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Properties for Close-to-Convex and Quasi-Convex Functions Using *q*-Linear Operator

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Abstract: In this work, we describe the $\mathfrak{q}$-analogue of a multiplier–Ruscheweyh operator of a specific family of linear operators $\mathfrak{I}^{s}_{\mathfrak{q},}(\nu,\tau)$, and we obtain findings related to geometric function theory (GFT) by utilizing approaches established through subordination and knowledge of $\mathfrak{q}$-calculus operators. By using this operator, we develop generalized classes of quasi-convex and close-to-convex functions in this paper. Additionally, the classes $\mathfrak{K}^{s}_{\mathfrak{q},}(\nu,\tau)(\varphi)$, $\mathfrak{Q}^{s}_{\mathfrak{q},}(\nu,\tau)(\varphi)$ are introduced. The invariance of these recently formed classes under the $\mathfrak{q}$-Bernardi integral operator is investigated, along with a number of intriguing inclusion relationships between them. Additionally, several unique situations and the beneficial outcomes of these studies are taken into account.

Keywords: analytic function; $\mathfrak{q}$-starlike functions; $\mathfrak{q}$-convex functions; $\mathfrak{q}$-close-to-convex functions; $\mathfrak{q}$-analogue Catas operator; $\mathfrak{q}$-analogue of Ruscheweyh operator

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1. Introduction

Let $\mathcal{A}$ denote the analytic function satisfying $\mathfrak{f}(0) = \mathfrak{f}'(0) - 1 = 0$ written as

$$\mathfrak{f}(\xi) = \xi + \sum_{\kappa=2}^{\infty} a_{\kappa}\xi^{\kappa}. \tag{1}$$

When two functions, $\mathfrak{f}$ and $\Im$, are subordinated, the result is $\mathfrak{f} \prec \Im$, which is defined as $\mathfrak{f}(\xi) = \Im(\chi(\xi))$, where $\chi(\xi)$ is the Schwartz function in $\mathfrak{U}$ (see [1–3]). Let $\mathfrak{S}$, $\mathfrak{S}^*$, $\mathfrak{C}$, $\mathfrak{K}$, and $\mathfrak{Q}$ represent the corresponding subclasses of $\mathcal{A}$ that are univalent, starlike, convex, close-to-convex, and quasi-convex functions.

One of the key properties of $\mathfrak{q}$-difference equations is their relationship with the theory of $\mathfrak{q}$-analogues, which is a development of conventional calculus that includes the $\mathfrak{q}$-analogues of traditional calculus operations. By utilizing $\mathfrak{q}$-difference equations, researchers and mathematicians are able to study a wider range of mathematical problems and patterns that would not be possible to analyze using traditional difference equations. This has led to advancements in fields such as number theory, quantum mechanics, and combinatorics. Applications of $\mathfrak{q}$-difference equations are located in a number of mathematical fields and applied sciences, including physics, economics, and computer science. The $\mathfrak{q}$-difference equations provide an effective instrument for studying GFT. Jackson was the first to apply $\mathfrak{q}$-difference equations in the context of GFT [4,5], following Carmichael [6], Mason [7], and Trijitzinsky [8]. Ismail et al. investigated and analyzed $\mathfrak{q}$-starlike functions [9]. Discussing the $\mathfrak{q}$-analogues of specific geometric function theory features led to an incredible discovery. Moreover, numerous authors have investigated various applications of $\mathfrak{q}$-calculus related to generalized subclasses of analytic functions; refer to [10–12]. An important contribution to the growth of this field of study has been the study of $\mathfrak{q}$-operators. Raducanu and Kanas [13] presented the Ruscheweyh derivative operator $\mathfrak{q}$-extension, and Noor et al. [14] and Arif et al. [15], respectively, defined the $\mathfrak{q}$-analogue of the Bernardi and Noor integral operators.

Jackson's $\mathfrak{q}$-difference operator $\eth_{\mathfrak{q}} : \mathcal{A} \to \mathcal{A}$ is defined by

$$\eth_{\mathfrak{q}}\mathfrak{f}(\xi) := \begin{cases} \frac{\mathfrak{f}(\xi)-\mathfrak{f}(\mathfrak{q}\xi)}{(1-\mathfrak{q})\xi} & (\xi \neq 0;\ 0 < \mathfrak{q} < 1) \\ \mathfrak{f}'(0) & (\xi = 0). \end{cases} \tag{2}$$

It has been revealed that for $\kappa \in \mathbb{N}$ and $\xi \in \mathfrak{U}$,

$$\eth_{\mathfrak{q}}\left\{\sum_{\kappa=1}^{\infty} a_\kappa \xi^\kappa\right\} = \sum_{\kappa=1}^{\infty} [\kappa]_{\mathfrak{q}} a_\kappa \xi^{\kappa-1}, \tag{3}$$

where

$$\begin{aligned} [\kappa]_{\mathfrak{q}} &= \frac{1-\mathfrak{q}^\kappa}{1-\mathfrak{q}} = 1 + \sum_{n=1}^{\kappa-1} \mathfrak{q}^n, \qquad [0]_{\mathfrak{q}} = 0, \\ [\kappa]_{\mathfrak{q}}! &= \begin{cases} [\kappa]_{\mathfrak{q}}[\kappa-1]_{\mathfrak{q}}\ldots\ldots\ldots[2]_{\mathfrak{q}}[1]_{\mathfrak{q}} & \kappa = 1,2,3,\ldots \\ 1 & \kappa = 0. \end{cases} \end{aligned} \tag{4}$$

The following fundamental laws apply to the $\mathfrak{q}$-difference operator:

$$\eth_{\mathfrak{q}}(\sigma\mathfrak{f}(\xi) \pm \iota\pi(\xi)) = \sigma\eth_{\mathfrak{q}}\mathfrak{f}(\xi) \pm \iota\eth_{\mathfrak{q}}\pi(\xi) \tag{5}$$

$$\eth_{\mathfrak{q}}(\mathfrak{f}(\xi)\pi(\xi)) = \mathfrak{f}(\mathfrak{q}\xi)\eth_{\mathfrak{q}}(\pi(\xi)) + \pi(\xi)\eth_{\mathfrak{q}}(\mathfrak{f}(\xi)) \tag{6}$$

$$\eth_{\mathfrak{q}}\left(\frac{\mathfrak{f}(\xi)}{\pi(\xi)}\right) = \frac{\eth_{\mathfrak{q}}(\mathfrak{f}(\xi))\pi(\xi) - \mathfrak{f}(\xi)\eth_{\mathfrak{q}}(\pi(\xi))}{\pi(\mathfrak{q}\xi)\pi(\xi)}, \quad \pi(\mathfrak{q}\xi)\pi(\xi) \neq 0 \tag{7}$$

$$\eth_{\mathfrak{q}}(\log \mathfrak{f}(\xi)) = \frac{\ln \mathfrak{q}}{\mathfrak{q}-1}\frac{\eth_{\mathfrak{q}}(\mathfrak{f}(\xi))}{\mathfrak{f}(\xi)}, \tag{8}$$

where $\mathfrak{f}, \pi \in \mathcal{A}$, and σ and ι are real or complex constants.

