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Existence, Nonexistence, and Multiplicity of Positive Solutions for Nonlocal Boundary Value Problems

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Abstract: This study investigates the nonlocal boundary value problem for generalized Laplacian equations involving a singular, possibly non-integrable weight function. By analyzing the asymptotic behaviors of the nonlinearity $f = f(s)$ near both $s = 0$ and $s = \infty$, we establish the existence, nonexistence, and multiplicity of positive solutions for all positive values of the parameter λ. Our proofs employ the fixed-point theorem of cone expansion and compression of norm type, a powerful tool for demonstrating the existence of solutions in cones, as well as the Leray–Schauder fixed-point theorem, which offers an alternative approach for proving the existence of solutions. Illustrative examples are provided to concretely demonstrate the applicability of our main results.

Keywords: generalized Laplacian equations; multiple positive solutions; singular weight function

MSC: 34B08; 34B10; 34B16; 34B18

1. Introduction

We consider the following singular φ-Laplacian problem with integral boundary conditions

$$\begin{cases} (p(t)\varphi(u'(t)))' + \lambda h(t) f(u(t)) = 0, \ \ t \in (0,1), \\ u(0) = \int_0^1 u(\xi) d\beta_1(\xi), u(1) = \int_0^1 u(\xi) d\beta_2(\xi). \end{cases} \tag{1}$$

Here, $\varphi : \mathbb{R} \to \mathbb{R}$ is an odd increasing homeomorphism, $p : [0,1] \to (0,\infty)$ is a continuous function, $\lambda \in \mathbb{R}_+ := [0,\infty)$ is a parameter, $h : (0,1) \to \mathbb{R}_+$ is a continuous function, $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function satisfying $f(s) > 0$ for $s > 0$, and the integrator functions β_i $(i = 1,2)$ are nondecreasing on $[0,1]$.

All integrals in (1) are meant in the sense of Riemann–Stieltjes. Throughout this paper, we assume the following hypotheses.

(K_1) There exist increasing homeomorphisms $\psi_1, \psi_2 : [0,\infty) \to [0,\infty)$ such that

$$\varphi(m)\psi_1(n) \le \varphi(mn) \le \varphi(m)\psi_2(n) \text{ for all } \ x, y \in [0,\infty). \tag{2}$$

(K_2) For $i = 1,2$, $\hat{\beta}_i := \beta_i(1) - \beta_i(0) \in [0,1)$.

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Let $\Psi : [0,\infty) \to [0,\infty)$ be an increasing homeomorphism. Then, we denote by $\mathcal{H}_\Psi$ the set

$$\left\{ J \in C((0,1),\mathbb{R}_+) : \int_0^1 \Psi^{-1}\left(\left|\int_s^{\frac{1}{2}} J(\tau)d\tau\right|\right)ds < \infty \right\}.$$

As is well known, it follows from (K_1) that

$$\varphi^{-1}(m)\psi_2^{-1}(n) \le \varphi^{-1}(mn) \le \varphi^{-1}(m)\psi_1^{-1}(n) \text{ for all } m,n \in \mathbb{R}_+ \tag{3}$$

and

$$L^1(0,1) \cap C(0,1) \subsetneq \mathcal{H}_{\psi_1} \subseteq \mathcal{H}_\varphi \subseteq \mathcal{H}_{\psi_2}$$

(see, e.g., Remark 1 in [1]).

Since Picone's pioneering work [2] in 1908, the study of nonlocal boundary value problems has attracted numerous researchers, who have been actively involved in this area from then until now. Nonlocal boundary value problems appear in various fields of applied mathematics and physics, modeling numerous phenomena across the applied mathematical sciences. These applications include, but are not limited to, beam deflection [3] and thermostatics [4]. A classic example is the vibration of a guy wire with a uniform cross-section composed of N sections of varying densities, which can be formulated as a multi-point boundary value problem [5]. Furthermore, many elastic stability problems also give rise to multi-point boundary value problems [6]. To gain a comprehensive understanding of the historical development of research in this area, we refer the reader to the survey papers [7,8]. Driven by these applications, the existence of positive solutions for nonlocal boundary value problems has been extensively investigated. For instance, Bachouche, Djebali, and Moussaoui [9] employed fixed-point theorems to establish the existence of multiple positive solutions for φ-Laplacian boundary value problems with linear bounded operator conditions under suitable assumptions on the nonlinearity $f = f(t,v,w)$ satisfying the L^1-Carathéodory condition. Goodrich [10] utilized perturbed Volterra integral operator equations to study the existence of positive solutions for the r-Laplacian differential equation with nonlocal boundary conditions. For the nonlinearity $f = f(t,v)$ satisfying $f(t,0) \not\equiv 0$, Kim [11] investigated the existence, nonexistence, and multiplicity of positive solutions to problem (1) by analyzing the unbounded solution continuum. Tariboon, Samadi, and Ntouyas [12] studied the existence and uniqueness of solutions for boundary value problems involving Hilfer generalized proportional fractional differential equations with multi-point boundary conditions, notably pioneering the investigation of such problems with order in (1,2]. For a two-term nonlinear fractional integro-differential equation with nonlocal boundary conditions and variable coefficients, Li [13] established the uniqueness of solutions by utilizing the Mittag–Leffler function, Babenko's method, and the Banach contraction principle.

The existence of positive solutions for the φ-Laplacian problem (1) in the case when either $f_0 = f_\infty = \infty$ or $f_0 = f_\infty = 0$ has been investigated by several authors. Here,

$$f_0 := \lim_{s\to 0} \frac{f(s)}{\varphi(s)} \text{ and } f_\infty := \lim_{s\to\infty} \frac{f(s)}{\varphi(s)}.$$

For instance, Agarwal, Lü, and O'Regan [14] established the existence of two positive solutions to problem (1) with $\varphi(s) = |s|^{r-2}s$ for some $r \in (1,\infty)$, $q \equiv 1$, $h \in \mathcal{H}_\varphi$, and $\hat{\beta}_1 = \hat{\beta}_2 = 0$. Subsequent studies by Wang [15] and Lee and Xu [16] extended the result to more general cases. Recently, Kim obtained the following result for problem (1).

Theorem 1 ([17], Theorem 1). *Assume that* (K_1), (K_2), *and* $h \in \mathcal{H}_{\psi_1} \setminus \{0\}$ *hold.*

(1) *If* $f_\infty = f_0 = 0$, *then there exists* $\lambda_* \in (0, \infty)$ *such that* (1) *has two positive solutions for any* $\lambda \in (\lambda_*, \infty)$.
(2) *If* $f_\infty = f_0 = \infty$, *then there exist* $\lambda^* \in (0, \infty)$ *such that problem* (1) *has two positive solutions for any* $\lambda \in (0, \lambda^*)$.

A limitation of Theorem 1 is its inability to determine whether positive solutions exist for all possible positive values of λ. Specifically, the result does not cover the intervals $[0, \lambda_*]$ or $[\lambda^*, \infty]$. Recently, Jeong and Kim [18] thoroughly analyzed the existence, nonexistence, and multiplicity of positive solutions under the restrictive condition of zero Dirichlet boundary conditions (i.e., $\hat{\beta}_1 = \hat{\beta}_2 = 0$). We aim to extend the result as in [18], as well as those presented in [14–17], by establishing the existence of positive solutions for a wider range of conditions. Specifically, we consider all positive values of the parameter λ and impose less restrictive assumptions on φ, p, h, and/or boundary conditions. Our main theorem is presented below.

Theorem 2. *Assume that* (K_1), (K_2), *and* $h \in \mathcal{H}_{\psi_1} \setminus \{0\}$ *hold.*

(1) *If* $f_\infty = f_0 = 0$, *then there exist* $\lambda_1^* \geq \lambda_*^1 > 0$ *such that problem* (1) *has two positive solutions for* $\lambda > \lambda_1^*$, *one positive solution for* $\lambda \in [\lambda_*^1, \lambda_1^*]$, *and no positive solutions for* $\lambda \in (0, \lambda_*^1)$.

(2) *If* $f_\infty = f_0 = \infty$, *then there exist* $\lambda_2^* \geq \lambda_*^2 > 0$ *such that problem* (1) *has two positive solutions for* $\lambda \in (0, \lambda_*^2)$, *one positive solution for* $\lambda \in [\lambda_*^2, \lambda_2^*]$, *and no positive solutions for* $\lambda > \lambda_2^*$.

