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Conformal Interactions of Osculating Curves on Regular Surfaces in Euclidean 3-Space

Yingxin Cheng [1], Yanlin Li [1,*], Pushpinder Badyal [2], Kuljeet Singh [2] and Sandeep Sharma [2]

[1] School of Mathematics, Hangzhou Normal University, Hangzhou 311121, China; 2022212501142@stu.hznu.edu.cn
[2] School of Mathematics, Shri Mata Vaishno Devi University, Katra 182320, India; 22dmt002@smvdu.ac.in (P.B.); 21dmt003@smvdu.ac.in (K.S.); sandeep.1980@smvdu.ac.in (S.S.)
* Correspondence: liyl@hznu.edu.cn

Abstract: Conformal maps preserve angles and maintain the local shape of geometric structures. The osculating curve plays an important role in analyzing the variations in curvature, providing a detailed understanding of the local geometric properties and the impact of conformal transformations on curves and surfaces. In this paper, we study osculating curves on regular surfaces under conformal transformations. We obtained the conditions required for osculating curves on regular surfaces R and $\tilde{R}$ to remain invariant when subjected to a conformal transformation $\psi : R \to \tilde{R}$. The results presented in this paper reveal the specific conditions under which the transformed curve $\tilde{\sigma} = \psi \circ \sigma$ preserves its osculating properties, depending on whether $\tilde{\sigma}$ is a geodesic, asymptotic, or neither. Furthermore, we analyze these conditions separately for cases with zero and non-zero normal curvatures. We also explore the behavior of these curves along the tangent vector T_σ and the unit normal vector P_σ.

Keywords: Darboux frame; conformal transformation; osculating curve; geodesic; asymptotic curve

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1. Introduction

A frame field is a fundamental concept in differential geometry that characterizes the local geometric properties of surfaces and curves [1–3]. At each point in a space, a set of linearly independent vectors spans the tangent space at that point. These frame fields can be classified into various types depending on the context of the study. In this study, we focus specifically on two types of frame fields, namely the Darboux frame and the Frenet frame.

The Darboux frame is used to study the geometry of a surface curve contained in three-dimensional Euclidean space $(\mathbb{E}^3)$. It is spanned by the tangent vector (T_σ) of the curve and two vectors derived from the surface, the normal vector (U_σ) to the surface and the principal normal vector (P_σ), which is perpendicular to both T_σ and U_σ. For more detail about Darboux frame vectors, one can refer to [4–6]. The French mathematician Jean Gaston Darboux, in his study on the theory of surfaces, introduced the idea of the Darboux frame [7]. As shown by Düldül et al. [8], further advancements expanded the Darboux frame into Euclidean 4-space and investigated its invariants. Furthermore, new characterizations of osculating curves based on the Darboux frame in $\mathbb{E}^3$ space were provided by a recent study of Isah et al. [9].

On the other hand, the local geometry of a space curve is described by the Frenet–Serret frame. It is composed of three unit vectors that are mutually orthogonal: the tangent vector (T_σ), which is along the direction of the motion of a curve at a given point; the normal vector (N_σ), which points towards the center of curvature of the curve at that point; and the binormal vector (B_σ), which is perpendicular to both the tangent and the normal vector. The planes spanned by these vectors in the Serret–Frenet frame are known as the osculating plane, the normal plane, and the rectifying plane [10–12].

The osculating plane, spanned by the tangent vector and the normal vector, is a plane tangent to the curve or surface at a given point. Similarly, the normal plane, spanned by the normal vector and the binormal vector, represents the plane normal to the surface at that point, whereas the rectifying plane, defined by the tangent vector and the binormal vector, represents a plane perpendicular to both the osculating plane and the normal plane. The curves whose position vectors lie in these planes are, respectively, known as the osculating curve [13–15], the normal curve [16–18], and the rectifying curve [19–21]. For more details about these curves, one can refer to [22–24].

The behavior of the position vector of a surface curve within Euclidean 3-space was described by Camci et al. [25], whereas Chen [26] examined the circumstances in which the position vector of a space curve always lies in its rectifying plane. Moreover, several characteristics of osculating curves were investigated by Ilarslan and Nesovic, contributing to the understanding of the Serret–Frenet frame [27–29]. Recently, Lone et al. [30–32] studied the geometric properties of normal and osculating curves in different spaces. The work on differential geometry serves as a valuable resource for understanding these concepts in depth [33–35].

The motivation for this research lies in exploring osculating curves under conformal transformations. We focus on understanding the behavior of osculating curves and their invariance properties under conformal transformations. We have investigated the conditions that maintain the invariance of osculating curves under conformal transformations. Also, we have discussed their behavior along the tangent vector $T_\sigma = a\theta_x + b\theta_y$ and along the unit normal vector $P_\sigma = U_\sigma \times T_\sigma$. This study extends the understanding of osculating curves within the framework of conformal transformations.

2. Preliminaries

Consider a unit speed curve on a regular surface R, $\sigma : I \subset \mathbb{R} \to \mathbb{E}^3$, which is continuous at least up to its fourth-order derivatives. Let $\sigma : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed curve on a regular surface R, exhibiting continuity at least up to its fourth-order derivatives. Let T_σ, N_σ, and B_σ represent the tangent, normal, and binormal vectors, respectively, at each point of the curve σ. These three vectors constitute the frame known as the Serret–Frenet frame, and they are related to each other by the Serret–Frenet equations given as follows:

$$\left.\begin{array}{rcl} T_\sigma'(w) &=& \kappa(w)N_\sigma(w), \\ N_\sigma'(w) &=& -\kappa(w)T_\sigma(w) + \tau(w)B_\sigma(w), \\ B_\sigma'(w) &=& -\tau(w)N_\sigma(w), \end{array}\right\} \tag{1}$$

where κ represents the curvature, and τ represents the torsion of the curve σ. $'$ denotes the derivative with respect to the arc parameter.

Similarly, at every point on a curve σ, we can also create another frame known as the Darboux frame, which is made up of three vectors, namely the tangent vector (T_σ), the normal vector (U_σ) to the surface, and the vector (P_σ), perpendicular to both T_σ and U_σ.

The relation between the Serret–Frenet frame vectors and the Darboux frame vectors is given as follows:

$$\begin{bmatrix} T_\sigma \\ P_\sigma \\ U_\sigma \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\Phi & \sin\Phi \\ 0 & -\sin\Phi & \cos\Phi \end{bmatrix} \cdot \begin{bmatrix} T_\sigma \\ N_\sigma \\ B_\sigma \end{bmatrix}, \quad (2)$$

where Φ is the angle between the vectors N_σ and P_σ.

Consider a parametrized curve $\sigma(w)$ that lies on a regular surface R. Let $\theta : \mathbb{R}^2 \to R$ be a coordinate chart on R, such that the curve $\sigma(w)$ is contained within the image of this coordinate chart. This means that we can describe the curve $\sigma(w)$ in terms of the coordinates provided by the chart θ. In particular, we can express the curve $\sigma(w)$ as follows:

$$\sigma(w) = \theta(x(w), y(w)), \quad (3)$$

where $x(w)$ and $y(w)$ are functions that map the parameter 'w' to the coordinate values on the surface R. This representation allows us to analyze the curve $\sigma(w)$ using the local coordinates of the surface R.

Differentiating (3) with respect to parameter 'w' of the curve, we obtain

$$T_\sigma(w) = \sigma'(w) = x'\theta_x + y'\theta_y, \quad (4)$$

represents the tangent vector to the curve. Here, θ_x and θ_y are the partial derivatives of the parameterized function θ with respect to the parameters x and y.

Hence, $\sigma'' = T'_\sigma$ is perpendicular to the tangent vector, which lies in the plane bounded by the vectors U_σ and P_σ. Thus, we can write it as a linear combination of U_σ and P_σ, i.e.,

$$\sigma'' = \kappa_n(w)U_\sigma(w) + \kappa_g(w)P_\sigma(w), \quad (5)$$

where κ_n represents the normal curvature and κ_g represents the geodesic curvature of the curve σ.

From (5), we obtain

$$\kappa_n(w) = \sigma'' \cdot U_\sigma(w) \qquad \textit{and} \qquad \kappa_g(w) = \sigma'' \cdot P_\sigma(w). \quad (6)$$

However, from the Serret–Frenet equation (Equation (1)), $\sigma''(w) = T'_\sigma(w) = \kappa(w)N_\sigma(w)$.

