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Jacobi–Jordan Conformal Algebras: Basics, Constructions and Related Structures

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Abstract: The main purpose of this paper is to introduce and investigate the notion of Jacobi–Jordan conformal algebras. They are a generalization of Jacobi–Jordan algebras which correspond to the case in which the formal parameter λ equals 0. We consider some related structures such as conformal modules, corresponding representations and $\mathcal{O}$ -operators. Therefore, conformal derivations from Jacobi–Jordan conformal algebras to their conformal modules are used to describe conformal derivations of Jacobi–Jordan conformal algebras of the semidirect product type. Moreover, we study a class of Jacobi–Jordan conformal algebras called quadratic Jacobi–Jordan conformal algebras, which are characterized by mock-Gel’fand–Dorfman bialgebras. Finally, the $\mathbb{C}[\partial]$ -split extending structures problem for Jacobi–Jordan conformal algebras is studied. Furthermore, we introduce an unified product of a given Jacobi–Jordan conformal algebra J and a given $\mathbb{C}[\partial]$ -module K . This product includes some other interesting products of Jacobi–Jordan conformal algebras such as the twisted product and crossed product. Using this product, a cohomological type object is constructed to provide a theoretical answer to the $\mathbb{C}[\partial]$ -split extending structures problem.

Keywords: Jacobi–Jordan conformal algebra; conformal module; mock-Gel’fand–Dorfman bialgebra; unified product; extending structure



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1. Introduction

In the jungle of non-associative algebras, Jacobi–Jordan algebras are rather special objects. A Jacobi–Jordan algebra is a vector space J together with a commutative bilinear map $\bullet : J \times J \rightarrow J$ such that

$$a \bullet (b \bullet c) + b \bullet (c \bullet a) + c \bullet (a \bullet b) = 0$$

for all $a, b, c \in J$. According to [1], where a detailed history of Jacobi–Jordan algebras is given and several conjectures are proposed, this family of algebras was first defined in [2], and since then it has been studied independently in various papers [3–14] under different names such as Lie–Jordan algebras, Jordan algebras of nilindex 3, pathological algebras, mock-Lie algebras and Jacobi–Jordan algebras. Throughout this paper, we use the term Jacobi–Jordan algebras.

An algebraic formalization of the properties of the operator product expansion (OPE) in two-dimensional conformal field theory [15] gave rise to a new class of algebraic systems, vertex operator algebras [16,17]. The notion of Lie conformal algebra encodes the singular part of the OPE which is responsible for the commutator of two chiral fields [18]. Roughly speaking, Lie conformal algebras correspond to vertex algebras in the same way as Lie algebras correspond to their associative enveloping algebras.

The structure theory of finite (i.e., finitely generated as $\mathbb{C}[\partial]$ -modules) associative and Lie conformal algebras was developed in [19] and later generalized in [20] for pseudoalgebras over a wide class of cocommutative Hopf algebras. From the algebraic point of view, the notions of conformal algebra [19], their representations [21] and cohomologies [22–24] are higher-level analogues of the ordinary notions in the pseudo-tensor category [25] associated with the polynomial Hopf algebra (see [20] for a detailed explanation).

Some features of the structure theory of conformal algebras (and their representations) of infinite type were also considered in a series of works [26–33]. In this field, one of the most relevant problems is to describe the structure of conformal algebras with faithful irreducible representation of finite type (these algebras could be of infinite type themselves). In [26,29], conjectures on the structure of such algebras (associative and Lie) were stated. They were proved under some additional conditions in [19,26,33]. Another relevant problem is to classify simple and semisimple conformal algebras of linear growth (i.e., of Gel'fand–Kirillov dimension one). This problem was solved for finitely generated associative conformal algebras which contain a unit [30,31] or at least an idempotent [32,33]. A structure theory of associative conformal algebras with finite faithful representation similar to those examples of conformal algebras stated in these papers was developed in [34]. See also [35,36] for further results.

The split extending structures problem for some classical algebra objects such as groups, associative algebras, Hopf algebras, Lie algebras, Leibniz algebras and left-symmetric algebras has been studied in [37–42] respectively. In fact, this problem generalizes two important algebra problems. For associative and Lie conformal algebras, the first one is the $\mathbb{C}[\partial]$ -split extension problem introduced by Y.Y. Hong in [43,44], while the second one is the $\mathbb{C}[\partial]$ -split factorization problem.

The main purpose of this paper is to introduce and investigate the notion of Jacobi–Jordan conformal algebras. It is organized as follows. In Section 2, we introduce a definition of Jacobi–Jordan conformal algebras and give some basic results. We study $\mathcal{O}$ -operators on Jacobi–Jordan algebras and symplectic Jacobi–Jordan algebras. In Section 3, we provide a characterization of quadratic Jacobi–Jordan conformal algebras through mock-Gel'fand–Dorfman bialgebras and present various constructions. The $\mathbb{C}[\partial]$ -split extending structures problem for Jacobi–Jordan conformal algebras is studied in Section 4. We introduce a definition of unified product of a given a Jacobi–Jordan conformal algebra and consider some interesting sub-classes like twisted and crossed products of Jacobi–Jordan conformal algebras.

Throughout the paper, all algebraic systems are supposed to be over a field $\mathbb{F}$ of characteristic 0. We denote the set of all nonnegative integers with $\mathbb{Z}_+$ and the set of all integers with $\mathbb{Z}$.

2. Jacobi–Jordan Conformal Algebras and Their Modules

In this section, we provide some basic notions of (left) anti-associative and Jacobi–Jordan conformal algebras along with their conformal modules. We introduce the notion of $\mathcal{O}$ -operators on Jacobi–Jordan conformal algebras with respect to a given module as a generalization of Rota–Baxter operators.

2.1. Definitions and Basic Results

A conformal algebra is a vector space A equipped with a linear map $\partial : A \rightarrow A$ and a countable family of bilinear operations $-_{(n)} : A \times A \rightarrow A, n \in \mathbb{Z}_+$, satisfying the following axioms:

- (i) For every $a, b \in A$, there exists $N \in \mathbb{Z}_+$ such that $a_{(n)}b = 0$ for all $n \geq N$;
- (ii) $\partial a_{(n)}b = -na_{(n-1)}b$;
- (iii) $a_{(n)}\partial b = \partial(a_{(n)}b) + na_{(n-1)}b$.

Every conformal algebra A is a left module over the polynomial algebra $\mathbb{F}[\partial]$. The structure of a conformal algebra on an $\mathbb{F}[\partial]$ -module A may be expressed by means of a single polynomial-valued map called a λ -product

$$-_{\lambda} : A \times A \rightarrow A[\lambda], \quad a_{\lambda}b = \sum_{n \in \mathbb{Z}_+} \lambda^{(n)} a_{(n)}b,$$

where λ is a formal variable and $\lambda^{(n)} = \lambda^n/n!$. In terms of the λ -product, the definition of a conformal algebra is as follows.

Definition 1. A conformal algebra is a module A over the polynomial algebra $\mathbb{F}[\partial]$ equipped with a polynomial-valued $\mathbb{F}$ -linear operation $A \times A \rightarrow A[\lambda]$ denoted by $a \times b \rightarrow a_{\lambda}b$, where λ is a formal variable (this operation is called the λ -product), satisfying the following axioms

$$\partial a_{\lambda}b = -\lambda a_{\lambda}b, \quad a_{\lambda}\partial b = (\lambda + \partial)a_{\lambda}b, \quad (\text{Conformal sesquilinearity}). \quad (1)$$

A conformal algebra is said to be finite if it is finitely generated as a module over $\mathbb{F}[\partial]$. The rank of a finite conformal algebra A is its rank as a $\mathbb{F}[\partial]$ -module.

Let $(A, \cdot_{\lambda})$ be a conformal algebra. Then, by conformal sesquilinearity, the following equalities hold true

$$(a_{-\lambda-\partial}b)_{\lambda+\mu}c = (a_{\mu}b)_{\lambda+\mu}c, \quad (2)$$

$$a_{\mu}(b_{-\lambda-\partial}c) = a_{\mu}(b_{-\lambda-\mu-\partial}c), \quad (3)$$

$$(a_{-\mu-\partial}b)_{-\lambda-\partial}c = (a_{-\lambda-\mu-\partial}b)_{-\lambda-\partial}c \quad (4)$$

for all $a, b, c \in A$.

We prove the first identity by the following computation.

$$\begin{aligned} (a_{-\lambda-\partial}b)_{\lambda+\mu}c &= \left(\sum_{k=0}^n \frac{(-\lambda-\partial)^k}{k!} a_{(k)}b \right)_{\lambda+\mu} c \\ &= \sum_{k=0}^n \frac{(-\lambda+\lambda+\mu)^k}{k!} ((a_{(k)}b)_{\lambda+\mu}c) \quad (\text{by conformal sesquilinearity}) \\ &= \left(\sum_{k=0}^n \frac{\mu^k}{k!} a_{(k)}b \right)_{\lambda+\mu} c = (a_{\mu}b)_{\lambda+\mu}c. \end{aligned}$$

Let M, N be $\mathbb{F}[\partial]$ -modules. A conformal linear map from M to N is a sequence $f = \{f_{(n)}\}_{n \geq 0}$ of $f_{(n)} \in \text{Hom}_{\mathbb{F}}(M, N)$ satisfying

$$\partial f_{(n)} - f_{(n)}\partial = -nf_{(n-1)}, \quad \forall n \in \mathbb{N}.$$

In particular, $f_{(0)}$ is an $\mathbb{F}[\partial]$ -module homomorphism. Let $f_\lambda = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} f_{(k)}$. Then, $f = \{f_{(n)}\}_{n \geq 0}$ is a conformal linear map if and only if $f_\lambda : M \rightarrow N[\lambda]$ is $\mathbb{F}$ -linear and $f_\lambda \partial = (\lambda + \partial)f_\lambda$.

We denote the set of conformal linear maps from M to N with $Chom(M, N)$. It turns out that $Chom(M, N)$ is a $\mathbb{F}[\partial]$ -module via the following:

$$\partial f_{(n)} = -n f_{(n-1)}, \quad \text{equivalently,} \quad \partial f_\lambda = -\lambda f_\lambda.$$

The composition $fg : L \rightarrow N$ of conformal linear maps $f : M \rightarrow N$ and $g : L \rightarrow M$ is given by

$$(fg)_{(n)} = \sum_{k=0}^n \binom{n}{k} f_{(k)} g_{(n-k)}, \quad \text{equivalently,} \quad (f_\lambda g)_{\lambda+\mu} = f_\lambda g_\mu.$$

If M is a finitely generated $\mathbb{F}[\partial]$ -module, then $Cend(M) := Chom(M, M)$ is an associative conformal algebra with respect to the above composition. Thus, $Cend(M)$ becomes a Lie conformal algebra, denoted as $gc(M)$, with respect to the following λ -bracket:

$$[f_\lambda g]_\mu = f_\lambda g_{\mu-\lambda} - g_{\mu-\lambda} f_\lambda, \quad \text{equivalently,} \quad [f_\lambda g] = f_\lambda g - g_{-\lambda-\partial} f.$$

Define the conformal dual of an $\mathbb{F}[\partial]$ -module M as $M^{*c} = Chom(M, \mathbb{F})$; that is,

$$M^{*c} = \{f : M \rightarrow \mathbb{F}[\lambda] \mid f \text{ is } \mathbb{F}\text{-linear and } f_\lambda(\partial m) = \lambda f_\lambda(m)\}.$$

Lemma 1. Let $f, g \in Cend(M)$, then, for any $m \in M$, we have

- (1) $f_\lambda(g_{-\mu-\partial} m) = (f_\lambda g)_{-\mu-\partial} m$,
- (2) $f_{-\lambda-\partial}(g_\mu m) = (f_{-\mu-\partial} g)_{-\lambda-\partial+\mu} m$,
- (3) $f_{-\lambda-\partial}(g_{\mu-\partial} m) = (f_{-\lambda-\partial+\mu} g)_{-\mu-\partial} m$.

