

Article

A Study of Geodesic (E, F) -Preinvex Functions on Riemannian Manifolds

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Abstract: In this manuscript, we define the (E, F) -invex set, (E, F) -invex functions, and (E, F) -preinvex functions on Euclidean space, i.e., simply vector space. We extend these concepts on the Riemannian manifold. We also detail the fundamental properties of (E, F) -preinvex functions and provide some examples that illustrate the concepts well. We have established a relation between (E, F) -invex and (E, F) -preinvex functions on Riemannian manifolds. We introduce the conditions $\mathcal{A}$ and define the (E, F) -proximal sub-gradient. (E, F) -preinvex functions are also used to demonstrate their applicability in optimization problems. In the last, we establish the points of extrema of a non-smooth (E, F) -preinvex functions on (E, F) -invex subset of the Riemannian manifolds by using the (E, F) -proximal sub-gradient.

Keywords: Riemannian manifolds; geodesic (E, F) -invex sets; functions; optimization

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1. Introduction

One of the most crucial aspects of mathematical programming is the search for convexity and generalized convexity because these concepts play a vital role in optimization theory. Several notable generalizations of convexity have been made over the recent few decades (see [1–8]). Youness provided the concept of E -convex sets and E -convex functions. He also explored the E -convex programming problems in his paper [9]. Yang and Chen found some results to be incorrect in the paper written by [9]. Then, they modified those results (see [10,11]). Jian [12] came up with the concept of (E, F) -convex sets and functions with exciting applications regarding generalizations.

Recently, Wedad Saleh explored the concept of (E, F) -convex sets and functions on the Riemannian manifold and developed some fruitful results in his paper [13].

One can find the clear idea and development of the geodesic theory of convexity in the books authored by Udriste [14] and Rapcsák [15]. And, for more details on E -convex sets and functions, we refer [13,16], from where the idea of (E, F) -convex sets and functions was mainly generated. Convexity and its generalizations on the Riemannian manifolds have received more attention from researchers during the recent few decades, and a vast amount of literature has been developed on the subject (see [17–26]).

Motivated by the above works and results, particularly in [12,13], we provide the definitions of (E, F) -invex sets, (E, F) -invex functions and (E, F) -preinvex functions. We

explore this concept on Riemannian manifolds, which is the generalization of (E, F) -convexity sets and functions. We establish a relation between (E, F) -invex and (E, F) -preinvex functions on Riemannian manifolds. We introduce the conditions $\mathcal{A}$ and define the (E, F) -proximal sub-gradient. To explore and demonstrate its applicability on problems of optimization, (E, F) -preinvexity is applied. In the last section of the present paper, we have obtained extremum points for a non-smooth (E, F) -preinvex function defined on the (E, F) -invex subset of the Riemannian manifold, which became possible due to use of the (E, F) -proximal sub-gradient.

2. Preliminaries

Definition 1 ([13]). Let $\mathcal{A}$ be a non-empty subset of $\mathcal{R}^n$, and let $E, F : \mathcal{A} \rightarrow \mathcal{R}^n$ be two maps. If $r_1, s_1 \in \mathcal{A}$ and $\mu \in [0, 1]$, $F(s_1) + \mu(E(r_1) - F(s_1)) \in \mathcal{A}$, then $\mathcal{A}$ is known as an (E, F) -convex set.

In this section, we recall some well-known definitions on Riemannian manifolds.

Let $\mathcal{O}$ be a smooth Riemannian manifold (C^∞), which means that $\langle \cdot, \cdot \rangle_s$, the metric on the tangent space $T_s \mathcal{O} \times T_s \mathcal{O}$ of manifold $\mathcal{O}$, induces a norm $\| \cdot \|_s$. If z and w are two points on $\mathcal{O}$ and $\beta : [c, d] \rightarrow \mathcal{O}$ is a piecewise smooth curve connecting $\beta(c) = z$ to $\beta(d) = w$, its length $L(\beta)$ is given by

$$L(\beta) = \int_c^d \|\beta'(t)\|_{\beta'(t)} dt.$$

Now, we define

$$d(z, w) = \inf\{L(\gamma) : \beta \text{ is a piecewise } C^1 \text{ curve connecting } z \text{ and } w\}$$