Jackson, in [5], investigated the $\mathfrak{q}$-integral of $\mathfrak{f}$ as

$$\int_0^\xi \mathfrak{f}(t)\eth_{\mathfrak{q}}t = \xi(1-\mathfrak{q})\sum_{\kappa=0}^{\infty} \mathfrak{q}^\kappa \mathfrak{f}(\xi\mathfrak{q})$$

and

$$\lim_{q\to 1-}\int_0^\xi \mathfrak{f}(t)\eth_{\mathfrak{q}}t = \int_0^\xi \mathfrak{f}(t)\eth t,$$

where $\int_0^\xi \mathfrak{f}(t)\eth t$, is the standard integral.

Aouf and Madian studied the $\mathfrak{q}$-calculus Cătas operator in [16] as $\mathfrak{I}_{\mathfrak{q}}^s(\nu,\tau) : \mathcal{A} \to \mathcal{A}$ $(s \in \mathbb{N}_0, \tau, \nu \geq 0, 0 < \mathfrak{q} < 1)$

$$\mathfrak{I}_{\mathfrak{q}}^{s}(\nu,\tau)\mathfrak{f}(\xi) = \xi+\sum_{\kappa=2}^{\infty}\left(\frac{[1+\tau]_{\mathfrak{q}}+\nu([\kappa+\tau]_{\mathfrak{q}}-[1+\tau]_{\mathfrak{q}})}{[1+\tau]_{\mathfrak{q}}}\right)^{s}a_{\kappa}\xi^{\kappa}$$
$$(s \in \mathbb{N}_0, \tau,\nu\geq 0, 0<\mathfrak{q}<1).$$

Aldweby and Darus also looked at the $\mathfrak{q}$-Ruscheweyh operator $\mathfrak{R}_{\mathfrak{q}}^{\rho}\mathfrak{f}(\xi)$ in 2014 [17]

$$\mathfrak{R}_{\mathfrak{q}}^{\rho}\mathfrak{f}(\xi)=\xi+\sum_{\kappa=2}^{\infty}\frac{[\kappa+\rho-1]_{\mathfrak{q}}}{[\rho]_{\mathfrak{q}}![\kappa-1]_{\mathfrak{q}}!}a_{\kappa}\xi^{\kappa},\ (\rho\geq 0, 0<\mathfrak{q}<1)$$

where $[a]_{\mathfrak{q}}$ and $[a]_{\mathfrak{q}}!$ are defined in (4).

Let

$$\mathfrak{f}_{\mathfrak{q},\nu,\tau}^{s}(\xi)=\xi+\sum_{\kappa=2}^{\infty}\left(\frac{[1+\tau]_{\mathfrak{q}}+\nu([\kappa+\tau]_{\mathfrak{q}}-[1+\tau]_{\mathfrak{q}})}{[1+\tau]_{\mathfrak{q}}}\right)^{s}\xi^{\kappa}.$$

Now, we define a new function $\mathfrak{f}_{\mathfrak{q},\nu,\tau}^{s,\rho}(\xi)$ as

$$\mathfrak{f}_{\mathfrak{q},\nu,\tau}^{s}(\xi)*\mathfrak{f}_{\mathfrak{q},\nu,\tau}^{s,\rho}(\xi)=\xi+\sum_{\kappa=2}^{\infty}\frac{[\kappa+\rho-1]_{\mathfrak{q}}!}{[\rho]_{\mathfrak{q}}![\kappa-1]_{\mathfrak{q}}!}\xi^{\kappa}.$$

Using the operator $\mathfrak{I}_{\mathfrak{q},\rho}^{s}(\nu,\tau)$, an expanded multiplier operator was defined in [12] as follows.

Definition 1 ([12])**.** *For $s\in\mathbb{N}_0, \tau,\nu,\rho\geq 0$, and $0<\mathfrak{q}<1$ with the operator's assistance $\mathfrak{f}_{\mathfrak{q},\nu,\tau}^{s,\rho}(\xi)$ we define the new linear extended multiplier by the $\mathfrak{q}$-Ruscheweyh operator and the $\mathfrak{q}$-Cătas operator, $\mathfrak{I}_{\mathfrak{q},\rho}^{s}(\nu,\tau):\mathcal{A}\to\mathcal{A}$ as*

$$\mathfrak{I}_{\mathfrak{q},\rho}^{s}(\nu,\tau)\mathfrak{f}(\xi)=\mathfrak{f}_{\mathfrak{q},\nu,\tau}^{s,\rho}(\xi)*\mathfrak{f}(\xi) \tag{9}$$

for $\mathfrak{f}\in\mathcal{A}$ and (5), we have

$$\mathfrak{I}_{\mathfrak{q},\rho}^{s}(\nu,\tau)\mathfrak{f}(\xi)=\xi+\sum_{\kappa=2}^{\infty}\psi_{\mathfrak{q}}^{*s}(\kappa,\nu,\tau)\frac{[\kappa+\rho-1]_{\mathfrak{q}}!}{[\rho]_{\mathfrak{q}}![\kappa-1]_{\mathfrak{q}}!}a_{\kappa}\xi^{\kappa}, \tag{10}$$

where

$$\psi_{\mathfrak{q}}^{*s}(\kappa,\nu,\tau)=\left(\frac{[1+\tau]_{\mathfrak{q}}}{[1+\tau]_{\mathfrak{q}}+\nu([\kappa+\tau]_{\mathfrak{q}}-[1+\tau]_{\mathfrak{q}})}\right)^{s}.$$

(10) is used to infer the following:

$$\xi\mathfrak{d}_{\mathfrak{q}}\left(\mathfrak{I}_{\mathfrak{q},\rho}^{s+1}(\nu,\tau)\mathfrak{f}(\xi)\right) = \frac{[\tau+1]_{\mathfrak{q}}}{\nu\mathfrak{q}^{\tau}}\mathfrak{I}_{\mathfrak{q},\rho}^{s}(\nu,\tau)\mathfrak{f}(\xi)-\left(\frac{[\tau+1]_{\mathfrak{q}}}{\nu\mathfrak{q}^{\tau}}-1\right)\mathfrak{I}_{\mathfrak{q},\rho}^{s+1}(\nu,\tau)\mathfrak{f}(\xi),$$
$$(\nu > 0), \tag{11}$$
$$\mathfrak{q}^{\rho}\xi\mathfrak{d}_{\mathfrak{q}}\left(\mathfrak{I}_{\mathfrak{q},\rho}^{s}(\nu,\tau)\mathfrak{f}(\xi)\right) = [\rho+1]_{\mathfrak{q}}\mathfrak{I}_{\mathfrak{q},\rho+1}^{s}(\nu,\tau)\mathfrak{f}(\xi)-[\rho]_{\mathfrak{q}}\mathfrak{I}_{\mathfrak{q},\rho}^{s}(\nu,\tau)\mathfrak{f}(\xi). \tag{12}$$

The classes $\mathfrak{S}\mathcal{T}_{\mathfrak{q}}$ and $\mathfrak{C}\mathcal{V}_{\mathfrak{q}}$ of $\mathfrak{q}$-starlike and $\mathfrak{q}$-convex functions, respectively, were generalized by Agarwal and Sahoo [18] in 2017 in the following form:

$$\mathfrak{S}\mathcal{T}_{\mathfrak{q}}(\gamma)=\left\{\mathfrak{f}\in\mathcal{A}:\left|\frac{\frac{\xi\mathfrak{d}_{\mathfrak{q}}(\mathfrak{f}(\xi))}{\mathfrak{f}(\xi)}-\gamma}{1-\gamma}-\frac{1}{1-\mathfrak{q}}\right|<\frac{1}{1-\mathfrak{q}}\right\} \tag{13}$$

and

$$\mathfrak{C}\mathcal{V}_{\mathfrak{q}}(\gamma)=\{\mathfrak{f}\in\mathcal{A}:\xi\mathfrak{d}_{\mathfrak{q}}(\mathfrak{f}(\xi))\in\mathfrak{S}\mathcal{T}_{\mathfrak{q}}(\gamma)\}, \tag{14}$$

where $\gamma \in [0,1)$, $\mathfrak{q} \in (0,1)$, and $\xi \in \mathfrak{U}$.