The proof follows from a straightforward computation.

Definition 2. Let A be a conformal algebra. A conformal linear map $d \in Cend(A)$ is a conformal anti-derivation on A if for all $a, b \in A$,

$$d_\lambda(a_\mu b) = -(d_\lambda a)_{\lambda+\mu} b - a_\mu(d_\lambda b). \quad (5)$$

Now, we introduce the notion of Jacobi–Jordan conformal algebras. They generalize Jacobi–Jordan algebras, which correspond to $\lambda = 0$. On the other hand, this variety can be viewed as a variety of commutative Leibniz conformal algebras.

Definition 3. A Jacobi–Jordan conformal algebra is a conformal algebra $(A, \cdot_\lambda)$ such that the following conditions hold:

$$a_\lambda b = b_{-\lambda-\partial} a, \quad (6)$$

$$a_\lambda(b_\mu c) + (a_\lambda b)_{\lambda+\mu} c + b_\mu(a_\lambda c) = 0, \quad (7)$$

for any $a, b, c \in A$.

Remark 1. If we take the above definition, $\lambda = \mu = 0$ and $\partial = 0$, then we recover the definition of a Jacobi–Jordan algebra.

We consider a conformal linear map $d_\lambda : A \rightarrow \text{Cend}(A)$ defined by $d_\lambda(a)(b) = a_\lambda b$. Then, Equation (7) is equivalent to $d_\lambda(a)$ being a conformal anti-derivation for any $a \in A$.

Example 1. Consider one of the simplest (through important) examples of Jacobi–Jordan conformal algebras. Let $(A, \cdot)$ be a Jacobi–Jordan algebra. Then, we naturally define a Jacobi–Jordan conformal algebra $\text{Cur}A = \mathbb{F}[\partial] \otimes A$ with the λ -product

$$a_\lambda b = a \cdot b, \quad \forall a, b \in A.$$

$\text{Cur}A$ is called the current Jacobi–Jordan conformal algebra associated with A .

Example 2. Let A be an associative commutative algebra and $(B, \cdot_\lambda)$ be a Jacobi–Jordan conformal algebra. Then, the tensor product $A \otimes B$ becomes a Jacobi–Jordan conformal algebra via

$$\partial(a \otimes x) = a \otimes \partial x, \quad (a \otimes x)_\lambda(b \otimes y) = (ab) \otimes (x_\lambda y).$$

Definition 4. An anti-associative conformal algebra is a conformal algebra $(A, \cdot_\lambda)$ satisfying

$$(a_\lambda b)_{\lambda+\mu} c + a_\lambda (b_\mu c) = 0, \quad (8)$$

for all $a, b, c \in A$.

Similarly, as left symmetric conformal algebras are closely related to Lie conformal algebras, we introduce the notion of left anti-symmetric conformal algebras, whose commutator gives rise to a Jacobi–Jordan conformal algebra.

Definition 5. A conformal algebra $(A, \cdot_\lambda)$ is called a left anti-symmetric conformal algebra if for all $a, b, c \in A$ one has

$$(a_\lambda b)_{\lambda+\mu} c + a_\lambda (b_\mu c) + (b_\mu a)_{\lambda+\mu} c + b_\mu (a_\lambda c) = 0. \quad (9)$$

It is obvious that every anti-associative conformal algebra is left anti-symmetric. A conformal algebra $(A, \cdot_\lambda)$ is said to be Jacobi–Jordan-admissible if $(A, \circ_\lambda)$ is a Jacobi–Jordan conformal algebra, where $a \circ_\lambda b = a_\lambda b + b_{-\lambda-\partial} a$, for all $a, b \in A$.

Proposition 1. Let $(A, \cdot_\lambda)$ be a left anti-symmetric conformal algebra. Then, A is a Jacobi–Jordan-admissible conformal algebra.

Proof. Let $a, b, c \in A$. Then, $b \circ_{-\lambda-\partial} a = b_{-\lambda-\partial} a + a_\lambda b = a \circ_\lambda b$. On the other hand,

$$\begin{aligned} & a \circ_\lambda (b \circ_\mu c) + (a \circ_\lambda b) \circ_{\lambda+\mu} c + b \circ_\mu (a \circ_\lambda c) \\ &= a \circ_\lambda (b_\mu c + c_{-\mu-\partial} b) + (a_\lambda b + b_{-\lambda-\partial} a) \circ_{\lambda+\mu} c + b \circ_\mu (a_\lambda c + c_{-\lambda-\partial} a) \\ &= a_\lambda (b_\mu c + c_{-\mu-\partial} b) + (b_\mu c + c_{-\mu-\partial} b)_{-\lambda-\partial} a + (a_\lambda b + b_{-\lambda-\partial} a)_{\lambda+\mu} c \\ &\quad + c_{-\lambda-\mu-\partial} (a_\lambda b + b_{-\lambda-\partial} a) + b_\mu (a_\lambda c + c_{-\lambda-\partial} a) + (a_\lambda c + c_{-\lambda-\partial} a)_{-\mu-\partial} b \\ &= (a_\lambda b)_{\lambda+\mu} c + a_\lambda (b_\mu c) + (b_{-\lambda-\partial} a)_{\lambda+\mu} c + b_\mu (a_\lambda c) \\ &\quad + (a_\lambda c)_{-\mu-\partial} b + a_\lambda (c_{-\mu-\partial} b) + (c_{-\lambda-\partial} a)_{-\mu-\partial} b + c_{-\lambda-\mu-\partial} (a_\lambda b) \\ &= 0. \end{aligned}$$

Then, $(A, \circ_\lambda)$ is a Jacobi–Jordan conformal algebra. $\square$

Corollary 1. Any anti-associative conformal algebra is a Jacobi–Jordan-admissible conformal algebra.

Similarly, we have the following proposition.

Proposition 2. Any Jacobi–Jordan conformal algebra is Jacobi–Jordan-admissible.

Now, we introduce a notion of representation of a Jacobi–Jordan conformal algebra and give some examples.

Definition 6. An $\mathbb{F}[\partial]$ -module M is a conformal module of a Jacobi–Jordan conformal algebra A if there is an $\mathbb{F}$ -linear map $\pi : A \rightarrow \text{Cend}(M)$ satisfying the following condition: For all $a, b \in A$,

$$\pi(\partial a)_\lambda = -\lambda \pi(a)_\lambda, \quad (10)$$

$$\pi(a_\lambda b)_{\lambda+\mu} = -\pi(a)_\lambda \pi(b)_\mu - \pi(b)_\mu \pi(a)_\lambda. \quad (11)$$

Proposition 3. Given a finite dimensional complex Jacobi–Jordan algebra A , let $\rho : A \rightarrow \text{End}(M)$ be a finite dimensional representation of A . Then, the free $\mathbb{C}[\partial]$ -module $\mathbb{C}[\partial] \otimes M$ is a conformal module of $\text{Cur } A$, where the module structure $\pi : \text{Cur } A \rightarrow \text{Cend}(\mathbb{C}[\partial] \otimes M)$ is given by

$$\pi(f(\partial) \otimes a)_\lambda (g(\partial) \otimes m) = f(-\lambda)g(\lambda + \partial) \otimes (\rho(a)m).$$

Proof. For any $a, b \in A, m \in M$ and $f(\partial), g(\partial), h(\partial) \in \mathbb{F}[\partial]$, we have

$$\begin{aligned} \pi(f(\partial)a)_\lambda \pi(g(\partial)b)_\mu (h(\partial)m) &= \pi(f(\partial)a)_\lambda (g(-\mu)h(\partial + \mu)\rho(b)m) \\ &= f(-\lambda)g(-\mu)h(\partial + \lambda + \mu)\rho(a)\rho(b)m, \end{aligned}$$

and

$$\begin{aligned} \pi(g(\partial)b)_\mu \pi(f(\partial)a)_\lambda (h(\partial)m) &= \pi(g(\partial)b)_\mu (f(-\lambda)h(\partial + \lambda)\rho(a)m) \\ &= g(-\mu)f(-\lambda)h(\partial + \lambda + \mu)\rho(b)\rho(a)m. \end{aligned}$$

Then,

$$\begin{aligned} \pi(f(\partial)a_\lambda g(\partial)b)_{\lambda+\mu} (h(\partial)m) &= \pi(f(-\lambda)g(\partial + \lambda)ab)_{\lambda+\mu} (h(\partial)m) \\ &= f(-\lambda)g(-\mu)h(\lambda + \mu + \partial)\rho(ab)m \\ &= -f(-\lambda)g(-\mu)h(\lambda + \mu + \partial)(\rho(a)\rho(b)m + \rho(b)\rho(a)m) \\ &= -\pi(f(\partial)a)_\lambda \pi(g(\partial)b)_\mu (h(\partial)m) - \pi(g(\partial)b)_\mu \pi(f(\partial)a)_\lambda (h(\partial)m). \end{aligned}$$

Hence, the proof is complete. $\square$

Proposition 4. Let $(A, \cdot_\lambda)$ be a Jacobi–Jordan conformal algebra, M an A -conformal module and $\pi : A \rightarrow \mathbb{F}[\partial] \otimes \text{Cend}(M) : r \mapsto \pi(r)_\lambda$ the corresponding representation. Define the following λ -product on $A \oplus M$:

$$(a + m)_\lambda (b + n) = a_\lambda b + \pi(a)_\lambda n + \pi(b)_{-\lambda - \partial} m. \quad (12)$$

Then, $A \oplus M$ becomes a Jacobi–Jordan conformal algebra with respect to the above λ -product, called the semidirect product of A and M , and denoted as $A \ltimes M$. Moreover, M is an abelian ideal of $A \ltimes M$.