for any $z, w \in \mathcal{O}$. The original topology on $\mathcal{O}$ is then induced by a distance d . A Levi-Civita connection, denoted by $\nabla_A B$, for any vector fields $A, B \in T\mathcal{O}$ also known as a covariant derivative, is known to exist uniquely on every Riemannian manifold. We also recall that a geodesic is a C^∞ smooth path β whose tangent is parallel along the path β , i.e., β satisfies the equation $\nabla_{\frac{d\beta(t)}{dt}} \frac{d\beta(t)}{dt} = 0$. Any path β connecting z and w in $\mathcal{O}$, such that $L(\beta) = d(z, w)$, is a geodesic, and it is called a minimizing geodesic. Recall that for a given curve $\beta : J \rightarrow \mathcal{O}$, a point $t_0 \in J$, and a vector $u_0 \in T_{\beta(t_0)} \mathcal{O}$, there exists exactly one parallel vector field $U(t)$ along $\beta(t)$ s.t. $U(t_0) = u_0$. Moreover, the mapping defined by $u_0 \mapsto U(t)$ is a linear isometry between the tangent spaces $T_{\beta(t_0)}$ and $T_{\beta(t)}$, for each $t \in J$. We denote this mapping by $P_{t_0, \beta}^t$ and we call it the parallel translation from $T_{\beta(t_0)} \mathcal{O}$ to $T_{\beta(t)} \mathcal{O}$ along the curve β . A simply connected complete Riemannian manifold with non-positive sectional curvature is called a Hadamard manifold.

3. (E, F) -Invex Sets and (E, F) -Preinvex Functions

Definition 2. Let $E, F : \mathcal{A} \rightarrow \mathcal{R}^n$ be functions, defined on non-empty subset $\mathcal{A}$ of $\mathcal{R}^n$, and let $G : \mathcal{R}^n \times \mathcal{R}^n \rightarrow \mathcal{R}^n$ be another function. If $\forall r_1, s_1 \in \mathcal{A}$ and $\mu \in [0, 1]$, $F(s_1) + \mu G(E(r_1), F(s_1)) \in \mathcal{A}$, then $\mathcal{A}$ is known as an (E, F) -invex set.

Example 1. Let $\mathcal{A} = [0, 4] \cup [6, 8]$ be a non-empty subset of $\mathcal{R}$, and let $E, F : \mathcal{A} \rightarrow \mathcal{R}$ be two maps defined as

$$F(s_1) = \begin{cases} \frac{1}{2} & \text{if } 1 \leq s_1 \leq 3 \\ 1, & \text{if } s_1 < 1 \\ 0, & \text{if } s_1 > 3 \end{cases}, \quad E(r_1) = \begin{cases} 3, & \text{if } 1 \leq r_1 \leq 3 \\ 2, & \text{if } r_1 < 1 \\ 9, & \text{if } r_1 > 3, \end{cases}$$

and let be $G : \mathcal{R} \times \mathcal{R} \rightarrow \mathcal{R}$ a bi-function defined as $G(E(r_1), F(s_1)) = F(s_1)$.

Now, for $r_1, s_1 > 3$ and $\mu = 1$, we have $F(s_1) + \mu(E(r_1) - F(s_1)) = 9 \notin \mathcal{A}$.

Then, it is obvious that the set $\mathcal{A}$ is not a (E, F) -convex set with respect to the maps E and F . Now,

$$F(s_1) + \mu G(E(r_1), F(s_1)) = F(s_1) + \mu F(s_1) = (1 + \mu)F(s_1). \quad (1)$$

It is clear that $\forall r_1, s_1$ and $\mu \in [0, 1]$. Thus, we have $F(s_1) + \mu G(E(r_1), F(s_1)) \in \mathcal{A}$. Hence, the set $\mathcal{A}$ is an (E, F) -invex set.

Remark 1. If we take $E = F$ in the Definition (2), then the set $\mathcal{A}$ becomes an E -invex set.

Remark 2. If we take $G(E(r_1), F(s_1)) = E(r_1) - F(s_1)$ in the Definition (2), then the set $\mathcal{A}$ becomes an (E, F) -convex set.

Remark 3. If we take the $E = F$ identity map in the Definition (2), then the set $\mathcal{A}$ becomes an invex set.

Remark 4. If we take E and F to be the identity maps on $\mathcal{A}$ and $G(E(r_1), F(s_1)) = E(r_1) - F(s_1)$ in the Definition (2), then the set $\mathcal{A}$ becomes a convex set.

