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Second-Order Neutral Differential Equations with Sublinear Neutral Terms: New Criteria for the Oscillation

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Abstract: This paper aims to study the oscillatory behavior of second-order neutral differential equations. Using the Riccati substitution technique, we introduce new oscillation criteria that essentially improve some related criteria from the literature. We provide some examples and compare the results in this paper with earlier results to illustrate the importance of our results.

Keywords: oscillation criteria; Riccati substitution; canonical case

MSC: 34C10; 34K11

1. Introduction

In this paper, we are concerned with the oscillation criteria for the following second-order neutral differential equation (NDE):

$$\left(r(s)\left(z'(s)\right)^{\mu}\right)' + q(s)\varkappa^{\mu}(\varepsilon(s)) = 0, \tag{1}$$

where $s \geq s_0$ and $z(s) := \varkappa^{\gamma}(s) + \mathcal{P}(s)\varkappa^{\gamma}(\mathcal{J}(s))$. Also, we assume that

[N_1] $0 < \gamma \leq 1$ and μ are quotient of odd positive integers;
[N_2] $r \in C^1([s_0, \infty), (0, \infty))$, $r'(s) \geq 0$;
[N_3] $\mathcal{P}, q \in C([s_0, \infty), [0, \infty))$, $\mathcal{P}(s) > 1$ and $q(s)$ is not identically zero for large s;
[N_4] $\mathcal{J}, \varepsilon \in C^1([s_0, \infty), (0, \infty))$, $\mathcal{J}' > 0$, $\varepsilon' > 0$, $\mathcal{J}(s) \leq s$, $\varepsilon(s) \leq s$, and $\lim_{s\to\infty} \mathcal{J}(s) = \lim_{s\to\infty} \varepsilon(s) = \infty$.

If

$$\int_{s_0}^{\infty} r^{-1/\mu}(\xi)\mathrm{d}\xi = \infty, \tag{2}$$

then we say that (1) satisfies the canonical case.

By a solution of (1), we mean a function $\varkappa \in C^1[s_\varkappa, \infty)$, $s_\varkappa \geq s_0$, which has the property $r(s)(z'(s))^{\mu} \in C^1[s_\varkappa, \infty)$, and satisfies (1) on $[s_\varkappa, \infty)$. We consider only those solutions $\varkappa$ of (1) which satisfy $\sup\{|\varkappa(s)| : s \geq s_y\} > 0$, for all $s_y \geq s_\varkappa$. A solution $\varkappa$ of (1) is said to be non-oscillatory if it is positive or negative; otherwise, it is said to be oscillatory.

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Mathematics has been instrumental not just as a tool for other sciences but also in driving the advancement of various disciplines. A key factor in this progress is the development of differential equations (DEs). DEs are grounded in calculus theory and include both ordinary DEs and partial DEs. Owing to its robust practical applications, deep theoretical foundations, and strong connections with other areas of mathematics, it has evolved into a significant and widely used branch of the field.

An NDE is a type of DE where the highest-order derivative of the solution appears both with and without delay. This type of equation is frequently encountered in a variety of physical problems and electronic applications, as noted in [1]. Given their practical significance, the oscillation of NDEs of different orders has been the focus of extensive study in recent decades. For recent contributions on this topic, see references [2–7].

Second-order DEs are used in many fields, including population ecology, economics, electronics, mechanics, and contemporary control theory. First-order derivatives and second-order terms devoid of derivative components can both be included in nonlinear second-order NDEs. These equations are useful for evaluating dynamic systems with delay effects since many of them have delay features. For contributions on this topic, see references [8–11].

We know that the oscillatory and asymptotic behavior of differential equations is studied in the canonical or non-canonical case. Therefore, we will present the results of some previous studies on the oscillations of the second-order DEs for both cases.

First, some previous studies have been interested in studying the oscillatory behavior of DEs in the non-canonical case

$$\int_{s_0}^{\infty} r^{-1/\mu}(\xi)\mathrm{d}\xi < \infty.$$

We mention the following for clarification:

Agarwal et al. [12] have concluded more effective conditions for the oscillation of DE solutions. They proved that DE (1) is oscillatory if

$$\limsup_{s\to\infty}\int_{s_0}^{s}\left(\rho(\xi)q(\xi)(1-\mathcal{P}(\varepsilon(\xi)))^{\mu}-\frac{\left((\rho'(\xi))_{+}\right)^{\mu+1}r(\mathcal{J}(\xi))}{(\mu+1)^{\mu+1}\rho^{\mu}(\xi)(\mathcal{J}'(\xi))^{\mu}}\right)\mathrm{d}\xi=\infty, \tag{3}$$

and

$$\limsup_{s\to\infty}\int_{s_0}^{s}\left(\psi_1(\xi)-\frac{\delta(\xi)r(\xi)\left((\varphi(\xi))_{+}\right)^{\mu+1}}{(\mu+1)^{\mu+1}}\right)\mathrm{d}\xi=\infty, \tag{4}$$

hold, where

$$\psi_1(s)=\delta(s)\left(q(s)\left(1-\mathcal{P}(\varepsilon(s))\frac{\pi(\mathcal{J}(\varepsilon(s)))}{\pi(\varepsilon(s))}\right)^{\mu}+\frac{1-\mu}{r^{1/\mu}(s)\pi^{\mu+1}(s)}\right),$$

and

$$\varphi(s)=\frac{\delta'(s)}{\delta(s)}+\frac{1+\mu}{r^{1/\mu}(s)\pi(s)} \text{ and } (\varphi(s))_{+}=\max\{\varphi(s),0\}.$$

Bohner et al. [13] supplemented and simplified the results of Agarwal et al. [12] without verifying the extra condition (3). They proved that (1) is oscillatory if

$$\liminf_{s\to\infty}\int_{\varepsilon(s)}^{s}\left(\frac{1}{r(s)}\int_{s_1}^{s}q(s)\left(1-\mathcal{P}(\varepsilon(s))\frac{\pi(\mathcal{J}(\varepsilon(s)))}{\pi(\varepsilon(s))}\right)^{\mu}\right)^{1/\mu}\mathrm{d}\xi>\frac{1}{\mathrm{e}}.$$