In [11], the condition $f(t, v) > 0$ for all $(t, v) \in [0, 1] \times \mathbb{R}_+$ was imposed on the nonlinearity $f = f(t, v)$ to ensure that all non-negative solutions are positive ones. However, in our study, we relax this restriction by allowing $f(v)$ to satisfy $f(0) = 0$, which introduces the possibility of a trivial solution $u \equiv 0$ for every $\lambda \in \mathbb{R}_+$. To address the challenges posed by this relaxed assumption, we employ a combination of the fixed-point theorem of cone expansion and compression of norm type and the Leray–Schauder fixed-point theorem.

The subsequent sections of this paper are organized as follows. In Section 2, we provide a brief overview of existing results, laying the groundwork for the main theorem. In Section 3, we establish key lemmas and present the proof of Theorem 2, accompanied by illustrative examples. Finally, in Section 4, we conclude by summarizing our principal findings and highlighting potential directions for future research.

2. Preliminaries

Throughout this section, we assume that (K_1), (K_2), and $h \in \mathcal{H}_\varphi \setminus \{0\}$ hold. The usual maximum norm in a Banach space $C[0, 1]$ is denoted by

$$\|v\|_\infty := \max\{|v(t)| : t \in [0, 1]\} \text{ for } v \in C[0, 1].$$

Let

$$c_h := \inf\{t \in (0, 1) : h(t) > 0\}, \, d_h := \sup\{t \in (0, 1) : h(t) > 0\},$$

$$\bar{c}_h := \sup\{t \in (0, 1) : h(z) > 0 \text{ for all } z \in (c_h, t)\},$$

$$\bar{d}_h := \inf\{t \in (0, 1) : h(z) > 0 \text{ for all } z \in (t, d_h)\},$$

$$\gamma_h^1 := \frac{1}{4}(3c_h + \bar{c}_h), \, \gamma_h^2 := \frac{1}{4}(\bar{d}_h + 3d_h) \text{ and } \gamma_h^* := \frac{1}{2}(\gamma_h^1 + \gamma_h^2).$$

Then, since $h : (0, 1) \to \mathbb{R}_+$ is a continuous function with $h \not\equiv 0$, we have two cases: either

$$(i)\ 0 \le c_h < \bar{c}_h \le \bar{d}_h < d_h \le 1$$

or

$$(ii)\ 0 \le c_h = \bar{d}_h < d_h \le 1 \text{ and } 0 \le c_h < \bar{c}_h = d_h \le 1.$$

Hence,

$$h(t) > 0 \text{ for } t \in (c_h, \bar{c}_h) \cup (\bar{d}_h, d_h), \text{ and } 0 \le c_h < \gamma_h^1 < \gamma_h^2 < d_h \le 1. \tag{4}$$

Let $p_h := p_1 \min\{\gamma_h^1, 1-\gamma_h^2\} \in (0,1)$. Here,

$$p_0 := \min\{p(t) : t \in [0,1]\} > 0 \text{ and } p_1 := \psi_2^{-1}\left(\frac{1}{\|p\|_\infty}\right)\left[\psi_1^{-1}\left(\frac{1}{p_0}\right)\right]^{-1} \in (0,1].$$

Define $\mathcal{P}$ as the set of all non-negative continuous functions u satisfying

$$u(t) \ge p_h \|u\|_\infty \text{ for } t \in [\gamma_h^1, \gamma_h^2].$$

Then, $\mathcal{P}$ is a cone in $C[0,1]$. For $\alpha > 0$, let

$$\mathcal{P}_\alpha := \{u \in \mathcal{K} \ : \ \|u\|_\infty < \alpha\}, \partial\mathcal{P}_\alpha := \{u \in \mathcal{P} \ : \ \|u\|_\infty = \alpha\},$$

and $\overline{\mathcal{P}}_\alpha := \mathcal{P}_\alpha \cup \partial\mathcal{P}_\alpha$.

For $g \in \mathcal{H}_\varphi$, consider the following problem:

$$\begin{cases} (p(t)\varphi(u'(t)))' + g(t) = 0, \ \ t \in (0,1), \\ u(0) = \int_0^1 u(\xi)d\beta_1(\xi), u(1) = \int_0^1 u(\xi)d\beta_2(\xi). \end{cases} \tag{5}$$

Define a function $S : \mathcal{H}_\varphi \to C[0,1]$ by, for $g \in \mathcal{H}_\varphi$,

$$S(g)(t) = \begin{cases} (1-\hat{\beta}_1)^{-1}\int_0^1 \int_0^\xi I_g(s,\sigma)ds d\beta_1(\xi) + \int_0^t I_g(s,\sigma)ds, & \text{if } 0 \le t \le \sigma, \\ -(1-\hat{\beta}_2)^{-1}\int_0^1 \int_\xi^1 I_g(s,\sigma)ds d\beta_2(\xi) - \int_t^1 I_g(s,\sigma)ds, & \text{if } \sigma \le t \le 1, \end{cases} \tag{6}$$

where

$$I_g(s,x) := \varphi^{-1}\left(\frac{1}{p(s)}\int_s^x g(\varrho)d\varrho\right) \text{ for } s, x \in (0,1)$$

and $\sigma = \sigma(g)$ is a constant satisfying

$$\begin{aligned} & (1-\hat{\beta}_1)^{-1}\int_0^1 \int_0^\xi I_g(s,\sigma)ds d\beta_1(\xi) + \int_0^\sigma I_g(s,\sigma)ds \\ = \ & -(1-\hat{\beta}_2)^{-1}\int_0^1 \int_\xi^1 I_g(s,\sigma)ds d\beta_2(\xi) - \int_\sigma^1 I_g(s,\sigma)ds. \end{aligned} \tag{7}$$

From the definition of I_g, it follows that

$$I_g(s,x) \ge 0 \text{ for } 0 < s \le x < 1 \text{ and } I_g(s,x) \le 0 \text{ for } 0 < x \le s < 1. \tag{8}$$

For any $g \in \mathcal{H}_\varphi$ and any σ satisfying (7), $S(g)$ is monotone increasing on $[0,\sigma)$ and monotone decreasing on $(\sigma,1]$. Note that $\sigma = \sigma(g)$ is not necessarily unique, but $S(g)$ is independent of the choice of σ satisfying (7) (see Remark 2 in [19]).

Lemma 1 (Lemma 2 in [19]). *Assume that* (K_1), (K_2), *and* $g \in \mathcal{H}_\varphi$ *hold. Then,* $S(g)$ *is a unique solution to problem* (5) *satisfying the following properties:*

(1) $S(g)(t) \ge \min\{S(g)(0), S(g)(1)\} \ge 0$ *for* $t \in [0,1]$;
(2) *for any* $g \not\equiv 0$, $\max\{S(g)(0), S(g)(1)\} < \|S(g)\|_\infty$;

(3) σ *is a constant satisfying* (7) *if and only if* $S(g)(\sigma) = \|S(g)\|_\infty$;
(4) $S(g)(t) \geq p_1 \min\{t, 1-t\}\|S(g)\|_\infty$ *for* $t \in [0,1]$ *and* $S(g) \in \mathcal{P}$.

Define a function $G : \mathbb{R}_+ \times \mathcal{P} \to C(0,1)$ by

$$G(\lambda, u)(t) := \lambda h(t) f(u(t)) \text{ for } (\lambda, u) \in \mathbb{R}_+ \times \mathcal{P} \text{ and } t \in (0,1).$$

Clearly, because $h \in \mathcal{H}_\varphi$, $G(\lambda, u) \in \mathcal{H}_\varphi$ for any $(\lambda, u) \in \mathbb{R}_+ \times \mathcal{P}$. Let us define an operator $T : \mathbb{R}_+ \times \mathcal{P} \to \mathcal{P}$ by

$$T(\lambda, u) := S(G(\lambda, u)) \text{ for } (\lambda, u) \in \mathbb{R}_+ \times \mathcal{P}.$$

By Lemma 1 (4), $T(\mathbb{R}_+ \times \mathcal{P}) \subseteq \mathcal{P}$, and consequently, T is well defined.

Remark 1. (1) *Problem* (1) *has a solution if and only if* $T(\lambda, u) = u$ *for some* $(\lambda, u) \in \mathbb{R}_+ \times \mathcal{P}$.

(2) *From* $T(0, u) = 0$ *for any* $u \in \mathcal{P}$, *it follows that* 0 *is a unique solution to problem* (1) *with* $\lambda = 0$.

