi_n \leq \phi^n(\varpi_0), \quad \forall\ n \in \mathbb{N}. \tag{3}$$

For all $n, m \in \mathbb{N}$ with $n < m$, employing (3) and the triangle inequality, we conclude

$$\begin{aligned}
\varpi(\mathsf{u}_n, \mathsf{u}_m) &\leq \varpi_n + \varpi_{n+1} + \varpi_{n+2} + \cdots + \varpi_{m-1} \\
&\leq \phi^n(\varpi_0) + \phi^{n+1}(\varpi_0) + \phi^{n+2}(\varpi_0) + \cdots + \phi^{m-1}(\varpi_0) \\
&= \sum_{\kappa=n}^{m-1} \phi^\kappa(\varpi_0) \\
&\leq \sum_{\kappa \geq n} \phi^\kappa(\varpi_0) \\
&\to 0 \text{ as } n \text{ (and hence } m) \to \infty.
\end{aligned}$$

Thus, $\{\mathsf{u}_n\}$ is Cauchy. Since $\{\mathsf{u}_n\}$ is $\mathbf{R}$-preserving also, using the $\mathbf{R}$-completeness of $\mathbf{U}$, $\exists\, \mathsf{u}^* \in \mathbf{U}$ with $\mathsf{u}_n \xrightarrow{\varpi} \mathsf{u}^*$.

Step–5. We prove that u^* serves as a fixed point of $\mathcal{Q}$. In view of (d), assume first that $\mathcal{Q}$ is $\mathbf{R}$-continuous. As $\{\mathsf{u}_n\}$ is an $\mathbf{R}$-preserving sequence with $\mathsf{u}_n \xrightarrow{\varpi} \mathsf{u}^*$, therefore we have

$$\mathsf{u}_{n+1} = \mathcal{Q}(\mathsf{u}_n) \xrightarrow{\varpi} \mathcal{Q}(\mathsf{u}^*)$$

implying thereby, $\mathcal{Q}(\mathsf{u}^*) = \mathsf{u}^*$.

Secondly, if $\mathbf{R}$ is ϖ-self-closed, then as $\{\mathsf{u}_n\}$ is an $\mathbf{R}$-preserving sequence with $\mathsf{u}_n \xrightarrow{\varpi} \mathsf{u}^*$, $\exists$ a subsequence $\{\mathsf{u}_{n_k}\}$ of $\{\mathsf{u}_n\}$ verifying $[\mathsf{u}_{n_k}, \mathsf{u}^*] \in \mathbf{R}$, $\forall\, k \in \mathbb{N}$. Set $\sigma_n := \varpi(\mathsf{u}^*, \mathsf{u}_n)$. If $\mathcal{Q}(\mathsf{u}^*) = \mathsf{u}^*$, then it is complete. If $\mathcal{Q}(\mathsf{u}^*) \neq \mathsf{u}^*$, then applying assumption (e) and using Proposition 3 and $[\mathsf{u}_{n_k}, \mathsf{u}^*] \in \mathbf{R}$, we find

$$\begin{aligned}
\varpi(\mathsf{u}_{n_k+1}, \mathcal{Q}\mathsf{u}^*) &= \varpi(\mathcal{Q}\mathsf{u}_{n_k}, \mathcal{Q}\mathsf{u}^*) \\
&\leq \phi(\varpi(\mathsf{u}_{n_k}, \mathsf{u}^*)) + \min\{\theta(\varpi(\mathsf{u}^*, \mathcal{Q}\mathsf{u}_{n_k})), \varpi(\mathsf{u}_{n_k}, \mathcal{Q}\mathsf{u}^*)\} \\
&= \phi(\sigma_{n_k}) + \min\{\theta(\sigma_{n_k+1}), \varpi(\mathsf{u}_{n_k}, \mathcal{Q}\mathsf{u}^*)\}. \qquad (4)
\end{aligned}$$

Now, $\mathsf{u}_{n_k} \xrightarrow{\varpi} \mathsf{u}^*$ implies that $\sigma_{n_k} \longrightarrow 0^+$ in $\mathbb{R}^+$, if $k \to \infty$. Taking $k \to \infty$ in (4) and by Remark 1 and the definition of Θ, we conclude

$$\begin{aligned}
\lim_{k\to\infty} \varpi(\mathsf{u}_{n_k+1}, \mathcal{Q}\mathsf{u}^*) &\leq \lim_{k\to\infty} \phi(\sigma_{n_k}) + \min\left\{\lim_{k\to\infty} \theta(\sigma_{n_k+1}), \lim_{k\to\infty} \varpi(\mathsf{u}_{n_k}, \mathcal{Q}\mathsf{u}^*)\right\} \\
&= \lim_{t\to 0^+} \phi(t) + \min\left\{\lim_{t\to 0^+} \theta(t), \lim_{k\to\infty} \varpi(\mathsf{u}_{n_k}, \mathcal{Q}\mathsf{u}^*)\right\} \\
&= 0
\end{aligned}$$

so that $\mathsf{u}_{n_k+1} \xrightarrow{\varpi} \mathcal{Q}(\mathsf{u}^*)$ implying thereby $\mathcal{Q}(\mathsf{u}^*) = \mathsf{u}^*$. Thus, u^* is a fixed point of $\mathcal{Q}$. □

Theorem 5. *Under the premises (a)–(d) of Theorem 4, assuming that*

(f) $\exists\, \phi \in \Phi$ *and* $\theta \in \Theta$ *verifying*

$$\begin{aligned}
\varpi(\mathcal{Q}u, \mathcal{Q}v) \leq\ & \phi(\varpi(u, v)) + \min\{\theta(\varpi(v, \mathcal{Q}u)), \theta(\varpi(u, \mathcal{Q}v)), \theta(\varpi(u, \mathcal{Q}u)), \theta(\varpi(v, \mathcal{Q}v))\}, \\
& \forall\, (u, v) \in \mathbf{R}
\end{aligned}$$

and

(g) $\mathcal{Q}(\mathbf{U})$ *is* $\mathbf{R}^s$*-directed,*

then $\mathcal{Q}$ *possesses a unique fixed point.*

Proof. It is clear that if (f) holds, then premise (e) of Theorem 4 holds. By Theorem 4, $F(\mathcal{Q}) \neq \emptyset$. Take $\mathsf{v}, \mathsf{w} \in F(\mathcal{Q})$ so that

$$\mathcal{Q}^n(\mathsf{v}) = \mathsf{v} \text{ and } \mathcal{Q}^n(\mathsf{w}) = \mathsf{w}, \quad \forall\, n \in \mathbb{N}. \tag{5}$$