Therefore,

$$\begin{aligned} \kappa_n(w) = \kappa(w)N_\sigma(w) \cdot U_\sigma(w), \qquad &\textit{and} \qquad \kappa_g(w) = \kappa(w)N_\sigma(w) \cdot P_\sigma(w). \\ \Rightarrow \kappa_n(w) = \kappa(w)\sin\Phi, \qquad &\textit{and} \qquad \kappa_g(w) = \kappa(w)\cos\Phi. \end{aligned} \quad (7)$$

The unit normal vector U_σ, which is orthogonal to the surface R, can be expressed as follows:

$$U_\sigma(w) = \frac{\theta_x \times \theta_y}{\|\theta_x \times \theta_y\|} = \frac{\theta_x \times \theta_y}{\sqrt{EG - F^2}}. \quad (8)$$

Here, the terms $E = \theta_x \cdot \theta_x$, $F = \theta_x \cdot \theta_y$, and $G = \theta_y \cdot \theta_y$ represent the coefficients of the first fundamental form of surfaces [36].

Since the vector P_σ is perpendicular to the plane spanned by U_σ and T_σ, it is obtained by taking the cross product of U_σ and the tangent vector T_σ.
Equations (4) and (8) yield

$$P_\sigma(w) = \frac{1}{\sqrt{EG - F^2}}(Ex'\theta_y + F(y'\theta_y - x'\theta_x) - Gy'\theta_x). \tag{9}$$

Definition 1 ([36])**.** *Let R and $\tilde{R}$ be two regular surfaces. A function $\psi : R \to \tilde{R}$ is called a local isometry if for every point $q \in R$ and for all tangent vectors t_1 and t_2 at q. The inner product of the metric tensors t_1 and t_2 under ψ at $\psi(q)$ is identical to the inner product of t_1 and t_2 at q. i.e.,*

$$\langle \psi_*(t_1), \psi_*(t_2) \rangle_{\psi(q)} = \langle t_1, t_2 \rangle_q.$$

If, in addition, ψ is a bijection, then it is called an isometry, and the surfacesR and $\tilde{R}$ are said to be isometric.

An important property of an isometry $\psi : R \to \tilde{R}$ is that it preserves the coefficients of the first fundamental form. If E, F, and G denote the coefficients of the first fundamental form for R, and $\tilde{E}$, $\tilde{F}$, $\tilde{G}$ denote the corresponding coefficients for $\tilde{R}$, then

$$E = \tilde{E}, \qquad F = \tilde{F}, \qquad G = \tilde{G}. \tag{10}$$

Definition 2. *Let R and $\tilde{R}$ be two regular surfaces, and a function $\psi : R \to \tilde{R}$ is called a conformal map if, for every point $q \in R$ and for all tangent vectors t_1 and t_2 at q, the inner product of the metric tensors t_1 and t_2 under ψ at $\psi(q)$ is proportional to the inner product of t_1 and t_2 at q. Formally, this is written as follows:*

$$\langle \psi_*(t_1), \psi_*(t_2) \rangle_{\psi(q)} = \zeta \langle t_1, t_2 \rangle_q.$$

Here, ζ represents a differential function and is known as a scaling factor or a dilation factor.

A conformal transformation is basically a combination of an isometric map and a dilation factor; if the dilation factor is one, then a conformal map becomes an isometric map. In geometric terms, a conformal map maintains the angle both in direction and magnitude but not necessarily the lengths. For more information on conformal maps, one may refer to [37–39]. In the case of a conformal map, the coefficients of the first fundamental form are proportional, i.e.,

$$\zeta E = \tilde{E}, \quad \zeta F = \tilde{F}, \quad \zeta G = \tilde{G}. \tag{11}$$

Definition 3. *A unit speed curve σ, whose position vectors always lie in the orthogonal complement of the binormal vector, i.e., on the plane orthogonal to the binormal vector, is known as an osculating curve. In other words, an osculating curve is characterized by its position vector lying in an osculating plane and satisfying the following equation*

$$\sigma(w) \quad = \quad \alpha(w)T_\sigma(w) + \beta(w)N_\sigma(w), \tag{12}$$

for some smooth functions $\alpha(w)$ and $\beta(w)$.

Since there exists a relationship between the Serret–Frenet frame vectors and the Darboux frame vectors, if we convert the Serret–Frenet vector $N_\sigma(w)$ in (12) to the Darboux frame vectors from relation (2), we obtain

$$\sigma(w) \quad = \quad \alpha(w)T_\sigma(w) + \beta(w)P_\sigma(w)\cos\Phi - \beta(w)U_\sigma(w)\sin\Phi, \tag{13}$$

which represents an osculating curve in the Darboux frame.

Using values of Darboux vectors P_σ and U_σ from Equations (8) and (9) in (13), we obtain a general equation of an osculating curve, i.e.,

$$\begin{aligned} \sigma(w) \quad = \quad & \alpha(w)\{x'\theta_x + y'\theta_y\} + \frac{\beta(w)\kappa_g(w)}{\kappa(w)}\frac{1}{\sqrt{EG-F^2}}\{Ex'\theta_y \\ & +F(y'\theta_y - x'\theta_x) - Gy'\theta_x\} - \frac{\beta(w)\kappa_n(w)}{\kappa(w)}\frac{\theta_x \times \theta_y}{\sqrt{EG-F^2}}. \end{aligned} \tag{14}$$

Throughout this paper, we will use this equation to derive our main results. This equation of an osculating curve does not exhibit a geodesic or an asymptotic character.

If the geodesic curvature κ_g becomes zero, then the osculating curve σ is said to have a geodesic character. In that case, $\kappa_n = \kappa$ (from Equation (7)).

Thus, Equation (14) becomes

$$\sigma(w) \quad = \quad \alpha(w)\{x'\theta_x + y'\theta_y\} - \frac{\beta(w)}{\sqrt{EG-F^2}}\theta_x \times \theta_y, \tag{15}$$

and, if normal curvature κ_n becomes zero, then the osculating curve σ is said to have an asymptotic character. Hence, from Equation (7), we obtain $\kappa_g = \kappa$.

Thus, Equation (14) becomes

$$\begin{aligned} \sigma(w) \quad = \quad & \alpha(w)\{x'\theta_x + y'\theta_y\} + \frac{\beta(w)}{\sqrt{EG-F^2}}\{Ex'\theta_y \\ & +F(y'\theta_y - x'\theta_x) - Gy'\theta_x\}. \end{aligned} \tag{16}$$

3. Osculating Curves with Respect to the Conformal Transformation

In Theorems 1 and 2, we explore the conditions under which osculating curves remain invariant under conformal transformations. The first theorem considers the case where the osculating curve possesses a non-asymptotic character, while the second theorem addresses the case where the osculating curve exhibits an asymptotic character.

Theorem 1. *Let $\psi : R \to \tilde{R}$ be a conformal transformation, where R and $\tilde{R}$ are regular surfaces and $\sigma(w)$ is a non-asymptotic osculating curve on R (i.e., $\kappa_n \neq 0$). Then, $\tilde{\sigma} = \psi \circ \sigma$ is an osculating curve on $\tilde{R}$ in any of the following conditions:*

(a) $\tilde{\sigma}$ *is a geodesic curve on* $\tilde{R}$ *and satisfies* $\tilde{\sigma}(w) + \frac{\beta(w)\kappa_g(w)}{\kappa(w)}\tilde{P}_\sigma = \zeta\psi_*(\sigma(w))$.

(b) $\tilde{\sigma}$ *is an asymptotic curve on* $\tilde{R}$ *and satisfies* $\tilde{\sigma}(w) = \zeta\psi_*(\sigma(w)) + \frac{\beta(w)\kappa_n(w)}{\kappa(w)}\tilde{U}_\sigma$.

(c) $\tilde{\sigma}$ *is neither geodesic nor asymptotic curve on* $\tilde{R}$ *such that* $\tilde{\sigma}(w) = \zeta\psi_*(\sigma(w))$.

Proof. Let $\psi : R \to \tilde{R}$ be a conformal transformation, where R and $\tilde{R}$ are regular surfaces and $\sigma(w)$ is a non-asymptotic osculating curve on R, i.e., $\kappa_n \neq 0$.