Now, we will highlight the studies that have been concerned with the oscillation of solutions of DEs in the canonical case

$$\int_{s_0}^{\infty} r^{-1/\mu}(\xi)\mathrm{d}\xi = \infty.$$

Sun and Meng [14] established some oscillation criteria of

$$\left(r(s)\left(\varkappa'(s)\right)^{\mu}\right)' + q(s)\varkappa^{\mu}(\varepsilon(s)) = 0. \tag{5}$$

They proved using the Riccati transformation that (5) is oscillatory if

$$\int_{s_1}^{\infty}\left(\theta_{s_1}^{\mu}(\varepsilon(\xi))q(\xi) - \frac{\mu^{\mu+1}\varepsilon'(\xi)}{(\mu+1)^{\mu+1}\theta_{s_1}(\varepsilon(\xi))r^{1/\mu}(\varepsilon(\xi))}\right)\mathrm{d}\xi = \infty.$$

where

$$\theta_{\xi}(s) := \int_{\xi}^{s} r^{-1/\mu}(\xi)\mathrm{d}\xi.$$

In the same way, Xu and Meng [15] focused on studying the oscillatory and asymptotic behavior of

$$\left(r(s)\left(\left(\varkappa(s)+\mathcal{P}(s)\varkappa(\mathcal{J}(s))\right)'\right)^{\mu}\right)' + q(s)\varkappa^{\mu}(\varepsilon(s)) = 0, \tag{6}$$

and obtained new results that are extensions of the results of [14]. They proved that if

$$\int_{s_1}^{\infty}\left(\theta_{s_1}^{\mu}(\varepsilon(\xi))\Phi(\xi) - \frac{\mu^{\mu+1}\varepsilon'(\xi)}{(\mu+1)^{\mu+1}\theta_{s_1}(\varepsilon(\xi))r^{1/\mu}(\varepsilon(\xi))}\right)\mathrm{d}\xi = \infty,$$

then (6) is oscillatory, where $\Phi(s) := q(s)(1-\mathcal{P}(\varepsilon(s)))^{\mu}$ and $0 \le \mathcal{P}(s) \le \mathcal{P}_0 < 1$.

By using the omparison theory, Baculikova and Dzurina [16] proved that

$$\left(r(s)\left(\left(\varkappa(s)+\mathcal{P}(s)\varkappa(\mathcal{J}(s))\right)'\right)^{\mu}\right)' + q(s)\varkappa^{\beta}(\varepsilon(s)) = 0,\ 0 \le \mathcal{P}(s) \le \mathcal{P}_0 < \infty, \tag{7}$$

is oscillatory if $\mu \ge \beta$, $\mathcal{J}\circ\varepsilon = \varepsilon\circ\mathcal{J}$, $\varepsilon(s) \le \mathcal{J}(s)$ and

$$\liminf_{s\to\infty}\int_{\mathcal{J}^{-1}(\varepsilon(s))}^{s}\Phi_2(\xi)\left(\int_{\xi_1}^{\varepsilon(\xi)} r^{-1/\mu}(u)\mathrm{d}u\right)^{\beta}\mathrm{d}\xi > \left(1+\frac{\mathcal{P}_0^{\beta}}{\mathcal{J}_0}\right)^{\beta/\mu}\frac{1}{\kappa \mathrm{e}}, \tag{8}$$

where $\mathcal{J}'(s) \ge \mathcal{J}_0 > 0$,

$$(\mathcal{J}\circ\varepsilon)(s) = \mathcal{J}(\varepsilon(s)),$$

$$\Phi_2(s) := \min\{q(s), q(\mathcal{J}(s))\},$$

and

$$\kappa := \begin{cases} 1 & \text{if } 0 < \beta \le 1; \\ 2^{1-\beta} & \text{if } \beta > 1. \end{cases}$$

Grace et al. [17] developed new oscillation criteria for (6). These criteria improved the results in [15,16,18]. Some of these criteria are

(1) By using comparison theory,

$$\liminf_{s\to\infty}\int_{\varepsilon(s)}^{s}\Phi(\xi)\left(\theta_{s_1}^{*}(\varepsilon(\xi))\right)^{\mu}\mathrm{d}\xi > \frac{1}{\mathrm{e}},$$

where

$$\theta_{s_1}^{*}(s) := \theta_{s_1}(s) + \frac{1}{\mu}\int_{s_1}^{s}\theta_{s_1}(u)\theta_{s_1}^{\mu}(u)\mathrm{d}u;$$

(2) By using the Riccati transformation,

$$\limsup_{s\to\infty}\int_{s_1}^{\infty}\left(\phi(\xi)\exp\left(-\int_{\varepsilon(\xi)}^{\xi}\frac{\mathrm{d}u}{r^{1/\mu}(u)\theta_{s_1}^{*}(u)}\right)-\frac{r(\xi)(\phi'_{+}(\xi))^{\mu+1}}{(\mu+1)^{\mu+1}\phi^{\mu}(\xi)}\right)\mathrm{d}\xi=\infty,$$

where $0\le\mathcal{P}(s)\le\mathcal{P}_0<1$, $\phi\in C^1([s_0,\infty),(0,\infty))$ and

$$\phi'_{+}(s):=\max\{\phi'(s),0\}.$$

Recently, Moaaz et al. [19] improved the results in [16,17,20]. They proved that (6) is oscillatory if $\mathcal{J}\circ\varepsilon=\varepsilon\circ\mathcal{J}$, $\varepsilon(s)\le\mathcal{J}(s)$ and

$$\limsup_{s\to\infty}\int_{s_1}^{s}2^{1-\mu}\phi(\xi)q(\xi)-\left(1+\frac{\mathcal{P}_0^{\mu}}{\mathcal{J}_0}\right)\frac{1}{(\mu+1)^{\mu+1}}\frac{r(\varepsilon(\xi))(\phi'(\xi))^{\mu+1}}{(\varepsilon'(\xi)\phi(\xi))^{\mu}}\mathrm{d}\xi=\infty,\tag{9}$$

where $\mu\ge1$, $\varepsilon'>0$, $\mathcal{J}'(s)\ge\mathcal{J}_0>0$, $\phi\in C^1([s_0,\infty),(0,\infty))$ and