As $\mathsf{v}, \mathsf{w} \in \mathcal{Q}(\mathbf{U})$, by the hypothesis (g), $\exists\, \mathsf{u} \in \mathbf{U}$ with $[\mathsf{v}, \mathsf{u}] \in \mathbf{R}$ and $[\mathsf{w}, \mathsf{u}] \in \mathbf{R}$. By the $\mathcal{Q}$-closedness of $\mathbf{R}$ and Proposition 2, we attain

$$[\mathcal{Q}^n\mathsf{v}, \mathcal{Q}^n\mathsf{u}] \in \mathbf{R} \quad \text{and} \quad [\mathcal{Q}^n\mathsf{w}, \mathcal{Q}^n\mathsf{u}] \in \mathbf{R}, \quad \forall\, n \in \mathbb{N}. \tag{6}$$

Denote $\delta_n := \varpi(\mathcal{Q}^n\mathsf{v}, \mathcal{Q}^n\mathsf{u})$. We prove that

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \varpi(\mathcal{Q}^n\mathsf{v}, \mathcal{Q}^n\mathsf{u}) = 0. \tag{7}$$

Using (5), (6), assumption (f), and Proposition 4, we find

$$\begin{aligned} \varpi(\mathcal{Q}^{n+1}\mathsf{v}, \mathcal{Q}^{n+1}\mathsf{u}) &\leq \phi(\varpi(\mathcal{Q}^n\mathsf{v}, \mathcal{Q}^n\mathsf{u})) + \min\{\theta(\varpi(\mathcal{Q}^n\mathsf{u}, \mathcal{Q}^{n+1}\mathsf{v})), \theta(\varpi(\mathcal{Q}^n\mathsf{v}, \mathcal{Q}^{n+1}\mathsf{u})), \\ &\quad \theta(\varpi(\mathcal{Q}^n\mathsf{v}, \mathcal{Q}^{n+1}\mathsf{v})), \theta(\varpi(\mathcal{Q}^n\mathsf{u}, \mathcal{Q}^{n+1}\mathsf{u}))\}, \\ &= \phi(\varpi(\mathcal{Q}^n\mathsf{v}, \mathcal{Q}^n\mathsf{u})), \qquad \text{as } \varpi(\mathcal{Q}^n\mathsf{v}, \mathcal{Q}^{n+1}\mathsf{v}) = \varpi(\mathsf{v}, \mathsf{v}) = 0 \end{aligned}$$

so that

$$\delta_{n+1} \leq \phi(\delta_n). \tag{8}$$

If $\exists\, n_0 \in \mathbb{N}$ with $\delta_{n_0} = 0$, then $\mathcal{Q}^{n_0}(\mathsf{v}) = \mathcal{Q}^{n_0}(\mathsf{u})$ implying thereby $\mathcal{Q}^{n_0+1}(\mathsf{v}) = \mathcal{Q}^{n_0+1}(\mathsf{u})$. Thus, we find $\delta_{n_0+1} = 0$. By easy induction on n, it yields $\delta_n = 0$, $\forall\, n \geq n_0$, implying thereby $\lim_{n\to\infty} \delta_n = 0$. If $\delta_n > 0$, $\forall\, n \in \mathbb{N}$, then by the increasingness of ϕ in (8), it yields

$$\delta_{n+1} \leq \phi(\delta_n) \leq \phi^2(\delta_{n-1}) \leq \cdots \leq \phi^n(\delta_1)$$

so that

$$\delta_{n+1} \leq \phi^n(\delta_1).$$

Using $n \to \infty$ in the above and utilizing the definition of ϕ, we obtain

$$\lim_{n\to\infty} \delta_{n+1} \leq \lim_{n\to\infty} \phi^n(\delta_1) = 0.$$

The validation of (7) is thus performed in every scenario. Likewise, we can confirm that

$$\lim_{n\to\infty} \varpi(\mathcal{Q}^n\mathsf{w}, \mathcal{Q}^n\mathsf{u}) = 0. \tag{9}$$

From (7), (9), and the triangle inequality, it yields

$$\varpi(\mathsf{v}, \mathsf{w}) = \varpi(\mathcal{Q}^n\mathsf{v}, \mathcal{Q}^n\mathsf{w}) \leq \varpi(\mathcal{Q}^n\mathsf{v}, \mathcal{Q}^n\mathsf{u}) + \varpi(\mathcal{Q}^n\mathsf{u}, \mathcal{Q}^n\mathsf{w}) \to 0 \quad \text{as} \quad n \to \infty$$

so $\mathsf{v} = \mathsf{w}$. The conclusion is now achieved. □

4. Consequences

In this part, we derive certain well-known fixed point outcomes using our findings.

Corollary 1 (Khan [10]). *Assume that $(\mathbf{U}, \varpi)$ is an MS assigned with a BR $\mathbf{R}$ and $\mathcal{Q} : \mathbf{U} \to \mathbf{U}$ refers to a map. Also, we have the following:*

(a) $(\mathbf{U}, \varpi)$ *refers to an* $\mathbf{R}$*-complete MS.*
(b) $\exists\, u_0 \in \mathbf{U}$ *verifying* $(u_0, \mathcal{Q}u_0) \in \mathbf{R}$.
(c) $\mathbf{R}$ *is* $\mathcal{Q}$*-closed.*

(d) $\mathcal{Q}$ *refers to* **R***-continuous or* **R** *refers to* ϖ*-self-closed.*
(e) $\exists\, c \in [0,1)$ *and* $\ell \in \mathbb{R}^+$ *verifying*

$$\varpi(\mathcal{Q}u, \mathcal{Q}v) \leq c \cdot \varpi(u, v) + \ell \cdot \varpi(v, \mathcal{Q}u), \quad \forall\ (u, v) \in \mathbf{R}.$$

Then, $\mathcal{Q}$ *owns a fixed point.*

Proof. If we take $\phi(t) = ct$ (where $c \in [0,1)$) and $\theta(t) = \ell t$ (where $\ell \in \mathbb{R}^+$) in Theorem 4, then the contraction inequality (e) of the above corollary holds, and hence the conclusion is immediate. □

Corollary 2 (Aljawi and Uddin [12])**.** *Assume that* $(\mathbf{U}, \varpi)$ *is an MS assigned with a BR* **R** *and* $\mathcal{Q} : \mathbf{U} \to \mathbf{U}$ *refers to a map. Also, we have the following:*

(a) $(\mathbf{U}, \varpi)$ *refers to an* **R***-complete MS.*
(b) $\exists\, u_0 \in \mathbf{U}$ *verifying* $(u_0, \mathcal{Q}u_0) \in \mathbf{R}$.
(c) **R** *is* $\mathcal{Q}$*-closed.*
(d) $\mathcal{Q}$ *refers to* **R***-continuous or* **R** *refers to* ϖ*-self-closed.*
(e) $\exists\, \phi \in \Phi$ *and* $\theta \in \Theta$ *verifying*

$$\varpi(\mathcal{Q}u, \mathcal{Q}v) \leq \phi(\varpi(u, v)) + \theta(\varpi(v, \mathcal{Q}u)), \quad \forall\ (u, v) \in \mathbf{R}.$$