Suppose that $\tilde{\sigma}$ is a geodesic curve on $\tilde{R}$ and satisfies $\tilde{\sigma}(w) + \frac{\beta(w)\kappa_g(w)}{\kappa(w)}\tilde{P}_\sigma = \zeta\psi_*(\sigma(w))$, which implies

$$\begin{aligned}
\tilde{\sigma}(w) &= \alpha(w)(x'\zeta\psi_*\theta_x + y'\zeta\psi_*\theta_y) + \frac{\beta(w)\kappa_g(w)}{\kappa(w)}\frac{1}{\zeta\sqrt{EG-F^2}}\zeta^2\{Ex'\psi_*\theta_y \\
&\quad + F(y'\psi_*\theta_y - x'\psi_*\theta_x) - Gy'\psi_*\theta_x\} - \frac{\beta(w)\kappa_n(w)}{\kappa(w)}\zeta\psi_* U_\sigma - \frac{\beta(w)\kappa_g(w)}{\kappa(w)}\tilde{P}_\sigma. \\
\Rightarrow \tilde{\sigma}(w) &= \alpha(w)(x'\zeta\psi_*\theta_x + y'\zeta\psi_*\theta_y) + \frac{\beta(w)\kappa_g(w)}{\kappa(w)}\frac{1}{\sqrt{\zeta E\zeta G - (\zeta F)^2}}\{\zeta Ex'\zeta\psi_*\theta_y \\
&\quad + \zeta F(y'\zeta\psi_*\theta_y - x'\zeta\psi_*\theta_x) - \zeta Gy'\zeta\psi_*\theta_x\} - \frac{\beta(w)\kappa_n(w)}{\kappa(w)}\zeta\psi_* U_\sigma \\
&\quad - \frac{\beta(w)\kappa_g(w)}{\kappa(w)}\tilde{P}_\sigma. \qquad (17)
\end{aligned}$$

From (11) and (17), it follows that

$$\begin{aligned}
\tilde{\sigma}(w) &= \alpha(w)(x'\tilde{\theta}_x + y'\tilde{\theta}_y) + \frac{\beta(w)\kappa_g(w)}{\kappa(w)}\frac{1}{\sqrt{\tilde{E}\tilde{G}-\tilde{F}^2}}\{\tilde{E}x'\tilde{\theta}_y + \tilde{F}(y'\tilde{\theta}_y \\
&\quad - x'\tilde{\theta}_x) - \tilde{G}y'\tilde{\theta}_x\} - \frac{\beta(w)\kappa_n(w)}{\kappa(w)}\tilde{U}_\sigma - \frac{\beta(w)\kappa_g(w)}{\kappa(w)}\tilde{P}_\sigma. \\
\Rightarrow \tilde{\sigma}(w) &= \alpha(w)(x'\tilde{\theta}_x + y'\tilde{\theta}_y) - \frac{\beta(w)\kappa_n(w)}{\kappa(w)}\tilde{U}_\sigma. \\
\Rightarrow \tilde{\sigma}(w) &= \tilde{\alpha}(w)(x'\tilde{\theta}_x + y'\tilde{\theta}_y) - \tilde{\beta}(w)\tilde{U}_\sigma,
\end{aligned}$$

which represents an osculating curve exhibiting a geodesic character on the surface $\tilde{R}$, where $\tilde{\alpha}(w) = \alpha(w)$ and $\tilde{\beta}(w) = \frac{\beta(w)\kappa_n(w)}{\kappa(w)}$.

Suppose that $\tilde{\sigma}$ is an asymptotic curve on $\tilde{R}$ and satisfies $\tilde{\sigma}(w) = \zeta\psi_*(\sigma(w)) + \frac{\beta(w)\kappa_n(w)}{\kappa(w)}\tilde{U}_\sigma$, which implies

$$\begin{aligned}
\tilde{\sigma}(w) &= \alpha(w)(x'\zeta\psi_*\theta_x + y'\zeta\psi_*\theta_y) + \frac{\beta(w)\kappa_g(w)}{\kappa(w)}\frac{1}{\zeta\sqrt{EG-F^2}}\zeta^2\{Ex'\psi_*\theta_y \\
&\quad + F(y'\psi_*\theta_y - x'\psi_*\theta_x) - Gy'\psi_*\theta_x\} - \frac{\beta(w)\kappa_n(w)}{\kappa(w)}\zeta\psi_* U_\sigma + \frac{\beta(w)\kappa_n(w)}{\kappa(w)}\tilde{U}_\sigma. \\
\Rightarrow \tilde{\sigma}(w) &= \alpha(w)(x'\zeta\psi_*\theta_x + y'\zeta\psi_*\theta_y) + \frac{\beta(w)\kappa_g(w)}{\kappa(w)}\frac{1}{\sqrt{\zeta E\zeta G - (\zeta F)^2}}\{\zeta Ex'\zeta\psi_*\theta_y \\
&\quad + \zeta F(y'\zeta\psi_*\theta_y - x'\zeta\psi_*\theta_x) - \zeta Gy'\zeta\psi_*\theta_x\} - \frac{\beta(w)\kappa_n(w)}{\kappa(w)}\zeta\psi_* U_\sigma \\
&\quad + \frac{\beta(w)\kappa_n(w)}{\kappa(w)}\tilde{U}_\sigma. \qquad (18)
\end{aligned}$$

From (11) and (18), it follows that

$$\begin{aligned}
\tilde{\sigma}(w) &= \alpha(w)(x'\tilde{\theta}_x + y'\tilde{\theta}_y) + \frac{\beta(w)\kappa_g(w)}{\kappa(w)}\frac{1}{\sqrt{\tilde{E}\tilde{G}-\tilde{F}^2}}\{\tilde{E}x'\tilde{\theta}_y + \tilde{F}(y'\tilde{\theta}_y \\
&\quad - x'\tilde{\theta}_x) - \tilde{G}y'\tilde{\theta}_x\} - \frac{\beta(w)\kappa_n(w)}{\kappa(w)}\tilde{U}_\sigma + \frac{\beta(w)\kappa_n(w)}{\kappa(w)}\tilde{U}_\sigma. \\
\Rightarrow \tilde{\sigma}(w) &= \alpha(w)(x'\tilde{\theta}_x + y'\tilde{\theta}_y) + \frac{\beta(w)\kappa_g(w)}{\kappa(w)}\frac{1}{\sqrt{\tilde{E}\tilde{G}-\tilde{F}^2}}\{\tilde{E}x'\tilde{\theta}_y + \tilde{F}(y'\tilde{\theta}_y \\
&\quad - x'\tilde{\theta}_x) - \tilde{G}y'\tilde{\theta}_x\}. \\
\Rightarrow \tilde{\sigma}(w) &= \tilde{\alpha}(w)(x'\tilde{\theta}_x + y'\tilde{\theta}_y) + \tilde{\beta}(w)\frac{1}{\sqrt{\tilde{E}\tilde{G}-\tilde{F}^2}}\{\tilde{E}x'\tilde{\theta}_y + \tilde{F}(y'\tilde{\theta}_y - x'\tilde{\theta}_x) \\
&\quad - \tilde{G}y'\tilde{\theta}_x\},
\end{aligned}$$

which represents an osculating curve exhibiting an asymptotic character on the surface $\tilde{R}$, where $\alpha(w) = \tilde{\alpha}(w)$ and $\tilde{\beta}(w) = \frac{\beta(w)\kappa_g(w)}{\kappa(w)}$.