(3) *For* $(\lambda, u) \in \mathbb{R}_+ \times \mathcal{P}$, *by Lemma 1* (3),

$$\|T(\lambda, u)\|_\infty = T(\lambda, u)(\sigma).$$

(4) *By Lemma 1 (4), if* u *is a nonzero solution to problem* (1) *with* $\lambda > 0$, *then* u *is a positive solution, i.e.,* $u(t) > 0$ *for* $t \in (0,1)$.

Lemma 2 ([19], Lemma 4). *Assume that* (K_1), (K_2), *and* $h \in \mathcal{H}_\varphi \setminus \{0\}$ *hold. Then, the operator* $T : \mathbb{R}_+ \times \mathcal{K} \to \mathcal{K}$ *is completely continuous.*

Finally, we introduce the fixed-point theorem of cone expansion and compression of norm type and the Leray–Schauder fixed-point theorem.

Theorem 3 ([20]). *Let* $(\mathcal{Z}, \|\cdot\|)$ *be a Banach space, and let* $\mathcal{Q}$ *be a cone in* $\mathcal{Z}$. *Assume that* Γ_1 *and* Γ_2 *are open subsets of* $\mathcal{Z}$ *with* $0 \in \Gamma_1$ *and* $\overline{\Gamma}_1 \subset \Gamma_2$. *Let* $H : \mathcal{Q} \cap (\overline{\Gamma}_2 \setminus \Gamma_1) \to \mathcal{Q}$ *be a completely continuous operator such that if either*
$\|Hu\| \leq \|u\|$ *for* $u \in \mathcal{Q} \cap \partial\Gamma_1$ *and* $\|Hu\| \geq \|u\|$ *for* $u \in \mathcal{Q} \cap \partial\Gamma_2$ *or*
$\|Hu\| \geq \|u\|$ *for* $u \in \mathcal{Q} \cap \partial\Gamma_1$ *and* $\|Hu\| \leq \|u\|$ *for* $u \in \mathcal{Q} \cap \partial\Gamma_2$,
then H *has a fixed point in* $\mathcal{Q} \cap (\overline{\Gamma}_2 \setminus \Gamma_1)$.

Theorem 4 ([21]). *Let* X *be a Banach space, and let* $\mathcal{Q}$ *be a closed, convex, and bounded set in* X. *Assume that* $V : \mathcal{Q} \to \mathcal{Q}$ *is completely continuous. Then,* V *has a fixed point in* $\mathcal{Q}$.

3. Proof of Main Results

Lemma 3. *Assume that* (K_1), (K_2), $h \in \mathcal{H}_{\psi_1} \setminus \{0\}$, *and* $f_\infty = f_0 = 0$ *hold. Let* $J = [a, b]$ *be a compact interval with* $0 < a < b$. *Then there exist positive constants* c_J *and* C_J *such that* $c_J \leq \|u\|_\infty \leq C_J$ *for any positive solution* u *to problem* (1) *with* $\lambda \in J$.

Proof. Let $A := (2b+1)^{-1}\psi_1(C_*^{-1}) > 0$. Here, $\beta_* := \max\{(1-\hat{\beta}_1)^{-1}, (1-\hat{\beta}_2)^{-1}\}$ and

$$C_* := \beta_* \max\left\{ \int_0^{\frac{1}{2}} \psi_1^{-1}\left(p_0^{-1} \int_s^{\frac{1}{2}} h(\varrho) d\varrho \right) ds, \ \int_{\frac{1}{2}}^1 \psi_1^{-1}\left(p_0^{-1} \int_{\frac{1}{2}}^s h(\varrho) d\varrho \right) ds \right\} > 0.$$

First, we show the existence of c_I satisfying $\|u\|_\infty \geq c_I > 0$ for any positive solution u to problem (1) with $\lambda \in J$. In contrast, we assume that there exists a sequence $\{(\lambda_n, u_n)\}$ such that u_n is a positive solution to problem (1) with $\lambda = \lambda_n \in J$ and $\|u_n\|_\infty \to 0$ as $n \to \infty$.

Since $f_0 = 0$, there exists $\delta > 0$ such that $f(s) \le A\varphi(s)$ for $s \in [0, \delta]$. Since $\|u_n\|_\infty \to 0$ as $n \to \infty$, there exists $N > 0$ such that $\|u_N\|_\infty < \delta$ and

$$f(u_N(t)) \le A\varphi(u_N(t)) \le A\varphi(\|u_N\|_\infty) \text{ for all } t \in [0,1]. \tag{9}$$

We restrict our attention to the case where $\sigma_N > \frac{1}{2}$, because the case where $\sigma_N \le \frac{1}{2}$ can be treated analogously. From the definition of T and (8), it follows that

$$\begin{aligned}
& T(\lambda_N, u_N)(\sigma_N) \\
= \; & -(1-\hat{\beta}_2)^{-1} \int_0^1 \int_\xi^1 I_{G(\lambda_N,u_N)}(s,\sigma_N) ds d\beta_2(\xi) - \int_{\sigma_N}^1 I_{G(\lambda_N,u_N)}(s,\sigma_N) ds \\
= \; & -(1-\hat{\beta}_2)^{-1} \left[\int_0^1 \int_\xi^1 I_{G(\lambda_N,u_N)}(s,\sigma_N) ds d\beta_2(\xi) + \left(1 - \int_0^1 d\beta_2(r)\right) \int_{\sigma_N}^1 I_{G(\lambda_N,u_N)}(s,\sigma_N) ds \right] \\
= \; & -(1-\hat{\beta}_2)^{-1} \left[\int_0^1 \int_\xi^{\sigma_N} I_{G(\lambda_N,u_N)}(s,\sigma_N) ds d\beta_2(\xi) + \int_{\sigma_N}^1 I_{G(\lambda_N,u_N)}(s,\sigma_N) ds \right] \\
= \; & (1-\hat{\beta}_2)^{-1} \Big[- \int_0^{\sigma_N} \int_\xi^{\sigma_N} I_{G(\lambda_N,u_N)}(s,\sigma_N) ds d\beta_2(\xi) + \int_{\sigma_N}^1 \int_{\sigma_N}^\xi I_{G(\lambda_N,u_N)}(s,\sigma_N) ds d\beta_2(\xi) \\
& \qquad - \int_{\sigma_N}^1 I_{G(\lambda_N,u_N)}(s,\sigma_N) ds \Big] \\
\le \; & -(1-\hat{\beta}_2)^{-1} \int_{\sigma_N}^1 I_{G(\lambda_N,u_N)}(s,\sigma_N) ds \\
= \; & (1-\hat{\beta}_2)^{-1} \int_{\sigma_N}^1 \varphi^{-1}\left(\frac{1}{p(s)} \int_{\sigma_N}^s \lambda_N h(\varrho) f(u_N(\varrho)) d\varrho \right) ds.
\end{aligned} \tag{10}$$

Then, by (3) and (9) and the definition of A,

$$\begin{aligned}
\|u_N\|_\infty &= T(\lambda_N, u_N)(\sigma_N) \\
&\le \beta_* \int_{\sigma_N}^1 \varphi^{-1}\left(\frac{1}{p(s)} \int_{\sigma_N}^s \lambda_N h(\varrho) f(u_N(\varrho)) d\varrho \right) ds \\
&\le \beta_* \int_{\frac{1}{2}}^1 \varphi^{-1}\left(p_0^{-1} \int_s^{\frac{1}{2}} h(\varrho) d\varrho A\varphi(\|u_N\|_\infty) \right) ds \\
&\le \beta_* \int_{\frac{1}{2}}^1 \psi_1^{-1}\left(p_0^{-1} \int_s^{\frac{1}{2}} h(\varrho) d\varrho \right) ds \varphi^{-1}(A\varphi(\|u_N\|_\infty)) \\
&\le C_* \psi_1^{-1}(A) \|u_N\|_\infty < \|u_N\|_\infty.
\end{aligned}$$

This is a contradiction, and thus there exists $c_J > 0$ such that $\|u\|_\infty \ge c_J > 0$ for any positive solution u to problem (1) with $\lambda \in J$.