In [19], the class $\mathfrak{K}_{\mathfrak{q}}(\gamma)$ of $\mathfrak{q}$-close-to-convex functions of order γ was defined as

$$\mathfrak{K}_{\mathfrak{q}}(\gamma) = \left\{ \mathfrak{f} \in \mathcal{A} \colon \left| \frac{\frac{\xi \mathfrak{d}_{\mathfrak{q}}(\mathfrak{f}(\xi))}{\mathfrak{g}(\xi)} - \gamma}{1-\gamma} - \frac{1}{1-\mathfrak{q}} \right| < \frac{1}{1-\mathfrak{q}} \right\}, \tag{15}$$

where $\mathfrak{g} \in \mathfrak{S}\mathcal{T}_{\mathfrak{q}}(\gamma)$, $\gamma \in [0,1)$, $\mathfrak{q} \in (0,1)$, and $\xi \in \mathfrak{U}$.

From [19], $\mathfrak{K}_{\mathfrak{q}}(0) = \mathfrak{K}_{\mathfrak{q}}$ is provided as

$$\mathfrak{K}_{\mathfrak{q}} = \left\{ \mathfrak{f} \in \mathcal{A} \colon \left| \frac{\xi \mathfrak{d}_{\mathfrak{q}}(\mathfrak{f}(\xi))}{\mathfrak{g}(\xi)} - \frac{1}{1-\mathfrak{q}} \right| < \frac{1}{1-\mathfrak{q}} \right\}, \tag{16}$$

equivalently, $\mathfrak{f} \in \mathfrak{K}_{\mathfrak{q}}$ iff

$$\frac{\xi \mathfrak{d}_{\mathfrak{q}}(\mathfrak{f}(\xi))}{\mathfrak{g}(\xi)} \prec \frac{1+\xi}{1-\mathfrak{q}\xi} \tag{17}$$

where $\mathfrak{g} \in \mathfrak{S}\mathcal{T}_{\mathfrak{q}}(0) = \mathfrak{S}\mathcal{T}_{\mathfrak{q}}$, $\mathfrak{q} \in (0,1)$, and $\xi \in \mathfrak{U}$.

Consider Φ to be the class of univalent convex functions φ such that $Re(\varphi(\xi)) > 0$ in $\mathfrak{U}$ and $\varphi(0) = 1$.

Ali et al. [12] defined the class $\mathfrak{S}\mathcal{T}^{s}_{\mathfrak{q},\rho}(\nu,\tau)(\varphi)$, by using the operators given by (10) as follows:

$$\mathfrak{S}\mathcal{T}^{s}_{\mathfrak{q},\rho}(\nu,\tau)(\varphi) = \left\{ \mathfrak{f} \in \mathcal{A} : \mathfrak{I}^{s}_{\mathfrak{q},\rho}(\nu,\tau)\mathfrak{f}(\xi) \in \mathfrak{S}\mathcal{T}_{\mathfrak{q}}(\varphi) \right\}$$

where

$$\mathfrak{S}\mathcal{T}_{\mathfrak{q}}(\varphi) = \left\{ \mathfrak{f} \in \mathcal{A} \colon \frac{\xi \mathfrak{d}_{\mathfrak{q}} \mathfrak{f}(\xi)}{\mathfrak{f}(\xi)} \prec \varphi(\xi) \right\}.$$

Inspired by [12], we formulate the following definitions.

Definition 2. *$\mathfrak{f} \in \mathbf{A}$, $\varphi \in \Phi$, and $\mathfrak{q} \in (0,1)$. Then, $\mathfrak{f} \in \mathfrak{K}_{\mathfrak{q}}(\varphi)$ iff*

$$\frac{\xi \mathfrak{d}_{\mathfrak{q}} \mathfrak{f}(\xi)}{\mathfrak{g}(\xi)} \prec \varphi(\xi)$$

for some $\mathfrak{g} \in \mathfrak{S}\mathcal{T}_{\mathfrak{q}}(\varphi)$.

Definition 3. *$\mathfrak{f} \in \mathbf{A}$, $\varphi \in \Phi$, and $\mathfrak{q} \in (0,1)$. Then, $\mathfrak{f} \in \mathfrak{K}^{s}_{\mathfrak{q},\rho}(\nu,\tau)(\varphi)$ iff*

$$\frac{\xi \mathfrak{d}_{\mathfrak{q}} \mathfrak{I}^{s}_{\mathfrak{q},\rho}(\nu,\tau)\mathfrak{f}(\xi)}{\mathfrak{I}^{s}_{\mathfrak{q},\rho}(\nu,\tau)\mathfrak{g}(\xi)} \prec \varphi(\xi),$$

for some $\mathfrak{g} \in \mathfrak{S}\mathcal{T}^{s}_{\mathfrak{q},\rho}(\nu,\tau)(\varphi)$ with $s \in \mathbb{N}_0, \tau, \nu \geq 0, 0 < \mathfrak{q} < 1$, when $|\xi| < 1$, and $Re(s) > 1$ when $|\xi| = 1$.

Similar to the classes mentioned before, we define

$$\mathfrak{Q}_{\mathfrak{q}}(\varphi) = \{ \mathfrak{f} \in \mathcal{A} \colon \xi \mathfrak{d}_{\mathfrak{q}} \mathfrak{f}(\xi) \in \mathfrak{K}_{\mathfrak{q}}(\varphi) \},$$

and

$$\mathfrak{f} \in \mathfrak{Q}^{s}_{\mathfrak{q},\rho}(\nu,\tau)(\varphi) \text{ iff } \xi \mathfrak{d}_{\mathfrak{q}} \mathfrak{f}(\xi) \in \mathfrak{K}^{s}_{\mathfrak{q},\rho}(\nu,\tau)(\varphi).$$

The aforementioned classes reduce to specific classes of analytic functions for varying values of $\mathfrak{q}$, s, ρ,τ,ν, and φ. For instance,

(i) $\mathfrak{K}^{s}_{\mathfrak{q},\rho}\left(\frac{1+\{1-\gamma(1+\mathfrak{q})\}\xi}{1-\mathfrak{q}\xi}\right) = \mathfrak{K}^{s}_{\mathfrak{q},\rho}(\gamma)$ and $\mathfrak{Q}^{s}_{\mathfrak{q},\rho}\left(\frac{1+\{1-\gamma(1+\mathfrak{q})\}\xi}{1-\mathfrak{q}\xi}\right) = \mathfrak{Q}^{s}_{\mathfrak{q},\rho}(\gamma)$.