Definition 3. A real-valued function $\mathcal{H} : \mathcal{A} \rightarrow \mathcal{R}$ defined on the (E, F) -invex set $\mathcal{A}$ is known as an (E, F) -preinvex function if

$$\mathcal{H}(F(s_1) + \mu G(E(r_1), F(s_1))) \leq \mu \mathcal{H}(E(r_1)) + (1 - \mu) \mathcal{H}(F(s_1)) \quad (2)$$

holds for $\forall r_1, s_1 \in \mathcal{A}$ and $\forall \mu \in [0, 1]$.

Theorem 1. Let $\mathcal{A}$ be a non-empty (E, F) -invex subset of $\mathcal{R}^n$. Let functions $\mathcal{H}_i$, $i = 1, 2, \dots, n$ be (E, F) -preinvex functions on the (E, F) -invex set $\mathcal{A}$ and $\alpha_i \geq 0$, $i = 1, 2, \dots, n$. Then, $\sum_{i=1}^n \alpha_i \mathcal{H}_i$ is also an (E, F) -preinvex function on the (E, F) -invex set $\mathcal{A}$.

Definition 4. Let $\mathcal{H}$ be a real-valued differentiable function on the open interval $\mathcal{A}$ of $\mathcal{R}^n$. The function $\mathcal{H}$ is known as an (E, F) -invex on $\mathcal{A}$ if there we have the maps $E, F : \mathcal{A} \rightarrow \mathcal{R}^n$ and a bi-function $G : \mathcal{R}^n \times \mathcal{R}^n \rightarrow \mathcal{R}^n$ such that the inequality

$$\mathcal{H}(E(r_1)) - \mathcal{H}(F(s_1)) \geq G(E(r_1), F(s_1))^T \nabla \mathcal{H}(F(s_1))$$

holds for each $r_1, s_1 \in \mathcal{A}$ and $\mu \in [0, 1]$.

Now, we will explore all the above definitions in Riemannian manifolds.

4. Geodesic (E, F) -Invex Sets and (E, F) -Preinvex Functions on Riemannian Manifolds

Definition 5. Let $\mathcal{B}$ be a non-empty subset of the Riemannian manifold $\mathcal{O}$, and let $E, F : \mathcal{B} \rightarrow \mathcal{O}$, $G : \mathcal{O} \times \mathcal{O} \rightarrow T\mathcal{O}$ be functions such that for every $E(r_1), F(s_1) \in \mathcal{O}$, $G(E(r_1), F(s_1)) \in T_{F(s_1)}\mathcal{O}$. Then, the subset $\mathcal{B}$ of $\mathcal{O}$ is said to be **geodesic (E, F) -invex** if, for each $E(r_1), F(s_1) \in \mathcal{B}$, there exists exactly one geodesic $\beta_{E(r_1), F(s_1)} : [0, 1] \rightarrow \mathcal{O}$ such that

$$\beta_{E(r_1), F(s_1)}(0) = F(s_1), \beta'_{E(r_1), F(s_1)}(0) = G(E(r_1), F(s_1)), \beta_{E(r_1), F(s_1)}(s) \in \mathcal{B}, \forall s \in [0, 1].$$

Now, we define the (E, F) -invex function on an open geodesic (E, F) -invex subset of a Riemannian manifold.

Definition 6. Let $\mathcal{B}$ be a non-empty open subset of the Riemannian manifold $\mathcal{O}$, with Riemannian metric g and $\mathcal{B}$ being a geodesic invex set. A function $\mathcal{H} : \mathcal{B} \rightarrow \mathcal{R}$ is said to be (E, F) -**invex** on $\mathcal{B}$ if the following inequality holds:

$$\mathcal{H}(E(r_1)) - \mathcal{H}(F(s_1)) \geq \left\langle \nabla \mathcal{H}_{F(s_1)}, G(E(r_1), F(s_1)) \right\rangle \quad \forall E(r_1), F(s_1) \in \mathcal{B}.$$

Definition 7. Let $\mathcal{B}$ be a non-empty geodesic (E, F) -invex subset of the Riemannian manifold $\mathcal{O}$. A function $\mathcal{H} : \mathcal{B} \rightarrow \mathcal{R}$ is said to be (E, F) -**preinvex** if, for each $E(r_1), F(s_1) \in \mathcal{B}$ and $\forall s \in [0, 1]$,

$$\mathcal{H}(\beta_{E(r_1), F(s_1)}(s)) \leq s\mathcal{H}(E(r_1)) + (1-s)\mathcal{H}(F(s_1)) \quad (3)$$

holds, where $\beta_{E(r_1), F(s_1)}$ is the unique geodesic in Definition 5. If the above inequality (3) is strict for $E(r_1) \neq F(s_1)$ and $s \in (0, 1)$, then $\mathcal{H}$ is said to be a strict (E, F) -preinvex function.