Next, we show the existence of C_J satisfying $\|u\|_\infty \le C_J$ for any positive solution u to problem (1) with $\lambda \in J$. It follows from $f_\infty = 0$ that there exists $s_A > 0$ such that $f(s) \le A\varphi(s)$ for $s \in [s_A, \infty)$. Let $C_A = \max\{f(s) : s \in [0, s_A]\} > 0$. Then,

$$f(s) \le C_A + A\varphi(s) \text{ for } s \in \mathbb{R}_+. \tag{11}$$

In contrast, we assume that there exists a sequence $\{(\lambda_n, u_n)\}$ such that u_n is a positive solution to problem (1) with $\lambda = \lambda_n \in J$ and $\|u_n\|_\infty \to \infty$ as $n \to \infty$. Then, for sufficiently large $N > 0$, $C_A \le A\varphi(\|u_N\|_\infty)$, and by (11),

$$f(u_N(t)) \le 2A\varphi(\|u_N\|_\infty) \text{ for } t \in [0,1]. \tag{12}$$

Let σ_N denote a positive real number such that $\|u_N\|_\infty = u_N(\sigma_N)$. We restrict our attention to the case where $\sigma_N \le \frac{1}{2}$, because the case where $\sigma_N > \frac{1}{2}$ can be treated analogously. From (8), it follows that

$$\begin{aligned}
& T(\lambda_N, u_N)(\sigma_N) \\
= \;& (1-\hat{\beta}_1)^{-1} \int_0^1 \int_0^\xi I_{G(\lambda_N,u_N)}(s,\sigma_N) ds d\beta_1(\xi) + \int_0^{\sigma_N} I_{G(\lambda_N,u_N)}(s,\sigma_N) ds \\
= \;& (1-\hat{\beta}_1)^{-1} \Big[\int_0^1 \int_0^\xi I_{G(\lambda_N,u_N)}(s,\sigma_N) ds d\beta_1(\xi) + \Big(1 - \int_0^1 d\beta_1(r)\Big) \int_0^{\sigma_N} I_{G(\lambda_N,u_N)}(s,\sigma_N) ds \Big] \\
= \;& (1-\hat{\beta}_1)^{-1} \Big[\int_0^1 \int_{\sigma_N}^\xi I_{G(\lambda_N,u_N)}(s,\sigma_N) ds d\beta_1(\xi) + \int_0^{\sigma_N} I_{G(\lambda_N,u_N)}(s,\sigma_N) ds \Big] \\
= \;& (1-\hat{\beta}_1)^{-1} \Big[- \int_0^{\sigma_N} \int_\xi^{\sigma_N} I_{G(\lambda_N,u_N)}(s,\sigma_N) ds d\beta_1(\xi) + \int_{\sigma_N}^1 \int_{\sigma_N}^\xi I_{G(\lambda_N,u_N)}(s,\sigma_N) ds d\beta_1(\xi) \\
& \quad + \int_0^{\sigma_N} I_{G(\lambda_N,u_N)}(s,\sigma_N) ds \Big] \\
\le \;& \beta_* \int_0^{\sigma_N} I_{G(\lambda_N,u_N)}(s,\sigma_N) ds. \qquad (13)
\end{aligned}$$

Then, by (3) and (12) and the definition of A,

$$\begin{aligned}
\|u_N\|_\infty &= T(\lambda_N, u_N)(\sigma_N) \\
&\le \beta_* \int_0^{\sigma_N} \varphi^{-1}\Big(\frac{1}{p(s)} \int_s^{\sigma_N} \lambda_N h(\varrho) f(u_N(\varrho)) d\varrho \Big) ds \\
&\le \beta_* \int_0^{\frac{1}{2}} \varphi^{-1}\Big(p_0^{-1} \int_s^{\frac{1}{2}} h(\varrho) d\varrho 2bA\varphi(\|u_N\|_\infty) \Big) ds \\
&\le \beta_* \int_0^{\frac{1}{2}} \psi_1^{-1}\Big(p_0^{-1} \int_s^{\frac{1}{2}} h(\varrho) d\varrho \Big) ds \varphi^{-1}(2bA\varphi(\|u_N\|_\infty)) \\
&\le C_* \psi_1^{-1}(2bA)\|u_N\|_\infty < \|u_N\|_\infty.
\end{aligned}$$

This is a contradiction, and thus there exists $C_J > 0$ such that $\|u\|_\infty \le C_J$ for any positive solution u to problem (1) with $\lambda \in J$. □

Lemma 4. *Assume that* (K_1), (K_2), $h \in \mathcal{H}_{\psi_1} \setminus \{0\}$, *and* $f_\infty = f_0 = 0$ *hold. Then, there exists* $\underline{\lambda} > 0$ *such that problem* (1) *has no positive solutions for* $\lambda \in [0, \underline{\lambda})$.

Proof. Let λ be a positive constant satisfying that problem (1) has a positive solution u_λ, and let σ be a constant satisfying $u_\lambda(\sigma) = \|u_\lambda\|_\infty$. Since $f_0 = f_\infty = 0$, there exists $C_0 > 0$ such that $f(s) \le C_0\varphi(s)$ for $s \in \mathbb{R}_+$, and

$$f(u_\lambda(\varrho)) \le C_0\varphi(u_\lambda(\varrho)) \le C_0\varphi(\|u_\lambda\|_\infty) \text{ for all } \varrho \in [0,1]. \qquad (14)$$

Let

$$h_\gamma := \max\Big\{ \int_0^{\gamma_h} \psi_1^{-1}\big(\int_s^{\gamma_h} h(\varrho)d\varrho\big)ds, \int_{\gamma_h}^1 \psi_1^{-1}\Big(\int_{\gamma_h}^s h(\varrho)d\varrho\Big)ds \Big\} > 0.$$

We only consider the case $\sigma \le \gamma_h$, since the case $\sigma > \gamma_h$ can be proved similarly. By the similar argument as in (3), (13), and (14),

$$\begin{aligned}
\|u_\lambda\|_\infty &= u_\lambda(\sigma) \\
&\le \beta_* \int_0^{\sigma} \varphi^{-1}\Big(\frac{1}{p(s)} \int_s^{\sigma} \lambda h(\varrho) f(u_\lambda(\varrho)) d\varrho \Big) ds
\end{aligned}$$

$$\begin{aligned} &\leq \beta_* \int_0^{\gamma_h} \varphi^{-1}\left(\int_s^{\gamma} h(\varrho) d\varrho p_0^{-1} \lambda C_0 \varphi(\|u_\lambda\|_\infty)\right) ds \\ &\leq \beta_* h_\gamma \varphi^{-1}(p_0^{-1}\lambda C_0 \varphi(\|u_\lambda\|_\infty)) \\ &\leq \beta_* h_\gamma \psi_1^{-1}(p_0^{-1}\lambda C_0)\|u_\lambda\|_\infty. \end{aligned}$$

Consequently,

$$\lambda \geq p_0 C_0^{-1} \psi_1(\beta_*^{-1} h_\gamma^{-1}) =: \underline{\lambda},$$

and problem (1) has no positive solutions for $\lambda \in (0, \underline{\lambda})$. □

Lemma 5. *Assume that* (K_1), (K_2), $h \in \mathcal{H}_{\psi_1} \setminus \{0\}$, *and* $f_\infty = 0$ *hold. If* (1) *has a positive solution at* $\lambda = \lambda_1$, *then* (1) *has at least one positive solution for all* $\lambda \in [\lambda_1, \infty)$.

Proof. Let u_1 be a positive solution to problem (1) with $\lambda = \lambda_1$ and let $\lambda \in (\lambda_1, \infty)$ be fixed. Consider the following modified problem

$$\begin{cases} (p(t)\varphi(u'(t)))' + \lambda h(t) f(\gamma_1(t, u(t))) = 0, \ \ t \in (0,1), \\ u(0) = \int_0^1 u(\xi) d\beta_1(\xi), u(1) = \int_0^1 u(\xi) d\beta_2(\xi), \end{cases} \tag{15}$$

where $\gamma_1 : [0,1] \times \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function defined by, for $(t,s) \in [0,1] \times \mathbb{R}_+$,

$$\gamma_1(t,s) = \begin{cases} s, & \text{if } \ s \geq u_1(t), \\ u_1(t), & \text{if } \ 0 \leq s < u_1(t). \end{cases}$$

Define $H_1 : \mathcal{P} \to \mathcal{P}$ by $H_1(u) = S(G_1(u))$ for $u \in \mathcal{P}$, where $G_1(u)(t) = \lambda h(t) f(\gamma_1(t, u(t)))$ for $u \in \mathcal{P}$ and $t \in (0,1)$. Since $G_1(u) \in \mathcal{H}_{\psi_1}$ for any $u \in \mathcal{P}$, by Lemma 1, H_1 is well defined. It is easy to see that H_1 is completely continuous on $\mathcal{K}$, and u is a solution to problem (15) if and only if $u = H_1 u$.