Proof. For all $a, b, c \in A$ and $m, n, p \in M$, we have

$$\begin{aligned}\partial(a+m)_\lambda(b+n) &= \partial a_\lambda b + \pi(\partial a)_\lambda n + \pi(b)_{-\lambda-\partial} \partial m \\ &= -\lambda(a_\lambda b + \pi(a)_\lambda n + \pi(b)_{-\lambda-\partial} m) = -\lambda(a+m)_\lambda(b+n).\end{aligned}$$

Using a similar computation, one can easily check that

$$(a+m)_\lambda \partial(b+n) = (\lambda + \partial)(a+m)_\lambda(b+n).$$

Moreover,

$$(b+n)_{-\lambda-\partial}(a+m) = (b_{-\lambda-\partial} a + \pi(b)_{-\lambda-\partial} m + \pi(a)_\lambda m) = (a+m)_\lambda(b+n).$$

In addition,

$$\begin{aligned}(a+m)_\lambda((b+n)_\mu(c+p)) &= (a+m)_\lambda(b_\mu c) + \pi(b)_\mu p + \pi(c)_{-\mu-\partial} n \\ &= (a_\lambda(b_\mu c) + \pi(a)_\lambda(\pi(b)_\mu p) + \pi(a)_\lambda(\pi(c)_{-\mu-\partial} n) + \pi(b_\mu c)_{-\lambda-\partial} m), \\ &\quad \left((a+m)_\lambda(b+n)\right)_{\lambda+\mu}(c+p) \\ &= (a_\lambda b + \pi(a)_\lambda n + \pi(b)_{-\lambda-\partial} m)_{\lambda+\mu}(c+p) \\ &= ((a_\lambda b)_{\lambda+\mu} c + \pi(a_\lambda b)_{\lambda+\mu} p + \pi(c)_{-\lambda-\mu-\partial}(\pi(a)_\lambda n) + \pi(c)_{-\lambda-\mu-\partial}(\pi(b)_{-\lambda-\partial} m))\end{aligned}$$

and

$$(b+n)_\mu((a+m)_\lambda(c+p)) = (b_\mu(a_\lambda c) + \pi(b)_\mu(\pi(a)_\lambda p) + \pi(b)_\mu(\pi(c)_{-\lambda-\partial} m) + \pi(a_\lambda c)_{-\mu-\partial} n).$$

Then,

$$\begin{aligned}& (a+m)_\lambda((b+n)_\mu(c+p)) + \left((a+m)_\lambda(b+n)\right)_{\lambda+\mu}(c+p) + (b+n)_\mu((a+m)_\lambda(c+p)) \\ &= (a_\lambda(b_\mu c) + (a_\lambda b)_{\lambda+\mu} c + b_\mu(a_\lambda c) + X + Y + Z) = (0 + X + Y + Z),\end{aligned}$$

where

$$\begin{aligned}X &= \pi(a)_\lambda(\pi(b)_\mu p) + \pi(a_\lambda b)_{\lambda+\mu} p + \pi(b)_\mu(\pi(a)_\lambda p), \\ Y &= \pi(a)_\lambda(\pi(c)_{-\mu-\partial} n) + \pi(c)_{-\lambda-\mu-\partial}(\pi(a)_\lambda n) + \pi(a_\lambda c)_{-\mu-\partial} n, \\ Z &= \pi(b_\mu c)_{-\lambda-\partial} m + \pi(c)_{-\lambda-\mu-\partial}(\pi(b)_{-\lambda-\partial} m) + \pi(b)_\mu(\pi(c)_{-\lambda-\partial} m).\end{aligned}$$

According to Lemma 1 and identity (11), one can check that $X = Y = Z = 0$. $\square$

Proposition 5. Let (M, π) be a finite dimensional representation of a Jacobi–Jordan conformal algebra $(A, \cdot_\lambda)$. Let π^* be an $\mathbb{F}[\partial]$ -module homomorphism from A to $\text{Cend}(M^{*c})$ defined by

$$(\pi^*(a)_\lambda f)_\mu m = f_{\mu-\lambda}(\pi(a)_\lambda m), \quad \forall a \in A, f \in M^{*c}, m \in M. \quad (13)$$

Then, (M^{*c}, π^*) is a representation of A .

Proof. Let $a, b \in A, m \in M$ and $f \in M^{*c}$. Then, we have

$$(\pi^*(\partial a)_\lambda f)_\mu m = f_{\mu-\lambda}(\pi(\partial a)_\lambda m) = -\lambda f_{\mu-\lambda}(\pi(a)_\lambda m) = -\lambda(\pi^*(a)_\lambda f)_\mu m.$$

Then, (10) holds. To prove (11), we compute as follows

$$\begin{aligned} (\pi^*(a_\lambda b)_{\lambda+\mu} f)_v m &= f_{v-\lambda-\mu}(\pi(a_\lambda b)_{\lambda+\mu} m) \\ &= -f_{v-\lambda-\mu}(\pi(a)_\lambda \pi(b)_\mu m) - f_{v-\lambda-\mu}(\pi(b)_\mu \pi(a)_\lambda m). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \pi^*(a)_\lambda (\pi^*(b)_\mu f)_v m &= (\pi^*(b)_\mu f)_{v-\lambda} (\pi(a)_\lambda m) = f_{v-\lambda-\mu}(\pi(b)_\mu \pi(a)_\lambda m), \\ \pi^*(b)_\mu (\pi^*(a)_\lambda f)_v m &= (\pi^*(a)_\lambda f)_{v-\mu} (\pi(b)_\mu m) = f_{v-\lambda-\mu}(\pi(a)_\lambda \pi(b)_\mu m). \end{aligned}$$

Therefore

$$(\pi^*(a_\lambda b)_{\lambda+\mu} f)_v m = -\pi^*(a)_\lambda (\pi^*(b)_\mu f)_v m - \pi^*(b)_\mu (\pi^*(a)_\lambda f)_v m,$$

which implies that (M^{*c}, π^*) is a representation of A . $\square$

2.2. $\mathcal{O}$ -Operators on Jacobi–Jordan Conformal Algebras

Definition 7. Let $(A, \cdot_\lambda)$ be a Jacobi–Jordan conformal algebra and $\pi : A \rightarrow \text{Cend}(M)$ be a representation. An $\mathbb{F}[\partial]$ -module homomorphism $T : M \rightarrow A$ satisfying

$$Tu_\lambda Tv = T(\pi(Tu)_\lambda v + \pi(Tv)_{-\lambda-\partial} u), \quad \forall u, v \in M, \quad (14)$$

is called an $\mathcal{O}$ -operator.

If $M = A$ and π is the adjoint representation, then T is called a Rota–Baxter operator of weight 0 on A denoted by R , and the condition (14) can be rewritten as

$$R(a)_\lambda R(b) = R(R(a)_\lambda b + a_\lambda R(b)), \quad \forall a, b \in A. \quad (15)$$

More generally, we can extend the notion of weighted Rota–Baxter operators of Jacobi–Jordan algebras to the conformal case by considering the following weighted Rota–Baxter identity:

$$R(a)_\lambda R(b) = R(R(a)_\lambda b + a_\lambda R(b)) + \alpha R(a_\lambda b), \quad \forall a, b \in A. \quad (16)$$

Theorem 1. Let A be a Jacobi–Jordan conformal algebra and $\pi : A \rightarrow \text{Cend}(M)$ be a representation of A . Suppose $T : M \rightarrow A$ is an $\mathcal{O}$ -operator associated with π . Then, the following λ -product:

$$u *_\lambda v = \pi(Tu)_\lambda v, \quad \forall u, v \in M, \quad (17)$$

endows M with a left anti-symmetric conformal algebra structure. Therefore, M is a Jacobi–Jordan conformal algebra which is the sub-adjacent Jacobi–Jordan conformal algebra of this left anti-symmetric conformal algebra, and $T : M \rightarrow A$ is a homomorphism of Jacobi–Jordan conformal algebras. Moreover, $\{T(M) \subset A\}$ is a Jacobi–Jordan conformal subalgebra of A and there is also a natural left anti-symmetric conformal algebra structure on $T(M)$ defined as follows:

$$T(u)_\lambda T(v) = T(u *_\lambda v) = T(\pi(Tu)_\lambda v), \quad \forall u, v \in M.$$

In addition, the sub-adjacent Jacobi–Jordan conformal algebra of this left anti-symmetric conformal algebra is a subalgebra of A and $T : M \rightarrow A$ is a homomorphism of left anti-symmetric conformal algebras.

Proof. For all $u, v, w \in M$, we have

$$\begin{aligned}\partial u *_{\lambda} v &= \pi(T(\partial u))_{\lambda} v = \pi(\partial Tu)_{\lambda} v = -\lambda \pi(Tu)_{\lambda} v = -\lambda u *_{\lambda} v, \\ u *_{\lambda} \partial v &= \pi(Tu)_{\lambda} (\partial v) = (\partial + \lambda) \pi(Tu)_{\lambda} v = (\partial + \lambda) u *_{\lambda} v.\end{aligned}$$

Furthermore, we have

$$\begin{aligned}& (u *_{\lambda} v) *_{\lambda+\mu} w + u *_{\lambda} (v *_{\mu} w) \\&= (\pi(Tu)_{\lambda} v) *_{\lambda+\mu} w + u *_{\lambda} (\pi(Tv)_{\mu} w) \\&= \pi(T(\pi(Tu)_{\lambda} v))_{\lambda+\mu} w + \pi(Tu)_{\lambda} \pi(Tv)_{\mu} w \\&= \pi(T(u)_{\lambda} T(v))_{\lambda+\mu} w - \pi(T(\pi(Tv)_{-\lambda-\partial} u))_{\lambda+\mu} w - \pi(T(u)_{\lambda} b T(v))_{\lambda+\mu} w - \pi(Tv)_{\mu} \pi(Tu)_{\lambda} w \\&= -\pi(T(\pi(Tv)_{-\lambda-\partial} u))_{\lambda+\mu} w - \pi(Tv)_{\mu} \pi(Tu)_{\lambda} w \\&= -\pi(T(\pi(Tv)_{\mu} u))_{\lambda+\mu} w - \pi(Tv)_{\mu} \pi(Tu)_{\lambda} w \\&= -(v *_{\mu} u) *_{\lambda+\mu} w - v *_{\mu} (u *_{\lambda} w).\end{aligned}$$

Hence, $(M, *_{\lambda})$ is a left anti-symmetric conformal algebra. $\square$

Corollary 2. Let A be a Jacobi–Jordan conformal algebra and $T : A \rightarrow A$ be a Rota–Baxter operator of weight zero. Then, there is a left anti-symmetric conformal algebra structure on A with the following λ -product:

$$a \circ_{\lambda} b = T(a)_{\lambda} b, \quad a, b \in A.$$

Corollary 3. Let $(A, \cdot_{\lambda})$ be a Jacobi–Jordan conformal algebra. There is a compatible left anti-symmetric conformal algebra structure on A if and only if there exists a bijective $\mathcal{O}$ -operator $T : M \rightarrow A$ associated with some representation (M, π) of A .

Proof. Suppose that there exists a bijective $\mathcal{O}$ -operator $T : M \rightarrow A$ of $(A, \cdot_{\lambda})$ associated with a representation (M, π) . Then, by Theorem 1, with straightforward checking, we have that

$$a *_{\lambda} b = T(\pi(a)_{\lambda} T^{-1}(b)), \quad \forall a, b \in A,$$

defines a compatible left anti-symmetric conformal algebra structure on A .

Conversely, suppose that there is a compatible left anti-symmetric conformal algebra structure on A . Then, the identity map $Id : A \rightarrow A$ is a bijective $\mathcal{O}$ -operator of A associated with the adjoint representation. $\square$

2.3. Symplectic Jacobi–Jordan Conformal Algebras

Next, we introduce a definition of symplectic Jacobi–Jordan conformal algebras and show a relationship with left anti-symmetric conformal algebras. First, let us recall some notations of conformal bilinear forms.

Let V be an $\mathbb{F}[\lambda]$ -module. A conformal bilinear form on V is an $\mathbb{F}$ -linear map $\phi_{\lambda} : V \otimes V \rightarrow \mathbb{F}[\lambda]$ satisfying the following conditions

$$\phi_{\lambda}(\partial u, v) = -\lambda \phi_{\lambda}(u, v), \quad \phi_{\lambda}(u, \partial v) = \lambda \phi_{\lambda}(u, v). \quad (18)$$

A conformal bilinear form is called skew-symmetric if $\phi_{\lambda}(u, v) = -\phi_{-\lambda}(v, u)$. Suppose that there is a conformal bilinear form on a $\mathbb{F}[\partial]$ -module V . Then, we have an $\mathbb{F}[\partial]$ -module homomorphism $T : V \rightarrow V^{*c}$, $u \mapsto T_u$ given by

$$(T_u)_{\lambda} v = \phi_{\lambda}(u, v), \quad u, v \in V.$$

A conformal bilinear form is called non-degenerate if T is an isomorphism of $\mathbb{F}[\partial]$ -modules between V and V^{*c} .