Suppose $\tilde{\sigma}$ is neither geodesic nor asymptotic curve on $\tilde{R}$ such that $\tilde{\sigma}(w) = \zeta\psi_*(\sigma(w))$, which implies

$$\begin{aligned}
\tilde{\sigma}(w) &= \alpha(w)(x'\zeta\psi_*\theta_x + y'\zeta\psi_*\theta_y) + \frac{\beta(w)\kappa_g(w)}{\kappa(w)}\frac{1}{\zeta\sqrt{EG-F^2}}\zeta^2\{Ex'\psi_*\theta_y \\
&\quad + F(y'\psi_*\theta_y - x'\psi_*\theta_x) - Gy'\psi_*\theta_x\} - \frac{\beta(w)\kappa_n(w)}{\kappa(w)}\zeta\psi_*U_\sigma. \\
\Rightarrow \tilde{\sigma}(w) &= \alpha(w)(x'\zeta\psi_*\theta_x + y'\zeta\psi_*\theta_y) + \frac{\beta(w)\kappa_g(w)}{\kappa(w)}\frac{1}{\sqrt{\zeta E\zeta G - (\zeta F)^2}}\{\zeta Ex'\zeta\psi_*\theta_y \\
&\quad + \zeta F(y'\zeta\psi_*\theta_y - x'\zeta\psi_*\theta_x) - \zeta Gy'\zeta\psi_*\theta_x\} - \frac{\beta(w)\kappa_n(w)}{\kappa(w)}\zeta\psi_*U_\sigma. \qquad (19)
\end{aligned}$$

From (11) and (19), it follows that

$$\begin{aligned}
\tilde{\sigma}(w) &= \alpha(w)(x'\tilde{\theta}_x + y'\tilde{\theta}_y) + \frac{\beta(w)\kappa_g(w)}{\kappa(w)}\frac{1}{\sqrt{\tilde{E}\tilde{G} - \tilde{F}^2}}\{\tilde{E}x'\tilde{\theta}_y \\
&\quad + \tilde{F}(y'\tilde{\theta}_y - x'\tilde{\theta}_x) - \tilde{G}y'\tilde{\theta}_x\} - \frac{\beta(w)\kappa_n(w)}{\kappa(w)}\tilde{U}_\sigma. \\
\Rightarrow \tilde{\sigma}(w) &= \tilde{\alpha}(w)(x'\tilde{\theta}_x + y'\tilde{\theta}_y) + \frac{\tilde{\beta}(w)\tilde{\kappa}_g(w)}{\tilde{\kappa}(w)}\frac{1}{\sqrt{\tilde{E}\tilde{G} - \tilde{F}^2}}\{\tilde{E}x'\tilde{\theta}_y \\
&\quad + \tilde{F}(y'\tilde{\theta}_y - x'\tilde{\theta}_x) - \tilde{G}y'\tilde{\theta}_x\} - \frac{\tilde{\beta}(w)\tilde{\kappa}_n(w)}{\tilde{\kappa}(w)}\tilde{U}_\sigma, \qquad (20)
\end{aligned}$$

which represents an osculating curve on the surface $\tilde{R}$ that neither exhibits a geodesic character nor an asymptotic character, where $\tilde{\alpha}(w) = \alpha(w)$, $\frac{\tilde{\beta}(w)\tilde{\kappa}_g(w)}{\tilde{\kappa}(w)} = \frac{\beta(w)\kappa_g(w)}{\kappa(w)}$ and $\frac{\tilde{\beta}(w)\tilde{\kappa}_n(w)}{\kappa(w)} = \frac{\beta(w)\kappa_n(w)}{\kappa(w)}$. □

Theorem 2. *Let $\psi : R \to \tilde{R}$ be a conformal transformation, where R and $\tilde{R}$ are regular surfaces and $\sigma(w)$ is an asymptotic osculating curve on R (i.e., $\kappa_n = 0$). Then, $\tilde{\sigma} = \psi \circ \sigma$ is an osculating curve on $\tilde{R}$ if any of the following conditions are satisfied:*

(a) $\tilde{\sigma}$ is an asymptotic curve on $\tilde{R}$ and satisfies $\tilde{\sigma}(w) = \zeta\psi_*(\sigma(w))$.

(b) $\tilde{\sigma}$ is not an asymptotic curve on $\tilde{R}$ and satisfies $\tilde{\sigma}(w) + \frac{\tilde{\beta}(w)\tilde{\kappa}_n(w)}{\tilde{\kappa}(w)}\tilde{U}_\sigma = \zeta\psi_*(\sigma(w))$.

Proof. The process for proving this theorem is the same as that used for proving Theorem 1. All that is needed to obtain the expected results is to set $k_n = 0$ in the proof of Theorem 1. □

After analyzing the conditions under which an osculating curve remains invariant under a conformal transformation, we now extend our analysis to discuss its behavior along the tangent vector, $T_\sigma(w) = a\theta_x + b\theta_y$, and along the unit normal vector $P_\sigma = U_\sigma \times T_\sigma$. In the following theorems, Theorems 3 and 4, we examine their behavior along T_σ by considering the cases of non-zero and zero normal curvature, respectively. Subsequently, in Theorems 5 and 6, we will discuss their behavior along the unit normal vector P_σ by considering their non-asymptotic and asymptotic characteristics, respectively.

Theorem 3. *Let $\psi : R \to \tilde{R}$ be a conformal transformation, where R and $\tilde{R}$ are regular surfaces, and let σ be a non-asymptotic osculating curve on R (i.e., $\kappa_n \neq 0$), and let $\tilde{\sigma}$ be its corresponding osculating curve on the transformed surface $\tilde{R}$. Then, we have the following:*

(a) *If $\tilde{\sigma}$ exhibits a geodesic character on $\tilde{R}$, then*

$$\tilde{\sigma}(w)\cdot\tilde{T}_{\sigma}(w)=\zeta\sigma(w)\cdot T_{\sigma}(w)+\frac{\beta(w)\kappa_g(w)}{\kappa(w)}\sqrt{\tilde{E}\tilde{G}-(\tilde{F})^2}(ay'-bx').$$

(b) *If $\tilde{\sigma}$ shows an asymptotic character on $\tilde{R}$, then*

$$\tilde{\sigma}(w)\cdot\tilde{T}_{\sigma}(w)=\zeta\sigma(w)\cdot T_{\sigma}(w).$$

(c) *If $\tilde{\sigma}$ neither shows a geodesic character nor an asymptotic character on $\tilde{R}$, then*

$$\tilde{\sigma}(w)\cdot\tilde{T}_{\sigma}(w)=\zeta\sigma(w)\cdot T_{\sigma}(w),$$

where $T_{\sigma}(w)=a\theta_x+b\theta_y$ denotes a tangent vector at point $\sigma(w)$ on surface R.

Proof. Let $\psi: R\to\tilde{R}$ be a conformal transformation, and let σ and $\tilde{\sigma}$ be osculating curves on R and $\tilde{R}$, respectively, with $\kappa_n\neq 0$. Then,

$$\tilde{\sigma}(w)\cdot\tilde{T}_{\sigma}(w)-\zeta\sigma(w)\cdot T_{\sigma}(w)=a(\tilde{\sigma}(w)\cdot\tilde{\theta}_x-\zeta\sigma(w)\cdot\theta_x)+b(\tilde{\sigma}(w)\cdot\tilde{\theta}_y-\zeta\sigma(w)\cdot\theta_y).\quad(21)$$

Thus, in order to prove required results, we have to calculate the values of $(\tilde{\sigma}(w)\cdot\tilde{\theta}_x-\zeta\sigma(w)\cdot\theta_x)$ and $(\tilde{\sigma}(w)\cdot\tilde{\theta}_y-\zeta\sigma(w)\cdot\theta_y)$.

Now, from Equation (14), we obtain

$$\begin{aligned}\sigma(w)\cdot\theta_x &= \alpha(w)(x'\theta_x\cdot\theta_x+y'\theta_y\cdot\theta_x)-\frac{\beta(w)\kappa_n(w)}{\kappa(w)}\frac{1}{\sqrt{EG-F^2}}(\theta_x\times\theta_y)\cdot\theta_x\\ &\quad+\frac{\beta(w)\kappa_g(w)}{\kappa(w)}\frac{1}{\sqrt{EG-F^2}}\{Ex'\theta_y\cdot\theta_x+Fy'\theta_y\cdot\theta_x-Fx'\theta_x\cdot\theta_x\\ &\quad-Gy'\theta_x\cdot\theta_x\}.\\ \Rightarrow\sigma(w)\cdot\theta_x &= \alpha(w)(x'E+y'F)+\frac{\beta(w)\kappa_g(w)}{\kappa(w)}\frac{1}{\sqrt{EG-F^2}}\{x'EF+y'F^2-x'EF\\ &\quad-y'EG\}.\\ \Rightarrow\sigma(w)\cdot\theta_x &= \alpha(w)(x'E+y'F)+\frac{\beta(w)\kappa_g(w)}{\kappa(w)}\frac{1}{\sqrt{EG-F^2}}(F^2-EG)y'.\end{aligned}\quad(22)$$

Similarly,

$$\sigma(w)\cdot\theta_y = \alpha(w)(x'F+y'G)+\frac{\beta(w)\kappa_g(w)}{\kappa(w)}\frac{1}{\sqrt{EG-F^2}}(EG-F^2)x'.\quad(23)$$

In order to prove (a), let us suppose that a curve $\tilde{\sigma}(w)$ exhibits geodesic characteristics on $\tilde{R}$.