First, we show the existence of a solution to problem (15).

(i) Assume that f is bounded on $\mathbb{R}_+$. From the definition of γ_1 and the continuity of f, it follows that there exists $r > 0$ such that $||H_1(u)||_\infty < r$ for all $u \in \mathcal{P}$, and $H_1(\mathcal{P}_r) \subseteq \mathcal{P}_r$. Then, by Theorem 4, there exists $u \in \mathcal{P}_r$ such that $H_1(u) = u$, and consequently, problem (15) has a non-negative solution u.

(ii) Assume that f is unbounded on $\mathbb{R}_+$. Let $\delta \in (0, \lambda^{-1}\psi_1(C_*^{-1}))$ be given. Here, C_* is the constant in the proof of Lemma 3. Since $f_\infty = 0$, there exists $s_1 > 0$ such that

$$f(s) \leq \delta \varphi(s) \text{ for all } (t,s) \in [0,1] \times [s_1, \infty). \tag{16}$$

Since f is unbounded on $\mathbb{R}_+$ and $\gamma_1(t,s) = s$ for $(t,s) \in [0,1] \times [\|u_1\|_\infty, \infty)$, there exists $s_2 > 0$ such that $s_2 > s_1$ and

$$f(\gamma_1(t,s)) \leq f(s_2) \ \text{ for all } (t,s) \in [0,1] \times [0, s_2]. \tag{17}$$

Let $u \in \mathcal{K}_{s_2}$ be given. Then, by (16) and (17),

$$f(\gamma_1(t, u(\varrho))) \leq f(s_2) \leq \delta \varphi(s_2) \text{ for all } \varrho \in [0,1]. \tag{18}$$

Let σ denote a positive constant satisfying $\|H_1(u)\|_\infty = H_1(u)(\sigma)$. We restrict our attention to the case where $\sigma > \frac{1}{2}$, since the case where $\sigma \leq \frac{1}{2}$ can be treated analogously. Then, by the similar argument as in (10),

$$\|H_1(u)\|_\infty \leq (1 - \hat{\beta}_2)^{-1} \int_\sigma^1 \varphi^{-1}\left(\frac{1}{p(s)} \int_\sigma^s \lambda h(\varrho) f(\gamma_1(\varrho, u(\varrho))) d\varrho\right) ds.$$

From (3) and (18) and the choice of δ, it follows that

$$\begin{aligned}\|H_1(u)\|_\infty &\leq \beta_* \int_{\frac{1}{2}}^{1} \varphi^{-1}\left(p_0^{-1}\int_{\frac{1}{2}}^{s} h(\varrho)d\varrho[\delta\lambda\varphi(s_2)]\right)ds\\ &\leq C_*\varphi^{-1}(\epsilon\lambda\varphi(s_2)) \leq C_*\psi_1^{-1}(\delta\lambda)s_2 < s_2.\end{aligned}$$

By Theorem 4, there exists $u \in \mathcal{P}_{s_2}$ such that $H_1(u) = u$, and consequently, problem (15) has a non-negative solution u.

Finally, we prove that if u is a non-egative solution to problem (15), then $u(t) \geq u_1(t)$ for $t \in [0,1]$. If it is true, by the definition of γ_1, we can conclude that problem (1) has a positive solution u for all $\lambda \in [\lambda_1, \infty)$, and thus, the proof is complete.

Assume, on the contrary, that there exists a solution u to problem (15) such that $u(t) \ngeq u_1(t)$ for $t \in [0,1]$. Let $x(t) = u_1(t) - u(t)$ for $t \in [0,1]$. We claim that $x(0) \leq 0$. If not, $x(0) > 0$ and there exists $t^* \in [0,1)$ such that $x(t^*) = \max\{x(t) : t \in [0,1]\} > 0$. Then,

$$0 < x(0) = \int_0^1 x(\xi)d\beta_1(\xi) \leq x(t^*)\hat{\beta}_1 < x(t^*) \text{ and } x(1) = \int_0^1 x(\xi)d\beta_2(\xi) < x(t^*),$$

which imply that $t^* \in (0,1)$ and there exists $t_* \in [0,t^*)$ such that

$$0 < x(t_*) < x(t) \leq x(t^*) \text{ for } t \in (t_*, t^*] \text{ and } x'(t^*) = 0.$$

Since $\gamma_1(t,u(t)) = u_1(t)$ for $t \in [t_*, t^*]$,

$$-(p(t)\varphi(u'(t)))' = \lambda h(t)f(u_1(t)) \geq \lambda_1 h(t)f(u_1(t)) = -(p(t)\varphi(u_1'(t)))' \text{ for } t \in [t_*, t^*]. \tag{19}$$

For $t \in (t_*, t^*)$, integrating (19) from t to t^*, $p(t)\varphi(u'(t)) \geq p(t)\varphi(u_1'(t))$. Since $p(t) > 0$ for all $t \in [0,1]$ and φ is increasing on $\mathbb{R}$,

$$u'(t) \geq u_1'(t) \text{ for } t \in (t_*, t^*). \tag{20}$$

Integrating (20) from t_* to t^*, $x(t^*) = u_1(t^*) - u(t^*) \leq u_1(t_*) - u(t_*) = x(t_*)$, which contradicts the choice of t^* and t_*. Consequently, $x(0) \leq 0$. Similarly, we can show that $x(1) \leq 0$.

Since $x(0) = u_1(0) - u(0) \leq 0$, $x(1) = u_1(1) - u(0) \leq 0$ and $x(t) = u_1(t) - u(t) \ngeq 0$ for $t \in [0,1]$, there exists a subinterval $(t_1, t_2) \subseteq (0,1)$ such that $x(t) = u_1(t) - u(t) > 0$ for $t \in (t_1, t_2)$ and $u_1(t_1) - u(t_1) = u_1(t_2) - u(t_2) = 0$. From the fact $u_1 - u \in C[0,1]$, it follows that there exists $\bar{t} \in (t_1, t_2)$ such that $u_1(\bar{t}) - u(\bar{t}) = \max\{u_1(t) - u(t) : t \in [t_1, t_2]\} > 0$ and $u_1'(\bar{t}) = u'(\bar{t})$. For $t \in (t_1, t_2)$,

$$-(p(t)\varphi(u'(t)))' = \lambda h(t)f(\gamma_1(t,u(t))) = \lambda h(t)f(u_1(t)) \geq \lambda_1 h(t)f(u_1(t)) = -(p(t)\varphi(u_1'(t)))',$$

i.e.,

$$-(p(t)\varphi(u'(t)))' \geq -(p(t)\varphi(u_1'(t)))' \text{ for } t \in (t_1, t_2). \tag{21}$$

For $t \in (t_1, \bar{t})$, integrating (21) from t to $\bar{t}$, $p(t)\varphi(u'(t)) \geq p(t)\varphi(u_1'(t))$. Since $p(t) > 0$ for all $t \in [0,1]$ and φ is increasing on $\mathbb{R}$,

$$u'(t) \geq u_1'(t) \text{ for } t \in (t_1, \bar{t}). \tag{22}$$

Integrating (22) from t_1 to $\bar{t}$, $u_1(\bar{t}) - u(\bar{t}) \leq 0$, which contradicts the choice of $\bar{t}$. Thus, the proof is complete. □

Lemma 6. *Assume that* (K_1), (K_2), $h \in \mathcal{H}_{\psi_1} \setminus \{0\}$, *and* $f_0 = f_\infty = \infty$ *hold. Let* $J = [a,b]$ *be a compact interval with* $0 < a < b$. *Then, there exist positive constants* c_J *and* C_J *such that* $c_J \leq \|u\|_\infty \leq C_J$ *for any positive solution* u *to problem* (1) *with* $\lambda \in J$.

Proof. Let $C^* \in (\psi_2(B^{-1}), \infty)$ be given. Here,

$$B = \min\left\{\gamma_h^1 \psi_2^{-1}\left(a\|p\|_\infty^{-1}\int_{\gamma_h^1}^{\gamma_h^*} h(\varrho)d\varrho\right), (1-\gamma_h^2)\psi_2^{-1}\left(a\|p\|_\infty^{-1}\int_{\gamma_h^*}^{\gamma_h^2} h(\varrho)d\varrho\right)\right\}.$$

Recall $\gamma_h^* = 2^{-1}(\gamma_h^1 + \gamma_h^2) > 0$ and note that $B > 0$ by (4).

