

Article

Geometry of LP-Sasakian Manifolds Admitting a General Connection

Rajesh Kumar ¹, Lalitluangkima Chawngthu ², Oğuzhan Bahadır ³ and Meraj Ali Khan ^{4,*}¹ Department of Mathematics, Pachhunga University College, Mizoram University, Aizawl 796001, India; rajesh@pucollege.edu.in² Department of Mathematics and Computer Science, Mizoram University, Tanhril, Aizawl 796004, India; mzu2102195@mzu.edu.in³ Department of Mathematics, Faculty of Sciences, Kahramanmaraş Sutcu Imam University, Kahramanmaraş 46100, Turkey; obahadir@ksu.edu.tr⁴ Department of Mathematics and Statistics, College of Science, Imam Mohammad Ibn Saud Islamic University (IMSIU), Riyadh 11566, Saudi Arabia

* Correspondence: mskhan@imamu.edu.sa

Abstract: This paper concerns certain properties of projective curvature tensor, conharmonic curvature tensor, quasi-conharmonic curvature tensor, and Ricci semi-symmetric conditions with respect to the general connection in an LP-Sasakian manifold. We also provide the applications of LP-Sasakian manifolds admitting general connections in the context of the general theory of relativity.

Keywords: LP-Sasakian manifold; general connection; projective curvature tensor; conharmonic curvature tensor; quasi-conharmonic curvature tensor; Ricci semi-symmetric manifold

MSC: 53C15; 53C25; 53C40



Academic Editor: Mića S. Stanković

Received: 20 February 2025

Revised: 3 March 2025

Accepted: 6 March 2025

Published: 7 March 2025

Citation: Kumar, R.; Chawngthu, L.; Bahadır, O.; Khan, M.A. Geometry of LP-Sasakian Manifolds Admitting a General Connection. *Mathematics* **2025**, *13*, 902. <https://doi.org/10.3390/math13060902>

Copyright: © 2025 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

1. Introduction

In the realm of differential geometry, the study of manifolds with specialized geometric forms is basic for grasping complex spaces that appear in physics, especially in general relativity and cosmology. One such form is the Para-Sasakian manifold, which like Sasakian manifolds, provides an interesting framework for geometries with some symmetry properties. When these forms are combined with a Lorentzian metric, they bring to light Lorentzian Para-Sasakian manifolds. These manifolds have a fascinating relation between the geometric properties of both Riemannian and Lorentzian spaces. Lorentzian Para-Sasakian manifolds are remarkable in the study of spacetime symmetries and cosmological models. Their structure allows for the discovery of physical systems where time-like and space-like directions reveal a specific association, which is crucial in several areas of theoretical physics. These manifolds appeared in 1989 and were introduced by Matsumoto. In 1992, Mihai and Rosca independently introduced the idea of Lorentzian para-Sasakian manifolds in classical analysis [1,2].

In 2019, Biswas and Baishya [3,4] introduced the notion of a new connection, called general connection in the context of Sasakian geometry and discussed several cases. They defined general connection D^G in the following manner:

$$D_{\mathcal{X}_1}^G \mathcal{X}_2 = D_{\mathcal{X}_1} \mathcal{X}_2 + a[(D_{\mathcal{X}_1} \eta)(\mathcal{X}_2)\zeta - \eta(\mathcal{X}_2)D_{\mathcal{X}_1} \zeta] + b\eta(\mathcal{X}_1)\phi \mathcal{X}_2 \quad (1)$$

for any $\varkappa_1, \varkappa_2 \in \chi(M)$, where a, b are real constants, ϕ is a $(1, 1)$ -type tensor field, ζ is a vector field, η is a 1-form, $\chi(M)$ is the set of all vector fields on M and D is the Levi-Civita connection.

From (1), we have the following particular cases:

- (i) if $a = 0, b = -1$, then (1) reduces to the quarter-symmetric metric connection [5,6];
- (ii) if $a = 1, b = 0$, then (1) reduces to the Schouten–Van Kampen connection [7];
- (iii) if $a = 1, b = -1$, then (1) reduces to the Tanaka Webster connection [8];
- (iv) if $a = 1, b = 1$, then (1) reduces to the Zamkovoy connection [9].

Recently, Bhatt and Chanyal [10] studied η -Ricci solitons on Sasakian manifolds admitting general connection, ref. [11] investigated conformal Ricci soliton in Sasakian manifolds admitting general connection. Moreover, ref. [12] studied Ricci solitons on concircularly flat and W_i -flat Sasakian manifolds admitting a general connection.

In a Lorentzian para-Sasakian manifold (briefly LP-Sasakian manifold) M of dimension n ($n > 2$), the projective curvature tensor $\bar{P}$ and the conharmonic curvature tensor $\bar{C}$ in accordance with general connection D^G are given by

$$\bar{P}(\varkappa_1, \varkappa_2)\varkappa_3 = \bar{R}(\varkappa_1, \varkappa_2)\varkappa_3 - \frac{1}{n-1}[\bar{S}(\varkappa_2, \varkappa_3)\varkappa_1 - \bar{S}(\varkappa_1, \varkappa_3)\varkappa_2], \quad (2)$$

$$\begin{aligned} \bar{C}(\varkappa_1, \varkappa_2)\varkappa_3 = & \bar{R}(\varkappa_1, \varkappa_2)\varkappa_3 - \frac{1}{n-2}[\bar{S}(\varkappa_2, \varkappa_3)\varkappa_1 - \bar{S}(\varkappa_1, \varkappa_3)\varkappa_2 \\ & + g(\varkappa_2, \varkappa_3)\bar{Q}\varkappa_1 - g(\varkappa_1, \varkappa_3)\bar{Q}\varkappa_2] \end{aligned} \quad (3)$$

for any $\varkappa_1, \varkappa_2, \varkappa_3 \in \chi(M)$, where $\bar{R}$, $\bar{S}$, $\bar{Q}$ are the Riemannian curvature tensor, the Ricci tensor and the Ricci operator in accordance with general connection D^G , respectively [13–16].

Definition 1. An n -dimensional LP-Sasakian manifold M is said to be a generalized η -Einstein manifold if the Ricci tensor of type $(0, 2)$ satisfies

$$S(\varkappa_2, \varkappa_3) = \alpha_1 g(\varkappa_2, \varkappa_3) + \alpha_2 \eta(\varkappa_2)\eta(\varkappa_3) + \alpha_3 \gamma(\varkappa_2, \varkappa_3) \quad (4)$$

for any $\varkappa_2, \varkappa_3 \in \chi(M)$, with $\chi(M)$ being the set of all vector fields of the manifold M , where α_1, α_2 and α_3 are scalars and γ is a 2-form.

The paper is designed as follows: After the introduction, some postulates of the LP-Sasakian manifold are given in Section 2. In Section 3, we have discussed the LP-Sasakian manifold admitting the general connection and obtaining certain curvature tensors in an LP-Sasakian manifold. Section 4 contains projectively flat, ζ -projectively flat and ϕ -projectively flat LP-Sasakian manifolds in accordance with the general connection. Section 5 discusses the conharmonically flat, ζ -conharmonically flat and ϕ -conharmonically flat LP-Sasakian manifolds in accordance with the general connection. Quasi-conharmonically flat LP-Sasakian manifolds in accordance with the general connection are discussed in Section 6. Section 7 deals with Ricci semi-symmetric LP-Sasakian manifold, reviewing the general connection. Section 8 deals with the applications of the general connection to the general theory of relativity by considering Einstein's field equation without cosmological constant. Finally, with the help of Koszul's formula, an example is given for the validation of the work.

2. Preliminaries

An n -dimensional differentiable manifold M is classified as an LP-Sasakian manifold if it admits a $(1, 1)$ -type tensor field ϕ , a vector field ζ , a 1-form η and a Lorentzian metric g that satisfy the following conditions:

$$\phi^2 \varkappa_1 = \varkappa_1 + \eta(\varkappa_1)\zeta, \quad \eta(\zeta) = -1, \quad \eta(\phi \varkappa_1) = 0, \quad \phi \zeta = 0, \quad (5)$$

$$g(\phi \varkappa_1, \phi \varkappa_2) = g(\varkappa_1, \varkappa_2) + \eta(\varkappa_1)\eta(\varkappa_2), \quad (6)$$

$$\eta(\varkappa_1) = g(\varkappa_1, \zeta), \quad g(\varkappa_1, \phi \varkappa_2) = g(\phi \varkappa_1, \varkappa_2), \quad (7)$$

$$D_{\varkappa_1} \zeta = \phi \varkappa_1, \quad (8)$$

$$(D_{\varkappa_1} \phi)(\varkappa_2) = \eta(\varkappa_2)\varkappa_1 + g(\varkappa_1, \varkappa_2)\zeta + 2\eta(\varkappa_1)\eta(\varkappa_2), \quad (9)$$

where D denotes the operator of covariant differentiation in accordance with the Lorentzian metric g .