$$q(\xi)=\min\{q(s),q(\mathcal{J}(s))\}.$$

Moreover, Santra et al. [21] studied the oscillatory behavior of

$$\left(r(s)(\varkappa(s)+\mathcal{P}(s)\varkappa(\mathcal{J}(s)))'\right)'+q(s)G_1(\varkappa(\varepsilon(s)))+q_1(s)G_2(\varkappa(\varsigma(s)))=0,\tag{10}$$

where $0\le\mathcal{P}(s)\le\mathcal{P}_0<\infty$, $\varsigma(s)\le s$, $\lim_{s\to\infty}\varsigma(s)=\infty$, $q,q_1\in C([s_0,\infty),[0,\infty))$, and

$$\frac{G_1(u_1(\varepsilon(s)))}{u_1}\ge k>0,\ \frac{G_2(u_2(\varsigma(s)))}{u_2}\ge k>0,$$

for $u_1,u_2\neq0$, k is a constant. They proved that (10) is oscillatory if $\varepsilon'>0$, $\varepsilon(s)\le\varsigma(s)$, $\mathcal{J}\circ\varepsilon=\varepsilon\circ\mathcal{J}$, $\mathcal{J}\circ\varsigma=\varsigma\circ\mathcal{J}$ and

$$\int_{s_0}^{\infty}\left(\delta_1(\xi)(C(\xi)+D(\xi))-\left(1+\frac{\mathcal{P}_0}{\mathcal{J}_0}\right)\frac{1}{4}\frac{r(\varepsilon(\xi))\left((\delta_1'(\xi))_{+}\right)^2}{\delta_1(\xi)\varepsilon'(\xi)}\right)\mathrm{d}\xi=\infty,\tag{11}$$

where

$$C(s)=\min\{q(s),q(\mathcal{J}(s))\},$$
$$D(s)=\min\{q_1(s),q_1(\mathcal{J}(s))\},$$
$$(\delta_1'(s))_{+}=\max\{\delta_1'(s),0\},$$

and $\delta_1(s)\in C^1([s_0,\infty),(0,\infty))$.

In this paper, we aim to contribute to the development of the oscillation theory for NDEs of the second-order, focusing on the study of equations with infinite coefficients $\mathcal{P}(s)>1$, finding a relationship between the solution $\varkappa(s)$ and the corresponding function $z(s)$ using some lemmas and some inequalities. We present new results for the oscillation (1) using the Riccati transform technique, which improves the related results mentioned in [16,19,21]. The approach used also allows us to remove some additional restrictions on the delay functions used in DE.

2. Main Results

We will mention some notations that we will need in the rest of this paper for simplicity.

$$G(s) = 1 - \frac{(\mathcal{H}(\varepsilon(s)))^{1/\lambda}}{(\mathcal{J}^{-1}(\varepsilon(s)))^{1/\lambda}\mathcal{P}(\mathcal{H}(\varepsilon(s)))},$$

where

$$\mathcal{H}(s) = \mathcal{J}^{-1}\Big(\mathcal{J}^{-1}((s))\Big),\ s \le \mathcal{J}^{-1}(s),$$

$$\mathcal{P}(\mathcal{H}(s)) > \frac{(\mathcal{H}(s))^{1/\lambda}}{(\mathcal{J}^{-1}(s))^{1/\lambda}\left(1 - \frac{(1-\gamma)\mathcal{P}\left(\mathcal{J}^{-1}(s)\right)}{K_1}\right)},\ K_1 > 0,$$

and

$$M(s) = \int_s^\infty \frac{q(\xi)}{\gamma^\mu}\left[\frac{G(\xi)}{\mathcal{P}(\mathcal{J}^{-1}(\varepsilon(\xi)))} - \frac{(1-\gamma)}{K_1}\right]^\mu \mathrm{d}\xi.$$

We start with these auxiliary lemmas, which are crucial to our main result proofs.

Lemma 1 ([22]). *If k is non-negative, then*

$$k^\gamma \le \gamma k + (1-\gamma) \text{ for } 0 < \gamma \le 1.$$

Lemma 2 ([23]). *Assume that the function f satisfies $f^{(i)}(s) > 0$, $i = 0, 1, ..., n$, and $f^{(n+1)}(s) < 0$; then,*

$$\frac{f(s)}{f'(s)} \ge \frac{\lambda s}{n}, \tag{12}$$

for every $\lambda \in (0, 1)$.

Lemma 3 ([24]). *Let $g(v) = Ev - W(v)^{(\mu+1)/\mu}$, where $E, W > 0$, μ is a quotient of odd positive integers. Then, g attains its maximum value on $\mathbb{R}^+$ at $v^* = (\mu E/((\mu+1)W))^\mu$ and*

$$\max_{v\in\mathbb{R}^+} g(v) = g(v^*) = \frac{\mu^\mu}{(\mu+1)^{(\mu+1)}}\frac{E^{\mu+1}}{W^\mu}.$$

Lemma 4 ([16]). *Assume that $\varkappa$ is an eventually positive solution of (1). Then,*

$$z(s) > 0,\ z'(s) > 0 \text{ and } \Big(r(s)\big(z'(s)\big)^\mu\Big)' \le 0, \tag{13}$$

for $s \ge s_1$, where s_1 is sufficiently large.

Theorem 1. *If*

$$\int_{s_0}^\infty \frac{q(\xi)}{\gamma^\mu}\left[\frac{G(\xi)}{\mathcal{P}(\mathcal{J}^{-1}(\varepsilon(\xi)))} - \frac{(1-\gamma)}{K_1}\right]^\mu \mathrm{d}\xi = \infty, \tag{14}$$

holds, where K_1 is a constant greater than zero, then (1) is oscillatory.