So, by using Equation (15), we obtain

$$\tilde{\sigma}(w)\cdot\tilde{\theta}_x = \tilde{\alpha}(w)(x'\tilde{E}+y'\tilde{F}),\quad(24)$$

and

$$\tilde{\sigma}(w)\cdot\tilde{\theta}_y = \tilde{\alpha}(w)(x'\tilde{F}+y'\tilde{G}).\quad(25)$$

From Equations (22) and (24), we obtain

$$\tilde{\sigma}(w)\cdot\tilde{\theta}_x-\zeta\sigma(w)\cdot\theta_x = \tilde{\alpha}(w)(x'\tilde{E}+y'\tilde{F})-\alpha(w)(x'\zeta E+y'\zeta F)$$

$$\begin{aligned}
&\quad -\frac{\beta(w)\kappa_g(w)}{\kappa(w)}\frac{1}{\sqrt{\zeta E\zeta G-(\zeta F)^2}}((\zeta F)^2\\
&\quad -\zeta E\zeta G)y'.\\
\Rightarrow \tilde{\sigma}(w)\cdot\tilde{\theta}_x-\zeta\sigma(w)\cdot\theta_x &= (\tilde{\alpha}(w)-\alpha(w))(x'\tilde{E}+y'\tilde{F})\\
&\quad -\frac{\beta(w)\kappa_g(w)}{\kappa(w)}\frac{1}{\sqrt{\tilde{E}\tilde{G}-\tilde{F}^2}}(\tilde{F}^2-\tilde{E}\tilde{G})y'. \qquad (26)
\end{aligned}$$

Similarly, from (23) and (25), we obtain

$$\begin{aligned}
\tilde{\sigma}(w)\cdot\tilde{\theta}_y-\zeta\sigma(w)\cdot\theta_y &= (\tilde{\alpha}(w)-\alpha(w))(x'\tilde{F}+y'\tilde{G})\\
&\quad -\frac{\beta(w)\kappa_g(w)}{\kappa(w)}\frac{1}{\sqrt{\tilde{E}\tilde{G}-\tilde{F}^2}}(\tilde{E}\tilde{G}-\tilde{F}^2)x'. \qquad (27)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\tilde{\sigma}(w)\tilde{T}_\sigma(w)-\zeta\sigma(w)\cdot T_\sigma(w) &= a\{(\tilde{\alpha}(w)-\alpha(w))(x'\tilde{E}+y'\tilde{F})\\
&\quad -\frac{\beta(w)\kappa_g(w)}{\kappa(w)}\frac{1}{\sqrt{\tilde{E}\tilde{G}-\tilde{F}^2}}(\tilde{F}^2-\tilde{E}\tilde{G})y'\}\\
&\quad +b\{(\tilde{\alpha}(w)-\alpha(w))(x'\tilde{F}+y'\tilde{G})\\
&\quad -\frac{\beta(w)\kappa_g(w)}{\kappa(w)}\frac{1}{\sqrt{\tilde{E}\tilde{G}-\tilde{F}^2}}(\tilde{E}\tilde{G}-\tilde{F}^2)x'\}.\\
\Rightarrow \tilde{\sigma}(w)\tilde{T}_\sigma(w)-\zeta\sigma(w)\cdot T_\sigma(w) &= (\tilde{\alpha}(w)-\alpha(w))\{a(x'\tilde{E}+y'\tilde{F})+b(x'\tilde{F}\\
&\quad +y'\tilde{G})\}+\frac{\beta(w)\kappa_g(w)}{\kappa(w)}\frac{1}{\sqrt{\tilde{E}\tilde{G}-\tilde{F}^2}}(\tilde{E}\tilde{G}\\
&\quad -\tilde{F}^2)(ay'-bx').
\end{aligned}$$

Since σ and $\tilde{\sigma}$ are osculating curves on R and $\tilde{R}$, respectively, then $\tilde{\alpha}(w)=\alpha(w)$.

Hence,

$$\tilde{\sigma}\cdot\tilde{T}_\sigma(w) = \zeta\sigma(w)\cdot T_\sigma(w)+\frac{\beta(w)\kappa_g(w)}{\kappa(w)}\sqrt{\tilde{E}\tilde{G}-\tilde{F}^2}\,(ay'-bx').$$

For (b), suppose that $\tilde{\sigma}$ exhibits an asymptotic character on $\tilde{R}$. Then,

$$\tilde{\sigma}(w)\cdot\tilde{\theta}_x = \tilde{\alpha}(w)(x'\tilde{E}+y'\tilde{F})+\tilde{\beta}(w)\frac{1}{\sqrt{\tilde{E}\tilde{G}-\tilde{F}^2}}(\tilde{F}^2-\tilde{E}\tilde{G})y', \qquad (28)$$

and

$$\tilde{\sigma}(w)\cdot\tilde{\theta}_y = \tilde{\alpha}(w)(x'\tilde{E}+y'\tilde{F})+\tilde{\beta}(w)\frac{1}{\sqrt{\tilde{E}\tilde{G}-\tilde{F}^2}}(\tilde{E}\tilde{G}-\tilde{F}^2)x'. \qquad (29)$$

From Equations (22) and (28), we obtain

$$\begin{aligned}
\tilde{\sigma}(w)\cdot\tilde{\theta}_x-\zeta\sigma(w)\cdot\theta_x &= \tilde{\alpha}(w)(x'\tilde{E}+y'\tilde{F})+\tilde{\beta}(w)\frac{1}{\sqrt{\tilde{E}\tilde{G}-\tilde{F}^2}}(\tilde{F}^2-\tilde{E}\tilde{G})y'\\
&\quad -\alpha(w)(x'\zeta E+y'\zeta F)-\frac{\beta(w)\kappa_g(w)}{\kappa(w)}\frac{1}{\sqrt{\zeta E\zeta G-(\zeta F)^2}}((\zeta F)^2\\
&\quad -\zeta E\zeta G)y'.\\
\Rightarrow \tilde{\sigma}(w)\cdot\tilde{\theta}_x-\zeta\sigma(w)\cdot\theta_x &= (\tilde{\alpha}(w)-\alpha(w))(x'\tilde{E}+y'\tilde{F})-(\tilde{\beta}(w)
\end{aligned}$$

$$-\frac{\beta(w)\kappa_g(w)}{\kappa(w)})\frac{1}{\sqrt{\tilde{E}\tilde{G}-\tilde{F}^2}}(\tilde{E}\tilde{G}-\tilde{F}^2)y'. \quad (30)$$

Similarly, from (23) and (29), we obtain

$$\begin{aligned}\tilde{\sigma}(w)\cdot\tilde{\theta}_y-\zeta\sigma(w)\cdot\theta_y &= \tilde{\alpha}(w)(x'\tilde{F}+y'\tilde{G})+\tilde{\beta}(w)\frac{1}{\sqrt{\tilde{E}\tilde{G}-\tilde{F}^2}}(\tilde{E}\tilde{G}-\tilde{F}^2)x' \\ &\quad -\alpha(w)(x'\zeta F+y'\zeta G)-\frac{\beta(w)\kappa_g(w)}{\kappa(w)}\frac{1}{\sqrt{\zeta E\zeta G-(\zeta F)^2}}(\zeta E\zeta G \\ &\quad -(\zeta F)^2)x'.\end{aligned}$$

$$\begin{aligned}\Rightarrow \tilde{\sigma}(w)\cdot\tilde{\theta}_y-\zeta\sigma(w)\cdot\theta_y &= (\tilde{\alpha}(w)-\alpha(w))(x'\tilde{F}+y'\tilde{G})+(\tilde{\beta}(w) \\ &\quad -\frac{\beta(w)\kappa_g(w)}{\kappa(w)})\frac{1}{\sqrt{\tilde{E}\tilde{G}-\tilde{F}^2}}(\tilde{E}\tilde{G}-\tilde{F}^2)x'. \quad (31)\end{aligned}$$