Let γ be a symmetric $(0, 2)$ -type tensor field such that $\gamma(\varkappa_1, \varkappa_2) = g(\varkappa_1, \phi \varkappa_2)$. Moreover, in an LP-Sasakian manifold M , the vector field η is closed, then we have

$$(D_{\varkappa_1} \eta)(\varkappa_2) = g(\varkappa_1, \phi \varkappa_2) = \gamma(\varkappa_1, \varkappa_2), \quad \gamma(\varkappa_1, \zeta) = 0 \quad (10)$$

for all $\varkappa_1, \varkappa_2$ on M .

Furthermore, in an LP-Sasakian manifold M , the following condition holds:

$$\eta(R(\varkappa_1, \varkappa_2)\varkappa_3) = g(\varkappa_2, \varkappa_3)\eta(\varkappa_1) - g(\varkappa_1, \varkappa_3)\eta(\varkappa_2), \quad (11)$$

$$R(\varkappa_1, \varkappa_2)\zeta = \eta(\varkappa_2)\varkappa_1 - \eta(\varkappa_1)\varkappa_2, \quad (12)$$

$$R(\zeta, \varkappa_2)\varkappa_3 = g(\varkappa_2, \varkappa_3)\zeta - \eta(\varkappa_3)\varkappa_2, \quad (13)$$

$$R(\zeta, \varkappa_2)\zeta = \eta(\varkappa_2)\zeta + \varkappa_2, \quad (14)$$

$$S(\varkappa_1, \zeta) = (n-1)\eta(\varkappa_1), \quad (15)$$

$$S(\phi \varkappa_1, \phi \varkappa_2) = S(\varkappa_1, \varkappa_2) + (n-1)\eta(\varkappa_1)\eta(\varkappa_2), \quad (16)$$

$$Q\zeta = (n-1)\zeta, \quad Q\phi = \phi Q, \quad S(\varkappa_1, \varkappa_2) = g(Q\varkappa_1, \varkappa_2), \quad S^2(\varkappa_1, \varkappa_2) = S(Q\varkappa_1, \varkappa_2) \quad (17)$$

for all $\varkappa_1, \varkappa_2 \in \chi(M)$, where R, S and Q are the curvature tensor, the Ricci tensor and the Ricci operator with respect to the Levi-Civita connection D , respectively.

3. Curvature Tensors in Accordance with General Connection in an LP-Sasakian Manifolds

With the help of (8) and (10) in (1), we obtain

$$D_{\varkappa_1}^G \varkappa_2 = D_{\varkappa_1} \varkappa_2 + a[g(\varkappa_1, \phi \varkappa_2)\zeta - \eta(\varkappa_2)\phi \varkappa_1] + b\eta(\varkappa_1)\phi \varkappa_2, \quad (18)$$

where a and b are real constants. The torsion tensor is given by

$$\tilde{T}(\varkappa_1, \varkappa_2) = (a+b)[\eta(\varkappa_1)\phi \varkappa_2 - \eta(\varkappa_2)\phi \varkappa_1]. \quad (19)$$

From (1) and (10), we obtain

$$(D_{\varkappa_1}^G g)(\varkappa_2, \varkappa_3) = -2b\eta(\varkappa_1)g(\varkappa_2, \phi \varkappa_3). \quad (20)$$

Replacing $\varkappa_2$ by ζ in (18), we obtain

$$D_{\varkappa_1}^G \zeta = (a+1)\phi \varkappa_1. \quad (21)$$

Again, from (7)–(9) and (18), we have

$$D_{\mathcal{X}_1}^G \phi \mathcal{X}_2 = (a+1)g(\mathcal{X}_1, \mathcal{X}_2)\zeta + (a+b+2)\eta(\mathcal{X}_1)\eta(\mathcal{X}_2)\zeta \\ + \phi(D_{\mathcal{X}_1} \mathcal{X}_2) + \eta(\mathcal{X}_2)\mathcal{X}_1 + b\eta(\mathcal{X}_1)\mathcal{X}_2, \quad (22)$$

$$D_{\mathcal{X}_1}^G g(\mathcal{X}_2, \mathcal{X}_3) = g(D_{\mathcal{X}_1} \mathcal{X}_2, \phi \mathcal{X}_3) + g(\mathcal{X}_2, D_{\mathcal{X}_1} \mathcal{X}_3), \quad (23)$$

$$D_{\mathcal{X}_1}^G g(\mathcal{X}_2, \phi \mathcal{X}_3) = g(D_{\mathcal{X}_1} \mathcal{X}_2, \phi \mathcal{X}_3) + g(\mathcal{X}_2, \phi D_{\mathcal{X}_1} \mathcal{X}_3) + g(\mathcal{X}_1, \mathcal{X}_3)\eta(\mathcal{X}_2) \\ + g(\mathcal{X}_1, \mathcal{X}_2)\eta(\mathcal{X}_3) + 2\eta(\mathcal{X}_1)\eta(\mathcal{X}_2)\eta(\mathcal{X}_3). \quad (24)$$

Let $\bar{R}$ be the Riemannian curvature tensor in accordance with general connection D^G defined by

$$\bar{R}(\mathcal{X}_1, \mathcal{X}_2)\mathcal{X}_3 = D_{\mathcal{X}_1}^G D_{\mathcal{X}_2}^G \mathcal{X}_3 - D_{\mathcal{X}_2}^G D_{\mathcal{X}_1}^G \mathcal{X}_3 - D_{[\mathcal{X}_1, \mathcal{X}_2]}^G \mathcal{X}_3. \quad (25)$$

Using (18), (20) and (22)–(24) in (25), we obtain

$$\bar{R}(\mathcal{X}_1, \mathcal{X}_2)\mathcal{X}_3 = R(\mathcal{X}_1, \mathcal{X}_2)\mathcal{X}_3 + (a+ab+b)[g(\mathcal{X}_1, \mathcal{X}_3)\eta(\mathcal{X}_2)\zeta - g(\mathcal{X}_2, \mathcal{X}_3)\eta(\mathcal{X}_1)\zeta] \\ + a(a+2)[g(\mathcal{X}_2, \phi \mathcal{X}_3)\phi \mathcal{X}_1 - g(\mathcal{X}_1, \phi \mathcal{X}_3)\phi \mathcal{X}_2] \\ + (ab+b-a)[\eta(\mathcal{X}_2)\eta(\mathcal{X}_3)\mathcal{X}_1 - \eta(\mathcal{X}_1)\eta(\mathcal{X}_3)\mathcal{X}_2]. \quad (26)$$

Contracting (26), we have

$$\bar{S}(\mathcal{X}_2, \mathcal{X}_3) = S(\mathcal{X}_2, \mathcal{X}_3) + (ab+b-a^2-a)g(\mathcal{X}_2, \mathcal{X}_3) + a(a+2)\lambda g(\mathcal{X}_2, \phi \mathcal{X}_3) \\ + [ab+b-a^2-a+(n-1)(ab+b-a)]\eta(\mathcal{X}_2)\eta(\mathcal{X}_3), \quad (27)$$

where $\lambda = \text{trace}\phi$.

By direct calculation, we are able to obtain the following relations:

$$\bar{S}(\zeta, \mathcal{X}_3) = (a+1)(1-b)(n-1)\eta(\mathcal{X}_3), \quad (28)$$

$$\bar{Q}\mathcal{X}_2 = Q\mathcal{X}_2 + [ab+b-a^2-a]\mathcal{X}_2 + a(a+2)\lambda\phi\mathcal{X}_2 \\ + [ab+b-a^2-a+(n-1)(ab+b-a)]\eta(\mathcal{X}_2)\zeta, \quad (29)$$

$$\bar{Q}\zeta = (a+1)(1-b)(n-1)\zeta, \quad (30)$$

$$\bar{r} = r - a^2(n-1) + a(a+2)\lambda^2, \quad (31)$$

$$\bar{R}(\mathcal{X}_1, \mathcal{X}_2)\zeta = (a-ab-b+1)[\eta(\mathcal{X}_2)\mathcal{X}_1 - \eta(\mathcal{X}_1)\mathcal{X}_2], \quad (32)$$

$$\bar{R}(\zeta, \mathcal{X}_2)\mathcal{X}_3 = (a+ab+b+1)g(\mathcal{X}_2, \mathcal{X}_3)\zeta + 2(a+1)b\eta(\mathcal{X}_2)\eta(\mathcal{X}_3)\zeta \\ + (ab+b-a-1)\eta(\mathcal{X}_3)\mathcal{X}_2 \quad (33)$$

$$\bar{R}(\mathcal{X}_1, \zeta)\mathcal{X}_3 = -(a+ab+b+1)g(\mathcal{X}_1, \mathcal{X}_3)\zeta - 2(a+1)b\eta(\mathcal{X}_1)\eta(\mathcal{X}_3)\zeta \\ + (a-ab-b+1)\eta(\mathcal{X}_3)\mathcal{X}_1 \quad (34)$$

for any $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3 \in \chi(M)$, where $\lambda = \text{trace}\phi$. Therefore, we can state the following:

Theorem 1. An n -dimensional LP-Sasakian manifold M equipped with general connection D^G satisfies the following:

- (i) $\bar{R}$ of D^G is specified by (26);
- (ii) $\bar{S}$ of D^G is specified by (27);
- (iii) $\bar{r}$ of D^G is specified by (31);
- (iv) $\bar{S}$ of D^G is symmetric;
- (v) $\bar{R}(\mathcal{X}_1, \mathcal{X}_2)\mathcal{X}_3 + \bar{R}(\mathcal{X}_2, \mathcal{X}_3)\mathcal{X}_1 + \bar{R}(\mathcal{X}_3, \mathcal{X}_1)\mathcal{X}_2 = 0$.