Then, $\mathcal{Q}$ *admits a fixed point.*

Proof. The contraction inequality (e) of the above corollary trivially follows from the contraction inequality (e) of Theorem 4 and hence the conclusion is immediate. □

Corollary 3 (Algehyne et al. [13])**.** *Assume that* $(\mathbf{U}, \varpi)$ *is an MS assigned with a BR* **R** *and* $\mathcal{Q} : \mathbf{U} \to \mathbf{U}$ *refers to a map. Also, we have the following:*

(a) $(\mathbf{U}, \varpi)$ *refers to an* **R***-complete MS.*
(b) $\exists\, u_0 \in \mathbf{U}$ *verifying* $(u_0, \mathcal{Q}u_0) \in \mathbf{R}$.
(c) **R** *is* $\mathcal{Q}$*-closed.*
(d) $\mathcal{Q}$ *refers to* **R***-continuous or* **R** *refers to* ϖ*-self-closed.*
(e) $\exists\, \phi \in \Phi$ *and* $\ell \in \mathbb{R}^+$ *verifying*

$$\varpi(\mathcal{Q}u, \mathcal{Q}v) \leq \phi(\varpi(u, v)) + \ell \cdot \varpi(v, \mathcal{Q}u), \quad \forall\ (u, v) \in \mathbf{R}.$$

Then, $\mathcal{Q}$ *owns a fixed point.*

Proof. If we put $\theta(t) = \ell t$ (where $\ell \in \mathbb{R}^+$) in Theorem 4, then the contraction inequality (e) of above the corollary holds, and hence the conclusion is immediate. □

Corollary 4 (Algehyne et al. [11])**.** *Assume that* $(\mathbf{U}, \varpi)$ *is an MS assigned with a BR* **R** *and* $\mathcal{Q} : \mathbf{U} \to \mathbf{U}$ *is a map. Also, we have the following:*

(a) $(\mathbf{U}, \varpi)$ *refers to an* **R***-complete MS.*
(b) $\exists\, u_0 \in \mathbf{U}$ *verifying* $(u_0, \mathcal{Q}u_0) \in \mathbf{R}$.
(c) **R** *is* $\mathcal{Q}$*-closed.*
(d) $\mathcal{Q}$ *refers to* **R***-continuous or* **R** *refers to* ϖ*-self-closed.*
(e) $\exists\, \phi \in \Phi$ *verifying*

$$\varpi(\mathcal{Q}u, \mathcal{Q}v) \leq \phi(\varpi(u, v)), \quad \forall\ (u, v) \in \mathbf{R}.$$

Then, $\mathcal{Q}$ *owns a fixed point.*

Proof. Setting $\theta(t) = 0$ in Theorem 4, we conclude that the contraction inequality (e) of the above corollary holds, and hence the conclusion is immediate. □

Corollary 5 (Turinici [20]). *Assume that* $(\mathbf{U}, \varpi)$ *is a complete MS and* $\mathcal{Q} : \mathbf{U} \to \mathbf{U}$ *is a map. If* $\exists$ $c \in [0, 1)$ *and* $\theta \in \Theta$, *it verifies*

$$\varpi(\mathcal{Q}u, \mathcal{Q}v) \leq c \cdot \varpi(u, v)) + \theta(\varpi(v, \mathcal{Q}u)), \quad \forall\ u, v \in \mathbf{U}.$$

Then, $\mathcal{Q}$ *admits a fixed point.*

Proof. Under the universal relation $\mathbf{R} = \mathbf{U}^2$ and for $\phi(t) = ct$ (where $c \in [0, 1)$), Theorem 4 deduces the result. □

Corollary 6 (Berinde [19]). *Assume that* $(\mathbf{U}, \varpi)$ *is a complete MS and* $\mathcal{Q} : \mathbf{U} \to \mathbf{U}$ *is a map. If* $\exists \phi \in \Phi$ *and* $\ell \in \mathbb{R}^+$, *it verifies*

$$\varpi(\mathcal{Q}u, \mathcal{Q}v) \leq \phi(\varpi(u, v)) + \ell \cdot \varpi(v, \mathcal{Q}u), \quad \forall\ u, v \in \mathbf{U}.$$

Then, $\mathcal{Q}$ *admits a fixed point.*

Proof. Under the universal relation $\mathbf{R} = \mathbf{U}^2$ and for $\theta(t) = \ell t$ (where $\ell \in \mathbb{R}^+$), Theorem 4 deduces the result. □

Corollary 7 (Babu et al. [17]). *Assume that* $(\mathbf{U}, \varpi)$ *is a complete MS assigned with a BR* $\mathbf{R}$ *and* $\mathcal{Q} : \mathbf{U} \to \mathbf{U}$ *is a map. If there exist* $c \in [0, 1)$ *and* $\ell \in \mathbb{R}^+$, *it verifies*

$$\varpi(\mathcal{Q}u, \mathcal{Q}v) \leq c \cdot \varpi(u, v) + \ell \cdot \min\{\varpi(v, \mathcal{Q}u), \varpi(u, \mathcal{Q}v), \varpi(u, \mathcal{Q}u), \varpi(v, \mathcal{Q}v)\},\ \forall\ u, v \in \mathbf{U}.$$

Then, $\mathcal{Q}$ *owns a unique fixed point.*

Proof. For the universal relation $\mathbf{R} = \mathbf{U}^2$, $\phi(t) = ct$ (where $c \in [0, 1)$) and $\theta(t) = \ell t$ (where $\ell \in \mathbb{R}^+$), Theorem 5 deduces the result. □

Corollary 8 (Pant [15]). *Assume that* $(\mathbf{U}, \varpi)$ *is a complete MS and* $\mathcal{Q} : \mathbf{U} \to \mathbf{U}$ *is a map. If* $\exists \phi \in \Phi$, *it verifies*

$$\varpi(\mathcal{Q}u, \mathcal{Q}v) \leq \phi(\varpi(u, v)), \quad \forall\ u, v \in \mathbf{U} \text{ with } [u \neq \mathcal{Q}(u) \text{ or } v \neq \mathcal{Q}(v)].$$

Then, $\mathcal{Q}$ *owns a fixed point.*

Proof. If we put $\mathbf{R} = \mathbf{U}^2$ and $\theta(t) = 0$ in Theorem 4, then the contraction inequality of the above corollary trivially follows from the contraction inequality (e) of Theorem 4, and hence the conclusion is immediate. □

Corollary 9 (Bianchini and Grandolfi [4]). *Assume that* $(\mathbf{U}, \varpi)$ *is a complete MS and* $\mathcal{Q} : \mathbf{U} \to \mathbf{U}$ *is a map. If* $\exists\ \phi \in \Phi$, *it verifies*

$$\varpi(\mathcal{Q}u, \mathcal{Q}v) \leq \phi(\varpi(u, v)), \quad \forall\ u, v \in \mathbf{U}.$$

Then, $\mathcal{Q}$ *owns a unique fixed point.*

Proof. If we put $\mathbf{R} = \mathbf{U}^2$ and $\theta(t) = 0$ in Theorem 4, then the contraction inequality of the above corollary trivially follows from the contraction inequality (e) of Theorem 4, and hence the conclusion is immediate. □

5. Illustrative Examples

The purpose of this section is to deliver a few examples of Theorems 4 and 5.