First, we show the existence of c_J satisfying $\|u\|_\infty \geq c_J > 0$ for any positive solution u to problem (1) with $\lambda \in J$. By contradiction, we assume that there exists a sequence $\{(\lambda_n, u_n)\}$ such that u_n is a positive solution to problem (1) with $\lambda = \lambda_n \in J$ and $\|u_n\|_\infty \to 0$ as $n \to \infty$. Since $f_0 = \infty$, there exists $s_0 > 0$ such that $f(s) \geq C^*\varphi(s)$ for $s \in [0, s_0]$. Since $\|u_n\|_\infty \to 0$ as $n \to \infty$, there exists $n > 0$ such that $\|u_n\|_\infty < s_0$ and

$$f(u_n(t)) \geq C^*\varphi(u_n(t)) \text{ for all } t \in [0,1]. \tag{23}$$

We restrict our attention to the case where $\sigma_n < \gamma_h^*$ because the case where $\sigma_n \geq \gamma_h^*$ can be treated analogously. Since $u_n(t) \geq u_n(\gamma_h^2)$ for $t \in [\sigma_n, \gamma_h^2]$, by (23),

$$\lambda_n h(\varrho) f(u_n(\varrho)) \geq aC^* h(\varrho)\varphi(u_n(\gamma_h^2)) \text{ for all } \varrho \in [\sigma_n, \gamma_h^2]. \tag{24}$$

By Lemma 1 (1) , $u_n(1) = T(\lambda_n, u_n)(1) \geq 0$, and thus, by (3) and (24) and the definition of C^*,

$$\begin{aligned}
u_n(\gamma_h^2) &= u_n(1) - \int_{\gamma_h^2}^1 \varphi^{-1}\left(\frac{1}{p(s)}\int_s^{\sigma_N} \lambda_n h(\varrho) f(u_n(\varrho))d\varrho\right)ds \\
&\geq \int_{\gamma_h^2}^1 \varphi^{-1}\left(\frac{1}{p(s)}\int_{\sigma_n}^{s} \lambda_n h(\varrho) f(u_n(\varrho))d\varrho\right)ds \\
&\geq \int_{\gamma_h^2}^1 \varphi^{-1}\left(a\|p\|_\infty^{-1}\int_{\gamma_h^*}^{\gamma_h^2} h(\varrho)d\varrho C^*\varphi(u_n(\gamma_h^2))\right)ds \\
&\geq (1-\gamma_h^2)\psi_2^{-1}\left(a\|p\|_\infty^{-1}\int_{\gamma_h^*}^{\gamma_h^2} h(\varrho)d\varrho\right)\varphi^{-1}\left(C^*\varphi(u_n(\gamma_h^2))\right) \\
&\geq B\psi_2^{-1}(C^*)u_n(\gamma_h^2) > u_n(\gamma_h^2),
\end{aligned}$$

which is a contradiction. Thus, there exists $c_J > 0$ such that $\|u\|_\infty \geq c_J > 0$ for any positive solution u to problem (1) with $\lambda \in J$.

Next, we show the existence of C_J satisfying $\|u\|_\infty \leq C_J$ for any positive solution u to problem (1) with $\lambda \in J$. By contradiction, we assume that there exists a sequence $\{(\lambda_n, u_n)\}$ such that u_n is a positive solution to problem (1) with $\lambda = \lambda_n \in J$ and $||u_n||_\infty \to \infty$ as $n \to \infty$.

By $f_\infty = \infty$, there exists $L > 0$ such that $f(s) \geq C^*\varphi(s)$ for $s > L$. For all n, $u_n \in \mathcal{P}$, and $u_n(t) \geq p_h\|u_n\|_\infty$ for $t \in [\gamma_h^1, \gamma_h^2]$. For sufficiently large $n > 0$, $\|u_n\|_\infty > p_h^{-1}L$ and $u_n(t) \geq L$ for $t \in [\gamma_h^1, \gamma_h^2]$. Thus,

$$f(u_n(t)) \geq C^*\varphi(u_n(t)) \text{ for all } t \in [\gamma_h^1, \gamma_h^2].$$

By the same reasoning as above, we can easily see that the choice of C^* leads a contradiction. Thus, there exists $C_J > 0$ such that $\|u\|_\infty \leq C_J$ for any positive solution u to problem (1) with $\lambda \in J$. □

Lemma 7. *Assume that* (K_1), (K_2), $h \in \mathcal{H}_{\psi_1} \setminus \{0\}$, *and* $f_\infty = f_0 = \infty$ *hold. Then, there exists* $\overline{\lambda} > 0$ *such that problem* (1) *has no positive solutions for* $\lambda \in (\overline{\lambda}, \infty)$.

Proof. Let λ be a positive constant satisfying that problem (1) has a positive solution u_λ, and let σ be a constant satisfying $u_\lambda(\sigma) = \|u_\lambda\|_\infty$. Since $f_0 = f_\infty = \infty$, there exists $C > 0$ such that $f(s) > C\varphi(s)$ for $s \in \mathbb{R}_+$. We only consider the case $\sigma \geq \gamma_h$, since the case $\sigma < \gamma_h$ can be proved similarly. Since $u_\lambda(t) \geq u_\lambda(\gamma_h^1)$ for $t \in [\gamma_h^1, \sigma]$,

$$f(u_\lambda(\varrho)) > C\varphi(u_\lambda(\gamma_h^1)) \text{ for } \varrho \in [\gamma_h^1, \gamma].$$

Then, by (3),

$$\begin{aligned} u_\lambda(\gamma_h^1) &\geq u_\lambda(0) + \int_0^{\gamma_h^1} \varphi^{-1}\left(\frac{1}{p(s)}\int_s^\sigma \lambda h(\varrho) f(u_\lambda(\varrho))d\varrho\right)ds \\ &\geq \int_0^{\gamma_h^1} \varphi^{-1}\left(\int_{\gamma_h^1}^{\gamma_h} h(\varrho)d\varrho \|p\|_\infty^{-1}\lambda C\varphi(u_\lambda(\gamma_h^1))\right)ds \\ &\geq \gamma_0\varphi^{-1}(h_0\|p\|_\infty^{-1}\lambda C\varphi(u_\lambda(\gamma_h^1))) \geq \gamma_0\psi_2^{-1}(h_0\|p\|_\infty^{-1}\lambda C)u_\lambda(\gamma_h^1). \end{aligned}$$

Here,

$$\gamma_0 := \min\{\gamma_h^1, 1-\gamma_h^2\} > 0 \text{ and } h_0 := \min\left\{\int_{\gamma_h^1}^{\gamma_h} h(\varrho)d\varrho, \int_{\gamma_h}^{\gamma_h^2} h(\varrho)d\varrho\right\} > 0.$$

Consequently,

$$\lambda \leq \|p\|_\infty (h_0 C)^{-1}\psi_2(\gamma_0^{-1}) =: \overline{\lambda},$$

and we can conclude that problem (1) has no positive solutions for $\lambda \in (\overline{\lambda}, \infty)$. □

Lemma 8. *Assume that* (K_1), (K_2), $h \in \mathcal{H}_{\psi_1} \setminus \{0\}$, *and* $f_0 = \infty$ *hold. If* (1) *has a positive solution at* $\lambda = \lambda_2$, *then* (1) *has at least one positive solution for all* $\lambda \in (0, \lambda_2]$.

