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The Adapted Solutions for Backward Stochastic Schrödinger Equations with Jumps

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Abstract: This study considers a class of backward stochastic semi-linear Schrödinger equations with Poisson jumps in $\mathbb{R}^d$ or in its bounded domain of a C^2 boundary, which is associated with a stochastic control problem of nonlinear Schrödinger equations driven by Lévy noise. The approach to establish the existence and uniqueness of solutions is mainly based on the complex Itô formula, the Galerkin's approximation method, and the martingale representation theorem.

Keywords: backward stochastic Schrödinger equation; weak solution; Poisson jump; Galerkin's finite-dimensional approximation

MSC: 60H05; 60H15; 60G99

1. Introduction

The backward stochastic differential equations (BSDEs) first appeared as the adjoint equation for a stochastic control problem and were investigated by Bismut [1]. Later, Pardoux and Peng [2] introduced the general case of BSDEs. Since then, there have been many extension forms of BSDEs, and the BSDE theory has been applied to optimal control [3], finance [4], partial differential equations [5,6], and numerical approximation [7].

Backward stochastic partial differential equations (BSPDEs), a natural mathematical extension of BSDEs, have wide-ranging applications in probability theory and stochastic processes. For instance, they arise in the optimal control of SDEs with incomplete information. They also serve as adjoint equations for Duncan–Mortensen–Zakai filtering equations (see, e.g., [8]) to formulate the stochastic maximum principle for optimal control, and they play a key role in the stochastic Feynman–Kac formula (see, e.g., [9]) in mathematical finance. A specific class of fully nonlinear BSPDEs, known as backward stochastic Hamilton–Jacobi–Bellman equations, naturally emerges in the dynamic programming theory for controlled non-Markovian processes (see [10]). Further developments can be found in [11–15] and the references therein.

In practice, many random phenomena exhibit discontinuous motion characteristics with jumps. To model such behavior, the Poisson jump process, which captures discontinuous random phenomena, was introduced. In 1994, Tang and Li [16] applied Peng's framework [2] to establish the first result on the existence of an adapted solution to a BSDE with jumps driven by a Poisson point process independent of Brownian motion for a fixed terminal time and with Lipschitz coefficients. In this context, if the Brownian motion represents financial market noise, the Poisson point process can be interpreted as the randomness of insurance claims. For related results, see [17–26] and the references therein. In particular, Ref. [27] first studied a class of BSPDEs with jumps, which appeared

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as adjoint equations in the maximum principle approach to the optimal control of systems described by SPDEs driven by Lévy processes; see [28,29] and the references therein for further developments.

In this study, we consider the following backward stochastic semi-linear Schrödinger equation with jumps (BSSEJs):

$$\begin{cases} \mathrm{i}dy_t + \Delta y_t dt = f(t, y_t, z_t, q_t)dt + z_t dW_t + \int_U q_t(x)\tilde{N}_p(dt, dx), t \in [0, T]; \\ y_T = \varphi, \end{cases} \tag{1}$$

where Δ is the Laplacian operator acting on the space variable. Here, $\mathrm{i} = \sqrt{-1}$ is the imaginary unit; $W = \{W_t\}_{t\geq 0}$ is a one-dimensional standard Wiener process on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$; $N_p(t, x)$ is the Poisson counting measure induced by a stationary Poisson point process; $p(x)$ takes values in a measurable space $(U, \mathcal{U})$ with the character measure $\lambda(dx)dt$; and $\tilde{N}_p(t, x) = N_p(t, x) - \lambda(dx)dt$ is the martingale measure, which is specified later.