Example 1. *Let* $\mathbf{U} := [-1, 4]$ *assigned with standard metric* ϖ. *Consider a BR on* $\mathbf{U}$: $\mathbf{R} = \{(u, v) \in \mathbf{U}^2 : u - v \geq 0\}$ *and the map* $\mathcal{Q} : \mathbf{U} \to \mathbf{U}$ *such that*

$$\mathcal{Q}(u) = \begin{cases} u/2 & \text{if } -1 \leq u \leq 2 \\ 1 & \text{if } \quad 2 < u \leq 4. \end{cases}$$

Then, the MS $(\mathbf{U}, \varpi)$ *is* $\mathbf{R}$*-complete, and the BR* $\mathbf{R}$ *is* $\mathcal{Q}$*-closed. The contraction inequality of Theorem 4 is readily exhibited for* $\phi(t) = t/2$ *and* $\theta(t) = \ln(1 + t)$. *Since the remaining presumptions of Theorems 4 are likewise true,* $\mathcal{Q}$ *permits a unique fixed point:* $u^* = 0$.

Example 2. *Let* $\mathbf{U} := [1, 3]$ *be assigned the standard metric* ϖ. *Consider a BR on* $\mathbf{U}$: $\mathbf{R} = \{(1, 1), (1, 2), (2, 1), (2, 2), (1, 3)\}$ *and the map* $\mathcal{Q} : \mathbf{U} \to \mathbf{U}$ *such that*

$$\mathcal{Q}(u) = \begin{cases} 1 & \text{if } 1 \leq u \leq 2 \\ 2 & \text{if } 2 < u \leq 3. \end{cases}$$

Then, the MS $(\mathbf{U}, \varpi)$ *is* $\mathbf{R}$*-complete, and the BR* $\mathbf{R}$ *is* $\mathcal{Q}$*-closed. The contraction inequality of Theorem 4 is readily exhibited for* $\phi(t) = t/3$ *and* $\theta(t) = 3t$.

Let $\{u_n\} \subset \mathbf{U}$ *be a* $\mathbf{R}$*-preserving sequence such that* $u_n \xrightarrow{\varpi} \omega$ *and* $(u_n, u_{n+1}) \in E(\mathbf{R})$, $\forall\ n \in \mathbb{N}$. *We observe that* $(u_n, u_{n+1}) \notin \{(1, 3)\}$ *implying thereby* $(u_n, u_{n+1}) \in \{(1, 1), (1, 2), (2, 1), (2, 2)\}$, $\forall\ n \in \mathbb{N}$ *so that* $\{u_n\} \subset \{1, 2\}$. *This yields* $[u_n, \omega] \in \mathbf{R}$ *and so* $\mathbf{R}$ *refers to* ϖ*-self-closed. Since the remaining presumptions of Theorems 4 are pertinent,* $\mathcal{Q}$ *permits a unique fixed point:* $u^* = 1$.

Example 3. *Let* $\mathbf{U} = [0, 1]$ *with standard metric* ϖ *and BR* $\mathbf{R} = \mathbb{R} \times \mathbb{Q}$. *Trivially,* $(\mathbf{U}, \varpi)$ *forms an* $\mathbf{R}$*-complete MS. Assume that* $\mathcal{Q}$ *refers to the identity map on* $\mathbf{U}$. *Then,* $\mathbf{R}$ *is* $\mathcal{Q}$*-closed, and* $\mathcal{Q}$ *is* $\mathbf{R}$*-continuous.*

For a fixed $\alpha \in [0, 1)$, *define* $\phi \in \Phi$ *and* $\theta \in \Theta$ *with* $\phi(t) = \alpha t$ *and* $\theta(t) = t - \alpha t$. *For any* $(u, v) \in \mathbf{R}$, *the contraction inequality of Theorem 4 holds. As one might expect, all the premises of Theorem 4 are met. Consequently,* $\mathcal{Q}$ *owns a fixed point. More importantly, this example fails the application of Theorem 5 such that* $F(\mathcal{Q}) = [0, 1]$.

6. Existence of a Solution to Boundary Value Problems

This part describes the availability of a unique solution for a first-order periodic BVP:

$$\begin{cases} \mathsf{X}'(\hbar) = \mathbb{F}(\hbar, \mathsf{X}(\hbar)), \quad \hbar \in [0, \ell] \\ \mathsf{X}(0) = \mathsf{X}(\ell), \end{cases} \tag{10}$$

where $\ell > 0$, and the function $\mathbb{F} : [0, \ell] \times \mathbb{R} \to \mathbb{R}$ remains continuous.

Definition 12 ([5]). $\chi \in \mathcal{C}^1[0, \ell]$ *is termed a lower solution of* (10) *if*

$$\begin{cases} \chi'(\hbar) \leq \mathbb{F}(\hbar, \chi(\hbar)), & \hbar \in [0, \ell] \\ \chi(0) \leq \chi(\ell). \end{cases}$$

Definition 13 ([5]). *$\chi \in \mathcal{C}^1[0, \ell]$ is termed an upper solution of (10) if*

$$\begin{cases} \chi'(\hbar) \geq \mathbb{F}(\hbar, \chi(\hbar)), & \hbar \in [0, \ell] \\ \chi(0) \geq \chi(\ell). \end{cases}$$

First, we establish the following outcome regarding the existence and uniqueness of a solution to problem (10) in the presence of a lower solution.

Theorem 6. *Along with the problem (10), if $\exists\ \varepsilon > 0$ and a (c)-comparison function ϕ with $\forall\ \alpha, \beta \in \mathbb{R}$ with $\alpha \leq \beta$ that*

$$0 \leq \mathbb{F}(\hbar, \beta) + \varepsilon\beta - [\mathbb{F}(\hbar, \alpha) + \varepsilon\alpha] \leq \varepsilon\phi(\beta - \alpha) \tag{11}$$

and also (10) possesses a lower solution, then $\exists$ a unique solution of problem (10).