Example 2. Let $\mathcal{O} = [-8, 8]$ be a non-empty subset of $\mathcal{R}$, and let $E, F : \mathcal{O} \rightarrow \mathcal{R}$ be two maps defined as

$$E(r_1) = 3r_1 \text{ and } F(s_1) = s_1,$$

and let $G : \mathcal{R} \times \mathcal{R} \rightarrow \mathcal{R}$, a point-to-point bi-function, be defined as

$$G(E(r_1), F(s_1)) = \frac{E(r_1)}{3} - F(s_1).$$

Now, consider the geodesic β as

$$\begin{aligned} \beta_{E(r_1), F(s_1)}(s) &= F(s_1) + s \left[\frac{E(r_1)}{3} - F(s_1) \right] \\ &= s_1 + s(r_1 - s_1). \end{aligned}$$

It can be easily seen that

$$\beta_{E(r_1), F(s_1)}(0) = F(s_1), \quad \beta'_{E(r_1), F(s_1)}(0) = G(E(r_1), F(s_1)), \quad \beta_{E(r_1), F(s_1)}(s) \in \mathcal{O}, \quad \forall s \in [0, 1].$$

So, $\mathcal{O}$ is a geodesic (E, F) -invex set. Now, we define a function $\mathcal{H} : \mathcal{O} \rightarrow \mathcal{R}$ as $\mathcal{H}(r_1) = r_1 + r_1^2$. Then, $\mathcal{H}$ is an (E, F) -preinvex function on the set $\mathcal{O}$.

Proposition 1. Let $\mathcal{B}$ be a non-empty geodesic (E, F) -invex subset of the Riemannian manifold $\mathcal{O}$. Consider the function $\mathcal{H} : \mathcal{B} \rightarrow \mathcal{R}$, an (E, F) -preinvex. Thus, we have

(a) Every lower level set of $\mathcal{H}$ defined by

$$P(\mathcal{H}, t) := \{E(r_1) \in \mathcal{B} \mid \mathcal{H}(E(r_1)) \leq t\}$$

is a geodesic (E, F) -invex set.

(b) An optimization problem is presented as follows:

$$\begin{aligned} &\min \mathcal{H}(w) \\ &\text{subject to } w \in \mathcal{B} \end{aligned} \quad (4)$$

where the set L of the solution of the problem (4) is a geodesic (E, F) -invex set. Also, if $\mathcal{H}$ is strictly an (E, F) -preinvex function, then $\mathcal{O}$ contains (at most) one point.

Proof. (a) Take $E(r_1), F(s_1) \in P(\mathcal{H}, t)$. Then, we have $\mathcal{H}(E(r_1)) \leq t$ and $\mathcal{H}(F(s_1)) \leq t$. As $\mathcal{B}$ is a geodesic (E, F) -invex set, there exists exactly one geodesic $\beta_{E(r_1), F(s_1)} : [0, 1] \rightarrow \mathcal{O}$ such that

$$\beta_{E(r_1), F(s_1)}(0) = F(s_1), \beta'_{E(r_1), F(s_1)}(0) = G(E(r_1), F(s_1)), \beta_{E(r_1), F(s_1)}(s) \in \mathcal{B}, \forall s \in [0, 1].$$

Since $\mathcal{H}$ is (E, F) -preinvex, then, by definition, we have

$$\begin{aligned} \mathcal{H}(\beta_{E(r_1), F(s_1)})(s) &\leq s\mathcal{H}(E(r_1)) + (1-s)\mathcal{H}(F(s_1)) \\ &\leq st + (1-s)t \\ &= t. \end{aligned}$$

Hence, $\beta_{E(r_1), F(s_1)}(s) \in P(\mathcal{H}, t)$ for any $s \in [0, 1]$.

(b) If we take $\beta := \inf_{E(r_1) \in \mathcal{B}} \mathcal{H}(E(r_1))$, then it is clear that $L = \cap_{t > \gamma} P(\mathcal{H}, t)$. Therefore, L is an intersection of a geodesic (E, F) -invex set. Hence, it is a geodesic (E, F) -invex set as well.