Definition 8. A Jacobi–Jordan conformal algebra A is called a symplectic Jacobi–Jordan conformal algebra if there is a non-degenerate skew-symmetric conformal bilinear form ϕ_λ on A satisfying

$$\phi_\lambda(a, b_\mu c) + \phi_\mu(b, a_\lambda c) + \phi_{\lambda+\mu}(a_\lambda b, c) = 0. \quad (19)$$

Example 3. Let (A, ϕ) be a symplectic Jacobi–Jordan algebra. Then, $(\text{Cur } A, \phi_\lambda)$ is a symplectic Jacobi–Jordan conformal algebra where ϕ_λ is defined by

$$\phi_\lambda(f(\partial)a, g(\partial)b) = f(-\lambda)g(\lambda)\phi(a, b), \quad \forall f(\partial), g(\partial) \in \mathbb{F}[\partial], a, b \in A.$$

The following result relates left anti-symmetric conformal algebras with symplectic Jacobi–Jordan conformal algebras.

Proposition 6. Let $(A, \cdot_\lambda, \phi_\lambda)$ be a symplectic Jacobi–Jordan conformal algebra. Then, there exists a compatible left anti-symmetric conformal algebra structure $\circ_\lambda$ on A given by

$$\phi_\mu(a \circ_\lambda b, c) = \phi_{\mu-\lambda}(b, a_\lambda c), \quad \forall a, b, c \in A. \quad (20)$$

Proof. The proof is straightforward. $\square$

3. Quadratic Jacobi–Jordan Conformal Algebras and Mock-Gel’fand–Dorfman Bialgebras

3.1. Mock-Gel’fand–Dorfman Bialgebras

In this section, we introduce a new class of algebras which will be useful to characterize quadratic Jacobi–Jordan conformal algebras. This class will be called mock-Gel’fand–Dorfman bialgebras. We first provide some relevant definitions and constructions.

Definition 9. An anti-Novikov algebra is a pair $(A, \circ)$, where A is a vector space and $\circ$ is an operation on A satisfying the following axioms:

$$a \circ (b \circ c) = -b \circ (a \circ c), \quad (21)$$

$$(a, b, c)_\circ^- + (a, c, b)_\circ^- = 0, \quad (22)$$

for any $a, b, c \in A$ and where $(a, b, c)_\circ^- = (a \circ b) \circ c + a \circ (b \circ c)$.

Lemma 2. Let $(A, \circ)$ be an anti-Novikov algebra. Then, $(A, *)$ is a Jacobi–Jordan algebra, where, for any $a, b \in A$, $a * b = a \circ b + b \circ a$.

Proposition 7. Let $(A, \cdot)$ be an anti-commutative anti-associative algebra and $d : A \rightarrow A$ be a derivation on A . Then, $(A, \circ)$ is an anti-Novikov algebra, where

$$a \circ b = d(a) \cdot b, \quad \forall a, b \in A.$$

Proof. Let $a, b, c \in A$. Then, we have

$$\begin{aligned} a \circ (b \circ c) &= d(a) \cdot (d(b) \cdot c) = -d(a) \cdot (c \cdot d(b)) = (d(a) \cdot c) \cdot d(b) \\ &= -d(b) \cdot (d(a) \cdot c) = -b \circ (a \circ c). \end{aligned}$$

On the other hand,

$$\begin{aligned}(a \circ b) \circ c + a \circ (b \circ c) &= d(d(a) \cdot b) \cdot c + d(a) \cdot (d(b) \cdot c) \\ &= (d^2(a) \cdot b) \cdot c + (d(a) \cdot d(b)) \cdot c + d(a) \cdot (d(b) \cdot c) \\ &= -d^2(a) \cdot (b \cdot c) = d^2(a) \cdot (c \cdot b).\end{aligned}$$

Since b and c have symmetric role, then we have

$$(a \circ c) \circ b + a \circ (c \circ b) = -d^2(a) \cdot (c \cdot b).$$

Therefore, $(A, \circ)$ is an anti-Novikov algebra. $\square$

Now, we introduce the notion of a mock-Gel'fand–Dorfman bialgebra.

Definition 10. A mock-Gel'fand–Dorfman bialgebra is a vector space A with two operations $\circ$ and $*$, such that $(A, \circ)$ is an anti-Novikov algebra, $(A, *)$ is a Jacobi–Jordan algebra and the following compatibility condition holds (for all $a, b, c \in A$):

$$a \circ (b * c) + a * (b \circ c) + (a * b) \circ c + b \circ (a * c) + b * (a \circ c) = 0. \quad (23)$$

Such an algebra will be denoted by $(A, *, \circ)$.

In the following, we shall simply call a mock-Gel'fand–Dorfman bialgebra a mock-GD bialgebra.

The following construction of mock-GD bialgebras is from anti-Novikov algebras, analogous to the fundamental construction of Gel'fand–Dorfman bialgebras from Novikov algebras via the commutator bracket.

Theorem 2. Let $(A, \circ)$ be an anti-Novikov algebra. Define a new product $*$ on A by

$$a * b = a \circ b + b \circ a, \text{ for } a, b \in A. \quad (24)$$

Then, $(A, *, \circ)$ is a mock-GD bialgebra.

Proof. By Lemma 2, $(A, *)$ is a Jacobi–Jordan algebra. So, we only prove the compatibility condition (23). For all $a, b, c \in A$, we have

$$\begin{aligned}& a \circ (b * c) + a * (b \circ c) + (a * b) \circ c + b \circ (a * c) + b * (a \circ c) \\ &= a \circ (b \circ c) + a \circ (c \circ b) + a \circ (b \circ c) + (b \circ c) \circ a + (a \circ b) \circ c \\ &\quad + (b \circ a) \circ c + b \circ (a \circ c) + b \circ (c \circ a) + b \circ (a \circ c) + (a \circ c) \circ b \\ &= (a \circ b) \circ c + a \circ (b \circ c) + (a \circ c) \circ b + a \circ (c \circ b) \\ &\quad + (b \circ a) \circ c + b \circ (a \circ c) + (b \circ c) \circ a + b \circ (c \circ a) \\ &\quad + a \circ (b \circ c) + b \circ (a \circ c) \\ &= 0.\end{aligned}$$

$\square$

Proposition 8. Let $(A, \cdot)$ be an anti-commutative anti-associative algebra and $d : A \rightarrow A$ be a derivation. Then, $(A, *, \circ)$ is a mock-GD algebra, where $*$ and $\circ$ are defined by

$$a \circ b = d(a) \cdot b, \quad a * b = d(a) \cdot b + d(b) \cdot a, \quad \forall a, b \in A. \quad (25)$$

Example 4. Let A be a 3-dimensional vector space with basis $\{e_1, e_2, e_3\}$. Define the following product on A : $e_1 \cdot e_2 = -e_2 \cdot e_1 = e_3$. Then, it is easy to check that $(A, \cdot)$ is an anti-commutative anti-associative algebra. Consider the linear map $d : A \rightarrow A$ defined by:

$$d(e_1) = e_1, \quad d(e_2) = 2e_2, \quad d(e_3) = 3e_3.$$

Then, d is a derivation on A . Therefore, according to the above proposition, $(A, *, \circ)$ is a mock-GD bialgebra, where

$$e_i \circ e_i = e_i \circ e_i = 0, \quad e_1 \circ e_2 = e_3, \quad e_2 \circ e_1 = -2e_3, \quad e_1 * e_2 = e_2 * e_1 = e_3.$$

3.2. Quadratic Jacobi–Jordan Conformal Algebras and Their Characterization

Definition 11. Let A be a Jacobi–Jordan conformal algebra. If there exists a vector space V such that $A = \mathbb{F}[\partial]V$ is an $\mathbb{F}[\partial]$ -module over V and for all $a, b \in V$ the λ -product is of the form

$$a_\lambda b = \partial u + \lambda v + w,$$

where $u, v, w \in V$, then A is called a quadratic Jacobi–Jordan conformal algebra.

We have the following characterization result.

Theorem 3. Let V be a vector space equipped with two operations $*$ and $\circ$. Let $A = \mathbb{F}[\partial]V$ be the free $\mathbb{F}[\partial]$ -module over V . Define the λ -product $\cdot_\lambda : A \otimes A \rightarrow A[\lambda]$ on A by

$$u_\lambda v = u * v + \partial(u \circ v) + \lambda(u \circ v - v \circ u), \quad u, v \in A. \quad (26)$$

Then, $(A, \cdot_\lambda)$ is a quadratic Jacobi–Jordan conformal algebra if and only if $(V, *, \circ)$ is a mock-GD algebra.

Proof. Suppose that A is a quadratic Jacobi–Jordan conformal algebra. By its definition, we set

$$u_\lambda v = u * v + \partial(u \circ v) + \lambda(u \circ v - v \circ u), \quad \forall u, v \in V, \quad (27)$$

where $*$ and $\circ$ are two $\mathbb{F}$ -bilinear maps from $V \times V \rightarrow V$. Next, we consider axioms (6) and (7). For all $a, b, c \in A$, we have

$$\begin{aligned} b_{-\lambda-\partial}a &= b * a + \partial(b \circ a) - (\lambda + \partial)(b \circ a - a \circ b) \\ &= b * a + \partial(a \circ b) + \lambda(a \circ b - b \circ a). \end{aligned}$$

In addition, we have $a_\lambda b = a * b + \partial(a \circ b) + \lambda(a \circ b - b \circ a)$. Since (6) holds, then

$$a * b = b * a. \quad (28)$$

On the other hand,

$$\begin{aligned} a_\lambda(b_\mu c) &= a_\lambda(b * c + \partial(b \circ c) + \mu(b \circ c - c \circ b)) \\ &= a_\lambda(b * c) + (\lambda + \partial)a_\lambda(b \circ c) + \mu a_\lambda(b \circ c - c \circ b) \\ &= a * (b * c) + \partial(a \circ (b * c)) + \lambda(a \circ (b * c) - (b * c) \circ a) \\ &\quad + (\lambda + \partial)a * (b \circ c) + (\lambda + \partial)\partial(a \circ (b \circ c)) + \lambda(\lambda + \partial)(a \circ (b \circ c) - (b \circ c) \circ a) \\ &\quad + \mu a * (b \circ c - c \circ b) + \mu \partial a \circ (b \circ c - c \circ b) + \mu \lambda(a \circ (b \circ c - c \circ b) - (b \circ c - c \circ b) \circ a) \\ &= a * (b * c) + \partial(a \circ (b * c) + a * (b \circ c)) + \lambda(a \circ (b * c) - (b * c) \circ a + a * (b \circ c)) \end{aligned}$$

$$\begin{aligned}
& +\mu a * (b \circ c - c \circ b) + \lambda \partial (2a \circ (b \circ c) - (b \circ c) \circ a) + \lambda \mu (a \circ (b \circ c - c \circ b) - (b \circ c - c \circ b) \circ a) \\
& + \mu \partial a \circ (b \circ c - c \circ b) + \partial^2 a \circ (b \circ c) + \lambda^2 (a \circ (b \circ c) - (b \circ c) \circ a).
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
b_\mu(a_\lambda c) &= b * (a * c) + \partial(b \circ (a * c) + b * (a \circ c)) + \mu(b \circ (a * c) - (a * c) \circ b + b * (a \circ c)) \\
&+ \lambda b * (a \circ c - c \circ a) + \mu \partial(2b \circ (a \circ c) - (a \circ c) \circ b) + \lambda \partial b \circ (a \circ c - c \circ a) \\
&+ \lambda \mu(b \circ (a \circ c - c \circ a) - (a \circ c - c \circ a) \circ b) + \partial^2 b \circ (a \circ c) + \mu^2(b \circ (a \circ c) - (a \circ c) \circ b).
\end{aligned}$$