Remark 1. We can see that in an n -dimensional LP-Sasakian manifold M admitting general connection D^G , the following properties are obtained from (1), (26), (27) and (31) under the following conditions:

Case 1	Results
$a = 0, b = -1$	Quarter-symmetric metric connection
$\bar{R}$	$R(\varkappa_1, \varkappa_2)\varkappa_3 + g(\varkappa_2, \varkappa_3)\eta(\varkappa_1)\zeta - g(\varkappa_1, \varkappa_3)\eta(\varkappa_2)\zeta + \eta(\varkappa_1)\eta(\varkappa_3)\varkappa_2 - \eta(\varkappa_2)\eta(\varkappa_3)\varkappa_1$
$\bar{S}$	$S(\varkappa_2, \varkappa_3) - g(\varkappa_2, \varkappa_3) - n\eta(\varkappa_2)\eta(\varkappa_3)$
$\bar{r}$	r
Case 2	Results
$a = 1, b = 0$	Schouten–Van Kampen connection
$\bar{R}$	$R(\varkappa_1, \varkappa_2)\varkappa_3 - g(\varkappa_2, \varkappa_3)\eta(\varkappa_1)\zeta + g(\varkappa_1, \varkappa_3)\eta(\varkappa_2)\zeta + 3g(\varkappa_2, \phi\varkappa_3)\phi\varkappa_1 - 3g(\varkappa_1, \phi\varkappa_3)\phi\varkappa_2 + \eta(\varkappa_1)\eta(\varkappa_3)\varkappa_2 - \eta(\varkappa_2)\eta(\varkappa_3)\varkappa_1$
$\bar{S}$	$S(\varkappa_2, \varkappa_3) - 2g(\varkappa_2, \varkappa_3) - (n+1)\eta(\varkappa_2)\eta(\varkappa_3) + 3\lambda g(\varkappa_2, \phi\varkappa_3)$
$\bar{r}$	$r - n + 1 + 3\lambda^2$
Case 3	Results
$a = 1, b = -1$	Tanaka Webster connection
$\bar{R}$	$R(\varkappa_1, \varkappa_2)\varkappa_3 + g(\varkappa_2, \varkappa_3)\eta(\varkappa_1)\zeta - g(\varkappa_1, \varkappa_3)\eta(\varkappa_2)\zeta + 3g(\varkappa_2, \phi\varkappa_3)\phi\varkappa_1 - 3g(\varkappa_1, \phi\varkappa_3)\phi\varkappa_2 + 3\eta(\varkappa_1)\eta(\varkappa_3)\varkappa_2 - 3\eta(\varkappa_2)\eta(\varkappa_3)\varkappa_1$
$\bar{S}$	$S(\varkappa_2, \varkappa_3) - 4g(\varkappa_2, \varkappa_3) - (3n+1)\eta(\varkappa_2)\eta(\varkappa_3) + 3\lambda g(\varkappa_2, \phi\varkappa_3)$
$\bar{r}$	$r - n + 1 + 3\lambda^2$
Case 4	Results
$a = 1, b = 1$	Zamkovoy connection
$\bar{R}$	$R(\varkappa_1, \varkappa_2)\varkappa_3 - 3g(\varkappa_2, \varkappa_3)\eta(\varkappa_1)\zeta + 3g(\varkappa_1, \varkappa_3)\eta(\varkappa_2)\zeta + 3g(\varkappa_2, \phi\varkappa_3)\phi\varkappa_1 - 3g(\varkappa_1, \phi\varkappa_3)\phi\varkappa_2 - \eta(\varkappa_1)\eta(\varkappa_3)\varkappa_2 + \eta(\varkappa_2)\eta(\varkappa_3)\varkappa_1$
$\bar{S}$	$S(\varkappa_2, \varkappa_3) + (n-1)\eta(\varkappa_2)\eta(\varkappa_3) + 3\lambda g(\varkappa_2, \phi\varkappa_3)$
$\bar{r}$	$r - n + 1 + 3\lambda^2$

4. Projectively Flat LP-Sasakian Manifolds in Accordance with General Connection

From (2), (26) and (27), we obtain

$$\begin{aligned}
 \bar{P}(\varkappa_1, \varkappa_2)\varkappa_3 = & R(\varkappa_1, \varkappa_2)\varkappa_3 + (a + ab + b)[g(\varkappa_1, \varkappa_3)\eta(\varkappa_2)\zeta - g(\varkappa_2, \varkappa_3)\eta(\varkappa_1)\zeta] \\
 & + a(a + 2)[g(\varkappa_2, \phi\varkappa_3)\phi\varkappa_1 - g(\varkappa_1, \phi\varkappa_3)\phi\varkappa_2] \\
 & + (ab + b - a)[\eta(\varkappa_2)\eta(\varkappa_3)\varkappa_1 - \eta(\varkappa_1)\eta(\varkappa_3)\varkappa_2] \\
 & - \frac{1}{n-1}[S(\varkappa_2, \varkappa_3)\varkappa_1 + (ab + b - a^2 - a)g(\varkappa_2, \varkappa_3)\varkappa_1 \\
 & + \{ab + b - a^2 - a + (n-1)(ab + b - a)\}\eta(\varkappa_2)\eta(\varkappa_3)\varkappa_1 \\
 & + a(a + 2)\lambda g(\varkappa_2, \phi\varkappa_3)\varkappa_1 - S(\varkappa_1, \varkappa_3)\varkappa_2 \\
 & - (ab + b - a^2 - a)g(\varkappa_1, \varkappa_3)\varkappa_2 \\
 & - \{ab + b - a^2 - a + (n-1)(ab + b - a)\}\eta(\varkappa_1)\eta(\varkappa_3)\varkappa_2 \\
 & - a(a + 2)\lambda g(\varkappa_1, \phi\varkappa_3)\varkappa_2].
 \end{aligned} \tag{35}$$

Let M be projectively flat with respect to D^G , i.e., $\bar{P} = 0$. Then, taking the inner product with $\varkappa_4$ in (35), we obtain

$$\begin{aligned}
R(\varkappa_1, \varkappa_2, \varkappa_3, \varkappa_4) = & (a + ab + b)[g(\varkappa_2, \varkappa_3)\eta(\varkappa_1)\eta(\varkappa_4) - g(\varkappa_1, \varkappa_3)\eta(\varkappa_2)\eta(\varkappa_4)] \\
& - a(a + 2)[g(\varkappa_2, \phi\varkappa_3)g(\phi\varkappa_1, \varkappa_4) - g(\varkappa_1, \phi\varkappa_3)g(\phi\varkappa_2, \varkappa_4)] \\
& - (ab + b - a)[\eta(\varkappa_2)\eta(\varkappa_3)g(\varkappa_1, \varkappa_4) - \eta(\varkappa_1)\eta(\varkappa_3)g(\varkappa_2, \varkappa_4)] \\
& + \frac{1}{n-1}[S(\varkappa_2, \varkappa_3)g(\varkappa_1, \varkappa_4) \\
& + (ab + b - a^2 - a)g(\varkappa_2, \varkappa_3)g(\varkappa_1, \varkappa_4) \\
& + \{ab + b - a^2 - a + (n-1)(ab + b - a)\}\eta(\varkappa_2)\eta(\varkappa_3)g(\varkappa_1, \varkappa_4) \\
& + a(a + 2)\lambda g(\varkappa_2, \phi\varkappa_3)g(\varkappa_1, \varkappa_4) - S(\varkappa_1, \varkappa_3)g(\varkappa_2, \varkappa_4) \\
& - (ab + b - a^2 - a)g(\varkappa_1, \varkappa_3)g(\varkappa_2, \varkappa_4) \\
& - \{ab + b - a^2 - a + (n-1)(ab + b - a)\}\eta(\varkappa_1)\eta(\varkappa_3)g(\varkappa_2, \varkappa_4) \\
& - a(a + 2)\lambda g(\varkappa_1, \phi\varkappa_3)g(\varkappa_2, \varkappa_4)].
\end{aligned} \tag{36}$$