Now, if $\mathcal{H}$ is strictly an (E, F) -preinvex function, then, for the geodesic (E, F) -invexity of L , there exists exactly one geodesic $\gamma_{E(r_1), F(s_1)} : [0, 1] \rightarrow \mathcal{O}$ such that

$$\gamma_{E(r_1), F(s_1)}(0) = F(s_1), \gamma'_{E(r_1), F(s_1)}(0) = G(E(r_1), F(s_1)), \gamma_{E(r_1), F(s_1)}(s) \in \mathcal{B}, \forall s \in [0, 1].$$

By the strictly (E, F) -preinvexity of $\mathcal{H}$, we have

$$\begin{aligned} \beta = \mathcal{H}(\gamma_{E(r_1), F(s_1)})(s) &< s\mathcal{H}(E(r_1)) + (1-s)\mathcal{H}(F(s_1)) \\ &= s\beta + (1-s)\beta \\ &= \beta \end{aligned}$$

which is not possible. $\square$

Theorem 2. Let $\mathcal{B}$ be a non-empty open geodesic (E, F) -invex subset of the Riemannian manifold $\mathcal{O}$. Consider that $\mathcal{H} : \mathcal{B} \rightarrow \mathcal{O}$ is a differentiable and (E, F) -preinvex function. Then, $\mathcal{H}$ is an (E, F) -invex function.

Proof. Since $\mathcal{B}$ is a geodesic (E, F) -invex set of the Riemannian manifold, there exists exactly one geodesic $\beta_{E(r_1), F(s_1)} : [0, 1] \rightarrow \mathcal{O}$ for every $E(r_1), F(s_1) \in \mathcal{B}$, such that

$$\beta_{E(r_1), F(s_1)}(0) = F(s_1), \beta'_{E(r_1), F(s_1)}(0) = G(E(r_1), F(s_1)), \beta_{E(r_1), F(s_1)}(s) \in \mathcal{B}, \forall s \in [0, 1].$$

Now, as $\mathcal{H}$ is the (E, F) -preinvex for $s \in (0, 1)$, we obtain

$$\mathcal{H}(\beta_{E(r_1), F(s_1)})(s) \leq s\mathcal{H}(E(r_1)) + (1-s)\mathcal{H}(F(s_1)).$$

This may also be written as

$$\mathcal{H}(\beta_{E(r_1), F(s_1)})(s) - \mathcal{H}(F(s_1)) \leq s[\mathcal{H}(E(r_1)) - \mathcal{H}(F(s_1))],$$

we divide both sides by s , and we obtain

$$\frac{\mathcal{H}(\beta_{E(r_1), F(s_1)})(s) - \mathcal{H}(F(s_1))}{s} \leq [\mathcal{H}(E(r_1)) - \mathcal{H}(F(s_1))].$$

Taking the limit as $s \rightarrow 0$, we have

$$\left\langle \nabla \mathcal{H}(\beta_{E(r_1), F(s_1)})(0), \beta'_{E(r_1), F(s_1)}(0) \right\rangle \leq \mathcal{H}(E(r_1)) - \mathcal{H}(F(s_1)).$$

Therefore,

$$\left\langle \nabla \mathcal{H}_{F(s_1)}, G(E(r_1), F(s_1)) \right\rangle \leq \mathcal{H}(E(r_1)) - \mathcal{H}(F(s_1)).$$

Hence, $\mathcal{H}$ is an (E, F) -invex function. $\square$

Definition 8. Let the functions $E, F : \mathcal{O} \rightarrow \mathcal{O}$ be defined on a Riemannian manifold $\mathcal{O}$. Then, we say that the function $G : \mathcal{O} \times \mathcal{O} \rightarrow T\mathcal{O}$ satisfies the condition $\mathcal{A}$, if, for every $E(r_1), F(s_1) \in \mathcal{O}$ and for the geodesic $\beta : [0, 1] \rightarrow \mathcal{O}$ satisfying $\beta_{E(r_1), F(s_1)}(0) = F(s_1)$, $\beta'_{E(r_1), F(s_1)}(0) = G(E(r_1), F(s_1))$, we have

$$\begin{aligned} (\mathcal{A}_1) \quad P_{s, \beta}^o[G(F(s_1), \beta(s))] &= -sG(E(r_1), F(s_1)) \\ (\mathcal{A}_2) \quad P_{s, \beta}^o[G(E(r_1), \beta(s))] &= (1-s)G(E(r_1), F(s_1)) \end{aligned}$$

for all $s \in [0, 1]$.