The stochastic Schrödinger equation, utilized to characterize the evolution of modulated wave envelopes or the transformation of a quantum state under random perturbations, finds extensive application across various disciplines, including plasma physics, nonlinear optics, hydrodynamics, and quantum field theory. Numerous scholars have extensively studied forward stochastic Schrödinger equations; see [30–32] and the references therein. Regarding stochastic nonlinear Schrödinger equations with jump noise, models of this nature were initially introduced in [33,34] to account for signal amplification in optical fibers at discrete, randomly occurring locations due to material inhomogeneities. For further advancements, readers are directed to [35–40]. Concerning the investigation of BSSEs, we refer to [41], where the existence and uniqueness of weak solutions were established under Lipschitz conditions.

The motivation for this research stems from the fact that system (1) exhibits a blend of properties characteristic of both parabolic and hyperbolic equations, making it equivalent to the following real BSPDE with jumps:

$$\begin{aligned} d\begin{pmatrix} \mathrm{Re}\,(y_t) \\ \mathrm{Im}\,(y_t) \end{pmatrix} &+ \begin{pmatrix} 0 & \Delta \\ -\Delta & 0 \end{pmatrix} \begin{pmatrix} \mathrm{Re}\,(y_t) \\ \mathrm{Im}\,(y_t) \end{pmatrix} dt \\ &= \begin{pmatrix} \mathrm{Im}\,(f(t, y_t, z_t, q_t)) \\ \mathrm{Re}\,(f(t, y_t, z_t, q_t)) \end{pmatrix} dt + \begin{pmatrix} \mathrm{Im}\,(z_t) \\ -\mathrm{Re}\,(z_t) \end{pmatrix} dW_t + \int_U \begin{pmatrix} \mathrm{Im}\,(q_t) \\ -\mathrm{Re}\,(q_t) \end{pmatrix} \tilde{N}_p(dt, dx), \end{aligned}$$

where $\mathrm{Re}(u)$ and $\mathrm{Im}(u)$ denote the real and imaginary parts of a complex number u, respectively. It is important to note that the differential operator $\begin{pmatrix} 0 & \Delta \\ -\Delta & 0 \end{pmatrix}$ does not possess symmetric positive definiteness and fails to meet the monotonicity in [27,29]. Consequently, the existing techniques within the infinite-dimensional framework, as discussed in these works, are insufficient for solving system (1). Therefore, it becomes imperative to establish the existence and uniqueness of solutions to the BSSEJ (1). Additionally, BSSEJs (1) hold significant interest because of their role as adjoint equations in the maximum principle approach to optimal control of stochastic Schrödinger equations, as seen in [42,43].

In this research, we aim to prove the existence and uniqueness of adapted solutions for BSSEJ (1). Given the low regularity of the solution, direct application of the Itô formula for a priori estimates is not feasible. To address this, we initially employ Galerkin's approximation method, which transforms the original system into a finite-dimensional one by leveraging the eigenvalue problem of the Laplacian operator in bounded domains. Notably, when applying Itô's complex formula to the finite-dimensional system, the higher-

order partial derivative term of y vanishes. Subsequently, we seek to derive uniform a priori estimates for the triplet (y, z, q) and demonstrate the existence and uniqueness of solutions through a combination of compactness arguments and the martingale representation theorem. For the case of the entire space $\mathbb{R}^d$, truncation and approximation techniques are utilized to achieve the desired results.

The structure of this research is as follows. Section 2 outlines the assumptions and states the main result, Theorem 1. Section 3 investigates the existence of adapted solutions for Galerkin's approximation system of Equation (1) and provides estimates for the approximation equation. Sections 4 and 5, respectively, prove the existence and uniqueness of solutions for (1).

2. Preliminaries

Before starting our main results, we give some necessary notions and definitions. Let $\mathbb{R}^d$ be a d-dimensional Euclidean space and D be the whole space $\mathbb{R}^d$ or its bounded domain of boundary $\partial D \in C^2$. Denote by H the complex Hilbert space $L^2(D;\mathbb{C})$ equipped with the inner product $(\cdot,\cdot)$ and the norm $\|\cdot\|$, which are defined as follows:

$$(u,v) := \int_D u(x)\overline{v(x)}dx, \quad \|u\| := (u,u)^{\frac{1}{2}}.$$

For $m \in \mathbb{N}$, let $H^m(D)$ be the Sobolev space $W^{m,2}(D)$ and $H_0^m(D) = W_0^{m,2}(D)$.