Setting $\varkappa_1 = \varkappa_4 = \zeta$ in (36) with (5) and (14), we get

$$S(\varkappa_2, \varkappa_3) = d_1 g(\varkappa_2, \varkappa_3) + d_2 \eta(\varkappa_2)\eta(\varkappa_3) + d_3 \lambda \gamma(\varkappa_2, \varkappa_3), \tag{37}$$

where $d_1 = -2ab - 2b + a^2 + bn + abn + n - 1$, $d_2 = -2ab - 2b + a^2 + bn + abn + bn$, $d_3 = -a(a + 2)$, $\gamma(\varkappa_2, \varkappa_3) = g(\phi\varkappa_2, \varkappa_3) = g(\varkappa_2, \phi\varkappa_3)$ and $\lambda = \text{trace}\phi$. Therefore, the following theorem can be stated as:

Theorem 2. *If an n -dimensional LP-Sasakian manifold M is projectively flat in accordance with general connection D^G , then it is found to be a generalized η -Einstein manifold.*

Again, setting $\varkappa_3 = \zeta$ in (35), we obtain

$$\begin{aligned}
\bar{P}(\varkappa_1, \varkappa_2)\zeta = & R(\varkappa_1, \varkappa_2)\zeta + (a + ab + b)[g(\varkappa_1, \zeta)\eta(\varkappa_2)\zeta - g(\varkappa_2, \zeta)\eta(\varkappa_1)\zeta] \\
& + a(a + 2)[g(\varkappa_2, \phi\zeta)\phi\varkappa_1 - g(\varkappa_1, \phi\zeta)\phi\varkappa_2] \\
& + (ab + b - a)[\eta(\varkappa_2)\eta(\zeta)\varkappa_1 - \eta(\varkappa_1)\eta(\zeta)\varkappa_2] \\
& - \frac{1}{n-1}[S(\varkappa_2, \zeta)\varkappa_1 + (ab + b - a^2 - a)g(\varkappa_2, \zeta)\varkappa_1 \\
& + \{ab + b - a^2 - a + (n-1)(ab + b - a)\}\eta(\varkappa_2)\eta(\zeta)\varkappa_1 \\
& + a(a + 2)\lambda g(\varkappa_2, \phi\zeta)\varkappa_1 - S(\varkappa_1, \zeta)\varkappa_2 \\
& - (ab + b - a^2 - a)g(\varkappa_1, \zeta)\varkappa_2 \\
& - \{ab + b - a^2 - a + (n-1)(ab + b - a)\}\eta(\varkappa_1)\eta(\zeta)\varkappa_2 \\
& - a(a + 2)\lambda g(\varkappa_1, \phi\zeta)\varkappa_2].
\end{aligned}$$

Inserting (2), (5), (7), (12) and (15) in the above equation, we obtain

$$P(\varkappa_1, \varkappa_2)\zeta = P(\varkappa_1, \varkappa_2)\zeta.$$

Therefore, the theorem states the following:

Theorem 3. *An n -dimensional LP-Sasakian manifold M is ζ -projectively flat with respect to the general connection D^G if and only if it is ζ -projectively flat with respect to the Levi-Civita connection D .*

Now, let us consider the ϕ -projectively flat LP-Sasakian manifold M in accordance with general connection D^G .

Definition 2. An n -dimensional LP-Sasakian manifold M is said to be ϕ -projectively flat in accordance with the general connection D^G if

$$g(\bar{P}(\phi\kappa_1, \phi\kappa_2)\phi\kappa_3, \phi\kappa_4) = 0 \quad (38)$$

for all the vector fields $\kappa_1, \kappa_2, \kappa_3, \kappa_4 \in \chi(M)$.

Theorem 4. If an n -dimensional LP-Sasakian manifold M ($n > 2$) is ϕ -projectively flat in accordance with the general connection D^G , then M is a generalized η -Einstein manifold.

Proof. Taking the inner product with κ_4 in (4), we get

$$g(\bar{R}(\phi\kappa_1, \phi\kappa_2)\phi\kappa_3, \phi\kappa_4) = \frac{1}{n-1} [\bar{S}(\phi\kappa_2, \phi\kappa_3)g(\phi\kappa_1, \phi\kappa_4) - \bar{S}(\phi\kappa_1, \phi\kappa_3)g(\phi\kappa_2, \phi\kappa_4)] \quad (39)$$

Let $\{v_1, v_2, v_3, \dots, v_{n-1}, \zeta\}$ be a local orthonormal basis of vector fields in M by using the fact that $\{\phi v_1, \phi v_2, \phi v_3, \dots, \phi v_{n-1}, \zeta\}$ is also a local orthonormal basis of vector fields in M . Setting $\kappa_2 = \kappa_3 = v_i$ in (39) and summing over i ($1 \leq i \leq n-1$), we obtain

$$\sum_{i=1}^{n-1} g(\bar{R}(\phi\kappa_1, \phi v_i)\phi v_i, \phi\kappa_4) = \frac{1}{n-1} \sum_{i=1}^{n-1} [\bar{S}(\phi v_i, \phi v_i)g(\phi\kappa_1, \phi\kappa_4) - \bar{S}(\phi\kappa_1, \phi v_i)g(\phi v_i, \phi\kappa_4)] \quad (40)$$

It can be easily seen that

$$\sum_{i=1}^{n-1} g(\phi v_i, \phi v_i) = n-1, \quad (41)$$

$$\sum_{i=1}^{n-1} \bar{S}(\phi\kappa_1, \phi v_i)g(\phi v_i, \phi\kappa_4) = \bar{S}(\phi\kappa_1, \phi\kappa_4), \quad (42)$$

$$\sum_{i=1}^{n-1} \bar{S}(\phi v_i, \phi v_i) = \bar{r}. \quad (43)$$

Now, using (41)–(43) in (40), we have

$$\bar{S}(\phi\kappa_1, \phi\kappa_4) = \frac{\bar{r}}{n} g(\phi\kappa_1, \phi\kappa_4). \quad (44)$$

From (6), (16), (31) and (44), we get

$$S(\kappa_1, \kappa_4) = f_1 g(\kappa_1, \kappa_4) + f_2 \eta(\kappa_1)\eta(\kappa_4) + f_3 \lambda \gamma(\kappa_1, \kappa_4), \quad (45)$$

where

$$f_1 = \frac{r - a^2(n-1) + a(a+2)\lambda^2}{n} - (ab + b - a^2 - a),$$

$$f_2 = \frac{r - a^2(n-1) + a(a+2)\lambda^2}{n} - (ab + b - a^2 - a + n - 1),$$

$$f_3 = -a(a+2),$$

$\lambda = \text{trace}\phi$ and $\gamma(\kappa_1, \kappa_4) = g(\phi\kappa_1, \kappa_4) = g(\kappa_1, \phi\kappa_4)$.

Thus, M is a generalized η -Einstein manifold. $\square$

5. Conharmonically Flat LP-Sasakian Manifolds in Accordance with General Connection

In view of (3), (26), (27) and (29), we obtain

$$\begin{aligned}\bar{C}(\mathfrak{x}_1, \mathfrak{x}_2)\mathfrak{x}_3 = & C(\mathfrak{x}_1, \mathfrak{x}_2)\mathfrak{x}_3 + (a + ab + b)[g(\mathfrak{x}_1, \mathfrak{x}_3)\eta(\mathfrak{x}_2)\zeta - g(\mathfrak{x}_2, \mathfrak{x}_3)\eta(\mathfrak{x}_1)\zeta] \\ & + a(a + 2)[g(\mathfrak{x}_2, \phi\mathfrak{x}_3)\phi\mathfrak{x}_1 - g(\mathfrak{x}_1, \phi\mathfrak{x}_3)\phi\mathfrak{x}_2] \\ & + (ab + b - a)[\eta(\mathfrak{x}_2)\eta(\mathfrak{x}_3)\mathfrak{x}_1 - \eta(\mathfrak{x}_1)\eta(\mathfrak{x}_3)\mathfrak{x}_2] \\ & - \frac{1}{n-2}[2(ab + b - a^2 - a)\{g(\mathfrak{x}_2, \mathfrak{x}_3)\mathfrak{x}_1 - g(\mathfrak{x}_1, \mathfrak{x}_3)\mathfrak{x}_2\} \\ & + \{ab + b - a^2 - a + (n-1)(ab + b - a)\}\{\eta(\mathfrak{x}_2)\mathfrak{x}_1 \\ & - \eta(\mathfrak{x}_1)\mathfrak{x}_2\}\eta(\mathfrak{x}_3) \\ & + \{ab + b - a^2 - a + (n-1)(ab + b - a)\}g(\mathfrak{x}_2, \mathfrak{x}_3)\eta(\mathfrak{x}_1)\zeta \\ & - \{ab + b - a^2 - a + (n-1)(ab + b - a)\}g(\mathfrak{x}_1, \mathfrak{x}_3)\eta(\mathfrak{x}_2)\zeta \\ & + a(a + 2)\lambda\{g(\mathfrak{x}_2, \phi\mathfrak{x}_3)\mathfrak{x}_1 - g(\mathfrak{x}_1, \phi\mathfrak{x}_3)\mathfrak{x}_2\} \\ & - a(a + 2)\lambda g(\mathfrak{x}_1, \mathfrak{x}_3)\phi\mathfrak{x}_2 - \lambda g(\mathfrak{x}_2, \mathfrak{x}_3)\phi\mathfrak{x}_1].\end{aligned}\quad (46)$$