Therefore,

$$\begin{aligned}\tilde{\sigma}(w)\cdot\tilde{T}_\sigma(w)-\zeta\sigma(w)\cdot T_\sigma(w) &= a\{(\tilde{\alpha}(w)-\alpha(w))(x'\tilde{E}+y'\tilde{F})-(\tilde{\beta}(w) \\ &\quad -\frac{\beta(w)\kappa_g(w)}{\kappa(w)})\frac{1}{\sqrt{\tilde{E}\tilde{G}-\tilde{F}^2}}(\tilde{E}\tilde{G}-\tilde{F}^2)y'\} \\ &\quad +b\{(\tilde{\alpha}(w)-\alpha(w))(x'\tilde{F}+y'\tilde{G})+(\tilde{\beta}(w) \\ &\quad -\frac{\beta(w)\kappa_g(w)}{\kappa(w)})\frac{1}{\sqrt{\tilde{E}\tilde{G}-\tilde{F}^2}}(\tilde{E}\tilde{G}-\tilde{F}^2)x'\}. \\ \Rightarrow \tilde{\sigma}(w)\cdot\tilde{T}_\sigma(w)-\zeta\sigma(w)\cdot T_\sigma(w) &= (\tilde{\alpha}(w)-\alpha(w))\{a(x'\tilde{E}+y'\tilde{F})+b(x'\tilde{F} \\ &\quad +y'\tilde{G})\}+(\tilde{\beta}(w)-\frac{\beta(w)\kappa_g(w)}{\kappa(w)}) \\ &\quad \frac{1}{\sqrt{\tilde{E}\tilde{G}-\tilde{F}^2}}(\tilde{E}\tilde{G}-\tilde{F}^2)(bx'-ay').\end{aligned}$$

Since $\sigma(w)$ and $\tilde{\sigma}(w)$ are normal curves on R and $\tilde{R}$, respectively, then $\tilde{\alpha}(w)=\alpha(w)$ and $\tilde{\beta}(w)=\frac{\beta(w)\kappa_g(w)}{\kappa(w)}$.

Hence,

$$\tilde{\sigma}(w)\cdot\tilde{T}_\sigma(w) = \zeta\sigma(w)\cdot T_\sigma(w).$$

For (c), suppose that $\tilde{\sigma}$ exhibits neither a geodesic character nor an asymptotic character on $\tilde{R}$. Then,

$$\tilde{\sigma}(w)\cdot\tilde{\theta}_x = \tilde{\alpha}(w)(x'\tilde{E}+y'\tilde{F})+\frac{\tilde{\beta}(w)\tilde{\kappa}_g(w)}{\tilde{\kappa}(w)}\frac{1}{\sqrt{\tilde{E}\tilde{G}-\tilde{F}^2}}(\tilde{F}^2-\tilde{E}\tilde{G})y', \quad (32)$$

and

$$\tilde{\sigma}(w)\cdot\tilde{\theta}_y = \tilde{\alpha}(w)(x'\tilde{F}+y'\tilde{G})+\frac{\tilde{\beta}(w)\tilde{\kappa}_g(w)}{\tilde{\kappa}(w)}\frac{1}{\sqrt{\tilde{E}\tilde{G}-\tilde{F}^2}}(\tilde{E}\tilde{G}-\tilde{F}^2)x'. \quad (33)$$

Therefore,

$$\tilde{\sigma}(w)\cdot\tilde{\theta}_x-\zeta\sigma(w)\cdot\theta_x = \tilde{\alpha}(w)(x'\tilde{E}+y'\tilde{F})+\frac{\tilde{\beta}(w)\tilde{\kappa}_g(w)}{\tilde{\kappa}(w)}\frac{1}{\sqrt{\tilde{E}\tilde{G}-\tilde{F}^2}}(\tilde{F}^2-\tilde{E}\tilde{G})y'$$

$$
\begin{aligned}
& -\alpha(w)(x'\zeta E + y'\zeta F) - \frac{\beta(w)\kappa_g(w)}{\kappa(w)}\frac{1}{\sqrt{\zeta E\zeta G - (\zeta F)^2}}((\zeta F)^2 \\
& -\zeta E\zeta G)y'. \\
\Rightarrow \tilde{\sigma}(w)\cdot\tilde{\theta}_x - \zeta\sigma(w)\cdot\theta_x \;=\;& (\tilde{\alpha}(w) - \alpha(w))(x'\tilde{E} + y'\tilde{F}) + (\frac{\tilde{\beta}(w)\tilde{\kappa_g}(w)}{\tilde{\kappa}(w)} \\
& -\frac{\beta(w)\kappa_g(w)}{\kappa(w)})\frac{1}{\sqrt{\tilde{E}\tilde{G} - \tilde{F}^2}}(\tilde{F}^2 - \tilde{E}\tilde{G})y'. \qquad (34)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\tilde{\sigma}(w)\cdot\tilde{\theta}_y - \zeta\sigma(w)\cdot\theta_y \;=\;& \tilde{\alpha}(w)(x'\tilde{F} + y'\tilde{G}) + \frac{\tilde{\beta}(w)\tilde{\kappa_g}(w)}{\tilde{\kappa}(w)}\frac{1}{\sqrt{\tilde{E}\tilde{G} - \tilde{F}^2}}(\tilde{E}\tilde{G} - \tilde{F}^2)x' \\
& -\alpha(w)(x'\zeta F + y'\zeta G) - \frac{\beta(w)\kappa_g(w)}{\kappa(w)}\frac{1}{\sqrt{\zeta E\zeta G - (\zeta F)^2}}((\zeta E\zeta G \\
& -(\zeta F)^2)x'. \\
\Rightarrow \tilde{\sigma}(w)\cdot\tilde{\theta}_y - \zeta\sigma(w)\cdot\theta_y \;=\;& (\tilde{\alpha}(w) - \alpha(w))(x'\tilde{F} + y'\tilde{G}) + (\frac{\tilde{\beta}(w)\tilde{\kappa_g}(w)}{\tilde{\kappa}(w)} \\
& -\frac{\beta(w)\kappa_g(w)}{\kappa(w)})\frac{1}{\sqrt{\tilde{E}\tilde{G} - \tilde{F}^2}}(\tilde{E}\tilde{G} - \tilde{F}^2)x'. \qquad (35)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\tilde{\sigma}(w)\tilde{T}_\sigma(w) - \zeta\sigma(w)\cdot T_\sigma(w) \;=\;& a\{(\tilde{\alpha}(w) - \alpha(w))(x'\tilde{E} + y'\tilde{F}) + (\frac{\tilde{\beta}(w)\tilde{\kappa_g}(w)}{\tilde{\kappa}(w)} \\
& -\frac{\beta(w)\kappa_g(w)}{\kappa(w)})\frac{1}{\sqrt{\tilde{E}\tilde{G} - \tilde{F}^2}}(\tilde{F}^2 - \tilde{E}\tilde{G})y'\} \\
& +b\{(\tilde{\alpha}(w) - \alpha(w))(x'\tilde{F} + y'\tilde{G}) + (\frac{\tilde{\beta}(w)\tilde{\kappa_g}(w)}{\tilde{\kappa}(w)} \\
& -\frac{\beta(w)\kappa_g(w)}{\kappa(w)})\frac{1}{\sqrt{\tilde{E}\tilde{G} - \tilde{F}^2}}(\tilde{E}\tilde{G} - \tilde{F}^2)x'\}. \\
\Rightarrow \tilde{\sigma}(w)\tilde{T}_\sigma(w) - \zeta\sigma(w)\cdot T_\sigma(w) \;=\;& (\tilde{\alpha}(w) - \alpha(w))\{a(x'\tilde{E} + y'\tilde{F}) + b(x'\tilde{F} + y'\tilde{G})\} \\
& +(\frac{\tilde{\beta}(w)\tilde{\kappa_g}(w)}{\tilde{\kappa}(w)} - \frac{\beta(w)\kappa_g(w)}{\kappa(w)})\frac{1}{\sqrt{\tilde{E}\tilde{G} - \tilde{F}^2}}(\tilde{E}\tilde{G} \\
& -\tilde{F}^2)(bx' - ay').
\end{aligned}
$$

Since $\sigma(w)$ and $\tilde{\sigma}(w)$ are normal curves on R and $\tilde{R}$, respectively, then $\tilde{\alpha}(w) = \alpha(w)$ and $\frac{\tilde{\beta}(w)\tilde{\kappa}_g(w)}{\tilde{\kappa}(w)} = \frac{\beta(w)\kappa_g(w)}{\kappa(w)}$.