Furthermore, we can obtain

$$\begin{aligned}
(a_\lambda b)_{\lambda+\mu} c &= (a * b) * c + \partial(a * b) \circ c + \lambda((a * b) \circ c - c \circ (a * b) - (b \circ a) * c) \\
&+ \mu((a * b) \circ c - c \circ (a * b) - (a \circ b) * c) - \lambda \partial(b \circ a) \circ c - \mu \partial(a \circ b) \circ c \\
&+ \lambda \mu((b \circ a - a \circ b) \circ c + c \circ (a \circ b + b \circ a)) - \mu^2((a \circ b) \circ c - c \circ (a \circ b)) \\
&+ \lambda^2(c \circ (b \circ a) - (b \circ a) \circ c).
\end{aligned}$$

According to the identity (7) and comparing the coefficients of $\lambda^2, \mu^2, \partial^2, \lambda\mu, \lambda\partial, \mu\partial, \lambda, \mu, \partial$ and $\lambda^0\mu^0\partial^0$, we obtain

$$0 = a * (b * c) + (a * b) * c + b * (a * c), \quad (29)$$

$$0 = a \circ (b * c) + a * (b \circ c) + b \circ (a * c) + b * (a \circ c) + (a * b) \circ c, \quad (30)$$

$$\begin{aligned}
0 &= a \circ (b * c) - (b * c) \circ a + a * (b \circ a) + b * (a \circ c - c \circ a) \\
&+ (a * b) \circ c - c \circ (a * b) - (b \circ a) * c,
\end{aligned} \quad (31)$$

$$\begin{aligned}
0 &= a * (b \circ c - c \circ b) + b \circ (a * c) - (a * c) \circ b + b * (a \circ c) \\
&+ (a * b) \circ c - c \circ (a * b) - (a \circ b) * c,
\end{aligned} \quad (32)$$

$$0 = 2a \circ (b \circ c) - (b \circ c) \circ a + b \circ (a \circ c - c \circ a) - (b \circ a) \circ c, \quad (33)$$

$$0 = a \circ (b \circ c - c \circ b) + 2b \circ (a \circ c) - (a \circ c) \circ b - (a \circ b) \circ c, \quad (34)$$

$$\begin{aligned}
0 &= a \circ (b \circ c - c \circ b) - (b \circ c - c \circ b) \circ a + b \circ (a \circ c - c \circ a) \\
&- (a \circ c - c \circ a) \circ b + (b \circ a - a \circ b) \circ c + c \circ (a \circ b + b \circ a),
\end{aligned} \quad (35)$$

$$0 = a \circ (b \circ c) - (b \circ c) \circ a + c \circ (b \circ a) - (b \circ a) \circ c, \quad (36)$$

$$0 = b \circ (a \circ c) - (a \circ c) \circ b + c \circ (a \circ b) - (a \circ b) \circ c, \quad (37)$$

$$0 = a \circ (b \circ c) + b \circ (a \circ c). \quad (38)$$

By (28) and (29), we deduce that $(V, *)$ is a Jacobi–Jordan algebra. Note that (36) and (37) are the same. So, if we take, for example, (37) with (38), we obtain that $(A, \circ)$ is an anti-Novikov algebra. On the other hand, identities (33)–(35) are equivalent to (37). Furthermore, equalities (31) and (32) are equivalent to (30), which is just the compatibility condition (23). Hence, $(A, *, \circ)$ is a mock-GD bialgebra.

Conversely, if $(A, *, \circ)$ is a mock-GD bialgebra, then by a similar computation, we can prove according to the above discussions that $(A, \cdot_\lambda)$ is a Jacobi–Jordan conformal algebra. The proof is finished. $\square$

Example 5. Consider the mock-GD bialgebra constructed in Example 4. Then, there is an associated Jacobi–Jordan conformal algebra whose λ -product is given by

$$e_1 \cdot_\lambda e_2 = (\partial + 3\lambda - 1)e_3, \quad e_2 \cdot_\lambda e_1 = -(2\partial + 3\lambda + 1)e_3,$$

where the non given products are zeros.

4. The $\mathbb{C}[\partial]$ -Split Extending Structures Problem

In this section, the $\mathbb{C}[\partial]$ -split extending structures problem for Jacobi–Jordan conformal algebras is studied. We introduce a definition of unified product of a given Jacobi–Jordan conformal algebra J and a given $\mathbb{C}[\partial]$ -module K . This product includes some other interesting products of Jacobi–Jordan conformal algebras such as the twisted product. Using this product, a cohomological-type object is constructed to provide a theoretical answer to the $\mathbb{C}[\partial]$ -split extending structures problem.

We introduce the following definition which is relevant for studying the $\mathbb{C}[\partial]$ -split extending structures problem.

Definition 12. Let $(J, \cdot_\lambda)$ be a Jacobi–Jordan conformal algebra, K a $\mathbb{C}[\partial]$ -module and $E = J \oplus K$, where the direct sum is the sum of $\mathbb{C}[\partial]$ -modules. Let $\varphi : E \rightarrow E$ be a $\mathbb{C}[\partial]$ -module homomorphism. We consider the following diagram:

$$\begin{array}{ccccc} J & \xrightarrow{i} & E & \xrightarrow{\pi} & K \\ \downarrow Id & & \downarrow \varphi & & \downarrow Id \\ J & \xrightarrow{i} & E & \xrightarrow{\pi} & K \end{array}$$

where $\pi : E \rightarrow K$ is the natural projection of $E = J \oplus K$ onto K and $i : J \rightarrow E$ is the inclusion map. If the left square (alternatively, the right square) of the above diagram are commutative, we say that $\varphi : E \rightarrow E$ stabilizes J (alternatively, co-stabilizes K).

Let $*_\lambda$ and $\square_\lambda$ be two Jacobi–Jordan conformal algebra structures on E both containing J as a Jacobi–Jordan conformal subalgebra. If there exists a Jacobi–Jordan conformal algebra isomorphism $\varphi : (E, *_\lambda) \rightarrow (E, \square_\lambda)$ which stabilizes J , then $*_\lambda$ and $\square_\lambda$ are called equivalent. In this case, we denote it by $(E, *_\lambda) \equiv (E, \square_\lambda)$. If there exists a Jacobi–Jordan conformal algebra isomorphism $\varphi : (E, *_\lambda) \rightarrow (E, \square_\lambda)$ which stabilizes J and co-stabilizes K , then $*_\lambda$ and $\square_\lambda$ are called cohomologous. In this case, we denote it by $(E, *_\lambda) \approx (E, \square_\lambda)$.

Obviously, $\equiv$ and $\approx$ are equivalence relations on the set of all Jacobi–Jordan conformal algebra structures on E containing J as a Jacobi–Jordan conformal subalgebra. We denote the set of all equivalence classes via $\equiv$ (alternatively, $\approx$) by $\text{CExt}_d(E, J)$ (alternatively, $\text{CExt}_d'(E, J)$). It is easy to see that $\text{CExt}_d(E, J)$ is the classifying object of the $\mathbb{C}[\partial]$ -split extending structures problem, and there exists a canonical projection $\text{CExt}_d'(E, J) \rightarrow \text{CExt}_d(E, J)$.

4.1. Unified Products of Jacobi–Jordan Conformal Algebras

In this section, a unified product of Jacobi–Jordan conformal algebras is defined and used to provide a theoretical answer to the $\mathbb{C}[\partial]$ -split extending structures problem for Jacobi–Jordan conformal algebras.

Definition 13. Let $(J, \cdot_\lambda)$ be a Jacobi–Jordan conformal algebra and K a $\mathbb{C}[\partial]$ -module. An extending datum of J by K is a system $\mathcal{U}(J, K) = \{\triangleleft_\lambda, \triangleright_\lambda, \omega_\lambda, \circ_\lambda\}$ consisting of four conformal bilinear maps

$$\begin{aligned} \triangleleft_\lambda : K \times J &\rightarrow K[\lambda], & \triangleright_\lambda : K \times J &\rightarrow J[\lambda], \\ \omega_\lambda : K \times K &\rightarrow J[\lambda], & \circ_\lambda : K \times K &\rightarrow K[\lambda]. \end{aligned}$$

Let $\mathcal{U}(J, K) = \{\triangleleft_\lambda, \triangleright_\lambda, \omega_\lambda, \circ_\lambda\}$ be an extending datum of J by K . $J \natural K$ denotes the $\mathbb{C}[\partial]$ -module $J \oplus K$ with the natural $\mathbb{C}[\partial]$ -module action $\partial(a, x) = (\partial a, \partial x)$ and the bilinear map $*_\lambda : (J \oplus K) \times (J \oplus K) \rightarrow (J \oplus K)[\lambda]$ defined by

$$(a, x) *_\lambda (b, y) = (a_\lambda b + x \triangleright_\lambda b + y \triangleright_{-\lambda-\partial} a + \omega_\lambda(x, y), x \triangleleft_\lambda b + y \triangleleft_{-\lambda-\partial} a + x \circ_\lambda y), \quad (39)$$

for all $a, b \in J, x, y \in K$. Since $\triangleleft_\lambda, \triangleright_\lambda, \omega_\lambda$ and $\circ_\lambda$ are conformal bilinear maps, the $*_\lambda$ -product defined by (39) satisfies conformal sesquilinearity. Then $J \natural K$ is called the unified product of J and K associated with $\mathcal{U}(J, K)$ if it is a Jacobi–Jordan conformal algebra with the $*_\lambda$ -product given by (39). In this case, the extending datum $\mathcal{U}(J, K)$ is called a Jacobi–Jordan conformal extending structure of J through K . We denote the set of all Jacobi–Jordan conformal extending structures of J through K with $\mathcal{S}(J, K)$.

The maps $\triangleleft_\lambda$ and $\triangleright_\lambda$ are called the actions of $\mathcal{U}(J, K)$ and ω_λ is called the cocycle of $\mathcal{U}(J, K)$.

Remark 2. Note that by Equation (39), the following identities hold in $J \natural J$ for any $a, b \in J$ and $x, y \in K$

$$(a, 0) *_\lambda (b, 0) = (a_\lambda b, 0), \quad (a, 0) *_\lambda (0, y) = (y \triangleright_{-\lambda-\partial} a, y \triangleleft_{-\lambda-\partial} a), \quad (40)$$

$$(0, x) *_\lambda (b, 0) = (x \triangleright_\lambda b, x \triangleleft_\lambda b), \quad (0, x) *_\lambda (0, y) = (\omega_\lambda(x, y), x \circ_\lambda y). \quad (41)$$

The next theorem provides the set of axioms that need to be fulfilled by an extending datum $\mathcal{U}(J, K)$ such that $J \natural K$ is a unified product.