Proof. Let $\varkappa(s)$ be a non-oscillatory solution of (1). From the definition of $z(s)$, we have

$$\begin{aligned}
\varkappa^\gamma(s) &= \frac{z\big(\mathcal{J}^{-1}(s)\big) - \varkappa^\gamma\big(\mathcal{J}^{-1}(s)\big)}{\mathcal{P}(\mathcal{J}^{-1}(s))} \\
&= \frac{z\big(\mathcal{J}^{-1}(s)\big)}{\mathcal{P}(\mathcal{J}^{-1}(s))} - \frac{1}{\mathcal{P}(\mathcal{J}^{-1}(s))}\left(\frac{z(\mathcal{H}(s)) - \varkappa^\gamma(\mathcal{H}(s))}{\mathcal{P}(\mathcal{H}(s))}\right) \\
&\ge \frac{z\big(\mathcal{J}^{-1}(s)\big)}{\mathcal{P}(\mathcal{J}^{-1}(s))} - \frac{z(\mathcal{H}(s))}{\mathcal{P}(\mathcal{J}^{-1}(s))\mathcal{P}(\mathcal{H}(s))}.
\end{aligned} \tag{15}$$

Now, from Lemma 1 and (15), we see that

$$\varkappa(s) \geq \frac{1}{\gamma}\left[\frac{z\left(\mathcal{J}^{-1}(s)\right)}{\mathcal{P}(\mathcal{J}^{-1}(s))} - \frac{z(\mathcal{H}(s))}{\mathcal{P}(\mathcal{J}^{-1}(s))\mathcal{P}(\mathcal{H}(s))} - (1-\gamma)\right]. \tag{16}$$

From Lemma 4, we see that $z'(s) > 0$ for $s \geq s_1$. Since $r > 0$ and $r' \geq 0$, then we have $z'' < 0$.

Now, using (12), we have

$$\frac{z(s)}{z'(s)} \geq \lambda s,$$

where $n = 1$. Thus, we see that

$$\begin{aligned}\left(\frac{z(s)}{s^{1/\lambda}}\right)' &= \frac{s^{1/\lambda}z'(s)}{\left(s^{1/\lambda}\right)^2} - \frac{1}{\lambda}\frac{s^{(1/\lambda)-1}z(s)}{\left(s^{1/\lambda}\right)^2} \\ &= \frac{\lambda s z'(s) - z(s)}{\lambda s^{(1/\lambda)+1}} \leq 0.\end{aligned}$$

By $\mathcal{J}(s) \leq s$ and $\mathcal{J}' \geq 0$, we obtain $\left(\mathcal{J}^{-1}(s)\right)' \geq 0$; thus, $\mathcal{J}^{-1}(s) \leq \mathcal{H}(s)$ and

$$\frac{z\left(\mathcal{J}^{-1}(s)\right)}{\left(\mathcal{J}^{-1}(s)\right)^{1/\lambda}} \geq \frac{z(\mathcal{H}(s))}{(\mathcal{H}(s))^{1/\lambda}}.$$

Using the above inequality and (16), we obtain

$$\begin{aligned}\varkappa(s) \geq\ & \frac{z\left(\mathcal{J}^{-1}(s)\right)}{\gamma}\left[\left(\frac{1}{\mathcal{P}(\mathcal{J}^{-1}(s))} - \frac{(\mathcal{H}(s))^{1/\lambda}}{\left(\mathcal{J}^{-1}(s)\right)^{1/\lambda}\mathcal{P}(\mathcal{J}^{-1}(s))\mathcal{P}(\mathcal{H}(s))}\right)\right. \\ & \left. - \frac{(1-\gamma)}{z(\mathcal{J}^{-1}(s))}.\right]\end{aligned} \tag{17}$$

However, since $z(s)$ is positive and increasing, then there is a constant $K_1 > 0$ such that

$$z(s) \geq K_1, \text{ for } s \geq s_2. \tag{18}$$

By substituting (18) into (17), we obtain

$$\varkappa(s) \geq \frac{z\left(\mathcal{J}^{-1}(s)\right)}{\gamma}\left[\left(\frac{1}{\mathcal{P}(\mathcal{J}^{-1}(s))} - \frac{(\mathcal{H}(s))^{1/\lambda}}{\left(\mathcal{J}^{-1}(s)\right)^{1/\lambda}\mathcal{P}(\mathcal{J}^{-1}(s))\mathcal{P}(\mathcal{H}(s))}\right) - \frac{(1-\gamma)}{K_1}\right]. \tag{19}$$

This with (1) yields

$$\left(r(s)\left(z'(s)\right)^{\mu}\right)' \leq -q(s)\frac{z^{\mu}\left(\mathcal{J}^{-1}(\varepsilon(s))\right)}{\gamma^{\mu}}\left[\frac{G(s)}{\mathcal{P}(\mathcal{J}^{-1}(\varepsilon(s)))} - \frac{(1-\gamma)}{K_1}\right]^{\mu}. \tag{20}$$

Define

$$u(s) = \frac{r(s)(z'(s))^{\mu}}{z^{\mu}(\mathcal{J}^{-1}(\varepsilon(s)))}. \tag{21}$$

Then, $u(s) > 0$, and we have

$$\begin{aligned} u'(s) &= -\frac{r(s)(z'(s))^{\mu}\left[\mu z^{\mu-1}(\mathcal{J}^{-1}(\varepsilon(s)))z'(\mathcal{J}^{-1}(\varepsilon(s)))(\mathcal{J}^{-1}(\varepsilon(s)))'\right]}{z^{2\mu}(\mathcal{J}^{-1}(\varepsilon(s)))} \\ &\quad + \frac{z^{\mu}(\mathcal{J}^{-1}(\varepsilon(s)))(r(s)(z'(s))^{\mu})'}{z^{2\mu}(\mathcal{J}^{-1}(\varepsilon(s)))} \\ &\leq -\frac{r(s)(z'(s))^{\mu}\left[\mu z'(\mathcal{J}^{-1}(\varepsilon(s)))(\mathcal{J}^{-1}(\varepsilon(s)))'\right]}{z^{\mu+1}(\mathcal{J}^{-1}(\varepsilon(s)))} \\ &\quad - \frac{q(s)}{\gamma^{\mu}}\left[\frac{G(s)}{\mathcal{P}(\mathcal{J}^{-1}(\varepsilon(s)))} - \frac{(1-\gamma)}{K_1}\right]^{\mu}. \end{aligned} \tag{22}$$

Since

$$\frac{r(s)(z'(s))^{\mu}\left[\mu z'(\mathcal{J}^{-1}(\varepsilon(s)))(\mathcal{J}^{-1}(\varepsilon(s)))'\right]}{z^{\mu+1}(\mathcal{J}^{-1}(\varepsilon(s)))} > 0,$$

we have

$$u'(s) \leq -\frac{q(s)}{\gamma^{\mu}}\left[\frac{G(s)}{\mathcal{P}(\mathcal{J}^{-1}(\varepsilon(s)))} - \frac{(1-\gamma)}{K_1}\right]^{\mu}. \tag{23}$$

By integrating (23) from s_3 to s and using (14), we obtain

$$u(s) \leq u(s_3) - \int_{s_3}^{s} \frac{q(\xi)}{\gamma^{\mu}}\left[\frac{G(\xi)}{\mathcal{P}(\mathcal{J}^{-1}(\varepsilon(\xi)))} - \frac{(1-\gamma)}{K_1}\right]^{\mu} \mathrm{d}\xi \to -\infty \text{ as } s \to \infty,$$

which contradicts the positivity of $u(s)$. Therefore, the proof is complete. □

We will now introduce some useful lemmas that will help us to get a new oscillation criterion for Equation (1) in case Condition (14) is not satisfied.