Proof. Rewrite the problem (10) as

$$\begin{cases} \mathsf{X}'(\hbar) + \varepsilon\mathsf{X}(\hbar) = \mathbb{F}(\hbar, \mathsf{X}(\hbar)) + \varepsilon\mathsf{X}(\hbar), & \forall\ \hbar \in [0, \ell] \\ \mathsf{X}(0) = \mathsf{X}(\ell). \end{cases} \tag{12}$$

Equation (12) is equivalent to the integral equation

$$\mathsf{X}(\hbar) = \int_0^\ell \mathbb{G}(\hbar, \xi)[\mathbb{F}(\xi, \mathsf{X}(\xi)) + \varepsilon\mathsf{X}(\xi)]d\xi, \tag{13}$$

where

$$\mathbb{G}(\hbar, \xi) = \begin{cases} \frac{e^{\varepsilon(\ell+\xi-\hbar)}}{e^{\varepsilon\ell}-1}, & 0 \leq \xi < \hbar \leq \ell \\ \frac{e^{\varepsilon(\xi-\hbar)}}{e^{\varepsilon\ell}-1}, & 0 \leq \hbar < \xi \leq \ell \end{cases}$$

defines the Green function. Consider the mapping $\mathcal{Q} : \mathcal{C}[0, \ell] \to \mathcal{C}[0, \ell]$ defined by

$$(\mathcal{Q}\mathsf{X})(\hbar) = \int_0^\ell \mathbb{G}(\hbar, \xi)[\mathbb{F}(\xi, \mathsf{X}(\xi)) + \varepsilon\mathsf{X}(\xi)]d\xi, \quad \forall\ \hbar \in [0, \ell]. \tag{14}$$

Thus, $\mathsf{X} \in \mathcal{C}[0, \ell]$ refers to a fixed point of $\mathcal{Q}$ iff $\mathsf{X} \in \mathcal{C}^1[0, \ell]$ solves (13) and, consequently, (10).

Invent a metric ϖ on $\mathcal{C}[0, \ell]$ by

$$\varpi(\mathsf{X}, \mathsf{Y}) = \sup_{\hbar \in [0,\ell]} |\mathsf{X}(\hbar) - \mathsf{Y}(\hbar)|, \quad \forall\ \mathsf{X}, \mathsf{Y} \in \mathcal{C}[0, \ell]. \tag{15}$$

Undertake a BR $\mathbf{R}$ on $\mathcal{C}[0, \ell]$ by

$$\mathbf{R} = \{(\mathsf{X}, \mathsf{Y}) : \mathsf{X}(\hbar) \leq \mathsf{Y}(\hbar),\ \forall\ \hbar \in [0, \ell]\}. \tag{16}$$

Now, we fulfill each of the predictions of Theorem 5.

(a) Evidently, $(\mathcal{C}[0,\ell],\varpi)$ serves as **R**-complete.

(b) If $\chi \in \mathcal{C}^1[0,\ell]$ refers to a lower solution of (10), then we conclude

$$\chi'(\hbar) + \varepsilon\chi(\hbar) \leq \mathbb{F}(\hbar,\chi(\hbar)) + \varepsilon\chi(\hbar), \ \ \forall\, \hbar \in [0,\ell].$$

The multiplication of $e^{\varepsilon\hbar}$ on both sides yields

$$(\chi(\hbar)e^{\varepsilon\hbar})' \leq [\mathbb{F}(\hbar,\chi(\hbar)) + \varepsilon\chi(\hbar)]e^{\varepsilon\hbar}, \ \ \forall\, \hbar \in [0,\ell]$$

yielding thereby

$$\chi(\hbar)e^{\varepsilon\hbar} \leq \chi(0) + \int_0^{\hbar} [\mathbb{F}(\xi,\chi(\xi)) + \varepsilon\chi(\xi)]e^{\varepsilon\xi}d\xi, \ \ \forall\, \hbar \in [0,\ell]. \tag{17}$$

With regard to $\chi(0) \leq \chi(\ell)$, we find

$$\chi(0)e^{\varepsilon\ell} \leq \chi(\ell)e^{\varepsilon\ell} \leq \chi(0) + \int_0^{\ell} [\mathbb{F}(\xi,\chi(\xi)) + \varepsilon\chi(\xi)]e^{\varepsilon\xi}d\xi$$

so that

$$\chi(0) \leq \int_0^{\ell} \frac{e^{\varepsilon\xi}}{e^{\varepsilon\ell}-1}[\mathbb{F}(\xi,\chi(\xi)) + \varepsilon\chi(\xi)]d\xi. \tag{18}$$

By (17) and (18), we find

$$\begin{aligned}
\chi(\hbar)e^{\varepsilon\hbar} &\leq \int_0^{\ell} \frac{e^{\varepsilon\xi}}{e^{\varepsilon\ell}-1}[\mathbb{F}(\xi,\chi(\xi)) + \varepsilon\chi(\xi)]d\xi + \int_0^{\hbar} e^{\varepsilon\xi}[\mathbb{F}(\xi,\chi(\xi)) + \varepsilon\chi(\xi)]d\xi \\
&= \int_0^{\hbar} \frac{e^{\varepsilon(\ell+\xi)}}{e^{\varepsilon\ell}-1}[\mathbb{F}(\xi,\chi(\xi)) + \varepsilon\chi(\xi)]d\xi + \int_{\hbar}^{\ell} \frac{e^{\varepsilon\xi}}{e^{\varepsilon\ell}-1}[\mathbb{F}(\xi,\chi(\xi)) + \varepsilon\chi(\xi)]d\xi
\end{aligned}$$

so that

$$\begin{aligned}
\chi(\hbar) &\leq \int_0^{\hbar} \frac{e^{\varepsilon(\ell+\xi-\hbar)}}{e^{\varepsilon\ell}-1}[\mathbb{F}(\xi,\chi(\xi)) + \varepsilon\chi(\xi)]d\xi + \int_{\hbar}^{\ell} \frac{e^{\varepsilon(\xi-\hbar)}}{e^{\varepsilon\ell}-1}[\mathbb{F}(\xi,\chi(\xi)) + \varepsilon\chi(\xi)]d\xi \\
&= \int_0^{\ell} \mathbb{G}(\hbar,\xi)[\mathbb{F}(\xi,\chi(\xi)) + \varepsilon\chi(\xi)]d\xi \\
&= (\mathcal{Q}\chi)(\hbar), \ \forall\, \hbar \in [0,\ell]
\end{aligned}$$

which yields that $(\chi, \mathcal{Q}\chi) \in \mathbf{R}$.

(c) Take $\mathsf{X},\mathsf{Y} \in \mathcal{C}[0,\ell]$ verifying $(\mathsf{X},\mathsf{Y}) \in \mathbf{R}$. Using (11), we obtain

$$\mathbb{F}(\hbar,\mathsf{X}(\hbar)) + \varepsilon\mathsf{X}(\hbar) \leq \mathbb{F}(\hbar,\mathsf{Y}(\hbar)) + \varepsilon\mathsf{Y}(\hbar), \quad \forall\, \hbar \in [0,\ell]. \tag{19}$$

Employing the fact $\mathbb{G}(\hbar,\xi) > 0$ ($\forall\ (\hbar,\xi) \in [0,\ell]\times[0,\ell]$), and by (14) and (19), we obtain

$$\begin{aligned}
(\mathcal{Q}\mathsf{X})(\hbar) &= \int_0^{\ell} \mathbb{G}(\hbar,\xi)[\mathbb{F}(\xi,\mathsf{X}(\xi)) + \varepsilon\mathsf{X}(\xi)]d\xi \\
&\leq \int_0^{\ell} \mathbb{G}(\hbar,\xi)[\mathbb{F}(\xi,\mathsf{Y}(\xi)) + \varepsilon\mathsf{Y}(\xi)]d\xi \\
&= (\mathcal{Q}\mathsf{Y})(\hbar), \quad \forall\, \hbar \in [0,\ell],
\end{aligned}$$

which, through (16), leads to $(\mathcal{Q}\mathsf{X}, \mathcal{Q}\mathsf{Y}) \in \mathbf{R}$ and so **R** is $\mathcal{Q}$-closed.