If M is a conharmonically flat (i.e., $\bar{C} = 0$) LP-Sasakian manifold in accordance with general connection, then from (3), we obtain

$$\bar{R}(\mathfrak{x}_1, \mathfrak{x}_2)\mathfrak{x}_3 = \frac{1}{n-2}[\bar{S}(\mathfrak{x}_2, \mathfrak{x}_3)\mathfrak{x}_1 - \bar{S}(\mathfrak{x}_1, \mathfrak{x}_3)\mathfrak{x}_2 + g(\mathfrak{x}_2, \mathfrak{x}_3)\bar{Q}\mathfrak{x}_1 - g(\mathfrak{x}_1, \mathfrak{x}_3)\bar{Q}\mathfrak{x}_2]. \quad (47)$$

Taking the inner product of (47) with vector field $\mathfrak{x}_4$, we obtain

$$\begin{aligned}g(\bar{R}(\mathfrak{x}_1, \mathfrak{x}_2)\mathfrak{x}_3, \mathfrak{x}_4) = & \frac{1}{n-2}[\bar{S}(\mathfrak{x}_2, \mathfrak{x}_3)g(\mathfrak{x}_1, \mathfrak{x}_4) - \bar{S}(\mathfrak{x}_1, \mathfrak{x}_3)g(\mathfrak{x}_2, \mathfrak{x}_4) \\ & + g(\mathfrak{x}_2, \mathfrak{x}_3)\bar{S}(\mathfrak{x}_1, \mathfrak{x}_4) - g(\mathfrak{x}_1, \mathfrak{x}_3)\bar{S}(\mathfrak{x}_2, \mathfrak{x}_4)],\end{aligned}\quad (48)$$

where $g(\bar{Q}\mathfrak{x}_1, \mathfrak{x}_4) = \bar{S}(\mathfrak{x}_1, \mathfrak{x}_4)$.

Performing an orthonormal frame field of M and contracting (48) over $\mathfrak{x}_1$ and $\mathfrak{x}_4$, gives

$$\bar{r} = 0. \quad (49)$$

Using (31) in (49), we have

$$r = a^2n - a^2 - a^2\lambda^2 - 2a\lambda^2. \quad (50)$$

Therefore, the following theorem arises:

Theorem 5. *If an n -dimensional LP-Sasakian manifold M ($n > 2$) is conharmonically flat in accordance with general connection D^G , then the scalar curvature is given by $\bar{r} = 0$, meaning that, $r = a^2n - a^2 - a^2\lambda^2 - 2a\lambda^2$.*

Theorem 6. *An n -dimensional LP-Sasakian manifold M is ζ -conharmonically flat in accordance with general connection D^G if and only if it is so in accordance with Levi-Civita connection D , assuming that the vector fields are horizontal vector fields.*

Proof. Now, we will show for the ζ -conharmonically flat LP-Sasakian manifold M in accordance with general connection D^G .

Replacing $\mathfrak{x}_3$ by ζ in (46) and with the help of (3) and (5), we obtain

$$\begin{aligned}\bar{C}(\varkappa_1, \varkappa_2)\zeta &= C(\varkappa_1, \varkappa_2)\zeta + \frac{a}{n-2} [\eta(\varkappa_2)\varkappa_1 - \eta(\varkappa_1)\varkappa_2] \\ &\quad + \frac{a(a+2)\lambda}{n-2} [\eta(\varkappa_1)\phi\varkappa_2 - \eta(\varkappa_2)\phi\varkappa_1] \\ &= C(\varkappa_1, \varkappa_2)\zeta,\end{aligned}\quad (51)$$

if $\varkappa_1, \varkappa_2$ are horizontal vector fields on M , and a is a real constant.

Hence, it is proven. $\square$

Now, $\varkappa_3$ is replaced with ζ in (3). Then, using (28) and (32), we have

$$\bar{C}(\varkappa_1, \varkappa_2)\zeta = \frac{(a+1)(b-1)}{n-2} [\eta(\varkappa_2)\varkappa_1 - \eta(\varkappa_1)\varkappa_2] + \frac{1}{n-2} [\eta(\varkappa_1)\bar{Q}\varkappa_2 - \eta(\varkappa_2)\bar{Q}\varkappa_1]. \quad (52)$$

If M is ζ -conharmonically flat in accordance with general connection D^G , then from (52), we obtain

$$(a+1)(b-1) [\eta(\varkappa_2)\varkappa_1 - \eta(\varkappa_1)\varkappa_2] = \eta(\varkappa_2)\bar{Q}\varkappa_1 - \eta(\varkappa_1)\bar{Q}\varkappa_2. \quad (53)$$

After taking the inner product with $\varkappa_4$ in the above equation, the new equation becomes

$$\begin{aligned}(a+1)(b-1) [\eta(\varkappa_2)g(\varkappa_1, \varkappa_4) - \eta(\varkappa_1)g(\varkappa_2, \varkappa_4)] &= \eta(\varkappa_2)\bar{S}(\varkappa_1, \varkappa_4) \\ &\quad - \eta(\varkappa_1)\bar{S}(\varkappa_2, \varkappa_4).\end{aligned}\quad (54)$$

Setting $\varkappa_2 = \zeta$ in (54), we obtain

$$\bar{S}(\varkappa_1, \varkappa_4) = (a+1)(1-b) [g(\varkappa_1, \varkappa_4) + n\eta(\varkappa_1)\eta(\varkappa_4)]. \quad (55)$$

Contracting (55) over $\varkappa_1$ and $\varkappa_4$, we obtain

$$\bar{r} = 0. \quad (56)$$

Theorem 7. An n -dimensional LP-Sasakian manifold M is ζ -conharmonically flat in accordance with general connection D^G , then the scalar curvature in accordance with general connection D^G vanishes.

Moreover, let us consider the ϕ -conharmonically flat LP-Sasakian manifold M in accordance with general connection D^G .

Definition 3. An n -dimensional LP-Sasakian manifold M is said to be ϕ -conharmonically flat in accordance with general connection D^G if

$$g(\bar{C}(\phi\varkappa_1, \phi\varkappa_2)\phi\varkappa_3, \phi\varkappa_4) = 0 \quad (57)$$

for all the vector fields $\varkappa_1, \varkappa_2, \varkappa_3, \varkappa_4 \in \chi(M)$.

Theorem 8. If an n -dimensional LP-Sasakian manifold M ($n > 2$) is ϕ -conharmonically flat in accordance with general connection D^G , then M is generalized η -Einstein manifold.