Proof. Let u_2 be a positive solution to problem (1) with $\lambda = \lambda_2$ and let $\lambda \in (0, \lambda_2)$ be fixed. Consider the following modified problem

$$\begin{cases} (p(t)\varphi(u'(t)))' + \lambda h(t) f(\gamma_2(t, u(t))) = 0, \ \ t \in (0,1), \\ u(0) = \int_0^1 u(\xi)d\beta_1(\xi), u(1) = \int_0^1 u(\xi)d\beta_2(\xi), \end{cases} \tag{25}$$

where $\gamma_2 : [0,1] \times \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function defined by, for $(t,s) \in [0,1] \times \mathbb{R}_+$,

$$\gamma_2(t,s) = \begin{cases} u_2(t), & \text{if } s > u_2(t), \\ s, & \text{if } 0 \leq s \leq u_2(t). \end{cases}$$

Define $H_2 : \mathcal{P} \to \mathcal{P}$ by $H_2(u) = S(G_2(u))$ for $u \in \mathcal{P}$, where $G_2(u)(t) = \lambda h(t) f(\gamma_2(t, u(t)))$ for $u \in \mathcal{P}$ and $t \in (0,1)$. Since $G_2(u) \in \mathcal{H}_{\psi_1}$ for any $u \in \mathcal{P}$, by Lemma 1, H_2 is well defined. It is easy to see that H_2 is completely continuous on $\mathcal{P}$, and u is a solution to problem (25) if and only if $u = H_2 u$. By the definition of γ_2, $f(\gamma_2(t,s)) \leq \max\{f(s) : 0 \leq s \leq \|u_2\|_\infty\} \in (0, \infty)$ for all $(t,s) \in [0,1] \times \mathbb{R}_+$ and there exists $s_1 > 0$ such that $||H_2(v)||_\infty < s_1$ for all $v \in \mathcal{P}$, which implies

$$\|H_2(v)\|_\infty \leq \|v\|_\infty \text{ for } v \in \partial\mathcal{P}_{s_1}. \tag{26}$$

Let $u_2^* = \min\{u_2(t) : t \in [\gamma_h^1, \gamma_h^2]\} > 0$. For $(t,s) \in [\gamma_h^1, \gamma_h^2] \times [0, u_2^*]$, $\gamma_2(t,s) = s$, and

$$\lim_{s \to 0^+} \frac{\min\{f(\gamma_2(t,s)) : t \in [\gamma_h^1, \gamma_h^2]\}}{\varphi(s)} = \lim_{s \to 0^+} \frac{f(s)}{\varphi(s)} = f_0 = \infty.$$

Let

$$\Lambda = \frac{\|p\|_\infty}{\lambda} \max\left\{ \psi_2\left(\left[p_h \int_{\gamma_h^1}^{\gamma_h^*} \psi_2^{-1}\left(\int_s^{\gamma_h^*} h(\varrho)d\varrho\right)ds\right]^{-1}\right), \psi_2\left(\left[p_h \int_{\gamma_h^*}^{\gamma_h^2} \psi_2^{-1}\left(\int_{\gamma_h^*}^{s} h(\varrho)d\varrho\right)ds\right]^{-1}\right)\right\}$$

be fixed. Then, there exists $s_* \in (0, u_2^*)$ such that

$$f(\gamma_2(t,s)) \geq \Lambda\varphi(s) \text{ for all } (t,s) \in [\gamma_h^1, \gamma_h^2] \times (0, s_*).$$

Take $s_2 \in (0, \min\{s_1, s_*\})$, and let $v \in \partial\mathcal{P}_{s_2}$ be given. Then $0 \leq v(t) \leq s_2 < s_*$ and

$$f(\gamma_2(t, v(t))) \geq \Lambda\varphi(v(t)) \geq \Lambda\varphi(p_h\|v\|_\infty) \text{ for all } t \in [\gamma_h^1, \gamma_h^2]. \tag{27}$$

Let σ denote a positive real number such that $H_2(v)(\sigma) = \|H_2(v)\|_\infty$. We have two cases: either $\sigma \geq \gamma_h^*$ or $\sigma < \gamma_h^*$. We restrict our attention to the case where $\sigma \geq \gamma_h^*$, since the case where $\sigma < \gamma_h^*$ can be treated similarly. By (3) and (27), it follows from $(H_2(v))(0) \geq 0$ that

$$\begin{aligned}
\|H_2(v)\|_\infty &\geq \int_0^\sigma \varphi^{-1}\left(\frac{1}{p(s)}\int_s^\sigma \lambda h(\varrho) f(\gamma_2(\varrho, v(\varrho)))d\varrho\right)ds \\
&\geq \int_{\gamma_h^1}^{\gamma_h^*} \varphi^{-1}\left(\|p\|_\infty^{-1}\lambda\Lambda\varphi(p_h\|v\|_\infty)\int_s^{\gamma_h^*} h(\varrho)d\varrho\right)ds \\
&\geq \int_{\gamma_h^1}^{\gamma_h^*} \psi_2^{-1}\left(\int_s^{\gamma_h} h(\varrho)d\varrho\right)ds\varphi^{-1}(\|p\|_\infty^{-1}\lambda\Lambda\varphi(p_h\|v\|_\infty)) \\
&\geq \int_{\gamma_h^1}^{\gamma_h^*} \psi_2^{-1}\left(\int_s^{\gamma_h} h(\varrho)d\varrho\right)ds\psi_2^{-1}(\|p\|_\infty^{-1}\lambda\Lambda)p_h\|v\|_\infty,
\end{aligned}$$

which implies, by the choice of Λ,

$$\|H_2(v)\|_\infty \geq \|v\|_\infty \text{ for } v \in \partial\mathcal{P}_{s_2}. \tag{28}$$

By (26) and (28), in view of Theorem 3, problem (25) has a non-negative solution $u \in \mathcal{P}$ with $\|s_2\|_\infty \leq \|u\|_\infty \leq \|s_1\|_\infty$. By Lemma 1 (4), u is a positive solution to problem (25).

By similar reasoning as that in the proof of Lemma 5, we can show that if u is a positive solution to problem (25), then $u(t) \leq u_2(t)$ for $t \in [0,1]$. Thus, by the definition of γ_2, we can conclude that problem (1) has a positive solution u for all $\lambda \in (0, \lambda_2]$. □

For the sake of completeness, we provide a proof of Theorem 2, which follows similar arguments to those given for Theorem 2 in [18].

Proof of Theorem 2. (1) Let $\lambda_1^* := \inf\{\nu :$ problem (1) has at least two positive solutions for $\lambda > \nu\}$ and $\lambda_*^1 := \inf\{\lambda :$ problem (1) has at least one positive solution $\}$. By Theorem 1 (1) and Lemma 4, λ_1^* and λ_*^1 are well defined and $\lambda_1^* \geq \lambda_*^1 \geq \underline{\lambda} > 0$. From Lemma 5, it follows that problem (1) has two positive solutions for $\lambda > \lambda_1^*$, one positive solution for $\lambda > \lambda_*^1$, and no positive solutions for $\lambda \in (0, \lambda_*^1)$. To complete the proof, it is enough to show that problem (1) has a positive solution for $\lambda = \lambda_*^1$. By the definition of λ_*^1, there exists a sequence $\{(\lambda_n, u_n)\}$ such that $\underline{\lambda} \leq \lambda_*^1 < \lambda_n \leq \lambda_*^1 + n^{-1}$ and u_n is a positive solution to problem (1) with $\lambda = \lambda_n$. Then, $\lambda_n \to \lambda_*^1$ as $n \to \infty$, and by Lemma 3, there exist positive constants c and C such that $c \leq \|u_n\|_\infty \leq C$ for all n. Since $\{u_n\}$ is bounded and $T = T(\lambda, u)$ is compact, there exist a subsequence $\{T(\lambda_{n_k}, u_{n_k})\}$ of $\{T(\lambda_n, u_n)\}$ and $u_* \in \mathcal{P}$ such that $T(\lambda_{n_k}, u_{n_k}) \to u_*$ as $n_k \to \infty$. Since $T(\lambda_n, u_n) = u_n$ for all n, $u_{n_k} \to u_*$ as $n_k \to \infty$. In view of the continuity of T,

$$u_* = \lim_{n_k \to \infty} u_{n_k} = \lim_{n_k \to \infty} T(\lambda_{n_k}, u_{n_k}) = T(\lambda_*, u_*).$$

Because $\|u_n\|_\infty \geq c > 0$, for all n, $u_* \not\equiv 0$. Thus, by Lemma 1 (4), (1) has a positive solution u_* for $\lambda = \lambda_*^1$.

(2) Let $\lambda_*^2 := \sup\{\nu :$ problem (1) has at least two positive solutions for $\lambda \in (0, \nu)\}$ and $\lambda_2^* := \sup\{\lambda :$ problem (1) has at least one positive solution $\}$. By Theorem 1 (2) and Lemma 7, λ_2^* and λ_*^2 are well defined and $\lambda^* \geq \lambda_* \geq \bar{\lambda} > 0$. From Lemma 8, it follows that problem (1) has two positive solutions for $\lambda \in (0, \lambda_*^2)$, one positive solution for $\lambda \in (0, \lambda_2^*)$, and no positive solutions for $\lambda > \lambda_2^*$. To complete the proof, it is enough to show that problem (1) has a positive solution for $\lambda = \lambda_2^*$. By the same reasoning as that in the proof of Theorem 2 (1), we can prove it, and thus, the proof is complete. □

We conclude this section by presenting examples to illustrate Theorem 2.