Theorem 4. Let J be a Jacobi–Jordan conformal algebra, K be a $\mathbb{F}[\partial]$ -module and $\mathcal{U}(J, K)$ be an extending datum of J by K . Then, $J \natural K$ is a Jacobi–Jordan conformal algebra if and only if the following conditions are satisfied for all $a, b \in J$ and $x, y \in K$

- (U₁) $\omega_\lambda(x, y) = \omega_{-\lambda-\partial}(y, x), \quad x \circ_\lambda y = y \circ_{-\lambda-\partial} x,$
- (U₂) $a_\lambda(x \triangleright_{-\mu-\partial} b) + (x \triangleleft_{-\mu-\partial} b) \triangleright_{-\lambda-\partial} a + x \triangleright_{-\lambda-\mu-\partial} (a_\lambda b) + b_\mu(x \triangleright_{-\lambda-\partial} a) + (x \triangleleft_{-\lambda-\partial} a) \triangleright_{-\mu-\partial} b = 0,$
- (U₃) $(x \triangleleft_{-\mu-\partial} b) \triangleleft_{-\lambda-\partial} a + x \triangleleft_{-\lambda-\mu-\partial} (a_\lambda b) + (x \triangleleft_{-\lambda-\partial} a) \triangleleft_{-\mu-\partial} b = 0,$
- (U₄) $(a_\lambda \omega_\mu(x, y) + (x \circ_\mu y) \triangleright_{-\lambda-\partial} a + (y \triangleright_{-\lambda-\mu-\partial} (x \triangleright_{-\lambda-\partial} a) + \omega_{\lambda+\mu}(x \triangleleft_{-\lambda-\partial} a, y) + (x \triangleright_\mu (y \triangleright_{-\lambda-\partial} a) + \omega_\mu(x, y \triangleleft_{-\lambda-\partial} a) = 0,$
- (U₅) $(x \circ_\mu y) \triangleleft_{-\lambda-\partial} a + y \triangleleft_{-\lambda-\mu-\partial} (x \triangleright_{-\lambda-\partial} a) + (x \triangleleft_{-\lambda-\partial} a) \circ_{\lambda+\mu} y + x \triangleleft_\mu (y \triangleright_{-\lambda-\partial} a) + x \circ_\mu (y \triangleleft_{-\lambda-\partial} a) = 0,$
- (U₆) $x \triangleright_\lambda \omega_\mu(y, z) + \omega_\lambda(x, y \circ_\mu z) + y \triangleright_{-\lambda-\mu-\partial} \omega_\lambda(x, y) + \omega_{\lambda+\mu}(x \circ_\lambda y, z) + y \triangleright_\mu \omega_\lambda(x, z) + \omega_\mu(y, x \circ_\lambda z) = 0,$
- (U₇) $x \triangleleft_\lambda \omega_\mu(y, z) + x \circ_\lambda (y \circ_\mu z) + y \triangleleft_{-\lambda-\mu-\partial} \omega_\lambda(x, y) + (x \circ_\lambda y)_{\lambda+\mu} z + y \triangleleft_\mu \omega_\lambda(x, z) + y \circ_\mu (x \circ_\lambda z) = 0.$

Before addressing the proof of the theorem, we make a few remarks on the compatibility in Theorem 4. Aside from the fact that K is not a Jacobi–Jordan conformal algebra, the identities (U₃) and (U₄) are exactly the compatibility defining a matched pair of Jacobi–Jordan conformal algebras. The compatibility condition (U₅) is called the twisted module condition for the action $\triangleleft_\lambda$; in the case when K is a Jacobi–Jordan conformal algebra, it measures how far $(J, \triangleleft_\lambda)$ is from being a left K -module. The condition (U₆) is called the twisted cocycle condition: if $\triangleleft_\lambda$ is the trivial action and $(K, \circ_\lambda)$ is a Jacobi–Jordan conformal algebra, then the compatibility condition (U₆) is exactly the 2-cocycle condition for Jacobi–Jordan conformal algebras. Finally, the identity (U₇) is called the twisted conformal Jacobi

condition: it measures how far $\circ_\lambda$ is from being a Jacobi–Jordan conformal structure on K . If either $\triangleright_\lambda$ or ω_λ is the trivial map, then (U_7) is equivalent to $\circ_\lambda$ being a Jacobi–Jordan conformal product on K .

Proof. Since $\triangleright_\lambda$, $\triangleleft_\lambda$, ω_λ and $\circ_\lambda$ are conformal linear maps, then the conformal sesquilinearity of $*_\lambda$ is naturally checked. Therefore, we need only to show that the commutativity condition Equation (6) and the Jacobi identity Equation (7) hold for $*_\lambda$ if and only if the identities (U_1) – (U_7) are satisfied.

First, we can easily prove that $*_\lambda$ is commutative, i.e., $(a, x) *_\lambda (b, y) = (b, y) *_{-\lambda-\partial} (a, x)$ if and only if $\omega_\lambda(x, y) = \omega_{-\lambda-\partial}(y, x)$ and $x \circ_\lambda y = y \circ_{-\lambda-\partial} x$; that is, (U_1) holds.

Then, $J\sharp K$ is a Jacobi–Jordan conformal algebra if and only if the conformal Jacobi-identity Equation (7) holds for the $*_\lambda$ -product, i.e.,

$$0 = (a, x) *_\lambda ((b, y) *_\mu (c, z)) + ((a, x) *_\lambda (b, y)) *_{\lambda+\mu} (c, z) + (b, y) *_\mu ((a, x) *_\lambda (c, z)), \quad (42)$$

for all $a, b, c \in J$ and $x, y, z \in K$. Since in $J\sharp K$, we have $(a, x) = (a, 0) + (0, x)$, it follows that (42) holds if and only if it holds for all generators of $J\sharp K$, i.e., the set $\{(a, 0) | a \in J\} \cup \{(0, x) | x \in K\}$. First, we should notice that (42) holds for the triple $(a, 0), (b, 0), (c, 0)$ as we have

$$\begin{aligned} & (a, 0) *_\lambda ((b, 0) *_\mu (c, 0)) + ((a, 0) *_\lambda (b, 0)) *_{\lambda+\mu} (c, 0) + (b, 0) *_\mu ((a, 0) *_\lambda (c, 0)) \\ &= (a_\lambda(b_\mu c) + (a_\lambda b)_{\lambda+\mu} c + b_\mu(a_\lambda c), 0) = (0, 0). \end{aligned}$$

Next, we prove that (42) holds for $(a, 0), (b, 0), (0, x)$ if and only if (U_2) and (U_3) hold. Indeed, we have

$$\begin{aligned} & (a, 0) *_\lambda ((b, 0) *_\mu (0, x)) + ((a, 0) *_\lambda (b, 0)) *_{\lambda+\mu} (0, x) + (b, 0) *_\mu ((a, 0) *_\lambda (0, x)) \\ &= (a, 0) *_\lambda (x \triangleright_{-\mu-\partial} b, x \triangleleft_{-\mu-\partial} b) + (a_\lambda b, 0) *_{\lambda+\mu} (0, x) + (b, 0) *_\mu (x \triangleright_{-\lambda-\partial} a, x \triangleleft_{-\lambda-\partial} a) \\ &= (a_\lambda(x \triangleright_{-\mu-\partial} b) + (x \triangleleft_{-\mu-\partial} b) \triangleright_{-\lambda-\partial} a, (x \triangleleft_{-\mu-\partial} b) \triangleleft_{-\lambda-\partial} a) \\ &+ (x \triangleright_{-\lambda-\mu-\partial} (a_\lambda b), x \triangleleft_{-\lambda-\mu-\partial} (a_\lambda b)) \\ &+ (b_\mu(x \triangleright_{-\lambda-\partial} a) + (x \triangleleft_{-\lambda-\partial} a) \triangleright_{-\mu-\partial} b, (x \triangleleft_{-\lambda-\partial} a) \triangleleft_{-\mu-\partial} b). \end{aligned}$$

The right hand side vanishes if and only if

$$\begin{aligned} & a_\lambda(x \triangleright_{-\mu-\partial} b) + (x \triangleleft_{-\mu-\partial} b) \triangleright_{-\lambda-\partial} a + x \triangleright_{-\lambda-\mu-\partial} (a_\lambda b) \\ &+ b_\mu(x \triangleright_{-\lambda-\partial} a) + (x \triangleleft_{-\lambda-\partial} a) \triangleright_{-\mu-\partial} b = 0 \end{aligned}$$

and

$$(x \triangleleft_{-\mu-\partial} b) \triangleleft_{-\lambda-\partial} a + x \triangleleft_{-\lambda-\mu-\partial} (a_\lambda b) + (x \triangleleft_{-\lambda-\partial} a) \triangleleft_{-\mu-\partial} b = 0.$$

Thus, (U_2) and (U_3) hold. Now, we show that (42) holds for $(a, 0), (0, x), (0, y)$ if and only if (U_4) and (U_5) hold. Indeed, we have

$$\begin{aligned} & (a, 0) *_\lambda ((0, x) *_\mu (0, y)) + ((a, 0) *_\lambda (0, x)) *_{\lambda+\mu} (0, y) + (0, x) *_\mu ((a, 0) *_\lambda (0, y)) \\ &= (a, 0) *_\lambda (\omega_\mu(x, y), x \circ_\mu y) + (x \triangleright_{-\lambda-\partial} a, x \triangleleft_{-\lambda-\partial} a) *_{\lambda+\mu} (0, y) \\ &+ (0, x) *_\mu (y \triangleright_{-\lambda-\partial} a, y \triangleleft_{-\lambda-\partial} a) \\ &= (a_\lambda \omega_\mu(x, y) + (x \circ_\mu y) \triangleright_{-\lambda-\partial} a, (x \circ_\mu y) \triangleleft_{-\lambda-\partial} a) \\ &+ (y \triangleright_{-\lambda-\mu-\partial} (x \triangleright_{-\lambda-\partial} a) + \omega_{\lambda+\mu}(x \triangleleft_{-\lambda-\partial} a, y), y \triangleleft_{-\lambda-\mu-\partial} (x \triangleright_{-\lambda-\partial} a) + (x \triangleleft_{-\lambda-\partial} a) \circ_{\lambda+\mu} y) \\ &+ (x \triangleright_\mu (y \triangleright_{-\lambda-\partial} a) + \omega_\mu(x, y \triangleleft_{-\lambda-\partial} a), x \triangleleft_\mu (y \triangleright_{-\lambda-\partial} a) + x \circ_\mu (y \triangleleft_{-\lambda-\partial} a)). \end{aligned}$$

Then, $(a, 0) *_{\lambda} ((0, x) *_{\mu} (0, y)) + ((a, 0) *_{\lambda} (0, x)) *_{\lambda+\mu} (0, y) + (0, x) *_{\mu} ((a, 0) *_{\lambda} (0, y)) = 0$ if and only if

$$(a_{\lambda} \omega_{\mu}(x, y) + (x \circ_{\mu} y) \triangleright_{-\lambda-\partial} a + (y \triangleright_{-\lambda-\mu-\partial} (x \triangleright_{-\lambda-\partial} a) + \omega_{\lambda+\mu}(x \triangleleft_{-\lambda-\partial} a, y) + (x \triangleright_{\mu} (y \triangleright_{-\lambda-\partial} a) + \omega_{\mu}(x, y \triangleleft_{-\lambda-\partial} a) = 0$$

and

$$(x \circ_{\mu} y) \triangleleft_{-\lambda-\partial} a + y \triangleleft_{-\lambda-\mu-\partial} (x \triangleright_{-\lambda-\partial} a) + (x \triangleleft_{-\lambda-\partial} a) \circ_{\lambda+\mu} y + x \triangleleft_{\mu} (y \triangleright_{-\lambda-\partial} a) + x \circ_{\mu} (y \triangleleft_{-\lambda-\partial} a) = 0.$$