Proof. Taking the inner product with $\varkappa_4$ in (3), we obtain

$$\begin{aligned}
g(\bar{R}(\phi\kappa_1, \phi\kappa_2)\phi\kappa_3, \phi\kappa_4) &= \frac{1}{n-2} [\bar{S}(\phi\kappa_2, \phi\kappa_3)g(\phi\kappa_1, \phi\kappa_4) \\
&\quad - \bar{S}(\phi\kappa_1, \phi\kappa_3)g(\phi\kappa_2, \phi\kappa_4) \\
&\quad + g(\phi\kappa_2, \phi\kappa_3)\bar{S}(\phi\kappa_1, \phi\kappa_4) \\
&\quad - g(\phi\kappa_1, \phi\kappa_3)\bar{S}(\phi\kappa_2, \phi\kappa_4)].
\end{aligned} \tag{58}$$

Let $\{v_1, v_2, v_3, \dots, v_{n-1}, \zeta\}$ be a local orthonormal basis of vector fields in M . Utilizing the fact that $\{\phi v_1, \phi v_2, \phi v_3, \dots, \phi v_{n-1}, \zeta\}$ also forms a local orthonormal basis of vector fields in M , we proceed by setting $\kappa_2 = \kappa_3 = v_i$ in Equation (58) and summing over i for $1 \leq i \leq n-1$, we obtain

$$\begin{aligned}
\sum_{i=1}^{n-1} g(\bar{R}(\phi\kappa_1, \phi v_i)\phi v_i, \phi\kappa_4) &= \frac{1}{n-2} \sum_{i=1}^{n-1} [\bar{S}(\phi v_i, \phi v_i)g(\phi\kappa_1, \phi\kappa_4) \\
&\quad - \bar{S}(\phi\kappa_1, \phi v_i)g(\phi v_i, \phi\kappa_4) + g(\phi v_i, \phi v_i)\bar{S}(\phi\kappa_1, \phi\kappa_4) \\
&\quad - g(\phi\kappa_1, \phi v_i)\bar{S}(\phi v_i, \phi\kappa_4)].
\end{aligned} \tag{59}$$

It is evident that

$$\sum_{i=1}^{n-1} g(\phi v_i, \phi v_i) = n-1, \tag{60}$$

$$\sum_{i=1}^{n-1} \bar{S}(\phi\kappa_1, \phi v_i)g(\phi v_i, \phi\kappa_4) = \bar{S}(\phi\kappa_1, \phi\kappa_4), \tag{61}$$

$$\sum_{i=1}^{n-1} \bar{S}(\phi v_i, \phi v_i) = \bar{r}. \tag{62}$$

Using (60)–(62) in (59), we obtain

$$\bar{S}(\phi\kappa_1, \phi\kappa_4) = \bar{r}g(\phi\kappa_1, \phi\kappa_4). \tag{63}$$

From (6), (16), (31) and (63), we obtain

$$S(\kappa_1, \kappa_4) = k_1g(\kappa_1, \kappa_4) + k_2\eta(\kappa_1)\eta(\kappa_4) + k_3\lambda\gamma(\kappa_1, \kappa_4), \tag{64}$$

where

$$\begin{aligned}
k_1 &= r - a^2(n-1) + a(a+2)\lambda^2 - (ab + b - a^2 - a), \\
k_2 &= r - a^2(n-1) + a(a+2)\lambda^2 - (ab + b - a^2 - a + n-1), \\
k_3 &= -a(a+2),
\end{aligned}$$

$\lambda = \text{trace}\phi$ and $\gamma(\kappa_1, \kappa_4) = g(\phi\kappa_1, \kappa_4) = g(\kappa_1, \phi\kappa_4)$. This verifies that M is generalized η -Einstein manifold. $\square$

6. Quasi-Conharmonically Flat LP-Sasakian Manifolds in Accordance with General Connection

Definition 4. An n -dimensional LP-Sasakian manifold M is said to be quasi-conharmonically flat in accordance with general connection D^G if

$$g(\bar{C}(\phi\kappa_1, \kappa_2)\kappa_3, \phi\kappa_4) = 0 \tag{65}$$

for all vector fields $\kappa_1, \kappa_2, \kappa_3, \kappa_4 \in \chi(M)$.

Theorem 9. *If an n -dimensional LP-Sasakian manifold M ($n > 2$) is quasi-conharmonically flat in accordance with general connection D^G , then its scalar curvature in accordance with general connection D^G vanishes.*

Proof. Taking the inner product with $\varkappa_4$ in (3), we obtain

$$g(\bar{R}(\phi \varkappa_1, \varkappa_2) \varkappa_3, \phi \varkappa_4) = \frac{1}{n-2} [\bar{S}(\varkappa_2, \varkappa_3)g(\phi \varkappa_1, \phi \varkappa_4) - \bar{S}(\phi \varkappa_1, \varkappa_3)g(\varkappa_2, \phi \varkappa_4) + g(\varkappa_2, \varkappa_3)\bar{S}(\phi \varkappa_1, \phi \varkappa_4) - g(\phi \varkappa_1, \varkappa_3)\bar{S}(\varkappa_2, \phi \varkappa_4)]. \quad (66)$$

Let $\{v_1, v_2, v_3, \dots, v_{n-1}, \zeta\}$ be a local orthonormal basis of vector fields in M . Using the fact that $\{\phi v_1, \phi v_2, \phi v_3, \dots, \phi v_{n-1}, \zeta\}$ also form a local orthonormal basis of vector fields in M , we proceed as follows. Setting $\varkappa_2 = \varkappa_3 = v_i$ in (66) and summing over i ($1 \leq i \leq n-1$), we obtain

$$\begin{aligned} \sum_{i=1}^{n-1} g(\bar{R}(\phi \varkappa_1, v_i) v_i, \phi \varkappa_4) &= \frac{1}{n-2} \sum_{i=1}^{n-1} [\bar{S}(v_i, v_i)g(\phi \varkappa_1, \phi \varkappa_4) \\ &\quad - \bar{S}(\phi \varkappa_1, v_i)g(v_i, \phi \varkappa_4) + g(v_i, v_i)\bar{S}(\phi \varkappa_1, \phi \varkappa_4) \\ &\quad - g(\phi \varkappa_1, v_i)\bar{S}(v_i, \phi \varkappa_4)]. \end{aligned} \quad (67)$$

It is readily apparent that

$$\sum_{i=1}^{n-1} g(\phi v_i, \phi v_i) = n, \quad (68)$$

$$\sum_{i=1}^{n-1} \bar{S}(\phi \varkappa_1, \phi v_i)g(\phi v_i, \phi \varkappa_4) = \bar{S}(\phi \varkappa_1, \phi \varkappa_4), \quad (69)$$

$$\sum_{i=1}^{n-1} \bar{S}(\phi v_i, \phi v_i) = \bar{r}. \quad (70)$$

Using (68)–(70) in (67), we obtain

$$\frac{\bar{r}}{n-2} g(\phi \varkappa_1, \phi \varkappa_4) = 0. \quad (71)$$

Since $g(\phi \varkappa_1, \phi \varkappa_4) \neq 0$. Then, the above equation reduces to

$$\bar{r} = 0. \quad (72)$$

Thus, the proof is completed. $\square$

7. Ricci Semi-Symmetric LP-Sasakian Manifolds in Accordance with General Connection

Numerous geometers have explored the characteristics of semi-symmetric and Ricci semi-symmetric manifolds, as documented in [17,18]. In this section, we will examine Ricci semi-symmetric LP-Sasakian manifolds in the context of a general connection D^G .

Definition 5. *An n -dimensional LP-Sasakian manifold M is said to be Ricci semi-symmetric in accordance with general connection D^G if the curvature tensor $\bar{R}$ in accordance with D^G satisfies*

$$(\bar{R}(\varkappa_1, \varkappa_2) \cdot \bar{S})(\varkappa_3, \varkappa_4) = 0, \quad (73)$$

where $\varkappa_1, \varkappa_2, \varkappa_3, \varkappa_4$ are horizontal vector fields on M .

Theorem 10. A Ricci semi-symmetric LP-Sasakian manifold M of dimension n ($n > 2$) equipped with general connection D^G is a generalized η -Einstein manifold.

Proof. Equation (73) can be written as

$$\bar{S}(\bar{R}(\varkappa_1, \varkappa_2)\varkappa_3, \varkappa_4) + \bar{S}(\varkappa_3, \bar{R}(\varkappa_1, \varkappa_2)\varkappa_4) = 0. \quad (74)$$

Setting $\varkappa_1 = \varkappa_4 = \zeta$ in (74), we obtain

$$\bar{S}(\bar{R}(\zeta, \varkappa_2)\varkappa_3, \zeta) + \bar{S}(\varkappa_3, \bar{R}(\zeta, \varkappa_2)\zeta) = 0. \quad (75)$$

Using (28) and (33) in (75), we have

$$(a - ab - b + 1)\bar{S}(\varkappa_2, \varkappa_3) = (a + ab + b + 1)(n - 1)(a + 1)(1 - b)[g(\varkappa_2, \varkappa_3) + 2(a + 1)b\eta(\varkappa_2)\eta(\varkappa_3)]. \quad (76)$$

Now, inserting (27) in (76), we obtain

$$S(\varkappa_2, \varkappa_3) = l_1 g(\varkappa_2, \varkappa_3) + l_2 \eta(\varkappa_2)\eta(\varkappa_3) + l_3 \lambda \gamma(\varkappa_2, \varkappa_3), \quad (77)$$

where

$$\begin{aligned} l_1 &= \frac{1}{a - ab - b + 1} [(a - ab - b + 1)(ab + b - a^2 - a) \\ &\quad - (a + ab + b + 1)(n - 1)(a + 1)(1 - b)], \\ l_2 &= \frac{1}{a - ab - b + 1} [(a - ab - b + 1)(ab + b - a^2 - a + abn + bn \\ &\quad - an - ab - b + a) - 2(a + ab + b + 1)(n - 1)(a + 1)(1 - b)], \\ l_3 &= a(a + 2), \end{aligned}$$

$\lambda = \text{trace} \phi$ and $\gamma(\varkappa_2, \varkappa_3) = g(\varkappa_1, \phi \varkappa_3)$. Therefore, we can say that M is a generalized η -Einstein manifold. $\square$

8. Applications to the General Theory of Relativity

The objective of this section is to explore the applications of the general connection D^G on an n -dimensional LP-Sasakian manifold M within the framework of the general theory of relativity. We assume that an LP-Sasakian manifold M of dimension n represents a fluid space-time and its Ricci tensor $\bar{S}$ defined with respect to D^G satisfies $\bar{S} = 0$.