(d) Let $\{\mathsf{X}_n\} \subset \mathcal{C}[0,\ell]$ be an **R**-preserving sequence that converges to $\mathsf{X} \in \mathcal{C}[0,\ell]$. Then, for every $\hbar \in [0,\ell]$, the sequence $\{\mathsf{X}_n(\hbar)\} \subset \mathbb{R}$ will remain monotonically increasing,

converging to $\mathsf{X}(\hbar)$, so $\forall\ n \in \mathbb{N}_0$ and $\forall\ \hbar \in [0,\ell]$, we attain $\mathsf{X}_n(\hbar) \leq \mathsf{X}(\hbar)$. Thus, by (16), we conclude that $(\mathsf{X}_n, \mathsf{X}) \in \mathbf{R}$, $\forall\ n \in \mathbb{N}$ so that $\mathbf{R}$ is ϖ-self-closed.

(f) Take $\mathsf{X}, \mathsf{Y} \in \mathcal{C}[0,\ell]$ with $(\mathsf{X}, \mathsf{Y}) \in \mathbf{R}$. By (11), (14), and (15), we have

$$\begin{aligned}
\varpi(\mathcal{Q}\mathsf{X}, \mathcal{Q}\mathsf{Y}) &= \sup_{\hbar\in[0,\ell]} |(\mathcal{Q}\mathsf{X})(\hbar) - (\mathcal{Q}\mathsf{Y})(\hbar)| = \sup_{\hbar\in[0,\ell]} \big((\mathcal{Q}\mathsf{Y})(\hbar) - (\mathcal{Q}\mathsf{X})(\hbar)\big) \\
&\leq \sup_{\hbar\in[0,\ell]} \int_0^\ell \mathbb{G}(\hbar,\xi)[\mathbb{F}(\xi,\mathsf{Y}(\xi)) + \varepsilon\mathsf{Y}(\xi) - \mathbb{F}(\xi,\mathsf{X}(\xi)) - \varepsilon\mathsf{X}(\xi)]d\xi \qquad (20) \\
&\leq \sup_{\hbar\in I} \int_0^\ell \mathbb{G}(\hbar,\xi)\varepsilon\phi(\mathsf{Y}(\xi) - \mathsf{X}(\xi))d\xi.
\end{aligned}$$

Now, $0 \leq \mathsf{Y}(\xi) - \mathsf{X}(\xi) \leq \varpi(\mathsf{X}, \mathsf{Y}))$. As ϕ is increasing, we obtain $\phi(\mathsf{Y}(\xi) - \mathsf{X}(\xi)) \leq \phi(\varpi(\mathsf{X}, \mathsf{Y}))$, so (20) reduces to

$$\begin{aligned}
\varpi(\mathcal{Q}\mathsf{X}, \mathcal{Q}\mathsf{Y}) &\leq \varepsilon\phi(\varpi(\mathsf{X}, \mathsf{Y})) \sup_{\hbar\in[0,\ell]} \int_0^\ell \mathbb{G}(\hbar,\xi)\varpi\xi \\
&= \varepsilon\phi(\varpi(\mathsf{X}, \mathsf{Y})) \sup_{\hbar\in[0,\ell]} \frac{1}{e^{\varepsilon\ell} - 1}\Big(\frac{1}{\varepsilon}e^{\varepsilon(\ell+\xi-\hbar)}\Big]_0^\hbar + \frac{1}{\varepsilon}e^{\varepsilon(\xi-\hbar)}\Big]_\hbar^\ell\Big) \\
&= \varepsilon\phi(\varpi(\mathsf{X}, \mathsf{Y}))\frac{1}{\varepsilon(e^{\varepsilon\ell} - 1)}(e^{\varepsilon\ell} - 1) \\
&= \phi(\varpi(\mathsf{X}, \mathsf{Y}))
\end{aligned}$$

so that

$$\varpi(\mathcal{Q}\mathsf{X}, \mathcal{Q}\mathsf{Y}) \leq \phi(\varpi(\mathsf{X}, \mathsf{Y})) + \min\{\theta(\varpi(\mathsf{Y}, \mathcal{Q}\mathsf{X})), \theta(\varpi(\mathsf{X}, \mathcal{Q}\mathsf{Y})), \theta(\varpi(\mathsf{X}, \mathcal{Q}\mathsf{X})), \theta(\varpi(\mathsf{Y}, \mathcal{Q}\mathsf{Y}))\},$$

$$\forall\ \mathsf{X}, \mathsf{Y} \in \mathcal{C}[0,\ell] \text{ such that } (\mathsf{X}, \mathsf{Y}) \in \mathbf{R}.$$

(g) Let $\mathsf{X}, \mathsf{Y} \in \mathcal{C}[0,\ell]$ be arbitrary. Then, we have $\omega := \max\{\mathcal{Q}\mathsf{X}, \mathcal{Q}\mathsf{Y}\} \in \mathcal{C}[0,\ell]$. As $(\mathcal{Q}\mathsf{X}, \omega) \in \mathbf{R}$ and $(\mathcal{Q}\mathsf{Y}, \omega) \in \mathbf{R}$, $\{\mathcal{Q}\mathsf{X}, \omega, \mathcal{Q}\mathsf{Y}\}$ refers to the path in $\mathbf{R}^s$ between $\mathcal{Q}(\mathsf{X})$ and $\mathcal{Q}(\mathsf{Y})$. Thus, $\mathcal{Q}(\mathcal{C}[0,\ell])$ is $\mathbf{R}^s$-directed, and by virtue of Theorem 5, $\mathcal{Q}$ enjoys a unique fixed point, yielding a unique solution of (10). □

Now, we establish the following outcome regarding the existence and uniqueness of a solution to problem (10) in the presence of an upper solution.

Theorem 7. *Along with the problem* (10)*, if* $\exists\ \varepsilon > 0$ *and a (c)-comparison function* ϕ *verifying* $\forall\ \alpha, \beta \in \mathbb{R}$ *with* $\alpha \leq \beta$ *that*

$$0 \leq \mathbb{F}(\hbar,\beta) + \varepsilon\beta - [\mathbb{F}(\hbar,\alpha) + \varepsilon\alpha] \leq \varepsilon\phi(\beta - \alpha) \qquad (21)$$

and (10) *possesses an upper solution, then* $\exists$ *a unique solution to problem* (10).