Lemma 5. *Let $\varkappa(s)$ be an eventually positive solution of (1) and $\varepsilon(s) \leq \mathcal{J}(s)$. Then,*

$$u'(s) \leq -\frac{q(s)}{\gamma^{\mu}}\left[\frac{G(s)}{\mathcal{P}(\mathcal{J}^{-1}(\varepsilon(s)))} - \frac{(1-\gamma)}{K_1}\right]^{\mu} - \frac{\mu(\mathcal{J}^{-1}(\varepsilon(s)))'}{r^{1/\mu}(s)} u^{(\mu+1)/\mu}(s). \tag{24}$$

Proof. Similar to the proof of Theorem 1, we obtain (22). By $\varepsilon(s) \leq \mathcal{J}(s)$ and $z''(s) < 0$, we have

$$z'\left(\mathcal{J}^{-1}(\varepsilon(s))\right) \geq z'(s),$$

from above inequality and (22), we see that

$$u'(s) \leq -\frac{q(s)}{\gamma^{\mu}}\left[\frac{G(s)}{\mathcal{P}(\mathcal{J}^{-1}(\varepsilon(s)))} - \frac{(1-\gamma)}{K_1}\right]^{\mu} - \frac{r(s)(z'(s))^{\mu+1}\left[\mu(\mathcal{J}^{-1}(\varepsilon(s)))'\right]}{z^{\mu+1}(\mathcal{J}^{-1}(\varepsilon(s)))}. \tag{25}$$

Using (21) and (25), we obtain

$$u'(s) \leq -\frac{q(s)}{\gamma^{\mu}}\left[\frac{G(s)}{\mathcal{P}(\mathcal{J}^{-1}(\varepsilon(s)))} - \frac{(1-\gamma)}{K_1}\right]^{\mu} - \frac{\mu(\mathcal{J}^{-1}(\varepsilon(s)))'}{r^{1/\mu}(s)} u^{(\mu+1)/\mu}(s).$$

Therefore, the proof is complete. □

Now, we define a sequence of functions $\{\omega_n\}_{n=0}^{\infty}$ by

$$\omega_0(s) = M(s),\ s \geq s_0,$$

and

$$\omega_n(s) = \int_s^\infty \frac{\mu\left(\mathcal{J}^{-1}(\varepsilon(\xi))\right)'}{r^{1/\mu}(\xi)} \omega_{n-1}^{(\mu+1)/\mu}(\xi)\mathrm{d}\xi + \omega_0(s),\ s \geq s_0,\ n = 1,2,3,.... \tag{26}$$

Using induction, we find that $\omega_n(s) \leq \omega_{n+1}(s)$, $s \geq s_0$, $n = 1,2,3,....$

Lemma 6. *Let $\varkappa(s)$ be an eventually positive solution of (1), let $u(s)$ and $\omega_n(s)$ be defined by (21) and (26), respectively, and let $\varepsilon(s) \leq \mathcal{J}(s)$. Then, $\omega_n(s) \leq u(s)$. Moreover, there exists a function $\omega(s) \in C([s,\infty),(0,\infty))$, such that $\lim_{s\to\infty} \omega_n(s) = \omega(s)$ for $s \geq \vartheta \geq s_0$ and*

$$\omega(s) = \int_s^\infty \frac{\mu\left(\mathcal{J}^{-1}(\varepsilon(\xi))\right)'}{r^{1/\mu}(\xi)} \omega^{(\mu+1)/\mu}(\xi)\mathrm{d}\xi + \omega_0(s),\ s \geq \vartheta. \tag{27}$$

Proof. Similar to the proof of Lemma 5, we obtain

$$u'(s) \leq -\frac{q(s)}{\gamma^\mu}\left[\frac{G(s)}{\mathcal{P}(\mathcal{J}^{-1}(\varepsilon(s)))} - \frac{(1-\gamma)}{K_1}\right]^\mu - \frac{\mu\left(\mathcal{J}^{-1}(\varepsilon(s))\right)'}{r^{1/\mu}(s)} u^{(\mu+1)/\mu}(s). \tag{28}$$

From the above inequality, we have $u'(s) < 0$. By integrating (28) from s to s', we have

$$\begin{aligned} u(s') - u(s) \quad \leq \quad & -\int_s^{s'} \frac{q(\xi)}{\gamma^\mu}\left[\frac{G(\xi)}{\mathcal{P}(\mathcal{J}^{-1}(\varepsilon(\xi)))} - \frac{(1-\gamma)}{K_1}\right]^\mu \mathrm{d}\xi \\ & - \int_s^{s'} \frac{\mu\left(\mathcal{J}^{-1}(\varepsilon(\xi))\right)'}{r^{1/\mu}(\xi)} u^{(\mu+1)/\mu}(\xi)\mathrm{d}\xi. \end{aligned} \tag{29}$$

Then, from (29), we see that

$$u(s') - u(s) + \int_s^{s'} \frac{\mu\left(\mathcal{J}^{-1}(\varepsilon(\xi))\right)'}{r^{1/\mu}(\xi)} u^{(\mu+1)/\mu}(\xi)\mathrm{d}\xi \leq 0. \tag{30}$$

Thus, we claim that

$$\int_s^\infty \frac{\mu\left(\mathcal{J}^{-1}(\varepsilon(\xi))\right)'}{r^{1/\mu}(\xi)} u^{(\mu+1)/\mu}(\xi)\mathrm{d}\xi < \infty. \tag{31}$$