(ii) $\mathfrak{K}^{s}_{\mathfrak{q},\rho}\left(\frac{1+\xi}{1-\mathfrak{q}\xi}\right) = \mathfrak{K}^{s}_{\mathfrak{q},\rho}$ and $\mathfrak{Q}^{s}_{\mathfrak{q},\rho}\left(\frac{1+\xi}{1-\mathfrak{q}\xi}\right) = \mathfrak{Q}^{s}_{\mathfrak{q},\rho}$.

(iii) $\mathfrak{K}^0_{\mathfrak{q},o}(\varphi) = \mathfrak{K}_{\mathfrak{q}}(\varphi)$ and $\mathfrak{Q}^0_{\mathfrak{q},o}(\varphi) = \mathfrak{Q}_{\mathfrak{q}}(\varphi)$.

(iv) $\mathfrak{K}_{\mathfrak{q}}\left(\frac{1+\{1-\gamma(1+\mathfrak{q})\}\xi}{1-\mathfrak{q}\xi}\right) = \mathfrak{K}_{\mathfrak{q}}(\gamma)$ (see [19]) and $\mathfrak{Q}_{\mathfrak{q}}\left(\frac{1+\{1-\gamma(1+\mathfrak{q})\}\xi}{1-\mathfrak{q}\xi}\right) = \mathfrak{Q}_{\mathfrak{q}}(\gamma)$.

(v) $\mathfrak{K}_{\mathfrak{q}}\left(\frac{1+\xi}{1-\mathfrak{q}\xi}\right) = \mathfrak{K}_{\mathfrak{q}}$ (see [19]) and $\mathfrak{Q}_{\mathfrak{q}}\left(\frac{1+\xi}{1-\mathfrak{q}\xi}\right) = \mathfrak{Q}_{\mathfrak{q}}$.

(vi) $\underset{\mathfrak{q}\to 1^-}{lim}\mathfrak{K}_{\mathfrak{q}} = \mathfrak{K}$ and $\underset{\mathfrak{q}\to 1^-}{lim}\mathfrak{Q}_{\mathfrak{q}} = \mathfrak{Q}$, the classes of close-to-convex and quasi-convex functions, respectively.

2. Preliminary Results

Lemma 1 ([20]). *Let $\pi(\xi)$ be convex in $\mathfrak{U}$ with $\pi(0) = 1$ and let $\Upsilon : \mathfrak{U} \to \mathbb{C}$ with $Re(\Upsilon(\xi)) > 0$ in $\mathfrak{U}$. If $y(\xi) = 1 + y_1\xi + y_2\xi^2 \ldots$, is analytic in $\mathfrak{U}$, then*

$$y(\xi) + \Upsilon(\xi).\xi\mathfrak{d}_{\mathfrak{q}}y(\xi) \prec \pi(\xi) \tag{18}$$

implies that $y(\xi) \prec \pi(\xi)$.

Lemma 2 ([12]). *Let $\varphi(\xi)$ be an analytic and convex univalent function with $\varphi(0) = 1$ and $Re(\varphi(\xi)) > 0$ for $\xi \in \mathfrak{U}$. Then, for positive real s and $\tau,\rho \geq 0, \nu > 0, 0 < \mathfrak{q} < 1$ with $[\tau+1]_{\mathfrak{q}} > \nu\mathfrak{q}^{\tau}$,*

$$\mathfrak{S}\mathcal{T}^s_{\mathfrak{q},\rho+1}(\nu,\tau)(\varphi) \subset \mathfrak{S}\mathcal{T}^s_{\mathfrak{q},\rho}(\nu,\tau)(\varphi) \subset \mathfrak{S}\mathcal{T}^{s+1}_{\mathfrak{q},\rho}(\nu,\tau)(\varphi).$$

3. Main Results

3.1. Inclusion Results

Theorem 1. *Consider $\varphi(\xi)$ to be an analytic and convex univalent function with $\varphi(0) = 1$ and $Re(\varphi(\xi)) > 0$ for $\xi \in \mathfrak{U}$. Then, for positive real s and $\tau,\rho \geq 0, \nu > 0, 0 < \mathfrak{q} < 1$ with $[\tau+1]_{\mathfrak{q}} > \nu\mathfrak{q}^{\tau}$,*

$$\mathfrak{K}^s_{\mathfrak{q},\rho+1}(\nu,\tau)(\varphi) \subset \mathfrak{K}^s_{\mathfrak{q},\rho}(\nu,\tau)(\varphi) \subset \mathfrak{K}^{s+1}_{\mathfrak{q},\rho}(\nu,\tau)(\varphi).$$

Proof. Let $\mathfrak{f} \in \mathfrak{K}^s_{\mathfrak{q},\rho}(\nu,\tau)(\varphi)$. Then, by definition, there is $\mathfrak{g} \in \mathfrak{S}\mathcal{T}^s_{\mathfrak{q},\rho}(\nu,\tau)(\varphi)$, satisfying

$$\frac{\xi\mathfrak{d}_{\mathfrak{q}}\left(\mathfrak{I}^s_{\mathfrak{q},\rho}(\nu,\tau)\mathfrak{f}(\xi)\right)}{\mathfrak{I}^s_{\mathfrak{q},\rho}(\nu,\tau)\mathfrak{g}(\xi)} \prec \varphi(\xi). \tag{19}$$

Consider

$$\frac{\xi\mathfrak{d}_{\mathfrak{q}}\left(\mathfrak{I}^{s+1}_{\mathfrak{q},\rho}(\nu,\tau)\mathfrak{f}(\xi)\right)}{\mathfrak{I}^{s+1}_{\mathfrak{q},\rho}(\nu,\tau)\mathfrak{g}(\xi)} = \mathfrak{p}(\xi), \tag{20}$$

where $\mathfrak{p}(\xi)$ is analytic in $\mathfrak{U}$ with $\mathfrak{p}(0) = 1$. Using the identity (11) and $\mathfrak{q}$-differentiating with respect to ξ, we have

$$\begin{aligned}\frac{[\tau+1]_{\mathfrak{q}}}{\nu\mathfrak{q}^{\varsigma}}\frac{\xi\mathfrak{d}_{\mathfrak{q}}\left(\mathfrak{I}^s_{\mathfrak{q},\rho}(\nu,\tau)\mathfrak{f}(\xi)\right)}{\mathfrak{I}^s_{\mathfrak{q},\rho}(\nu,\tau)\mathfrak{g}(\xi)} &= \left(\frac{[\tau+1]_{\mathfrak{q}}}{\nu\mathfrak{q}^{\varsigma}}-1\right)\frac{\xi\mathfrak{d}_{\mathfrak{q}}\left(\mathfrak{I}^{s+1}_{\mathfrak{q},\rho}(\nu,\tau)\mathfrak{f}(\xi)\right)}{\mathfrak{I}^s_{\mathfrak{q},\rho}(\nu,\tau)\mathfrak{g}(\xi)} + \\ &\quad \frac{\xi\mathfrak{d}_{\mathfrak{q}}\mathfrak{p}(\xi)\left(\mathfrak{I}^{s+1}_{\mathfrak{q},\rho}(\nu,\tau)\mathfrak{g}(\xi)\right) + \mathfrak{p}(\xi)\xi\mathfrak{d}_{\mathfrak{q}}\left(\mathfrak{I}^{s+1}_{\mathfrak{q},\rho}(\nu,\tau)\mathfrak{g}(\xi)\right)}{\mathfrak{I}^s_{\mathfrak{q},\rho}(\nu,\tau)\mathfrak{g}(\xi)}\end{aligned}$$