Proof. Consider $\mathcal{C}[0,\ell]$ with a metric ϖ and a mapping $\mathcal{Q} : \mathcal{C}[0,\ell] \to \mathcal{C}[0,\ell]$ to Theorem 5. Undertake a BR $\mathbf{S}$ on $\mathcal{C}[0,\ell]$ by

$$\mathbf{S} = \{(\mathsf{X}, \mathsf{Y}) : \mathsf{X}(\hbar) \geq \mathsf{Y}(\hbar),\ \forall\ \hbar \in [0,\ell]\}. \qquad (22)$$

Now, we fulfill each of the predictions of Theorem 5.

(a) Evidently, $(\mathcal{C}[0,\ell], \varpi)$ is $\mathbf{S}$-complete.

(b) If $\chi \in \mathcal{C}^1[0,\ell]$ refers to an upper solution of (10), then

$$\chi'(\hbar) + \varepsilon\chi(\hbar) \geq \mathbb{F}(\hbar, \chi(\hbar)) + \varepsilon\chi(\hbar),\ \ \forall\ \hbar \in [0,\ell].$$

The multiplication of $e^{\varepsilon\hbar}$ on both sides yields

$$(\chi(\hbar)e^{\varepsilon\hbar})' \geq [\mathbb{F}(\hbar,\chi(\hbar)) + \varepsilon\chi(\hbar)]e^{\varepsilon\hbar}, \ \ \forall\, \hbar \in [0,\ell]$$

yielding thereby

$$\chi(\hbar)e^{\varepsilon\hbar} \geq \chi(0) + \int_0^{\hbar} [\mathbb{F}(\xi,\chi(\xi)) + \varepsilon\chi(\xi)]e^{\varepsilon\xi}d\xi, \ \ \forall\, \hbar \in [0,\ell]. \tag{23}$$

With regard to $\chi(0) \geq \chi(\ell)$, we find

$$\chi(0)e^{\varepsilon\ell} \geq \chi(\ell)e^{\varepsilon\ell} \geq \chi(0) + \int_0^{\ell} [\mathbb{F}(\xi,\chi(\xi)) + \varepsilon\chi(\xi)]e^{\varepsilon\xi}d\xi$$

so that

$$\chi(0) \geq \int_0^{\ell} \frac{e^{\varepsilon\xi}}{e^{\varepsilon\ell}-1}[\mathbb{F}(\xi,\chi(\xi)) + \varepsilon\chi(\xi)]d\xi. \tag{24}$$

By (23) and (24), we find

$$\begin{aligned}
\chi(\hbar)e^{\varepsilon\hbar} &\geq \int_0^{\ell} \frac{e^{\varepsilon\xi}}{e^{\varepsilon\ell}-1}[\mathbb{F}(\xi,\chi(\xi)) + \varepsilon\chi(\xi)]d\xi + \int_0^{\hbar} e^{\varepsilon\xi}[\mathbb{F}(\xi,\chi(\xi)) + \varepsilon\chi(\xi)]d\xi \\
&= \int_0^{\hbar} \frac{e^{\varepsilon(\ell+\xi)}}{e^{\varepsilon\ell}-1}[\mathbb{F}(\xi,\chi(\xi)) + \varepsilon\chi(\xi)]d\xi + \int_{\hbar}^{\ell} \frac{e^{\varepsilon\xi}}{e^{\varepsilon\ell}-1}[\mathbb{F}(\xi,\chi(\xi)) + \varepsilon\chi(\xi)]d\xi
\end{aligned}$$

so that

$$\begin{aligned}
\chi(\hbar) &\geq \int_0^{\hbar} \frac{e^{\varepsilon(\ell+\xi-\hbar)}}{e^{\varepsilon\ell}-1}[\mathbb{F}(\xi,\chi(\xi)) + \varepsilon\chi(\xi)]d\xi + \int_{\hbar}^{\ell} \frac{e^{\varepsilon(\xi-\hbar)}}{e^{\varepsilon\ell}-1}[\mathbb{F}(\xi,\chi(\xi)) + \varepsilon\chi(\xi)]d\xi \\
&= \int_0^{\ell} \mathbb{G}(\hbar,\xi)[\mathbb{F}(\xi,\chi(\xi)) + \varepsilon\chi(\xi)]d\xi \\
&= (\mathcal{Q}\chi)(\hbar), \ \forall\, \hbar \in [0,\ell]
\end{aligned}$$

which yields $(\chi, \mathcal{Q}\chi) \in \mathbf{S}$.

(c) Take $\mathsf{X}, \mathsf{Y} \in \mathcal{C}[0,\ell]$ verifying $(\mathsf{X},\mathsf{Y}) \in \mathbf{S}$. Using (21), we obtain

$$\mathbb{F}(\hbar,\mathsf{X}(\hbar)) + \varepsilon\mathsf{X}(\hbar) \geq \mathbb{F}(\hbar,\mathsf{Y}(\hbar)) + \varepsilon\mathsf{Y}(\hbar), \quad \forall\, \hbar \in [0,\ell]. \tag{25}$$

By (14), (25) and due to $\mathbb{G}(\hbar,\xi) > 0$, $\forall\, (\hbar,\xi) \in [0,\ell]\times[0,\ell]$, we obtain

$$\begin{aligned}
(\mathcal{Q}\mathsf{X})(\hbar) &= \int_0^{\ell} \mathbb{G}(\hbar,\xi)[\mathbb{F}(\xi,\mathsf{X}(\xi)) + \varepsilon\mathsf{X}(\xi)]d\xi \\
&\geq \int_0^{\ell} \mathbb{G}(\hbar,\xi)[\mathbb{F}(\xi,\mathsf{Y}(\xi)) + \varepsilon\mathsf{Y}(\xi)]d\xi \\
&= (\mathcal{Q}\mathsf{Y})(\hbar), \quad \forall\, \hbar \in [0,\ell],
\end{aligned}$$

which, through (22), leads to $(\mathcal{Q}\mathsf{X}, \mathcal{Q}\mathsf{Y}) \in \mathbf{S}$ and so $\mathbf{S}$ is $\mathcal{Q}$-closed.

(d) Let $\{\mathsf{X}_n\} \subset \mathcal{C}[0,\ell]$ be an $\mathbf{S}$-preserving sequence that converges to $\mathsf{X} \in \mathcal{C}[0,\ell]$. Then, for every $\hbar \in [0,\ell]$, the sequence $\{\mathsf{X}_n(\hbar)\} \subset \mathbb{R}$ will remain monotonically decreasing converging to $\mathsf{X}(\hbar)$, so $\forall\, n \in \mathbb{N}_0$ and $\forall\, \hbar \in [0,\ell]$, we attain $\mathsf{X}_n(\hbar) \leq \mathsf{X}(\hbar)$. Thus, by (22), we conclude that $(\mathsf{X}_n, \mathsf{X}) \in \mathbf{S}$, $\forall\, n \in \mathbb{N}$ so that $\mathbf{S}$ is $\wp$-self-closed.