Otherwise, by (30), we see that

$$u(s') \leq u(s) - \int_s^{s'} \frac{\mu\left(\mathcal{J}^{-1}(\varepsilon(\xi))\right)'}{r^{1/\mu}(\xi)} u^{(\mu+1)/\mu}(\xi)\mathrm{d}\xi \to -\infty \text{ as } s' \to \infty,$$

We get a contradiction, where $u(s) > 0$. Since $u(s)$ is a positive and decreasing function, from (31), we obtain $\lim_{s\to\infty} u(s) = 0$. Therefore, from (29), we obtain

$$\begin{aligned} u(s) \quad \geq \quad & M(s) + \int_s^\infty \frac{\mu\left(\mathcal{J}^{-1}(\varepsilon(\xi))\right)'}{r^{1/\mu}(\xi)} u^{(\mu+1)/\mu}(\xi)\mathrm{d}\xi \\ = \quad & \omega_0(s) + \int_s^\infty \frac{\mu\left(\mathcal{J}^{-1}(\varepsilon(\xi))\right)'}{r^{1/\mu}(\xi)} u^{(\mu+1)/\mu}(\xi)\mathrm{d}\xi, \end{aligned} \tag{32}$$

and so

$$u(s) \geq M(s) = \omega_0(s).$$

Next, by induction, we have that $u(s) \geq \omega_n(s)$ for $s \geq s_0$, $n = 1,2,3,...$ Thus, since the sequence $\{\omega_n(s)\}_{n=0}^\infty$ is monotone increasing and bounded above, it converges to $\omega(s)$. By letting $n \to \infty$ in (26) and using Lebesgue's monotone convergence theorem, we arrive at (27). The proof is complete. □

Theorem 2. *Assume that (2) is satisfied and that (14) is not satisfied. If*

$$\liminf_{s\to\infty}\frac{1}{M(s)}\int_s^\infty \frac{\mu\left(\mathcal{J}^{-1}(\varepsilon(\xi))\right)'}{r^{1/\mu}(\xi)}M^{(\mu+1)/\mu}(\xi)\mathrm{d}\xi > \frac{\mu}{(\mu+1)^{(\mu+1)/\mu}}, \tag{33}$$

then (1) is oscillatory.

Proof. Let $\varkappa(s)$ be a non-oscillatory solution of (1). Then, by proceeding as in the proof of Lemmas 5 and 6, we obtain (32) and

$$\frac{u(s)}{M(s)} \geq 1+\frac{1}{M(s)}\int_s^\infty \frac{\mu\left(\mathcal{J}^{-1}(\varepsilon(\xi))\right)'}{r^{1/\mu}(\xi)}M^{(\mu+1)/\mu}(\xi)\left(\frac{u(\xi)}{M(\xi)}\right)^{(\mu+1)/\mu}\mathrm{d}\xi. \tag{34}$$

If we set $\hbar = \inf_{s\geq s_0} u(s)/M(s)$, then $\hbar \geq 1$. Hence, it follows from (33) and (34) that

$$\hbar > 1+\mu\left(\frac{\hbar}{\mu+1}\right)^{(\mu+1)/\mu}. \tag{35}$$

Or, equivalently,

$$\frac{\hbar}{\mu+1} > \frac{1}{\mu+1}+\frac{\mu}{\mu+1}\left(\frac{\hbar}{\mu+1}\right)^{(\mu+1)/\mu},$$

which is a contradiction of the acceptable value for $\hbar \geq 1$ and $\mu > 0$. The proof is complete. □

Theorem 3. *Suppose that $\varepsilon(s) \leq \mathcal{J}(s)$. If there exist a function $\psi(s) \in C^1([s_0,\infty),\mathbb{R}^+)$, such that*

$$\limsup_{s\to\infty}\int_{s_0}^s\left(\psi(\xi)\frac{q(\xi)}{\gamma^\mu}\left[\frac{G(\xi)}{\mathcal{P}(\mathcal{J}^{-1}(\varepsilon(\xi)))}-\frac{(1-\gamma)}{K_1}\right]^\mu - \frac{1}{(\mu+1)^{(\mu+1)}}\frac{(\psi'(\xi))^{\mu+1}r(\xi)}{\left(\psi(\xi)(\mathcal{J}^{-1}(\varepsilon(\xi)))'\right)^\mu}\right)\mathrm{d}\xi = \infty, \tag{36}$$

then (1) is oscillatory.

Proof. Let $\varkappa(s)$ be a non-oscillatory solution of (1). Then, from Lemma 5, we have that (24) holds, and so

$$\frac{q(s)}{\gamma^\mu}\left[\frac{G(s)}{\mathcal{P}(\mathcal{J}^{-1}(\varepsilon(s)))}-\frac{(1-\gamma)}{K_1}\right]^\mu \leq -u'(s)-\frac{\mu\left(\mathcal{J}^{-1}(\varepsilon(s))\right)'}{r^{1/\mu}(s)}u^{(\mu+1)/\mu}(s). \tag{37}$$

By multiplying (37) by $\psi(s)$ and integrating from s_1 to s, we have

$$\begin{aligned}\int_{s_1}^s \psi(\xi)\frac{q(\xi)}{\gamma^\mu}\left[\frac{G(\xi)}{\mathcal{P}(\mathcal{J}^{-1}(\varepsilon(\xi)))}-\frac{(1-\gamma)}{K_1}\right]^\mu \mathrm{d}\xi &\leq -\int_{s_1}^s \psi(\xi)u'(\xi)\mathrm{d}\xi - \int_{s_1}^s \psi(\xi)\frac{\mu\left(\mathcal{J}^{-1}(\varepsilon(\xi))\right)'}{r^{1/\mu}(\xi)}u^{(\mu+1)/\mu}(\xi)\mathrm{d}\xi.\\ &\leq -\psi(s)u(s)+\psi(s_1)u(s_1)+\int_{s_1}^s \psi'(\xi)u(\xi)\mathrm{d}\xi\\ &\quad -\int_{s_1}^s \psi(\xi)\frac{\mu\left(\mathcal{J}^{-1}(\varepsilon(\xi))\right)'}{r^{1/\mu}(\xi)}u^{(\mu+1)/\mu}(\xi)\mathrm{d}\xi.\end{aligned}$$