From (27) and (31), we obtain

$$\begin{aligned} S(\varkappa_1, \varkappa_2) &= (a - ab - b + a^2)g(\varkappa_1, \varkappa_2) - a(a + 2)\lambda g(\varkappa_1, \phi \varkappa_2) \\ &\quad + [a - ab - b + a^2 - (n - 1)(ab + b - a)]\eta(\varkappa_1)\eta(\varkappa_2), \end{aligned} \quad (78)$$

$$r = a^2(n - 1) - a(a + 2)\lambda^2. \quad (79)$$

The Einstein field equation in the absence of a cosmological constant is expressed by

$$S(\varkappa_1, \varkappa_2) - \frac{r}{2}g(\varkappa_1, \varkappa_2) = \kappa T(\varkappa_1, \varkappa_2) \quad (80)$$

for all $\varkappa_1, \varkappa_2 \in \chi(M)$, where T is the energy momentum tensor and κ is the Einstein's gravitational constant. If possible, we consider that the fluid space-time is a perfect fluid space-time, then the energy-momentum tensor T assumes the form

$$T(\varkappa_1, \varkappa_2) = pg(\varkappa_1, \varkappa_2) + (p + \mu)\eta(\varkappa_1)\eta(\varkappa_2) \quad (81)$$

for $\varkappa_1, \varkappa_2 \in \chi(M)$, where p is the isotropic pressure, μ is the energy density, and ζ is the fluid flow velocity, i.e., $g(\zeta, \zeta) = -1$ of the perfect fluid space-time.

Now, from (78), (79) and (81), Equation (80) becomes

$$\begin{aligned} & \frac{a(a+2)\lambda^2 - a^2(n-1) + 2(a-ab-b+a^2)}{2}g(\varkappa_1, \varkappa_2) \\ & + \{a-ab-b+a^2 - (n-1)(ab+b-a)\}\eta(\varkappa_1)\eta(\varkappa_2) \\ & - a(a+2)\lambda g(\varkappa_1, \phi\varkappa_2) \\ & = \kappa\{pg(\varkappa_1, \varkappa_2) + (p+\mu)\eta(\varkappa_1)\eta(\varkappa_2)\}. \end{aligned} \quad (82)$$

Setting $\varkappa_1 = \varkappa_2 = \zeta$ and using (5) and (7) in (82), we find

$$(n-1)(a^2 + 2a - 2ab - 2b) - a(a+2)\lambda^2 = 2\kappa\mu. \quad (83)$$

By selecting a frame field and subsequently contracting Equation (80), we obtain

$$a(n-2)(a - an + a\lambda^2 + 2\lambda^2) = 2\kappa[p(n-1) - \mu]. \quad (84)$$

From (83) and (84), we obtain

$$\frac{p}{\mu} = \frac{\Delta}{(n-1)\{(n-1)(a^2 + 2a - 2ab - 2b) - a(a+2)\lambda^2\}}, \quad (85)$$

where $\Delta = a(n-2)(a - an + a\lambda^2 + 2\lambda^2) + (n-1)(a^2 + 2a - 2ab - 2b) - a(a+2)\lambda^2$. Thus, (85) represents the equation of state for the perfect fluid space-time. As a special case, if we assume $\lambda = \text{trace } \phi = 0$ and $n = 4$, the equation of state for the perfect fluid space-time reduces to

$$\frac{p}{\mu} = \frac{1}{3} \left[\frac{a(2-a) - 2b(a+1)}{a(a-2) - 2b(a+1)} \right]. \quad (86)$$

If the real constants $a = 0, b = -1$ in (86), then Equation (86) reduces to the quarter-symmetric metric connection, which can be seen from (1). Therefore, we obtain

$$3p = \mu. \quad (87)$$

Thus, this indicates that the space-time is isotropic and homogeneous [19]. De et al. [20] also investigated the properties of isotropic and homogeneous space-times. Numerous researchers, including [20–23] have explored the characteristics of such space-times. Based on these findings, we can formulate the following theorem:

Theorem 11. Consider a perfect fluid space-time equipped with the general connection D^G , where the Ricci tensor associated with D^G vanishes. If this perfect fluid space-time satisfies Einstein field equations without a cosmological constant, then the equation of state for the space-time is given by

$$\frac{p}{\mu} = \frac{a(n-2)(a - an + a\lambda^2 + 2\lambda^2) + (n-1)(a^2 + 2a - 2ab - 2b) - a(a+2)\lambda^2}{(n-1)\{(n-1)(a^2 + 2a - 2ab - 2b) - a(a+2)\lambda^2\}}.$$

In particular, for $n = 4, \lambda = \text{trace } \phi = 0$, and the real constants $a = 0, b = -1$, the space-time is filled with an isotropic and homogeneous perfect fluid.

9. Example

Let $M = (x, y, z, u) \in \mathbf{R}^4 : u \neq 0$ be a 4-dimensional differentiable manifold, where $\mathbf{R}^4$ denotes the Euclidean space of dimension 4.

Let us assume that

$$v_1 = e^{x+u} \frac{\partial}{\partial x}, \quad v_2 = e^{y+u} \frac{\partial}{\partial y}, \quad v_3 = e^{z+u} \frac{\partial}{\partial z}, \quad v_4 = \frac{\partial}{\partial u}.$$

are the linearly independent vector fields, and therefore, they form a basis of M . The non-vanishing components of the Lie bracket are given by

$$[v_1, v_4] = -v_1, \quad [v_2, v_4] = -v_2, \quad [v_3, v_4] = -v_3.$$

Let the semi-Riemannian metric g of M be defined by

$$g(v_i, v_j) = 1 \quad \text{for } i, j = 1, 2, 3, \quad g(v_i, v_j) = 0 \quad \text{for } i \neq j$$

and

$$g(v_i, v_j) = -1 \quad \text{for } i = j = 4, \quad \text{where } i, j = 1, 2, 3, 4.$$

We assume that the associated 1-form corresponding to the unit timelike vector field $\zeta = v_4$ on M is related by the condition $g(\varkappa_1, v_4) = \eta(\varkappa_1)$. Let the $(1, 1)$ -tensor field ϕ be defined by

$$\phi(v_1) = -v_1, \quad \phi(v_2) = -v_2, \quad \phi(v_3) = -v_3, \quad \phi(v_4) = 0.$$

Then, it is clear that

$$\phi^2 \varkappa_1 = \varkappa_1 + \eta(\varkappa_1) v_4, \quad \eta(v_4) = -1$$

hold for all vector fields $\varkappa_1$. The manifold M , equipped with the compatible metric g , forms a 4-dimensional Lorentzian para-contact metric manifold. Using Koszul's formula, we obtain

$$2g(D_{\varkappa_1} \varkappa_2, \varkappa_3) = \varkappa_1 g(\varkappa_2, \varkappa_3) + \varkappa_2 g(\varkappa_3, \varkappa_1) - \varkappa_3 g(\varkappa_1, \varkappa_2) - g(\varkappa_1, [\varkappa_2, \varkappa_3]) \\ - g(\varkappa_2, [\varkappa_1, \varkappa_3]) + g(\varkappa_3, [\varkappa_1, \varkappa_2]).$$