Hence,

$$
\tilde{\sigma}(w)\cdot\tilde{T}_\sigma(w) = \zeta\sigma(w)\cdot T_\sigma(w).
$$

□

Theorem 4. *Let $\psi : R \to \tilde{R}$ be a conformal transformation, where R and $\tilde{R}$ are regular surfaces, and let σ be an asymptotic osculating curve on R (i.e., $\kappa_n = 0$) and let $\tilde{\sigma}$ be its corresponding osculating curve on the transformed surface $\tilde{R}$. Then, we have the following:*

(a) *If $\tilde{\sigma}$ exhibits an asymptotic character on $\tilde{R}$, then*

$$
\tilde{\sigma}(w)\cdot\tilde{T}_\sigma(w) = \zeta\sigma(w)\cdot T_\sigma(w).
$$

(b) *If $\tilde{\sigma}$ does not exhibits an asymptotic curve on $\tilde{R}$, then*

$$\tilde{\sigma}(w) \cdot \tilde{T}_\sigma(w) = \zeta\sigma(w) \cdot T_\sigma(w),$$

where $T_\sigma(w) = a\theta_x + b\theta_y$ denotes a tangent vector at point $\sigma(w)$ on surface R.

Proof. We can prove this theorem by setting $\kappa_n = 0$ in the proof of Theorem 3. □

Theorem 5. *Let $\psi : R \to \tilde{R}$ be a conformal transformation, where R and $\tilde{R}$ are regular surfaces, and let σ be a non-asymptotic osculating curve on R (i.e., $\kappa_n \neq 0$) and let $\tilde{\sigma}$ be its corresponding osculating curve on the transformed surface $\tilde{R}$. Then, we have the following:*

(a) *If $\tilde{\sigma}$ exhibits a geodesic character on $\tilde{R}$, then*

$$\tilde{\sigma}(w) \cdot \tilde{P}_\sigma(w) = \zeta\sigma(w) \cdot P_\sigma(w) - \frac{\beta(w)\kappa_g(w)}{\kappa(w)}((a\tilde{E} + b\tilde{F})x' + (a\tilde{F} + b\tilde{G})y').$$

(b) *If $\tilde{\sigma}$ exhibits an asymptotic character on $\tilde{R}$, then*

$$\tilde{\sigma}(w) \cdot \tilde{P}_\sigma(w) = \zeta\sigma(w) \cdot P_\sigma(w).$$

(c) *If $\tilde{\sigma}$ neither shows a geodesic character nor an asymptotic character on $\tilde{R}$, then*

$$\tilde{\sigma}(w) \cdot \tilde{P}_\sigma(w) = \zeta\sigma(w) \cdot P_\sigma(w),$$

where $P_\sigma(w) = U_\sigma(w) \times T_\sigma(w)$ in which $T_\sigma(w) = a\theta_x + b\theta_y$ denotes a tangent vector at point $\sigma(w)$ on surface R.

Proof. Let $F : R \to \tilde{R}$ be a conformal transformation and let σ and $\tilde{\sigma}$ be osculating curves on R and $\tilde{R}$ with $\kappa_n \neq 0$. Then,

$$\sigma(w) \cdot P_\sigma(w) = \frac{(aE + bF)}{\sqrt{EG - F^2}}\sigma(w) \cdot \theta_y - \frac{(aF + bG)}{\sqrt{EG - F^2}}\sigma(w) \cdot \theta_x. \tag{36}$$

Similarly,

$$\tilde{\sigma}(w) \cdot \tilde{P}_\sigma(w) = \frac{(a\tilde{E} + b\tilde{F})}{\sqrt{\tilde{E}\tilde{G} - \tilde{F}^2}}\tilde{\sigma}(w) \cdot \tilde{\theta}_y - \frac{(a\tilde{F} + b\tilde{G})}{\sqrt{\tilde{E}\tilde{G} - \tilde{F}^2}}\tilde{\sigma}(w) \cdot \tilde{\theta}_x. \tag{37}$$

Thus, from (36) and (37), we obtain

$$\begin{aligned}
\tilde{\sigma}(w) \cdot \tilde{P}_\sigma(w) - \zeta\sigma(w) \cdot P_\sigma(w) &= \frac{(a\tilde{E} + b\tilde{F})}{\sqrt{\tilde{E}\tilde{G} - \tilde{F}^2}}\tilde{\sigma}(w) \cdot \tilde{\theta}_y - \frac{(a\tilde{F} + b\tilde{G})}{\sqrt{\tilde{E}\tilde{G} - \tilde{F}^2}}\tilde{\sigma}(w) \cdot \tilde{\theta}_x \\
&\quad - \frac{(a\zeta E + b\zeta F)}{\sqrt{\zeta E\zeta G - (\zeta F)^2}}\zeta\sigma(w) \cdot \theta_y - \frac{(a\zeta F + b\zeta G)}{\sqrt{\zeta E\zeta G - (\zeta F)^2}}\zeta\sigma(w) \cdot \theta_x. \\
\Rightarrow \tilde{\sigma}(w) \cdot \tilde{P}_\sigma(w) - \zeta\sigma(w) \cdot P_\sigma(w) &= \frac{(a\tilde{E} + b\tilde{F})}{\sqrt{\tilde{E}\tilde{G} - \tilde{F}^2}}(\tilde{\sigma}(w) \cdot \tilde{\theta}_y - \zeta\sigma(w) \cdot \theta_y) \\
&\quad - \frac{(a\tilde{F} + b\tilde{G})}{\sqrt{\tilde{E}\tilde{G} - \tilde{F}^2}}(\tilde{\sigma}(w) \cdot \tilde{\theta}_x - \zeta\sigma(w) \cdot \theta_x).
\end{aligned} \tag{38}$$

Thus, in order to prove required results, we have to calculate the values of $(\tilde{\sigma}(w) \cdot \tilde{\theta}_x - \zeta\sigma(w) \cdot \theta_x)$ and $(\tilde{\sigma}(w) \cdot \tilde{\theta}_y - \zeta\sigma(w) \cdot \theta_y)$.

For (a), suppose that $\tilde{\sigma}$ is a geodesic curve on $\tilde{R}$. Then, from (26), (27) and (38), we obtain

$$\begin{aligned}\tilde{\sigma}(w)\cdot\tilde{P}_{\sigma}(w)-\zeta\sigma(w)\cdot P_{\sigma}(w) &= \frac{(a\tilde{E}+b\tilde{F})}{\sqrt{\tilde{E}\tilde{G}-\tilde{F}^2}}\{(\tilde{\alpha}(w)-\alpha(w))(x'\tilde{F}+y'\tilde{G})\\ &-\frac{\beta(w)\kappa_g(w)}{\kappa(w)}\frac{1}{\sqrt{\tilde{E}\tilde{G}-\tilde{F}^2}}(\tilde{E}\tilde{G}-\tilde{F}^2)x'\}\\ &-\frac{(a\tilde{F}+b\tilde{G})}{\sqrt{\tilde{E}\tilde{G}-\tilde{F}^2}}\{(\tilde{\alpha}(w)-\alpha(w))(x'\tilde{E}+y'\tilde{F})\\ &-\frac{\beta(w)\kappa_g(w)}{\kappa(w)}\frac{1}{\sqrt{\tilde{E}\tilde{G}-\tilde{F}^2}}(\tilde{F}^2-\tilde{E}\tilde{G})y'\}.\\ \Rightarrow \tilde{\sigma}(w)\cdot\tilde{P}_{\sigma}(w)-\zeta\sigma(w)\cdot P_{\sigma}(w) &= (\tilde{\alpha}(w)-\alpha(w))\frac{1}{\sqrt{\tilde{E}\tilde{G}-\tilde{F}^2}}\{(a\tilde{E}+b\tilde{F})(x'\tilde{F}\\ &+y'\tilde{G})+(a\tilde{F}+b\tilde{G})(x'\tilde{E}+y'\tilde{F})\}\\ &-\frac{\beta(w)\kappa_g(w)}{\kappa(w)}\{(a\tilde{E}+b\tilde{F})(x'\tilde{F}+y'\tilde{G})+(a\tilde{F}\\ &+b\tilde{G})(x'\tilde{E}+y'\tilde{F})\}.\end{aligned}$$

Since σ and $\tilde{\sigma}$ are osculating curves on R and $\tilde{R}$, respectively, then $\tilde{\alpha}(w)=\alpha(w)$.