Using Lemma 3 with $E = \psi'(\xi)$, $W = \psi(\xi)\frac{\mu\left(\mathcal{J}^{-1}(\varepsilon(\xi))\right)'}{r^{1/\mu}(\xi)}$, and $v = u(s)$, we have

$$\int_{s_1}^{s} \psi(\xi)\frac{q(\xi)}{\gamma^{\mu}}\left[\frac{G(\xi)}{\mathcal{P}(\mathcal{J}^{-1}(\varepsilon(\xi)))} - \frac{(1-\gamma)}{K_1}\right]^{\mu} \mathrm{d}\xi \\ \leq -\psi(s)u(s) + \psi(s_1)u(s_1) + \int_{s_1}^{s} \frac{1}{(\mu+1)^{(\mu+1)}} \frac{(\psi'(\xi))^{\mu+1} r(\xi)}{\left(\psi(\xi)(\mathcal{J}^{-1}(\varepsilon(\xi)))'\right)^{\mu}} \mathrm{d}\xi,$$

and so

$$\psi(s_1)u(s_1) \geq \int_{s_1}^{s} \psi(\xi)\frac{q(\xi)}{\gamma^{\mu}}\left[\frac{G(\xi)}{\mathcal{P}(\mathcal{J}^{-1}(\varepsilon(\xi)))} - \frac{(1-\gamma)}{K_1}\right]^{\mu} \mathrm{d}\xi \\ - \int_{s_1}^{s} \frac{1}{(\mu+1)^{(\mu+1)}} \frac{(\psi'(\xi))^{\mu+1} r(\xi)}{\left(\psi(\xi)(\mathcal{J}^{-1}(\varepsilon(\xi)))'\right)^{\mu}} \mathrm{d}\xi.$$

We arrive at a contradiction with (36). The proof is complete. □

Corollary 1. *Suppose that $\varepsilon(s) \leq \mathcal{J}(s)$. If there exists a function $\psi(s) \in C^1([s_0, \infty), \mathbb{R}^+)$, such that*

$$\limsup_{s\to\infty} \int_{s_0}^{s} \psi(\xi)\frac{q(\xi)}{\gamma^{\mu}}\left[\frac{G(\xi)}{\mathcal{P}(\mathcal{J}^{-1}(\varepsilon(\xi)))} - \frac{(1-\gamma)}{K_1}\right]^{\mu} \mathrm{d}\xi = \infty,$$

and

$$\limsup_{s\to\infty} \int_{s_0}^{s} \frac{(\psi'(\xi))^{\mu+1} r(\xi)}{\left(\psi(\xi)(\mathcal{J}^{-1}(\varepsilon(\xi)))'\right)^{\mu}} \mathrm{d}\xi < \infty,$$

then (1) is oscillatory.

Example 1. *Consider the DE*

$$\left(\left((\varkappa^{\gamma}(s) + \mathcal{P}_0\varkappa^{\gamma}(\mathcal{J}_0 s))'\right)^{\mu}\right)' + q_0 s^3 \varkappa^{\mu}(\varepsilon_0 s) = 0. \tag{38}$$

A comparison between (1) and (38) leads us to the conclusion that $r(s) = 1$, $\mathcal{J}(s) = \mathcal{J}_0 s$, $\varepsilon(s) = \varepsilon_0 s$, $\mathcal{P}(s) = \mathcal{P}_0$ and $q(s) = q_0 s^3$. Assume that $\lambda = 0.5$ and $K_1 = 1$. Then, we see that

$$\mathcal{J}^{-1}(s) = \frac{s}{\mathcal{J}_0},\ \mathcal{J}^{-1}(\varepsilon(s)) = \frac{\varepsilon_0 s}{\mathcal{J}_0},\ \mathcal{H}(s) = \frac{s}{(\mathcal{J}_0)^2},\ \mathcal{H}(\varepsilon(s)) = \frac{\varepsilon_0 s}{(\mathcal{J}_0)^2},$$

and

$$\begin{aligned} G(s) &= 1 - \frac{(\mathcal{H}(\varepsilon(s)))^{1/\lambda}}{(\mathcal{J}^{-1}(\varepsilon(s)))^{1/\lambda}\mathcal{P}(\mathcal{H}(\varepsilon(s)))} \\ &= 1 - \frac{1}{\mathcal{P}_0(\mathcal{J}_0)^2} \end{aligned}$$

where

$$\mathcal{P}_0 > \frac{1}{(\mathcal{J}_0)^2(1-(1-\gamma)\mathcal{P}_0)}. \tag{39}$$

As a result, we obtain

$$\int_{s_0}^{\infty} \frac{q(\xi)}{\gamma^{\mu}} \left[\frac{G(\xi)}{\mathcal{P}(\mathcal{J}^{-1}(\varepsilon(\xi)))} - \frac{(1-\gamma)}{K_1} \right]^{\mu} \mathrm{d}\xi$$
$$= \int_{s_0}^{\infty} \frac{q_0 \xi^3}{\gamma^{\mu}} \left[\frac{1}{\mathcal{P}_0} \left(1 - \frac{1}{\left(\mathcal{P}_0 (\mathcal{J}_0)^2 \right)} - \mathcal{P}_0 (1-\gamma) \right) \right]^{\mu} \mathrm{d}\xi = \infty,$$

where

$$\left(1 - \frac{1}{\left(\mathcal{P}_0 (\mathcal{J}_0)^2 \right)} - \mathcal{P}_0 (1-\gamma) \right) > 0.$$

Thus, by Theorem 1, we see that (38) is oscillatory.