It can easily verify that

$$D_{v_i} v_4 = -v_i \quad \text{for } i = 1, 2, 3,$$

$$D_{v_i} v_i = -2v_4 \quad \text{for } i = 1, 2, 3,$$

and

$$D_{v_i} v_j = 0 \quad \text{for } i \neq j, \quad \text{where } i = j = 4.$$

From the above analysis, we establish that $D_{\varkappa_1} v_4 = \phi \varkappa_1$ for all $\varkappa_1 \in \chi(M)$. This allows us to assert that (M, g) is a 4-dimensional LP-Sasakian manifold. The non-zero components of the curvature tensor R corresponding to the Levi-Civita connection are

$$R(v_1, v_2)v_1 = -v_2, \quad R(v_1, v_3)v_1 = -v_3, \quad R(v_1, v_4)v_1 = -v_4, \quad R(v_1, v_2)v_2 = v_1,$$

$$R(v_2, v_3)v_2 = -v_3, \quad R(v_2, v_4)v_2 = -v_4, \quad R(v_1, v_3)v_3 = v_1, \quad R(v_2, v_3)v_3 = v_2,$$

$$R(v_3, v_4)v_3 = -v_4, \quad R(v_1, v_4)v_4 = -v_1, \quad R(v_2, v_4)v_4 = -v_2, \quad R(v_3, v_4)v_4 = -v_3.$$

Additionally, the non-zero components of the Ricci tensor S are expressed as

$$S(v_i, v_i) = 3 \quad \text{for } i = 1, 2, 3 \quad \text{and} \quad S(v_j, v_j) = -3 \quad \text{for } j = 4.$$

Proceeding further, based on the preceding discussions, the general connection D^G on M defined by Equation (1) takes the following form

$$D_{v_i}^G v_i = -(a+2)v_4 \quad \text{for } i = 1, 2, 3, \quad D_{v_i}^G v_4 = -(a+1)v_i \quad \text{for } i = 1, 2, 3$$

and

$$D_{v_i}^G v_j = 0 \quad \text{for } i \neq j, \quad \text{where } i = j = 4.$$

The non-zero components of the curvature tensor R in accordance with the general connection D^G are given as

$$\bar{R}(v_i, v_4)v_i = -(a+ab+b+1)v_4 \quad \text{for } i = 1, 2, 3.$$

Additionally, we compute the Ricci tensor $\bar{S}$ with respect to the connection D^G as follows

$$\bar{S}(v_i, v_i) = 3+ab+b-a^2-a-a^2\lambda-2a\lambda \quad \text{for } i = 1, 2, 3$$

and

$$\bar{S}(v_j, v_j) = -3(a-b-ab+1) \quad \text{for } j = 4,$$

where a and b are real constants. From the above findings, the scalar curvatures regarding the Levi-Civita connection D and the general connection D^G are $r = 12$ and $\bar{r} = r - 3a^2$, respectively, knowing that $\lambda = \text{trace}\phi = 0$. Hence, these results verify Equation (31) and Theorem 5.

10. Conclusions

From all of the above results, we can conclude that an n -dimensional LP-Sasakian manifold admitting general connection D^G satisfies the following:

Conditions	Results
$\bar{P} = 0$	generalized η -Einstein manifold
ϕ -projectively flat	generalized η -Einstein manifold
$\bar{C} = 0$	$\bar{r} = 0$
ζ -conharmonically flat	$\bar{r} = 0$
ϕ -conharmonically flat	generalized η -Einstein manifold
quasi-conharmonically flat	$\bar{r} = 0$
$\bar{R}.\bar{S} = 0$	generalized η -Einstein manifold

From Section 8, we established that a perfect fluid space-time equipped with the general connection D^G , whose Ricci tensor associated with D^G vanishes and satisfies the Einstein field equation without cosmological constant. Under these conditions, the space-time is filled with an isotropic and homogeneous perfect fluid, particularly for $n = 4$, $\lambda = \text{trace}\phi = 0$, and the real constants $a = 0, b = -1$. Further, connections are fundamental in both general relativity and differential geometry for understanding the geometry of curved spaces. In general relativity, they are used to describe space-time curvature, the movement of particles along geodesics, parallel transport, and covariant derivatives, which help explain how matter and energy shape space-time. In differential geometry, connections are employed to define curvature, facilitate parallel transport, and allow differentiation of tensor fields on curved manifolds, extending the concept of differentiation to curved spaces. In both fields, connections play a key role in describing how geometric objects change as they move through curved spaces, enhancing our understanding of curvature, structure, and geometry.

Author Contributions: Conceptualization, R.K., L.C. and M.A.K.; methodology, O.B., M.A.K. and L.C.; investigation, R.K., O.B. and L.C.; writing—original draft preparation, M.A.K., O.B., R.K. and L.C.; writing—review and editing, O.B., M.A.K. and L.C. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported and funded by the Deanship of Scientific Research at Imam Mohammad Ibn Saud Islamic University (IMSIU) (grant number IMSIU-DDRSP2502).

Data Availability Statement: The paper is self-contained, and no external data are used.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Mihai, I.; Rosca, R. *On Lorentzian P-Sasakian Manifolds*; Classical Analysis; World Scientific Publishing: Singapore, 1992; pp. 155–169.
2. Matsumoto, K. On Lorentzian paracontact manifolds. *Bull. Yamagata Univ. Nat. Sci.* **1989**, *12*, 151–156.
3. Biswas, A.; Baishya, K.K. A general connection on Sasakian manifolds and the case of almost pseudo symmetric Sasakian manifolds. *Sci. Stud. Res. Ser. Math. Inform.* **2019**, *29*, 59–72.
4. Biswas, A.; Baishya, K.K. Study on generalized pseudo (Ricci) symmetric Sasakian manifold admitting general connection. *Bull. Transilv. Univ. Bras.* **2020**, *12*, 233–246.
5. Biswas, A.; Das, S.; Baishya, K.K. On Sasakian manifolds satisfying curvature restrictions with respect to quarter symmetric metric connection. *Sci. Stud. Res. Ser. Math. Inform.* **2018**, *28*, 29–40.
6. Golab, S. On semi-symmetric and quarter-symmetric linear connections. *Tensor NS* **1975**, *29*, 249–254.
7. Schouten, J.A.; Kampen, E.R.V. Zur Einbettungs-und Krümmungstheorie nichtholonomer. *Gebilde Math. Ann.* **1930**, *103*, 752–783. [[CrossRef](#)]
8. Tanno, S. The automorphism groups of almost contact Riemannian manifold. *Tohoku Math. J.* **1969**, *21*, 21–38. [[CrossRef](#)]
9. Zamkovoy, S. Canonical connections on paracontact manifolds. *Ann. Glob. Anal. Geom.* **2008**, *36*, 37–60. [[CrossRef](#)]
10. Bhatt, P.; Chanyal, S.K. η -Ricci solitons on Sasakian manifolds admitting general connection. *J. Mt. Res.* **2024**, *19*, 163–169. [[CrossRef](#)]
11. Kundu, R.; Das, A.; Biswas, A. Conformal Ricci soliton in Sasakian manifolds admitting general connection. *J. Hyperstruct.* **2024**, *13*, 46–61.
12. Altunbas, M. Ricci solitons on concircularly flat and W_1 -flat Sasakian manifolds admitting a general connection. *Malaya J. Mat.* **2025**, *13*, 8–14. [[CrossRef](#)]
13. Yano, K.; Bochner, S. *Curvature and Betti Numbers*; Annals of Mathematics Studies 32; Princeton University Press: Princeton, NJ, USA, 1953.
14. Ghosh, S. On a class of (k, μ) -contact manifolds. *Bull. Cal. Math. Soc.* **2010**, *102*, 219–226.
15. De, U.C.; Sengupta, J. On a Type of SemiSymmetric Metric Connection on an almost contact metric connection. *Facta Universitatis Ser. Math. Inform.* **2001**, *16*, 87–96.
16. Ishii, Y. On conharmonic transformations. *Tensor NS* **1957**, *11*, 73–80.
17. Patterson, E.M. Some theorems on Ricci-recurrent spaces. *J. Lond. Math. Soc.* **1952**, *27*, 287–295. [[CrossRef](#)]
18. Eriksson, I.; Senovilla, J.M.M. Note on (conformally) semi-symmetric spacetimes. *Class. Quantum Grav.* **2010**, *27*, 027001. [[CrossRef](#)]
19. Ellis, G.F.R. *Relativistic Cosmology in General Relativity and Cosmology*; Sachs, R.K., Ed.; Academic Press: London, UK, 1971.
20. De, U.C.; Mallick, S.L. On a type of spacetime. *Mathematica Moravica.* **2014**, *18*, 29–38. [[CrossRef](#)]
21. Mantica, C.A.; De, U.C.; Suh, Y.J.; Molinari, L.G. Perfect fluid spacetimes with harmonic generalized curvature tensor. *Osaka J. Math.* **2019**, *56*, 173–182.
22. Zengin, F.O. M -projectively flat spacetimes. *Math. Rep.* **2012**, *4*, 363–370.
23. Güler, S.; Demirbağ, S.A. A study of generalized quasi Einstein spacetimes with applications in general relativity. *Int. J. Theor. Phys.* **2016**, *55*, 548–562.

Disclaimer/Publisher’s Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.