(f) Take $\mathsf{X}, \mathsf{Y} \in \mathcal{C}[0,\ell]$ with $(\mathsf{X},\mathsf{Y}) \in \mathbf{S}$. By (21), (14) and (15), we have

$$\begin{aligned}\varpi(\mathcal{Q}\mathsf{X},\mathcal{Q}\mathsf{Y}) &= \sup_{\hbar\in[0,\ell]}|(\mathcal{Q}\mathsf{X})(\hbar)-(\mathcal{Q}\mathsf{Y})(\hbar)| = \sup_{\hbar\in[0,\ell]}\big((\mathcal{Q}\mathsf{X})(\hbar)-(\mathcal{Q}\mathsf{Y})(\hbar)\big)\\ &\leq \sup_{\hbar\in[0,\ell]}\int_0^\ell \mathbb{G}(\hbar,\xi)[\mathbb{F}(\xi,\mathsf{X}(\xi))+\varepsilon\mathsf{X}(\xi)-\mathbb{F}(\xi,\mathsf{Y}(\xi))-\varepsilon\mathsf{Y}(\xi)]d\xi \quad (26)\\ &\leq \sup_{\hbar\in I}\int_0^\ell \mathbb{G}(\hbar,\xi)\varepsilon\phi(\mathsf{X}(\xi)-\mathsf{Y}(\xi))d\xi.\end{aligned}$$

Now, $0\leq \mathsf{X}(\xi)-\mathsf{Y}(\xi)\leq \varpi(\mathsf{X},\mathsf{Y}))$. As ϕ is increasing, we obtain $\phi(\mathsf{X}(\xi)-\mathsf{Y}(\xi))\leq \phi(\varpi(\mathsf{X},\mathsf{Y}))$, so (26) reduces to

$$\begin{aligned}\varpi(\mathcal{Q}\mathsf{X},\mathcal{Q}\mathsf{Y}) &\leq \varepsilon\phi(\varpi(\mathsf{X},\mathsf{Y}))\sup_{\hbar\in[0,\ell]}\int_0^\ell \mathbb{G}(\hbar,\xi)\varpi\xi\\ &= \varepsilon\phi(\varpi(\mathsf{X},\mathsf{Y}))\sup_{\hbar\in[0,\ell]}\frac{1}{e^{\varepsilon\ell}-1}\Big(\frac{1}{\varepsilon}e^{\varepsilon(\ell+\xi-\hbar)}\Big]_0^\hbar+\frac{1}{\varepsilon}e^{\varepsilon(\xi-\hbar)}\Big]_\hbar^\ell\Big)\\ &= \varepsilon\phi(\varpi(\mathsf{X},\mathsf{Y}))\frac{1}{\varepsilon(e^{\varepsilon\ell}-1)}(e^{\varepsilon\ell}-1)\\ &= \phi(\varpi(\mathsf{X},\mathsf{Y}))\end{aligned}$$

so that

$$\varpi(\mathcal{Q}\mathsf{X},\mathcal{Q}\mathsf{Y})\leq\phi(\varpi(\mathsf{X},\mathsf{Y}))+\min\{\theta(\varpi(\mathsf{Y},\mathcal{Q}\mathsf{X})),\theta(\varpi(\mathsf{X},\mathcal{Q}\mathsf{Y})),\theta(\varpi(\mathsf{X},\mathcal{Q}\mathsf{X})),\theta(\varpi(\mathsf{Y},\mathcal{Q}\mathsf{Y}))\},$$

$$\forall\,\mathsf{X},\mathsf{Y}\in\mathcal{C}[0,\ell] \text{ such that } (\mathsf{X},\mathsf{Y})\in\mathbf{S}.$$

(g) Let $\mathsf{X},\mathsf{Y}\in\mathcal{C}[0,\ell]$ be arbitrary. Then, we have $\omega:=\min\{\mathcal{Q}\mathsf{X},\mathcal{Q}\mathsf{Y}\}\in\mathcal{C}[0,\ell]$. As $(\mathcal{Q}\mathsf{X},\omega)\in\mathbf{S}$ and $(\mathcal{Q}\mathsf{Y},\omega)\in\mathbf{S}$, $\{\mathcal{Q}\mathsf{X},\omega,\mathcal{Q}\mathsf{Y}\}$ refers to path in $\mathbf{S}^s$ between $\mathcal{Q}(\mathsf{X})$ and $\mathcal{Q}(\mathsf{Y})$. Thus, $\mathcal{Q}(\mathcal{C}[0,\ell])$ is $\mathbf{S}^s$-directed and hence by Theorem 5, $\mathcal{Q}$ enjoys a unique fixed point, which forms a unique solution of (10). □

7. Conclusions and Future Directions

We investigated certain outcomes in a relational MS under a nonlinear almost contraction map of the Pant type. Our outcomes exhibit a generalized contraction-inequality that only applies to the comparative elements. A few examples were also conducted to corroborate these findings. In order to highlight the importance of the theory and the scope of our outcomes, we analysed the reliability of a unique solution for BVP when it admits a lower or an upper solution.

The findings examined here complement, enhance, and integrate a number of previously established findings including those of Khan [10], Aljawi and Uddin [12], Algehyne et al. [13], Algehyne et al. [11], Turinici [20], Berinde [19], Babu et al. [17], Pant [15], and Bianchini and Grandolfi [4]. Nevertheless, a number of fixed point outcomes have been proved by employing the concept of directed graph (cf. [24]). With such results, it is assumed that the edge-set of the directed graph contains all loops; this yields that the edge-set forms a reflexive BR on the given MS. It turns out that Theorems 4 and 5 deduce the findings of Alfuraidan et al. [21].

Due to the significance of the relation-theoretic fixed point theory,we take into consideration the following prospective future works, which, by themselves, would be very notable and widely recognised areas:

1. To improve the properties on the involved test functions ϕ and θ;
2. To extend our outcomes for a couple of maps;

3. To prove the analogues of our findings to enlarged metrical structures, such as a semi-MS, quasi-MS, dislocated MS, G-MS, etc., endued with a BR;
4. To utilize our outcomes in the study of nonlinear matrix equations and nonlinear integral equations rather than BVP.

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Notations and Abbreviations

In this investigation, the following symbols and acronyms are employed.

$\mathbb{N}$	the set of positive integers
$\mathbb{R}$	the set of real numbers
$\mathbb{R}^+$	the set of non-negative real numbers
$\mathcal{C}(A)$	the family of continuous functions from a set A to $\mathbb{R}$
$\mathcal{C}^1(A)$	the family of continuously differentiable functions from a set A to $\mathbb{R}$
BR	binary relation
MS	metric space
BVP	boundary value problem
BCP	Banach contraction principle.

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