Hence,

$$\begin{aligned}\tilde{\sigma}\cdot\tilde{P}_{\sigma}(w) &= \zeta\sigma(w)\cdot P_{\sigma}(w)-\frac{\beta(w)\kappa_g(w)}{\kappa(w)}\{(a\tilde{E}+b\tilde{F})(x'\tilde{F}\\ &+y'\tilde{G})+(a\tilde{F}+b\tilde{G})(x'\tilde{E}+y'\tilde{F})\}.\end{aligned}$$

For (b), suppose that $\tilde{\sigma}$ is an asymptotic curve on $\tilde{R}$. Then, from (30), (31) and (38), we obtain

$$\begin{aligned}\tilde{\sigma}(w)\cdot\tilde{P}_{\sigma}(w)-\zeta\sigma(w)\cdot P_{\sigma}(w) &= \frac{(a\tilde{E}+b\tilde{F})}{\sqrt{\tilde{E}\tilde{G}-\tilde{F}^2}}\{(\tilde{\alpha}(w)-\alpha(w))(x'\tilde{F}+y'\tilde{G})\\ &+(\tilde{\beta}(w)-\frac{\beta(w)\kappa_g(w)}{\kappa(w)})\frac{1}{\sqrt{\tilde{E}\tilde{G}-\tilde{F}^2}}(\tilde{E}\tilde{G}\\ &-\tilde{F}^2)x'\}-\frac{(a\tilde{F}+b\tilde{G})}{\sqrt{\tilde{E}\tilde{G}-\tilde{F}^2}}\{(\tilde{\alpha}(w)-\alpha(w))(x'\tilde{E}+y'\tilde{F})\\ &-(\tilde{\beta}(w)-\frac{\beta(w)\kappa_g(w)}{\kappa(w)})\frac{1}{\sqrt{\tilde{E}\tilde{G}-\tilde{F}^2}}(\tilde{E}\tilde{G}-\tilde{F}^2)y'\}.\\ \Rightarrow \tilde{\sigma}(w)\cdot\tilde{P}_{\sigma}(w)-\zeta\sigma(w)\cdot P_{\sigma}(w) &= (\tilde{\alpha}(w)-\alpha(w))\frac{1}{\sqrt{\tilde{E}\tilde{G}-\tilde{F}^2}}\{(a\tilde{E}+b\tilde{F})(x'\tilde{F}+y'\tilde{G})\\ &+(a\tilde{F}+b\tilde{G})(x'\tilde{E}+y'\tilde{F})\}+\\ &(\tilde{\beta}(w)-\frac{\beta(w)\kappa_g(w)}{\kappa(w)})\{(a\tilde{E}+b\tilde{F})x'+(a\tilde{F}+b\tilde{G})y'\}.\end{aligned}$$

Since $\sigma(w)$ and $\tilde{\sigma}(w)$ are osculating curves on R and $\tilde{R}$, respectively, then $\tilde{\alpha}(w)=\frac{\alpha(w)\kappa_g(w)}{\kappa(w)}$ and $\tilde{\beta}(w)-\frac{\beta(w)\kappa_g(w)}{\kappa(w)}$. Hence,

$$\tilde{\sigma}(w)\cdot\tilde{P}_{\sigma}(w) = \zeta\sigma(w)\cdot P_{\sigma}(w).$$

For (c), suppose that $\tilde{\sigma}$ is neither a geodesic nor asymptotic curve on $\tilde{R}$. Then, from (34), (35) and (38), we obtain

$$\begin{aligned}\tilde{\sigma}(w)\cdot\tilde{P}_\sigma(w)-\zeta\sigma(w)\cdot P_\sigma(w) &= \frac{(a\tilde{E}+b\tilde{F})}{\sqrt{\tilde{E}\tilde{G}-\tilde{F}^2}}\{(\tilde{\alpha}(w)-\alpha(w))(x'\tilde{F}+y'\tilde{G})\\ &+(\frac{\tilde{\beta}(w)\tilde{\kappa}_g(w)}{\tilde{\kappa}(w)}-\frac{\beta(w)\kappa_g(w)}{\kappa(w)})\frac{1}{\sqrt{\tilde{E}\tilde{G}-\tilde{F}^2}}(\tilde{E}\tilde{G}\\ &-\tilde{F}^2)x'\}-\frac{(a\tilde{F}+b\tilde{G})}{\sqrt{\tilde{E}\tilde{G}-\tilde{F}^2}}\{(\tilde{\alpha}(w)-\alpha(w))(x'\tilde{E}+y'\tilde{F})\\ &+(\frac{\tilde{\beta}(w)\tilde{\kappa}_g(w)}{\tilde{\kappa}(w)}-\frac{\beta(w)\kappa_g(w)}{\kappa(w)})\frac{1}{\sqrt{\tilde{E}\tilde{G}-\tilde{F}^2}}(\tilde{F}^2-\tilde{E}\tilde{G})y'\}.\\ \Rightarrow \tilde{\sigma}(w)\cdot\tilde{P}_\sigma(w)-\zeta\sigma(w)\cdot P_\sigma(w) &= (\tilde{\alpha}(w)-\alpha(w))\frac{1}{\sqrt{\tilde{E}\tilde{G}-\tilde{F}^2}}\{(a\tilde{E}+b\tilde{F})(x'\tilde{F}+y'\tilde{G})\\ &-(a\tilde{F}+b\tilde{G})(x'\tilde{E}+y'\tilde{F})\}+(\frac{\tilde{\beta}(w)\tilde{\kappa}_g(w)}{\tilde{\kappa}(w)}\\ &-\frac{\beta(w)\kappa_g(w)}{\kappa(w)})\{(a\tilde{E}+b\tilde{F})x'+(a\tilde{F}+b\tilde{G})y'\}.\end{aligned}$$

Since $\sigma(w)$ and $\tilde{\sigma}(w)$ are osculating curves on R and $\tilde{R}$, respectively, then $\tilde{\alpha}(w)=\alpha(w)$ and $\frac{\tilde{\beta}(w)\tilde{\kappa}_g(w)}{\tilde{\kappa}(w)}=\frac{\beta(w)\kappa_g(w)}{\kappa(w)}$. Hence,

$$\tilde{\sigma}(w)\cdot\tilde{P}_\sigma(w) = \zeta\sigma(w)\cdot P_\sigma(w).$$

□

Theorem 6. *Let $\psi: R\to\tilde{R}$ be a conformal transformation, where R and $\tilde{R}$ are regular surfaces, and let σ be an asymptotic osculating curve on R (i.e., $\kappa_n=0$) and let $\tilde{\sigma}$ be its corresponding osculating curve on the transformed surface $\tilde{R}$. Then, we have the following:*

(a) *If $\tilde{\sigma}$ exhibits an asymptotic character on $\tilde{R}$, then $\tilde{\sigma}\cdot\tilde{P}_\sigma(w)=\zeta\sigma(w)\cdot P_\sigma(w)$.*

(b) *If $\tilde{\sigma}$ does not exhibits an asymptotic curve on $\tilde{R}$, then $\tilde{\sigma}(w)\cdot\tilde{P}_\sigma(w)=\zeta\sigma(w)\cdot P_\sigma(w)$,*

where $T_\sigma(w)=a\theta_x+b\theta_y$ denotes a tangent vector at point $\sigma(w)$ on surface R.

Proof. Refer to Theorem 5 for proof by setting $\kappa_n=0$. □

4. Conclusions

In this paper, we have studied the geometric features of osculating curves with respect to the conformal transformation. We have obtained the specific conditions under which osculating curves remain invariant under such transformations, depending on their properties and whether they are geodesic or asymptotic. We also proved these conditions separately for the cases with zero and non-zero normal curvatures. Furthermore, we have explored the behavior of osculating curves along the unit tangent and unit normal vectors with reference to the conformal transformations between the regular surfaces in Euclidean 3-space. In the future, we will try to obtain the geometric properties of Darboux osculating, Darboux normal, and Darboux rectifying curves together with the recent results in [40–42] and under conformal transformation between surfaces in Euclidean 4-space.

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