Example 1. *Consider the following problem*

$$\begin{cases} \left(\dfrac{1}{1+t^4}\varphi(u'(t))\right)' + \lambda h(t) f(u(t)) = 0, \ \ t \in (0,1), \\ u(0) = \int_0^1 u(\xi) d\beta_1(\xi), u(1) = \int_0^1 u(\xi) d\beta_2(\xi), \end{cases} \tag{29}$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ *is an odd increasing homeomorphism defined by*

$$\varphi(s) = s + s^2 \text{ for } s \in \mathbb{R}_+,$$

$h : (0,1) \to \mathbb{R}_+$ *is a continuous function defined by*

$$h(t) = 0 \text{ for } t \in [0, \tfrac{1}{8}] \text{ and } h(t) = (t - \tfrac{1}{8})(1-t)^{-\alpha} \text{ for } t \in (\tfrac{1}{8}, 1),$$

and

$$\beta_1(r) = \beta_2(r) = \frac{1}{3} r^2 \text{ for } r \in [0,1].$$

It is evident that $\hat{\beta}_1$ *and* $\hat{\beta}_2$ *are both equal to* $\frac{1}{3}$*, which lies within the interval* $[0,1)$*, and thus, the assumption* (K_2) *holds. Moreover, by defining functions*

$$\psi_1(y) = \min\{y, y^2\} \text{ and } \psi_2(y) = \max\{y, y^2\} \text{ for } y \in \mathbb{R}_+,$$

we can verify that the assumption (K_1) *is also met (see, e.g., [1]). Since the inverse function* ψ_1*, denoted by* ψ_1^{-1}*, is given by* $\psi_1^{-1}(s) = \max\{\sqrt{s}, s\}$ *for* $s \in \mathbb{R}_+$*, it follows that* $h \in \mathcal{H}_{\psi_1} \setminus L^1(0,1)$ *for any* $\alpha \in [1,2)$*.*

Let f_1 *and* f_2 *be continuous functions on* $\mathbb{R}_+$ *defined by*

$$f_1(s) = s^{c_1} \text{ for } s \in \mathbb{R}_+ \text{ and } f_2(s) = \begin{cases} s^{c_2}, & \text{for } s \in [0,1], \\ s^{c_3}, & \text{for } s \in (1, \infty). \end{cases}$$

Here, $c_1 \in (1,2)$*,* $c_2 \in (0,1)$*, and* $c_3 \in (2, \infty)$ *are fixed constants. Then,*

$$(f_1)_0 = (f_1)_\infty = 0 \text{ and } (f_2)_0 = (f_2)_\infty = \infty.$$

As a consequence of Theorem 2, we can conclude that there exist positive constants $\lambda_1^* \geq \lambda_*^1 > 0$ *and* $\lambda_2^* \geq \lambda_*^2 > 0$ *such that problem (29) with* $f = f_1$ *has two positive solutions for* $\lambda > \lambda_1^*$*, one positive solution for* $\lambda \in [\lambda_*^1, \lambda_1^*]$*, and no positive solutions for* $\lambda \in (0, \lambda_*^1)$*, and problem (29) with* $f = f_2$ *has two positive solutions for* $\lambda \in (0, \lambda_*^2)$*, one positive solution for* $\lambda \in [\lambda_*^2, \lambda_2^*]$*, and no positive solutions for* $\lambda > \lambda_2^*$*.*

4. Conclusions

This study delved into the existence, nonexistence, and multiplicity of positive solutions to problem (1) across all positive values of the parameter λ. Our analysis hinges on the application of two pivotal fixed-point theorems: the cone expansion and compression of the norm type theorem, and the Leray–Schauder fixed-point theorem.

Despite our progress in understanding problem (1), there are still open questions that warrant further exploration. Specifically, it is still unknown whether $\lambda_1^* = \lambda_*^1$ or $\lambda_*^2 = \lambda_2^*$ in Theorem 2. Exploring the behavior of solutions near the bifurcation points may provide deeper insights into the qualitative properties of the problem. While this work has established the foundation, a more comprehensive analysis of the bifurcation points is necessary to fully understand the existence of positive solutions and will be the focus of future studies.

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References

1. Jeong, J.; Kim, C.G. Existence of Positive Solutions to Singular Boundary Value Problems Involving φ-Laplacian. *Mathematics* **2019**, *7*, 654. [CrossRef]
2. Picone, M. Su un problema al contorno nelle equazioni differenziali lineari ordinarie del secondo ordine. *Ann. Della Sc. Norm. Super.-Pisa-Sci. Fis. Mat.* **1908**, *10*, 1–95.
3. Infante, G.; Pietramala, P. A cantilever equation with nonlinear boundary conditions. *Electron. J. Qual. Theory Differ. Equ.* **2009**, 1–14. [CrossRef]
4. Cabada, A.; Infante, G.; Tojo, F.A.F. Nonzero solutions of perturbed Hammerstein integral equations with deviated arguments and applications. *Topol. Methods Nonlinear Anal.* **2016**, *47*, 265–287. [CrossRef]
5. Moshinsky, M. Sobre los problemas de condiciones a la frontiera en una dimension de caracteristicas discontinuas. *Bol. Soc. Mat. Mex.* **1950**, *7*, 10–25.
6. Timoshenko, S. *Theory of Elastic Stability*; McGraw-Hill: New York, NY, USA, 1961.
7. Ntouyas, S. Nonlocal initial and boundary value problems: A survey. In *Handbook of Differential Equations: Ordinary Differential Equations*; Elsevier: Amsterdam, The Netherlands, 2006; Volume 2, pp. 461–557.
8. Ma, R. A survey on nonlocal boundary value problems. *Appl. Math. E-Notes* **2007**, *7*, 257–279.
9. Bachouche, K.; Djebali, S.; Moussaoui, T. ϕ-Laplacian BVPS with linear bounded operator conditions. *Arch. Math.* **2012**, *48*, 121–137. [CrossRef]
10. Goodrich, C.S. Perturbed integral operator equations of Volterra type with applications to p-Laplacian equations. *Mediterr. J. Math.* **2018**, *15*, 47. [CrossRef]
11. Kim, C.G. Existence and Multiplicity Results for Nonlocal Boundary Value Problems with Strong Singularity. *Mathematics* **2020**, *8*, 680. [CrossRef]
12. Tariboon, J.; Samadi, A.; Ntouyas, S.K. Nonlocal Boundary Value Problems for Hilfer Generalized Proportional Fractional Differential Equations. *Fractal Fract.* **2022**, *6*, 154. [CrossRef]
13. Li, C. Uniqueness of a nonlinear integro-differential equation with nonlocal boundary condition and variable coefficients. *Bound. Value Probl.* **2023**, *2023*, 26. [CrossRef] [PubMed]
14. Agarwal, R.P.; Lü, H.; O'Regan, D. Eigenvalues and the one-dimensional p-Laplacian. *J. Math. Anal. Appl.* **2002**, *266*, 383–400. [CrossRef]
15. Wang, H. On the structure of positive radial solutions for quasilinear equations in annular domains. *Adv. Differ. Equ.* **2003**, *8*, 111–128. [CrossRef]
16. Lee, Y.H.; Xu, X. Existence and multiplicity results for generalized Laplacian problems with a parameter. *Bull. Malays. Math. Sci. Soc.* **2020**, *43*, 403–424. [CrossRef]
17. Kim, C.G. Multiplicity of Positive Solutions to Nonlocal Boundary Value Problems with Strong Singularity. *Axioms* **2021**, *11*, 7. [CrossRef]
18. Jeong, J.; Kim, C.G. Existence, Nonexistence and Multiplicity of Positive Solutions for Generalized Laplacian Problems with a Parameter. *Mathematics* **2024**, *12*, 3668. [CrossRef]
19. Jeong, J.; Kim, C.G. Existence of Positive Solutions to Singular φ-Laplacian Nonlocal Boundary Value Problems when φ is a Sup-multiplicative-like Function. *Mathematics* **2020**, *8*, 420. [CrossRef]

20. Guo, D.J.; Lakshmikantham, V. *Nonlinear Problems in Abstract Cones*; Academic Press, Inc.: Boston, MA, USA, 1988.
21. Deimling, K. *Nonlinear Functional Analysis*; Springer: Berlin/Heidelberg, Germany, 1985. [CrossRef]

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