$$\frac{[\tau+1]_{\mathfrak{q}}}{\nu\mathfrak{q}^{\varsigma}}\frac{\xi\mathfrak{d}_{\mathfrak{q}}\left(\mathfrak{I}^s_{\mathfrak{q},\rho}(\nu,\tau)\mathfrak{f}(\xi)\right)}{\mathfrak{I}^s_{\mathfrak{q},\rho}(\nu,\tau)\mathfrak{g}(\xi)} = \frac{\frac{\left(\frac{[\tau+1]_{\mathfrak{q}}}{\nu\mathfrak{q}^{\varsigma}}-1\right)\xi\mathfrak{d}_{\mathfrak{q}}\left(\mathfrak{I}^{s+1}_{\mathfrak{q},\rho}(\nu,\tau)\mathfrak{f}(\xi)\right)+\mathfrak{p}(\xi)\xi\mathfrak{d}_{\mathfrak{q}}\left(\mathfrak{I}^{s+1}_{\mathfrak{q},\rho}(\nu,\tau)\mathfrak{g}(\xi)\right)+\xi\mathfrak{d}_{\mathfrak{q}}\mathfrak{p}(\xi)}{\mathfrak{I}^{s+1}_{\mathfrak{q},\rho}(\nu,\tau)\mathfrak{g}(\xi)}}{\frac{\mathfrak{I}^s_{\mathfrak{q},\rho}(\nu,\tau)\mathfrak{g}(\xi)}{\mathfrak{I}^{s+1}_{\mathfrak{q},\rho}(\nu,\tau)\mathfrak{g}(\xi)}}$$

Applying identity (11), we have

$$\xi\mathfrak{d}_{\mathfrak{q}}\left(\mathfrak{I}^{s+1}_{\mathfrak{q},\rho}(\nu,\tau)\mathfrak{g}(\xi)\right) = \frac{[\tau+1]_{\mathfrak{q}}}{\nu\mathfrak{q}^{\tau}}\mathfrak{I}^{s}_{\mathfrak{q},\rho}(\nu,\tau)\mathfrak{g}(\xi) - \left(\frac{[\tau+1]_{\mathfrak{q}}}{\nu\mathfrak{q}^{\tau}} - 1\right)\mathfrak{I}^{s+1}_{\mathfrak{q},\rho}(\nu,\tau)\mathfrak{g}(\xi),$$

then

$$\frac{\xi\mathfrak{d}_{\mathfrak{q}}\left(\mathfrak{I}^{s}_{\mathfrak{q},\rho}(\nu,\tau)\mathfrak{f}(\xi)\right)}{\mathfrak{I}^{s}_{\mathfrak{q},\rho}(\nu,\tau)\mathfrak{g}(\xi)} = \frac{\frac{\xi\mathfrak{d}_{\mathfrak{q}}\left(\xi\mathfrak{d}_{\mathfrak{q}}\left(\mathfrak{I}^{s+1}_{\mathfrak{q},\rho}(\nu,\tau)\mathfrak{f}(\xi)\right)\right)}{\mathfrak{I}^{s+1}_{\mathfrak{q},\rho}(\nu,\tau)\mathfrak{g}(\xi)} + \zeta_{\mathfrak{q}}\frac{\xi\mathfrak{d}_{\mathfrak{q}}\left(\mathfrak{I}^{s+1}_{\mathfrak{q},\rho}(\nu,\tau)\mathfrak{f}(\xi)\right)}{\mathfrak{I}^{s+1}_{\mathfrak{q},\rho}(\nu,\tau)\mathfrak{g}(\xi)}}{\frac{\xi\mathfrak{d}_{\mathfrak{q}}\left(\mathfrak{I}^{s+1}_{\mathfrak{q},\rho}(\nu,\tau)\mathfrak{g}(\xi)\right)}{\mathfrak{I}^{s+1}_{\mathfrak{q},\rho}(\nu,\tau)\mathfrak{g}(\xi)} + \zeta_{\mathfrak{q}}}, \tag{21}$$

where $\zeta_{\mathfrak{q}} = \left(\frac{[\tau+1]_{\mathfrak{q}}}{\nu\mathfrak{q}^{\varsigma}} - 1\right)$.

On $\mathfrak{q}$-differentiation of (20), we have

$$\frac{\xi\mathfrak{d}_{\mathfrak{q}}\left(\xi\mathfrak{d}_{\mathfrak{q}}\left(\mathfrak{I}^{s+1}_{\mathfrak{q},\rho}(\nu,\tau)\mathfrak{f}(\xi)\right)\right)}{\mathfrak{I}^{s+1}_{\mathfrak{q},\rho}(\nu,\tau)\mathfrak{g}(\xi)} = \mathfrak{p}(\xi)\lambda(\xi) + \xi\mathfrak{d}_{\mathfrak{q}}\mathfrak{p}(\xi), \tag{22}$$

where $\lambda(\xi) = \frac{\xi\mathfrak{d}_{\mathfrak{q}}\left(\mathfrak{I}^{s+1}_{\mathfrak{q},\rho}(\nu,\tau)\mathfrak{g}(\xi)\right)}{\mathfrak{I}^{s+1}_{\mathfrak{q},\rho}(\nu,\tau)\mathfrak{g}(\xi)}$.

From (21) and (22), we obtain

$$\frac{\xi\mathfrak{d}_{\mathfrak{q}}\left(\mathfrak{I}^{s}_{\mathfrak{q},\rho}(\nu,\tau)\mathfrak{f}(\xi)\right)}{\mathfrak{I}^{s}_{\mathfrak{q},\rho}(\nu,\tau)\mathfrak{g}(\xi)} = \mathfrak{p}(\xi) + \frac{\xi\mathfrak{d}_{\mathfrak{q}}\mathfrak{p}(\xi)}{\lambda(\xi) + \zeta_{\mathfrak{q}}}. \tag{23}$$

Consequently, from (19)

$$\mathfrak{p}(\xi) + \frac{\xi\mathfrak{d}_{\mathfrak{q}}\mathfrak{p}(\xi)}{\lambda(\xi) + \zeta_{\mathfrak{q}}} \prec \varphi(\xi). \tag{24}$$

Since $\mathfrak{g} \in \mathfrak{S}\mathcal{T}^{s}_{\mathfrak{q},\rho}(\nu,\tau)(\varphi)$, by Lemma 2, we conclude $\mathfrak{g} \in \mathfrak{S}\mathcal{T}^{s+1}_{\mathfrak{q},\rho}(\nu,\tau)(\varphi)$. This implies $\lambda(\xi) \prec \varphi(\xi)$. Therefore, $Re(\lambda(\xi)) > 0$ in $\mathfrak{U}$, and hence $Re(\frac{1}{\lambda(\xi)+\zeta_{\mathfrak{q}}}) > 0$ in $\mathfrak{U}$. Lemma 1 now yields the desired outcome.