Theorem 3. Let $\mathcal{B}$ be a non-empty open geodesic (E, F) -invex subset of the Riemannian manifold $\mathcal{O}$. Suppose that the function $\mathcal{H} : \mathcal{B} \rightarrow \mathcal{R}$ is differentiable. If $\mathcal{H}$ is an (E, F) -invex on $\mathcal{B}$ and $\mathcal{H}$ satisfies the condition $\mathcal{A}$, then $\mathcal{H}$ is an (E, F) -preinvex function on $\mathcal{B}$.

Proof. Since $\mathcal{B}$ is a geodesic (E, F) -invex set of the Riemannian manifold, then, for every $E(r_1), F(s_1) \in \mathcal{B}$, there exists exactly one geodesic $\beta_{E(r_1), F(s_1)} : [0, 1] \rightarrow \mathcal{O}$ such that

$$\beta_{E(r_1), F(s_1)}(0) = F(s_1), \beta'_{E(r_1), F(s_1)}(0) = G(E(r_1), F(s_1)), \beta_{E(r_1), F(s_1)}(s) \in \mathcal{B}, \forall s \in [0, 1].$$

Now, fix $s \in [0, 1]$ and set $F(\bar{s}_1) := \beta_{E(r_1), F(s_1)}(s)$. Then, we have

$$\mathcal{H}(E(r_1)) - \mathcal{H}(F(\bar{s}_1)) \geq \left\langle \nabla \mathcal{H}_{F(\bar{s}_1)}, G(E(r_1), F(\bar{s}_1)) \right\rangle \quad (5)$$

$$\mathcal{H}(E(s_1)) - \mathcal{H}(F(\bar{s}_1)) \geq \left\langle \nabla \mathcal{H}_{F(\bar{s}_1)}, G(E(s_1), F(\bar{s}_1)) \right\rangle. \quad (6)$$

Multiplying Inequality (5) by s and Inequality (6) by $1-s$, respectively, and then carrying out addition, will lead to

$$\begin{aligned} s\mathcal{H}(E(r_1)) + (1-s)\mathcal{H}(F(s_1)) - \mathcal{H}(F(\bar{s}_1)) &\geq \left\langle \nabla \mathcal{H}_{F(\bar{s}_1)}, sG(E(r_1), F(\bar{s}_1)) \right\rangle + (1-s)G(E(r_1), F(\bar{s}_1)). \end{aligned}$$

From Condition $\mathcal{A}$, we obtain

$$\begin{aligned} sG(E(r_1), F(\bar{s}_1)) + (1-s)G(E(r_1), F(\bar{s}_1)) &= s(1-s)P_{s, \beta}^o[G(E(r_1), F(s_1))] \\ &\quad + (1-s)(-s)[G(E(r_1), F(s_1))] \\ &= 0. \end{aligned}$$

Therefore, we obtain

$$s\mathcal{H}(E(r_1)) + (1-s)\mathcal{H}(F(s_1)) \geq \mathcal{H}(F(\bar{s}_1)).$$

Hence, $\mathcal{H}$ is an (E, F) -preinvex function on the (E, F) -invex set $\mathcal{O}$. $\square$

Theorem 4. Let $\mathcal{B}$ be a non-empty open geodesic (E, F) -invex subset of the Riemannian manifold $\mathcal{O}$. Suppose that the function $\mathcal{H} : \mathcal{B} \rightarrow \mathcal{R}$ is an (E, F) -preinvex. If $F(\bar{r}_1) \in \mathcal{B}$ is a local optimal solution to the problem

$$\begin{aligned} & \text{minimize } \mathcal{H}(F(r_1)) \\ & \text{subject to } F(r_1) \in \mathcal{B}, \end{aligned} \quad (7)$$

then $F(\bar{r}_1)$ is a global minimum in Problem (7).

Proof. Consider a local minimum $F(\bar{r}_1)$. Then, there exists a neighborhood $C_\epsilon(F(\bar{r}_1))$ such that

$$\mathcal{H}(F(\bar{r}_1)) \leq \mathcal{H}(E(r_1)), \quad \forall E(r_1) \in \mathcal{B} \cap C_\epsilon(F(\bar{r}_1)). \quad (8)$$

Conversely, we assume that $F(\bar{r}_1)$ is not a global minimum of $\mathcal{H}$. Then, there exists an element $E(s_1^*) \in \mathcal{B}$ s.t.