Example 2. *Consider the DE*

$$\left(\varkappa(s) + 16 \varkappa\left(\frac{s}{2} \right) \right)'' + \frac{q_0}{s^2} \varkappa\left(\frac{s}{4} \right) = 0. \tag{40}$$

A comparison between (1) and (38) leads us to the conclusion that $\mu = \gamma = 1$, $r(s) = 1$, $\mathcal{J}(s) = s/2$, $\varepsilon(s) = s/4$, $\mathcal{P}(s) = 16$ *and* $q(s) = q_0/s^2$. *Assume that* $\lambda = 0.5$ *and* $K_1 = 1$. *Then, we see that*

$$\mathcal{J}^{-1}(s) = 2s, \ \mathcal{J}^{-1}(\varepsilon(s)) = \frac{s}{2}, \ \mathcal{H}(s) = 4s, \ \mathcal{H}(\varepsilon(s)) = s,$$

$$\begin{aligned} G(s) &= 1 - \frac{(\mathcal{H}(\varepsilon(s)))^{1/\lambda}}{(\mathcal{J}^{-1}(\varepsilon(s)))^{1/\lambda} \mathcal{P}(\mathcal{H}(\varepsilon(s)))} \\ &= \frac{3}{4}. \end{aligned}$$

Let $\psi(\xi) = \xi$. *Then, we see that Condition (36) becomes*

$$\limsup_{s\to\infty} \int_{s_0}^{s} \left(\psi(\xi) \frac{q(\xi)}{\gamma^{\mu}} \left[\frac{G(\xi)}{\mathcal{P}(\mathcal{J}^{-1}(\varepsilon(\xi)))} - \frac{(1-\gamma)}{K_1} \right]^{\mu} \right.$$
$$\left. - \frac{1}{(\mu+1)^{(\mu+1)}} \frac{(\psi'(\xi))^{\mu+1} r(\xi)}{\left(\psi(\xi) (\mathcal{J}^{-1}(\varepsilon(\xi)))' \right)^{\mu}} \right) \mathrm{d}\xi$$
$$= \limsup_{s\to\infty} \int_{s_0}^{s} \left(\xi \frac{q_0}{\xi^2} \left[\frac{3}{4(16)} \right] - \frac{1}{2} \frac{1}{\xi} \right) \mathrm{d}\xi = \infty.$$

Thus, by Theorem 3, we see that (40) is oscillatory if $q_0 > 10.667$.

Now, we see that Condition (8) becomes

$$\liminf_{s\to\infty} \int_{\mathcal{J}^{-1}(\varepsilon(s))}^{s} \Phi_2(\xi) \left(\int_{\xi_1}^{\varepsilon(\xi)} r^{-1/\mu}(u) \mathrm{d}u \right)^{\beta} \mathrm{d}\xi = \liminf_{s\to\infty} \int_{s/2}^{s} \frac{q_0}{\xi^2} \left(\int_{s_1}^{s/4} \mathrm{d}u \right) \mathrm{d}\xi > \left(1 + \frac{16}{\frac{1}{2}} \right) \frac{1}{\mathrm{e}}.$$

Therefore, we see that (40) is oscillatory if $q_0 > 70.057$.

By choosing $\phi(s) = s$, *we see that Condition (9) becomes*

$$\limsup_{s\to\infty}\int_{s_1}^{s}2^{1-\mu}\phi(\xi)q(\xi)-\left(1+\frac{\mathcal{P}_0^{\mu}}{\mathcal{J}_0}\right)\frac{1}{(\mu+1)^{\mu+1}}\frac{r(\varepsilon(\xi))(\phi'(\xi))^{\mu+1}}{(\varepsilon'(\xi)\phi(\xi))^{\mu}}\mathrm{d}\xi$$
$$=\limsup_{s\to\infty}\int_{s_1}^{s}\xi\frac{q_0}{\xi^2}-\left(1+\frac{16}{\frac{1}{2}}\right)\frac{1}{\xi}\mathrm{d}\xi=\infty.$$

Therefore, we see that (40) is oscillatory if $q_0>33$.

Assume that $q_1(s)=0$ *in (10), and by choosing* $\delta_1(s)=s$, *we see that Condition (11) becomes*

$$\int_{s_0}^{\infty}\left(\delta_1(\xi)(C(\xi)+D(\xi))-\left(1+\frac{\mathcal{P}_0}{\mathcal{J}_0}\right)\frac{1}{4}\frac{r(\varepsilon(\xi))\left(\left(\delta_1'(\xi)\right)_+\right)^2}{\delta_1(\xi)\varepsilon'(\xi)}\right)\mathrm{d}\xi$$
$$=\int_{s_0}^{\infty}\left(\xi\frac{q_0}{\xi^2}-\left(1+\frac{16}{\frac{1}{2}}\right)\frac{1}{\xi}\right)\mathrm{d}\xi=\infty.$$

Therefore, we see that (40) is oscillatory if $q_0>33$.

Remark 1. *Below, we will mention the most important features of the results we obtained compared to some previous results:*
1. We notice that the oscillation results in [16,19,21] require the condition $\mathcal{J}\circ\varepsilon=\varepsilon\circ\mathcal{J}$. *However, in the results we obtained, we did not require this condition.*
2. We find that using the oscillation conditions for [16,19,21], Equation (40) is oscillatory if $q_0>70.057$, $q_0>33$ *and* $q_0>33$, *respectively. However, using the results we obtained, we found that Equation (40) is oscillatory if* $q_0>10.667$.

Therefore, the results we obtained improve the results of [16,19,21].

Remark 2. *Let*

$$\left(\varkappa(s)+16\varkappa\left(\frac{s}{2}\right)\right)''+\frac{15}{s^2}\varkappa\left(\frac{s}{4}\right)=0, \tag{41}$$

be a special case of Equation (40). We find that Conditions (8), (9) and (11) fail to study the oscillation of (41) because $q_0=15$. *Using our results, we see that (41) is oscillatory. Thus, our results improve the results of [16,19,21].*

3. Conclusions

We know that most of the work that has been concerned with studying the oscillatory behavior of second-order NDEs has focused on equations with finite coefficients, i.e., the cases where $-1<\mathcal{P}_0\le\mathcal{P}(s)\le 0$ and $0\le\mathcal{P}(s)\le\mathcal{P}_0<1$, and very little has been published about NDEs with infinite coefficients $\mathcal{P}(s)>1$. In this paper, we studied the oscillation of second-order NDEs in the case where $\mathcal{P}(s)>1$ and under Condition (2), and new oscillation criteria were obtained. Also, the obtained results improved some of the results in previous studies. As a future work, the same approach can be used to study the following:

1. Oscillation of third-order NDEs:

$$\left(r(s)(z''(s))^{\mu}\right)'+q(s)\varkappa^{\mu}(\varepsilon(s))=0;$$

2. Oscillation of higher-order NDEs:

$$\left(r(s)\left(z^{(n-1)}(s)\right)^{\mu}\right)'+q(s)\varkappa^{\mu}(\varepsilon(s))=0,$$

where $n > 3$ is a positive integer. The same approach can also be tried out for systems with multiple delays.

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