To prove the first part, let $\mathfrak{f} \in \mathfrak{K}^{s}_{\mathfrak{q},\rho+1}(\nu,\tau)(\varphi)$ and set

$$\chi(\xi) = \frac{\xi\mathfrak{d}_{\mathfrak{q}}\left(\mathfrak{I}^{s}_{\mathfrak{q},\rho}(\nu,\tau)\mathfrak{f}(\xi)\right)}{\mathfrak{I}^{s}_{\mathfrak{q},\rho}(\nu,\tau)\mathfrak{g}(\xi)},$$

where χ is analytic in $\mathfrak{U}$ and $\chi(0) = 1$. Then, using the same reasoning as previously presented with (12), it follows that $\chi \prec \varphi$. The proof is now finished. □

Theorem 2. *Assume that $\varphi(\xi)$ is an analytic and convex univalent function with $\varphi(0) = 1$ and $Re(\varphi(\xi)) > 0$ for $\xi \in \mathfrak{U}$. Then, for positive real s and $\tau,\rho \geq 0, \nu > 0, 0 < \mathfrak{q} < 1$ with $[\tau+1]_{\mathfrak{q}} > \nu\mathfrak{q}^{\tau}$,*

$$\mathfrak{Q}^{s}_{\mathfrak{q},\rho+1}(\nu,\tau)(\varphi) \subset \mathfrak{Q}^{s}_{\mathfrak{q},\rho}(\nu,\tau)(\varphi) \subset \mathfrak{Q}^{s+1}_{\mathfrak{q},\rho}(\nu,\tau)(\varphi).$$

Proof. Let $\mathfrak{f} \in \mathfrak{Q}^{s}_{\mathfrak{q},\rho}(\nu,\tau)(\varphi)$. We have

$$\begin{aligned}
\mathfrak{f} \in \ & \mathfrak{Q}^{s}_{\mathfrak{q},\rho}(\nu,\tau)(\varphi) \Leftrightarrow \mathfrak{I}^{s}_{\mathfrak{q},\rho}(\nu,\tau)\mathfrak{f}(\xi) \in \mathfrak{Q}_{\mathfrak{q}}(\varphi) \\
\Leftrightarrow \ & \xi\mathfrak{d}_{\mathfrak{q}}\left(\mathfrak{I}^{s}_{\mathfrak{q},\rho}(\nu,\tau)\mathfrak{f}(\xi)\right) \in \mathfrak{K}_{\mathfrak{q}}(\varphi) \\
\Leftrightarrow \ & \xi(\mathfrak{d}_{\mathfrak{q}}\mathfrak{f}) \in \mathfrak{K}^{s}_{\mathfrak{q},\rho}(\nu,\tau)(\varphi) \\
\Leftrightarrow \ & \xi(\mathfrak{d}_{\mathfrak{q}}\mathfrak{f}) \in \mathfrak{K}^{s+1}_{\mathfrak{q},\rho}(\nu,\tau)(\varphi) \\
\Leftrightarrow \ & \xi\mathfrak{d}_{\mathfrak{q}}\left(\mathfrak{I}^{s+1}_{\mathfrak{q},\rho}(\nu,\tau)\mathfrak{f}(\xi)\right) \in \mathfrak{K}_{\mathfrak{q}}(\varphi)
\end{aligned}$$

$$
\begin{aligned}
&\Leftrightarrow \ \mathfrak{I}^{s+1}_{\mathfrak{q},\rho}(\nu,\tau)(\xi(\mathfrak{d}_{\mathfrak{q}}\mathfrak{f})) \in \mathfrak{K}_{\mathfrak{q}}(\varphi)\\
&\Leftrightarrow \ \mathfrak{I}^{s+1}_{\mathfrak{q},\rho}(\nu,\tau)\mathfrak{f}(\xi) \in \mathfrak{Q}_{\mathfrak{q}}(\varphi)\\
&\Leftrightarrow \ \mathfrak{f} \in \mathfrak{Q}^{s+1}_{\mathfrak{q},\rho}(\nu,\tau)(\varphi).
\end{aligned}
$$

We can use arguments like the ones mentioned above to illustrate the first part. The proof is now finished. □

Remark 1. *We can conclude the following inclusions relations from Theorems 1 and 2*

$$
\begin{aligned}
\mathfrak{K}^{s}_{\mathfrak{q},\rho+m}(\nu,\tau)(\varphi) &\subset \mathfrak{K}^{s}_{\mathfrak{q},\rho+m-1}(\nu,\tau)(\varphi) \subset \ldots.. \subset \mathfrak{K}^{s}_{\mathfrak{q},\rho}(\nu,\tau)(\varphi),\\
\mathfrak{K}^{s}_{\mathfrak{q},\rho}(\nu,\tau)(\varphi) &\subset \mathfrak{K}^{s+1}_{\mathfrak{q},\rho}(\nu,\tau)(\varphi) \subset \ldots.. \subset \mathfrak{K}^{s+m}_{\mathfrak{q},\rho}(\nu,\tau)(\varphi),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{Q}^{s}_{\mathfrak{q},\rho+n}(\nu,\tau)(\varphi) &\subset \mathfrak{Q}^{s}_{\mathfrak{q},\rho+n-1}(\nu,\tau)(\varphi) \subset \ldots. \subset \mathfrak{Q}^{s}_{\mathfrak{q},\rho}(\nu,\tau)(\varphi),\\
\mathfrak{Q}^{s}_{\mathfrak{q},\rho}(\nu,\tau)(\varphi) &\subset \mathfrak{Q}^{s+1}_{\mathfrak{q},\rho}(\nu,\tau)(\varphi) \subset \ldots. \subset \mathfrak{Q}^{s+n}_{\mathfrak{q},\rho}(\nu,\tau)(\varphi) \quad (m,n \in \mathbb{N}_0).
\end{aligned}
$$

Corollary 1. *Consider s to be a positive real and $\tau,\rho \geq 0, \nu > 0, 0 < \mathfrak{q} < 1$ with $[\tau+1]_{\mathfrak{q}} > \nu\mathfrak{q}^{\tau}$. Then, for $\varphi(\xi) = \frac{1+(1-\gamma(1+\mathfrak{q}))\xi}{1-\mathfrak{q}\xi}$ $(0 \leq \gamma < 1)$,*

$$
\begin{aligned}
\mathfrak{K}^{s}_{\mathfrak{q},\rho+1}(\nu,\tau)\left(\frac{1+(1-\gamma(1+\mathfrak{q}))\xi}{1-\mathfrak{q}\xi}\right) &\subset \mathfrak{K}^{s}_{\mathfrak{q},\rho}(\nu,\tau)\left(\frac{1+(1-\gamma(1+\mathfrak{q}))\xi}{1-\mathfrak{q}\xi}\right)\\
&\subset \mathfrak{K}^{s+1}_{\mathfrak{q},\rho}(\nu,\tau)\left(\frac{1+(1-\gamma(1+\mathfrak{q}))\xi}{1-\mathfrak{q}\xi}\right),\\
\mathfrak{Q}^{s}_{\mathfrak{q},\rho+1}(\nu,\tau)\left(\frac{1+(1-\gamma(1+\mathfrak{q}))\xi}{1-\mathfrak{q}\xi}\right) &\subset \mathfrak{Q}^{s}_{\mathfrak{q},\rho}(\nu,\tau)\left(\frac{1+(1-\gamma(1+\mathfrak{q}))\xi}{1-\mathfrak{q}\xi}\right)\\
&\subset \mathfrak{Q}^{s+1}_{\mathfrak{q},\rho}(\nu,\tau)\left(\frac{1+(1-\gamma(1+\mathfrak{q}))\xi}{1-\mathfrak{q}\xi}\right).
\end{aligned}
$$