$$\mathcal{H}(E(s_1^*)) < \mathcal{H}(F(\bar{r}_1)).$$

As $\mathcal{B}$ is a geodesic (E, F) -invex set, there exists exactly one geodesic $\beta_{E(s_1^*), F(\bar{r}_1)} : [0, 1] \rightarrow \mathcal{O}$ such that

$$\beta_{E(s_1^*), F(\bar{r}_1)}(0) = F(\bar{r}_1), \quad \beta'_{E(s_1^*), F(\bar{r}_1)}(0) = G(E(s_1^*), F(\bar{r}_1)), \quad \beta_{E(s_1^*), F(\bar{r}_1)}(s) \in \mathcal{B}, \quad \forall s \in [0, 1].$$

Now, if we choose $\epsilon > 0$ to be small enough s.t. $d(\beta_{E(s_1^*), F(\bar{r}_1)}(s), F(\bar{r}_1)) < \epsilon$, then $\beta_{E(s_1^*), F(\bar{r}_1)}(s) \in C_\epsilon(F(\bar{r}_1))$. As $\mathcal{H}$ represents (E, F) -preinvexity, we have

$$\begin{aligned} \mathcal{H}(\beta_{E(s_1^*), F(\bar{r}_1)}(s)) & \leq s\mathcal{H}(E(s_1^*)) + (1-s)\mathcal{H}(F(\bar{r}_1)) \\ & < \mathcal{H}(F(\bar{r}_1)), \quad \forall s \in (0, 1). \end{aligned}$$

Therefore, for each $\beta_{E(s_1^*), F(\bar{r}_1)}(s) \in \mathcal{B} \cap C_\epsilon(F(\bar{r}_1))$, which is a contradiction to Inequality (8), our result is proved. $\square$

Definition 9. Let the functions $E, F : \mathcal{O} \rightarrow \mathcal{O}$ be defined on Riemannian manifold $\mathcal{O}$, and let function $\mathcal{H} : \mathcal{O} \rightarrow (-\infty, \infty]$ be a lower semi-continuous function. An element σ in $T_{F(s_1)}\mathcal{O}$ is an (E, F) -proximal sub-gradient of $\mathcal{H}$ at $F(s_1) \in \text{dom}(\mathcal{H})$, if $\exists$ the numbers $\mu > 0$ and $\lambda > 0$ s.t.

$$\mathcal{H}(E(r_1)) \geq \mathcal{H}(F(s_1)) + \langle \sigma, \exp_{F(s_1)}^{-1} E(r_1) \rangle_{F(s_1)} - \lambda d(E(r_1), F(s_1))^2,$$

$\forall E(r_1) \in N(F(s_1), \mu)$, and where $\text{dom}(\mathcal{H}) := \{E(r_1) \in \mathcal{O} : \mathcal{H}(E(r_1)) < \infty\}$. The set of all (E, F) -proximal sub-gradients of $\mathcal{H}$ at $F(s_1) \in \mathcal{O}$ is denoted by $\partial_{(E, F)}\mathcal{H}(F(s_1))$ and is called the (E, F) -proximal sub-differential of $\mathcal{H}$ at $F(s_1)$.

Theorem 5. Let $\mathcal{B}$ be a non-empty open geodesic (E, F) -invex subset of the Cartan–Hadamard manifold $\mathcal{O}$ with $G(E(r_1), F(s_1)) \neq 0$ for $E(r_1) \neq F(s_1)$. Also, let $\mathcal{H} : \mathcal{B} \rightarrow (-\infty, \infty]$ be a lower semi-continuous (E, F) -preinvex function. Assume the $F(s_1) \in \text{domain}$ to be $(\mathcal{H})$ and $\sigma \in \partial_{(E, F)}\mathcal{H}(F(s_1))$. Then, there exists a real number μ s.t.

$$\mathcal{H}(E(r_1)) \geq \mathcal{H}(F(s_1)) + \langle \sigma, G(E(r_1), F(s_1)) \rangle_{F(s_1)}, \quad \forall E(r_1) \in \mathcal{B} \cap N(F(s_1), \mu) \quad (9)$$

Proof. Using the definition of the (E, F) -proximal sub-gradient $\partial_{(E, F)}\mathcal{H}(F(s_1))$, the numbers $\mu > 0$ and $\lambda > 0$ exist s.t.

$$\mathcal{H}(E(r_1)) \geq \mathcal{H}(F(s_1)) + \langle \sigma, \exp_{F(s_1)}^{-1} E(r_1) \rangle_{F(s_1)} - \lambda d(E(r_1), F(s_1))^2, \quad (10)$$