Thus, (U_4) and (U_5) hold. Finally, we prove that (42) holds for $(0, x), (0, y), (0, z)$ if and only if (U_6) and (U_7) hold. Indeed, we have

$$\begin{aligned} & (0, x) *_{\lambda} ((0, y) *_{\mu} (0, z)) + ((0, x) *_{\lambda} (0, y)) *_{\lambda+\mu} (0, z) + (0, y) *_{\mu} ((0, x) *_{\lambda} (0, z)) \\ &= (0, x) *_{\lambda} (\omega_{\mu}(y, z), y \circ_{\mu} z) + (\omega_{\lambda}(x, y), x \circ_{\lambda} y) *_{\lambda+\mu} (0, z) + (0, y) *_{\mu} (\omega_{\lambda}(x, z), x \circ_{\lambda} z) \\ &= (x \triangleright_{\lambda} \omega_{\mu}(y, z) + \omega_{\lambda}(x, y \circ_{\mu} z), x \triangleleft_{\lambda} \omega_{\mu}(y, z) + x \circ_{\lambda} (y \circ_{\mu} z)) \\ &+ (y \triangleright_{-\lambda-\mu-\partial} \omega_{\lambda}(x, y) + \omega_{\lambda+\mu}(x \circ_{\lambda} y, z), y \triangleleft_{-\lambda-\mu-\partial} \omega_{\lambda}(x, y) + (x \circ_{\lambda} y)_{\lambda+\mu} z) \\ &+ (y \triangleright_{\mu} \omega_{\lambda}(x, z) + \omega_{\mu}(y, x \circ_{\lambda} z), y \triangleleft_{\mu} \omega_{\lambda}(x, z) + y \circ_{\mu} (x \circ_{\lambda} z)). \end{aligned}$$

Therefore, $(0, x) *_{\lambda} ((0, y) *_{\mu} (0, z)) + ((0, x) *_{\lambda} (0, y)) *_{\lambda+\mu} (0, z) + (0, y) *_{\mu} ((0, x) *_{\lambda} (0, z)) = 0$ if and only if

$$\begin{aligned} & x \triangleright_{\lambda} \omega_{\mu}(y, z) + \omega_{\lambda}(x, y \circ_{\mu} z) + y \triangleright_{-\lambda-\mu-\partial} \omega_{\lambda}(x, y) + \omega_{\lambda+\mu}(x \circ_{\lambda} y, z) \\ &+ y \triangleright_{\mu} \omega_{\lambda}(x, z) + \omega_{\mu}(y, x \circ_{\lambda} z) = 0 \end{aligned}$$

and

$$\begin{aligned} & x \triangleleft_{\lambda} \omega_{\mu}(y, z) + x \circ_{\lambda} (y \circ_{\mu} z) + y \triangleleft_{-\lambda-\mu-\partial} \omega_{\lambda}(x, y) + (x \circ_{\lambda} y)_{\lambda+\mu} z \\ &+ y \triangleleft_{\mu} \omega_{\lambda}(x, z) + y \circ_{\mu} (x \circ_{\lambda} z) = 0. \end{aligned}$$

Hence, the proof is finished. $\square$

Next, we use the following convention: If one of the maps $\triangleleft_{\lambda}, \triangleright_{\lambda}, \omega_{\lambda}$ or $\circ_{\lambda}$ of an extending datum is trivial, we will omit it from $\mathcal{U}(J, K) = \{\triangleleft_{\lambda}, \triangleright_{\lambda}, \omega_{\lambda}, \circ_{\lambda}\}$.

From now on, in light of Theorem 4, a Jacobi–Jordan conformal extending structure of J through K will be viewed as a system $\mathcal{U}(J, K) = \{\triangleleft_{\lambda}, \triangleright_{\lambda}, \omega_{\lambda}, \circ_{\lambda}\}$ satisfying the compatibility conditions (U_1) – (U_7) .

Example 6. Let $\mathcal{U}(J, K) = \{\triangleleft_{\lambda}, \triangleright_{\lambda}, \omega_{\lambda}, \circ_{\lambda}\}$ be an extending datum of a Jacobi–Jordan conformal algebra J through a $\mathbb{F}[\partial]$ -module K , where $\triangleright_{\lambda}, \omega_{\lambda}$ and $\circ_{\lambda}$ are trivial conformal bilinear maps. Then, the corresponding extending datum denoted by $\mathcal{U}(J, K) = \{\triangleleft_{\lambda}\}$ is a Jacobi–Jordan conformal extending structure of J by K if and only if K is a right K -module under $\triangleright_{\lambda}$. Therefore, the associated unified product is just the semi-direct product of J and K .

Given an extending structure $\mathcal{U}(J, K)$, it is obvious that J can be seen as a Jacobi–Jordan conformal subalgebra of $J \bowtie K$. Conversely, we will prove that any Jacobi–Jordan conformal algebra structure on a vector space E containing J as a subalgebra is isomorphic to a unified product.

Theorem 5. Let J be a Jacobi–Jordan conformal algebra and K an $\mathbb{F}[\partial]$ -module. Let $E = J \oplus K$, where the direct sum is the sum of $\mathbb{F}[\partial]$ -modules. Suppose that E has a Jacobi–Jordan conformal algebra structure $*_{\lambda}$ such that J is a subalgebra. Then, there exists a Jacobi–Jordan conformal extending structure $\mathcal{U}(J, K) = \{\triangleleft_{\lambda}, \triangleright_{\lambda}, \omega_{\lambda}, \circ_{\lambda}\}$ of J by K and an isomorphism of Jacobi–Jordan conformal algebra $E \cong J \natural K$ which stabilizes J and co-stabilizes K .

Proof. Since $E = J \oplus K$, there exists a natural $\mathbb{F}[\partial]$ -homomorphism $p : E \rightarrow J$ such that $p(a) = a$ for any $a \in J$. Then, we can define an extending datum $\mathcal{U}(J, K) = \{\triangleleft_{\lambda}, \triangleright_{\lambda}, \omega_{\lambda}, \circ_{\lambda}\}$ of J by K as follows ($a \in J$ and $x, y \in K$)

$$\triangleright_{\lambda} : K \times J \rightarrow J[\lambda], \quad x \triangleright_{\lambda} a = p(x *_{\lambda} a) \quad (43)$$

$$\triangleleft_{\lambda} : K \times J \rightarrow K[\lambda], \quad x \triangleleft_{\lambda} a = x *_{\lambda} a - p(x *_{\lambda} a) \quad (44)$$

$$\omega_{\lambda} : K \times K \rightarrow J[\lambda], \quad \omega_{\lambda}(x, y) = p(x *_{\lambda} y) \quad (45)$$

$$\circ_{\lambda} : K \times K \rightarrow K[\lambda], \quad x \circ_{\lambda} y = x *_{\lambda} y - p(x *_{\lambda} y). \quad (46)$$

Obviously, $\triangleleft_{\lambda}, \triangleright_{\lambda}, \omega_{\lambda}$ and $\circ_{\lambda}$ are four conformal bilinear maps. Next, we shall prove that $\mathcal{U}(J, K) = \{\triangleleft_{\lambda}, \triangleright_{\lambda}, \omega_{\lambda}, \circ_{\lambda}\}$ of J is a Jacobi–Jordan conformal extending structure and $E \cong J \natural K$ is a Jacobi–Jordan conformal algebra. Let $\psi : J \times K \rightarrow E$ by $\psi(a, x) = a + x$. It is easy to see that ψ is an $\mathbb{F}[\partial]$ -module isomorphism. Its inverse is $\psi^{-1} : E \rightarrow J \times K$ defined by $\psi^{-1}(e) = (p(e), e - p(e))$. Therefore, if ψ is also an isomorphism of Jacobi–Jordan conformal algebra, there exists a unique λ -product $\square_{\lambda}$ on $J \times K$ given by

$$(a, x) \square_{\lambda} (b, y) = \psi^{-1}(\psi(a, x) *_{\lambda} \psi(b, y)), \quad (47)$$

for all $a, b \in J$ and $x, y \in K$. Hence, to finish the proof, we only need to prove that the λ -product defined by (47) is just the one given by (39) associated with the above extending system $\mathcal{U}(J, K) = \{\triangleleft_{\lambda}, \triangleright_{\lambda}, \omega_{\lambda}, \circ_{\lambda}\}$. We check it as follows

$$\begin{aligned} & (a, x) \square_{\lambda} (b, y) \\ &= \psi^{-1}(\psi(a, x) *_{\lambda} \psi(b, y)) \\ &= \psi^{-1}((a + x) *_{\lambda} (b + y)) \\ &= \psi^{-1}(a *_{\lambda} b + a *_{\lambda} y + x *_{\lambda} b + x *_{\lambda} y) \\ &= (p(a *_{\lambda} b) + p(a *_{\lambda} y) + p(x *_{\lambda} b) + p(x *_{\lambda} y), \\ &\quad a *_{\lambda} b + a *_{\lambda} y + x *_{\lambda} b + x *_{\lambda} y - p(a *_{\lambda} b) - p(a *_{\lambda} y) - p(x *_{\lambda} b) - p(x *_{\lambda} y)) \\ &= (a_{\lambda} b + p(y *_{-\lambda-\partial} a) + p(x *_{\lambda} b) + p(x *_{\lambda} y), \\ &\quad y *_{-\lambda-\partial} a - p(y *_{-\lambda-\partial} a) + x *_{\lambda} b - p(x *_{\lambda} b) + x *_{\lambda} y - p(x *_{\lambda} y)) \\ &= (a_{\lambda} b + y \triangleright_{-\lambda-\partial} a + x \triangleright_{\lambda} b + \omega_{\lambda}(x, y), y \triangleleft_{-\lambda-\partial} a + x \triangleleft_{\lambda} b + x \circ_{\lambda} y) \\ &= (a, x) *_{\lambda} (b, y). \end{aligned}$$

Moreover, it is easy to see that ψ stabilizes J and co-stabilizes K . Then, the proof is finished. $\square$

Definition 14. Let J be a Jacobi–Jordan conformal algebra and K an $\mathbb{F}[\partial]$ -module. Two Jacobi–Jordan conformal algebra extending structures of J by K , $\mathcal{U}(J, K) = \{\triangleleft_{\lambda}, \triangleright_{\lambda}, \omega_{\lambda}, \circ_{\lambda}\}$ and $\mathcal{U}'(J, K) = \{\triangleleft'_{\lambda}, \triangleright'_{\lambda}, \omega'_{\lambda}, \circ'_{\lambda}\}$ are called equivalent, and we denote this by $\mathcal{U}(J, K) \equiv \mathcal{U}'(J, K)$ if there exists a pair of $\mathbb{F}[\partial]$ -module homomorphisms (r, s) , where $r : K \rightarrow J$ and $s \in \text{Aut}_{\mathbb{F}[\partial]}(K)$ such that $\{\triangleleft'_{\lambda}, \triangleright'_{\lambda}, \omega'_{\lambda}, \circ'_{\lambda}\}$ is implemented from $\{\triangleleft_{\lambda}, \triangleright_{\lambda}, \omega_{\lambda}, \circ_{\lambda}\}$ using (r, s) via the following:

$$x \triangleright_{-\lambda-\partial} a + r(x \triangleleft_{-\lambda-\partial} a) = a_{\lambda} r(x) + s(x) \triangleright'_{-\lambda-\partial} a, \quad (48)$$

$$s(x \triangleleft_{-\lambda-\partial} a) = s(x) \triangleleft'_{-\lambda-\partial} a, \quad (49)$$

$$\omega_\lambda(x, y) + r(x \circ_\lambda y) = r(x)_\lambda r(y) + s(x) \triangleright'_\lambda r(y) + s(y) \triangleright'_{-\lambda-\partial} r(x) + \omega'_\lambda(s(x), s(y)), \quad (50)$$

$$s(x \circ_\lambda y) = s(x) \triangleleft'_\lambda r(y) + s(y) \triangleleft'_{-\lambda-\partial} r(x) + s(x) \circ'_\lambda s(y), \quad (51)$$

for all $a \in J$ and $x, y \in K$. In particular, if $s = id$, $\mathcal{U}(J, K)$ and $\mathcal{U}'(J, K)$ are called cohomologous, we denote them by $\mathcal{U}(J, K) \approx \mathcal{U}'(J, K)$.