Furthermore, for $\gamma = 0$ and for $\gamma = 1$, we have

$$
\begin{aligned}
\mathfrak{K}^{s}_{\mathfrak{q},\rho+1}(\nu,\tau)\left(\frac{1+\mathfrak{q}\xi}{1-\mathfrak{q}\xi}\right) &\subset \mathfrak{K}^{s}_{\mathfrak{q},\rho}(\nu,\tau)\left(\frac{1+\mathfrak{q}\xi}{1-\mathfrak{q}\xi}\right) \subset \mathfrak{K}^{s+1}_{\mathfrak{q},\rho}(\nu,\tau)\left(\frac{1+\mathfrak{q}\xi}{1-\mathfrak{q}\xi}\right) \text{ and}\\
\mathfrak{K}^{s}_{\mathfrak{q},\rho+1}(\nu,\tau)\left(\frac{1+\xi}{1-\mathfrak{q}\xi}\right) &\subset \mathfrak{K}^{s}_{\mathfrak{q},\rho}(\nu,\tau)\left(\frac{1+\xi}{1-\mathfrak{q}\xi}\right) \subset \mathfrak{K}^{s+1}_{\mathfrak{q},\rho}(\nu,\tau)\left(\frac{1+\xi}{1-\mathfrak{q}\xi}\right),
\end{aligned}
$$

respectively.

The following conclusions can be shown by using the same arguments as previously.

3.2. Invariance of the Classes Under $\mathfrak{q}$-Bernardi Integral Operator

In this part, we employ a feature of $\mathfrak{q}$-calculus to the $\mathfrak{q}$-Bernardi integral operator for analytic functions, as stated:

$$
\begin{aligned}
\Im_{\mathfrak{q},\varsigma}\mathfrak{f}(\xi) &= \frac{[1+\varsigma]_{\mathfrak{q}}}{\xi^{\varsigma}} \int_0^{\xi} t^{\varsigma-1}\mathfrak{f}(t)\mathfrak{d}_{\mathfrak{q}}t \\
&= \sum_{\kappa=1}^{\infty}\left(\frac{[1+\varsigma]_{\mathfrak{q}}}{[\kappa+\varsigma]_{\mathfrak{q}}}\right)a_{\kappa}\xi^{\kappa}, \quad \varsigma = 1,2,3,\ldots.
\end{aligned} \tag{25}
$$

In (25), we observe that the $\mathfrak{q}$-Łibera integral operator is defined as follows for $\varsigma = 1$:

$$
\Im_{\mathfrak{q}}\mathfrak{f}(\xi) = \frac{[2]_{\mathfrak{q}}}{\xi} \int_0^{\xi} \mathfrak{f}(t)\mathfrak{d}_{\mathfrak{q}}t
$$

$$= \sum_{\kappa=1}^{\infty}\left(\frac{[2]_{\mathfrak{q}}(1-\mathfrak{q})}{1-\mathfrak{q}^{\kappa+1}}\right)a_{\kappa}\xi^{\kappa}, \quad (0<\mathfrak{q}<1).$$

For $0<\mathfrak{q}<1$, we have

$$\lim_{\mathfrak{q}\to 1^-}\mathfrak{I}_{\mathfrak{q},\varsigma}\mathfrak{f}(\xi) = \sum_{\kappa=1}^{\infty}\frac{(1+\varsigma)}{(\kappa+\varsigma)}a_{\kappa}\xi^{\kappa},$$
$$\lim_{\mathfrak{q}\to 1^-}\mathfrak{I}_{\mathfrak{q}}\mathfrak{f}(\xi) = \sum_{\kappa=1}^{\infty}\frac{2}{(\kappa+1)}a_{\kappa}\xi^{\kappa},$$

which are defined in [21].

Theorem 3. *Let $\mathfrak{f}\in\mathfrak{K}^{s}_{\mathfrak{q},\rho}(\nu,\tau)(\varphi)$, $\varphi(0)=1$, $\varsigma\geq -1$ and $Re(\varphi(\xi))>0$. Then, $\Im_{\mathfrak{q},\varsigma}\mathfrak{f}(\xi)\in\mathfrak{K}^{s}_{\mathfrak{q},\rho}(\nu,\tau)(\varphi)$, where $\mathfrak{I}_{\mathfrak{q},\varsigma}\mathfrak{f}(\xi)$ is called $\mathfrak{q}$-Bernardi integral operator, defined in (25).*

Proof. Consider $\mathfrak{f}\in\mathfrak{K}^{s}_{\mathfrak{q},\rho}(\nu,\tau)(\varphi)$. Then we want to show that $\Im_{\mathfrak{q},\varsigma}\mathfrak{f}(\xi)\in\mathfrak{K}^{s}_{\mathfrak{q},\rho}(\nu,\tau)(\varphi)$, where

$$\Im_{\mathfrak{q},\varsigma}\mathfrak{f}(\xi)=\frac{[1+\varsigma]_{\mathfrak{q}}}{\xi^{\varsigma}}\int_0^{\xi}t^{\varsigma-1}\mathfrak{f}(t)\mathfrak{d}_{\mathfrak{q}}t.$$

It was found in [12] that for $\mathfrak{g}\in\mathfrak{S}\mathcal{T}^{s}_{\mathfrak{q},\rho}(\nu,\tau)(\varphi)$

$$\mathfrak{g}_{\mathfrak{q},\varsigma}(\xi)=\frac{[1+\varsigma]_{\mathfrak{q}}}{\xi^{\varsigma}}\int_0^{\xi}t^{\varsigma-1}\mathfrak{g}(t)\mathfrak{d}_{\mathfrak{q}}t\in\mathfrak{S}\mathcal{T}^{s}_{\mathfrak{q},\rho}(\nu,\tau)(\varphi). \tag{26}$$

Consider

$$\frac{\xi\mathfrak{d}_{\mathfrak{q}}\Im_{\mathfrak{q},\varsigma}\mathfrak{f}(\xi)}{\mathfrak{g}_{\mathfrak{q},\varsigma}(\xi)}=\mathfrak{p}(\xi), \tag{27}$$

where $\mathfrak{p}(\xi)$ is analytic in $\mathfrak{U}$ with $\mathfrak{p}(0)=1$.

From [22], we obtain

$$\xi\mathfrak{d}_{\mathfrak{q}}\Im_{\mathfrak{q},\varsigma}\mathfrak{f}(\xi)=\left(1+\frac{[\varsigma]_{\mathfrak{q}}}{\mathfrak{q}^{\varsigma}}\right)\mathfrak{f}(\xi)-\frac{[\varsigma]_{\mathfrak{q}}}{\mathfrak{q}^{\varsigma}}\Im_{\mathfrak{q},\varsigma}\mathfrak{f}(\xi). \tag{28}$$

$\mathfrak{q}$-Differentiation yields

$$\left(1+\frac{[\varsigma]_{\mathfrak{q}}}{\mathfrak{q}^{\varsigma}}\right)\mathfrak{d}_{\mathfrak{q}}\mathfrak{f}(\xi)=\mathfrak{d}_{\mathfrak{q}}(\xi\mathfrak{d}_{\mathfrak{q}}\Im_{\mathfrak{q},\varsigma}\mathfrak{f}(\xi))+\frac{[\varsigma]_{\mathfrak{q}}}{\mathfrak{q}^{\varsigma}}\mathfrak{d}_{\mathfrak{q}}(\Im_{\mathfrak{q},\varsigma}\mathfrak{f}(\xi)). \tag{29}$$