$\forall E(r_1) \in N(F(s_1), \mu)$, specifically for $E(r_1) \in \mathcal{B} \cap N(F(s_1), \mu)$. As $\mathcal{B}$ is a geodesic (E, F) -invex set, there exists exactly one geodesic $\beta_{E(r_1), F(s_1)} : [0, 1] \rightarrow \mathcal{O}$ such that

$$\beta_{E(r_1), F(s_1)}(0) = F(s_1), \beta'_{E(r_1), F(s_1)}(0) = G(E(r_1), F(s_1)), \beta_{E(r_1), F(s_1)}(s) \in \mathcal{B}, \forall s \in [0, 1].$$

As $\mathcal{O}$ is a Cartan–Hadamard manifold, $\beta_{E(r_1), F(s_1)}(s) = \exp_{F(s_1)}(sG(E(r_1), F(s_1)))$ for any $s \in [0, 1]$ (see [24] p. 776). Now, take $r_1 = \frac{\mu}{\|G(E(r_1), F(s_1))\|_{F(s_1)}}$. Therefore, $\beta_{E(r_1), F(s_1)}(s) = \exp_{F(s_1)}(sG(E(r_1), F(s_1))) \in \mathcal{B} \cap N(F(s_1), \mu), \forall s \in (0, r_1)$.

As $\mathcal{H}$ is geodesic (E, F) -preinvex, we have

$$\mathcal{H}(\exp_{F(s_1)}(sG(E(r_1), F(s_1)))) \leq s\mathcal{H}(E(r_1)) + (1-s)\mathcal{H}(F(s_1)), \forall s \in (0, r_1) \quad (11)$$

Combining Inequalities (10) and (11), we have

$$\begin{aligned} \mathcal{H}(\exp_{F(s_1)}(sG(E(r_1), F(s_1)))) &\geq \mathcal{H}(F(s_1)) - \lambda d(\exp_{F(s_1)}(sG(E(r_1), F(s_1))), F(s_1))^2 \\ &\quad + \langle \sigma, \exp_{F(s_1)}^{-1} \exp_{F(s_1)}(sG(E(r_1), F(s_1))) \rangle_{F(s_1)} \\ &= \mathcal{H}(F(s_1)) + \langle \sigma, sG(E(r_1), F(s_1)) \rangle_{F(s_1)} \\ &\quad - \lambda d(\exp_{F(s_1)}(sG(E(r_1), F(s_1))), F(s_1))^2 \end{aligned} \quad (12)$$

$\forall E(r_1) \in N(F(s_1), \mu)$. As $\mathcal{O}$ can be a Cartan–Hadamard manifold, we have

$$\begin{aligned} d(\exp_{F(s_1)}(sG(E(r_1), F(s_1))), F(s_1))^2 &= \|sG(E(r_1), F(s_1))\|_{F(s_1)}^2 \\ &= s^2 \|G(E(r_1), F(s_1))\|_{F(s_1)}^2. \end{aligned} \quad (13)$$

By combining Inequalities (11) and (12) with Equation (13), we have

$$\begin{aligned} s\mathcal{H}(E(r_1)) + (1-s)\mathcal{H}(F(s_1)) &\geq \mathcal{H}(\exp_{F(s_1)}(sG(E(r_1), F(s_1)))) \\ &\geq \mathcal{H}(F(s_1)) + \langle \sigma, sG(E(r_1), F(s_1)) \rangle_{F(s_1)} \\ &\quad - \lambda s^2 \|G(E(r_1), F(s_1))\|_{F(s_1)}^2. \end{aligned}$$

$\implies$

$$s[\mathcal{H}(E(r_1)) - \mathcal{H}(F(s_1))] \geq s\langle \sigma, G(E(r_1), F(s_1)) \rangle_{F(s_1)} - \lambda s^2 \|G(E(r_1), F(s_1))\|_{F(s_1)}^2$$

$\implies$

$$\mathcal{H}(E(r_1)) - \mathcal{H}(F(s_1)) \geq \langle \sigma, G(E(r_1), F(s_1)) \rangle_{F(s_1)} - \lambda s \|G(E(r_1), F(s_1))\|_{F(s_1)}^2.$$

Now, taking limit $\lim s \rightarrow 0$ on both sides, we have

$$\mathcal{H}(E(r_1)) - \mathcal{H}(F(s_1)) \geq \langle \sigma, G(E(r_1), F(s_1)) \rangle_{F(s_1)}, \forall E(r_1) \in \mathcal{B} \cap N(F(s_1), \mu).$$

Therefore, the proof is completed. $\square$

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