Theorem 5 shows that the classification of all Jacobi–Jordan conformal algebra structures on E that contain J as a subalgebra comes down to the classification of the unified products $J \natural K$ for a given complement K of J in E . In order to describe the classifying sets $Jextd(E, J)$, we need the following:

Lemma 3. Suppose that $\mathcal{U}(J, K) = \{\triangleleft_\lambda, \triangleright_\lambda, \omega_\lambda, \circ_\lambda\}$ and $\mathcal{U}'(J, K) = \{\triangleleft'_\lambda, \triangleright'_\lambda, \omega'_\lambda, \circ'_\lambda\}$ are two Jacobi–Jordan conformal extending structures of J by K . Let $J \natural K$ and $J \natural' K$ be the corresponding unified products. Then, $J \natural K \equiv J \natural' K$ if and only if $\mathcal{U}(J, K) \equiv \mathcal{U}'(J, K)$ and $J \natural K \approx J \natural' K$ if and only if $\mathcal{U}(J, K) \approx \mathcal{U}'(J, K)$.

Proof. Let $\phi : J \natural K \rightarrow J \natural' K$ be an isomorphism of Jacobi–Jordan conformal algebras which stabilizes J . According to that ϕ stabilizes J , $\phi(a, 0) = (a, 0)$. Moreover, we can set $\phi(0, x) = (r(x), s(x))$, where $r : K \rightarrow J$ and $s : K \rightarrow K$ are two linear maps. Therefore, we obtain $\phi(a, x) = (a + r(x), s(x))$. Furthermore, it is easy to see that ϕ is an $\mathbb{F}[\partial]$ -module homomorphism if and only if r and s are two $\mathbb{F}[\partial]$ -module homomorphisms. Then, we shall prove that ϕ is a Jacobi–Jordan conformal algebra homomorphism, i.e.,

$$\phi((a, x) *_\lambda (b, y)) = \phi(a, x) *'_\lambda \phi(b, y), \quad (52)$$

for all $a, b \in J$ and $x, y \in K$ if and only if Equations (48)–(51) hold. It is enough to check that (52) holds for all generators of $J \natural K$. Obviously, (52) holds for the pair $(a, 0), (b, 0)$. Then, we consider (52) for the pair $(a, 0), (0, x)$. According to

$$\begin{aligned} \phi((a, 0) *_\lambda (0, x)) &= \phi(x \triangleright_{-\lambda-\partial} a, x \triangleleft_{-\lambda-\partial} a) \\ &= (x \triangleright_{-\lambda-\partial} a + r(x \triangleleft_{-\lambda-\partial} a), s(x \triangleleft_{-\lambda-\partial} a)) \end{aligned}$$

and

$$\begin{aligned} \phi(a, 0) *'_\lambda \phi(0, x) &= (a, 0) *'_\lambda (r(x), s(x)) \\ &= (a_\lambda r(x) + s(x) \triangleright'_{-\lambda-\partial} a, s(x) \triangleleft'_{-\lambda-\partial} a), \end{aligned}$$

we obtain that (52) holds for the pair $(a, 0), (0, x)$ if and only if (48) and (49) hold. Next, we consider (52) for the pair $(0, x), (0, y)$. We have

$$\begin{aligned} \phi((0, x) *_\lambda (0, y)) &= \phi(\omega_\lambda(x, y), x \circ_\lambda y) \\ &= (\omega_\lambda(x, y) + r(x \circ_\lambda y), s(x \circ_\lambda y)) \end{aligned}$$

and

$$\begin{aligned} \phi(0, x) *'_\lambda \phi(0, y) &= (r(x), s(x)) *'_\lambda (r(y), s(y)) \\ &= (r(x)_\lambda r(y) + s(x) \triangleright'_\lambda r(y) + s(y) \triangleright'_{-\lambda-\partial} r(x) + \omega'_\lambda(s(x), s(y)), \\ &\quad s(x) \triangleleft'_\lambda r(y) + s(y) \triangleleft'_{-\lambda-\partial} r(x) + s(x) \circ'_\lambda s(y)), \end{aligned}$$

which means that (52) holds if and only if (50) and (51) hold. $\square$

By putting together all the results proved in this section, we obtain the following theoretical answer to the extending structures problem for Jacobi–Jordan conformal algebras

Theorem 6. Let J be a Jacobi–Jordan conformal algebra and K be an $\mathbb{F}[\partial]$ -module. Let $E = J \oplus K$, where the direct sum is the sum of $\mathbb{F}[\partial]$ -modules. Then, we obtain:

1. Let $\mathcal{H}_J^2(K, J) := \mathcal{S}(J, K) / \equiv$. Then, the map

$$\mathcal{H}_J^2(K, J) \rightarrow \text{Ext}_d(E, J), \quad \overline{\mathcal{U}(J, K)} \mapsto (J \natural K, *_{\lambda})$$

is bijective, where $\overline{\mathcal{U}(J, K)}$ is the equivalence class of $\mathcal{U}(J, K)$ with respect to $\equiv$.

2. Let $\mathcal{H}^2(K, J) := \mathcal{S}(J, K) / \approx$. Then, the map

$$\mathcal{H}^2(K, J) \rightarrow \text{Ext}_d(E, J), \quad \overline{\overline{\mathcal{U}(J, K)}} \mapsto (J \natural K, *_{\lambda})$$

is bijective, where $\overline{\overline{\mathcal{U}(J, K)}}$ is the equivalence class of $\mathcal{U}(J, K)$ with respect to $\approx$.

4.2. Applications to Special Cases of Unified Products

In this section, we introduce some interesting products of Jacobi–Jordan conformal algebras such as twisted products, crossed products and bicrossed products, which are all special cases of unified products. These products will be useful for studying the structure theory of Jacobi–Jordan conformal algebras.

4.2.1. Twisted Products of Jacobi–Jordan Conformal Algebras

Let $\mathcal{U}(J, K) = \{\triangleleft_{\lambda}, \triangleright_{\lambda}, \omega_{\lambda}, \circ_{\lambda}\}$ be an extending datum of a Jacobi–Jordan conformal algebra J through an $\mathbb{F}[\partial]$ -module K , where $\triangleright_{\lambda}$ and $\triangleleft_{\lambda}$ are trivial conformal bilinear maps. Then, by Theorem 4, $\mathcal{U}(J, K) = \{\omega_{\lambda}, \circ_{\lambda}\}$ is a Jacobi–Jordan conformal extending structure of J by K if and only if $(K, \circ_{\lambda})$ is a Jacobi–Jordan conformal algebra and $\omega_{\lambda} : K \times K \rightarrow J[\lambda]$ satisfies

$$\omega_{\lambda}(x, y) = \omega_{-\lambda-\partial}(y, x), \quad (53)$$

$$\omega_{\lambda}(x, y \circ_{\mu} z) + \omega_{\lambda+\mu}(x \circ_{\lambda} y, z) + \omega_{\mu}(y, x \circ_{\lambda} z) = 0, \quad (54)$$

for all $a \in J$ and $x, y, z \in K$. The associated unified product $J \natural K$ is called the twisted product of J and K . The product on $J \natural K$ is given for all $a, b \in J$ and $x, y \in K$ by

$$(a, x) *_{\lambda} (b, y) = (a_{\lambda} b + \omega_{\lambda}(x, y), x \circ_{\lambda} y). \quad (55)$$

4.2.2. Crossed Products of Jacobi–Jordan Conformal Algebras

Let $\mathcal{U}(J, K) = \{\triangleleft_{\lambda}, \triangleright_{\lambda}, \omega_{\lambda}, \circ_{\lambda}\}$ be an extending datum of a Jacobi–Jordan conformal algebra J through an $\mathbb{F}[\partial]$ -module K , where $\triangleleft_{\lambda}$ is the trivial conformal bilinear map. Then, $\mathcal{U}(J, K)$ is a Jacobi–Jordan conformal extending structure of J through K if and only if $(K, \circ_{\lambda})$ is a Jacobi–Jordan conformal algebra and the following conditions are satisfied:

$$\omega_{\lambda}(x, y) = \omega_{-\lambda-\partial}(y, x), \quad (56)$$

$$a_{\lambda}(x \triangleright_{-\mu-\partial} b) + x \triangleright_{-\lambda-\mu-\partial} (a_{\lambda} b) + b_{\mu}(x \triangleright_{-\lambda-\partial} a) = 0, \quad (57)$$

$$a_{\lambda} \omega_{\mu}(x, y) + (x \circ_{\mu} y) \triangleright_{-\lambda-\partial} a + y \triangleright_{-\lambda-\mu-\partial} (x \triangleright_{-\lambda-\partial} a) + x \triangleright_{\mu} (y \triangleright_{-\lambda-\partial} a) = 0, \quad (58)$$

$$x \triangleright_{\lambda} \omega_{\mu}(y, z) + \omega_{\lambda}(x, y \circ_{\mu} z) + y \triangleright_{-\lambda-\mu-\partial} \omega_{\lambda}(x, y) + \omega_{\lambda+\mu}(x \circ_{\lambda} y, z) \quad (59)$$

$$+ y \triangleright_{\mu} \omega_{\lambda}(x, z) + \omega_{\mu}(y, x \circ_{\lambda} z) = 0 \quad (60)$$

for all $a, b \in J$ and $x, y, z \in K$. The associated unified product is denoted by $J \bowtie_{\triangleright}^{\omega} K$ and called the crossed product of J through K . The λ -product on $J \bowtie_{\triangleright}^{\omega} K$ is given for all $a, b \in J$ and $x, y \in K$, by

$$(a, x) *_{\lambda} (b, y) = (a_{\lambda} b + x \triangleright_{\lambda} b + y \triangleright_{-\lambda - \partial} a + \omega_{\lambda}(x, y), x \circ_{\lambda} y).$$

It is obvious that J is an ideal of $J \bowtie_{\triangleright}^{\omega} K$ and one can directly obtain by Theorem 4 the following result:

Proposition 9. *Let J be a Jacobi–Jordan conformal algebra and K a $\mathbb{C}[\partial]$ -module. Let $E = J \oplus K$, where the direct sum is the sum of $\mathbb{C}[\partial]$ -modules. Assume that E has a Jacobi–Jordan conformal algebra structure $*_{\lambda}$ such that J is an ideal of E . Then, E is isomorphic to a crossed product $J \bowtie_{\triangleright}^{\omega} K$ of Jacobi–Jordan conformal algebras.*

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