Similarly, from (26), we obtain

$$\left(1+\frac{[\varsigma]_{\mathfrak{q}}}{\mathfrak{q}^{\varsigma}}\right)\mathfrak{g}(\xi)=\xi\mathfrak{d}_{\mathfrak{q}}\mathfrak{g}_{\mathfrak{q},\varsigma}(\xi)+\frac{[\varsigma]_{\mathfrak{q}}}{\mathfrak{q}^{\varsigma}}\mathfrak{g}_{\mathfrak{q},\varsigma}(\xi). \tag{30}$$

From (29) and (30), we obtain

$$\frac{\mathfrak{d}_{\mathfrak{q}}\mathfrak{f}(\xi)}{\mathfrak{g}(\xi)}=\frac{\mathfrak{d}_{\mathfrak{q}}(\xi\mathfrak{d}_{\mathfrak{q}}\Im_{\mathfrak{q},\varsigma}\mathfrak{f}(\xi))+\frac{[\varsigma]_{\mathfrak{q}}}{\mathfrak{q}^{\varsigma}}\mathfrak{d}_{\mathfrak{q}}(\Im_{\mathfrak{q},\varsigma}\mathfrak{f}(\xi))}{\xi\mathfrak{d}_{\mathfrak{q}}\mathfrak{g}_{\mathfrak{q},\varsigma}(\xi)+\frac{[\varsigma]_{\mathfrak{q}}}{\mathfrak{q}^{\varsigma}}\mathfrak{g}_{\mathfrak{q},\varsigma}(\xi)},$$

equivalently

$$\frac{\xi\mathfrak{d}_{\mathfrak{q}}\mathfrak{f}(\xi)}{\mathfrak{g}(\xi)}=\frac{\frac{\xi\mathfrak{d}_{\mathfrak{q}}(\xi\mathfrak{d}_{\mathfrak{q}}\Im_{\mathfrak{q},\varsigma}\mathfrak{f}(\xi))}{\mathfrak{g}_{\mathfrak{q},\varsigma}(\xi)}+\frac{[\varsigma]_{\mathfrak{q}}}{\mathfrak{q}^{\varsigma}}\frac{\xi\mathfrak{d}_{\mathfrak{q}}(\Im_{\mathfrak{q},\varsigma}\mathfrak{f}(\xi))}{\mathfrak{g}_{\mathfrak{q},\varsigma}(\xi)}}{\frac{\xi\mathfrak{d}_{\mathfrak{q}}\mathfrak{g}_{\mathfrak{q},\varsigma}(\xi)}{\mathfrak{g}_{\mathfrak{q},\varsigma}(\xi)}+\frac{[\varsigma]_{\mathfrak{q}}}{\mathfrak{q}^{\varsigma}}}. \tag{31}$$

On $\mathfrak{q}$-differentiation of (27), and simple calculation implies

$$\frac{\xi\mathfrak{d}_{\mathfrak{q}}(\xi\mathfrak{d}_{\mathfrak{q}}\Im_{\mathfrak{q},\varsigma}\mathfrak{f}(\xi))}{\mathfrak{g}_{\mathfrak{q},\varsigma}(\xi)} = \mathfrak{p}(\xi).\mathfrak{p}_1(\xi) + \xi\mathfrak{d}_{\mathfrak{q}}\mathfrak{p}(\xi), \tag{32}$$

where $\mathfrak{p}_1(\xi) = \frac{\xi\mathfrak{d}_{\mathfrak{q}}\mathfrak{g}_{\mathfrak{q},\varsigma}(\xi)}{\mathfrak{g}_{\mathfrak{q},\varsigma}(\xi)}$.

Substituting (32) in (31), we obtain

$$\frac{\xi\mathfrak{d}_{\mathfrak{q}}\mathfrak{f}(\xi)}{\mathfrak{g}(\xi)} = \mathfrak{p}(\xi) + \frac{\xi\mathfrak{d}_{\mathfrak{q}}\mathfrak{p}(\xi)}{\mathfrak{p}_1(\xi) + \frac{[\varsigma]_{\mathfrak{q}}}{\mathfrak{q}^{\varsigma}}}. \tag{33}$$

Since $\mathfrak{f} \in \mathfrak{K}_{\mathfrak{q}}(\varphi)$, we can rewrite (33) as

$$\mathfrak{p}(\xi) + \frac{\xi\mathfrak{d}_{\mathfrak{q}}\mathfrak{p}(\xi)}{\mathfrak{p}_1(\xi) + \frac{[\varsigma]_{\mathfrak{q}}}{\mathfrak{q}^{\varsigma}}} \prec \varphi(\xi).$$

From (26), we determine that $\mathrm{Re}(\mathfrak{p}_1(\xi)) > 0$ in $\mathfrak{U}$ indicates $\mathrm{Re}\left(\frac{1}{\mathfrak{p}_1(\xi)+\frac{[\varsigma]_{\mathfrak{q}}}{\mathfrak{q}^{\varsigma}}}\right) > 0$ in $\mathfrak{U}$. Now, using Lemma 1 to get $\mathfrak{p}(\xi) \prec \varphi(\xi)$, and so $\frac{\xi\mathfrak{d}_{\mathfrak{q}}\Im_{\mathfrak{q},\varsigma}\mathfrak{f}(\xi)}{\mathfrak{g}_{\mathfrak{q},\varsigma}(\xi)} \prec \varphi(\xi)$. Hence, $\Im_{\mathfrak{q},\varsigma}\mathfrak{f}(\xi) \in \mathfrak{K}_{\mathfrak{q}}(\varphi)$. □

To prove the following theorem, we use the same justification.

Theorem 4. *Consider* $\mathfrak{f} \in \mathfrak{Q}^{s}_{\mathfrak{q},\rho}(\nu,\tau)(\varphi)$. *Then,* $\Im_{\mathfrak{q},\varsigma}\mathfrak{f}(\xi) \in \mathfrak{Q}^{s}_{\mathfrak{q},\rho}(\nu,\tau)(\varphi)$, *where* $\Im_{\mathfrak{q},\varsigma}\mathfrak{f}(\xi)$ *is defined by* (25).

Remark 2. *(i) If we assume* $\rho = 0$ *and* $\nu = 1$, *we can obtain the results studied by Daniel et al. ([20]; Theorems 1–3); (ii) we obtain all the conclusions that relate to all of the operators listed in the introduction by using the specialization of the parameters* s, ρ, ν, τ, *and* $\mathfrak{q}$.

4. Conclusions

This study relates new classes of analytic normalized functions in $\mathfrak{U}$ to its novel conclusions. Utilizing the concept of a $\mathfrak{q}$-difference operator, we create the $\mathfrak{q}$-analogue multiplier–Ruscheweyh operator $\mathfrak{I}^{s}_{\mathfrak{q},\rho}(\nu,\tau)$ to introduce several subclasses of univalent functions. Different subclasses are also introduced and studied using the $\mathfrak{q}$-analogues of the Ruscheweyh operator and the Cătas operator. For the newly established classes, we studied the inclusion outcomes and the integral preservation property. Other writers will be encouraged by this work to make contributions in this area in the future for numerous generalized subclasses of $\mathfrak{q}$-close-to-convex and quasi-convex univalents by employing a different operator.

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