Denote by $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ the real and imaginary parts of a complex z, respectively. $\mathrm{i} = \sqrt{-1}$ is the imaginary unit.

Let $(\Omega, \mathcal{F}, P)$ be a given complete probability space. $W := \{W_t\}_{t\geq 0}$ is a one-dimensional standard Wiener process on $(\Omega, \mathcal{F}, P)$. $p(x)$ is a stationary Poisson point process on $(\Omega, \mathcal{F}, P)$, and $N_p(t,x)$ is the Poisson counting measure induced by $p(x)$ taking values in a measurable space $(U, \mathcal{U})$ with the character measure $\lambda(dx)dt$. Then, we define the martingale measure by $\tilde{N}_p(t,x) = N_p(t,x) - \lambda(dx)dt$. We denote the natural filtration $(\mathcal{F}_t) = \sigma[W_s; s \leq t] \vee \sigma[N_p((0,s],A); s \leq t, A \in \mathcal{U}] \vee \mathcal{N}$, where $\mathcal{N}$ denote the collection of all $\mathcal{N}$-null sets in $\mathcal{F}$, and let $\mathcal{P}$ represent the σ-algebra of predictable sets on $\Omega \times [0,T]$ associated with the filtration $\{\mathcal{F}_t\}_{t\geq 0}$.

The subsequent step in the process is to introduce the definitions of certain spaces and norms that will be used throughout this work.

- $L^2(\mathcal{F}_T; H)$ =$\{H$-valued $\mathcal{F}_T$-measurable random variables ξ satisfying $E[\|\xi\|^2] < \infty\}$;
- $L^2(\mathcal{U}; H)$: H-valued $\mathcal{U}$-measurable random variables such that

  $$\int_U \|q\|^2 \lambda(dx) < \infty,$$

  then $\|q\|^2_{L^2(\mathcal{U};H)} = \int_U \|q\|^2\lambda(dx)$;
- $L^2(\mathcal{F}_t \times \mathcal{U}; H)$: H-valued $\mathcal{F}_t \times \mathcal{U}$-measurable random variables such that

  $$E\int_U \|q\|^2 \lambda(dx) < \infty;$$

- $L^2_{\mathcal{F}}([0,T]; H)$: H-valued $\mathcal{F}_t$-adapted random processes such that

  $$E[\int_0^T \|\varphi_t\|^2 dt] < \infty;$$

- $S^2_{\mathcal{F}}([0,T]; H)$: continuous processes in $L^2_{\mathcal{F}}([0,T]; H)$ such that

$$E[\sup_{0\le t\le T}\|\varphi_t\|^2]<\infty;$$

- $M^2_{\mathcal{F}}([0,T];H)$: H-valued $\mathcal{P}\times\mathcal{U}$-adapted processes such that

$$E[\int_0^T \|\varphi_t\|^2_{L^2(\mathcal{U};H)}dt]<\infty.$$

Definition 1 (The weak solution). *(y,z,q) is said to be a weak solution of BSSEJ (1) if $(y,z,q)\in S^2_{\mathcal{F}}([0,T];H)\times L^2_{\mathcal{F}}([0,T];H)\times M^2_{\mathcal{F}}([0,T];H)$ such that for any $\eta\in H^1_0(D)\cap H^2(D)$,*

$$\begin{aligned}&\mathrm{i}(\varphi,\eta)-\mathrm{i}(y_t,\eta)+\int_t^T(y_s,\Delta\eta)ds\\&=\int_t^T(f(s,y_s,z_s,q_s),\eta)ds+\int_t^T(z_s,\eta)dW_s+\int_t^T\int_U(q_s(x),\eta)\tilde{N}_p(ds,dx).\end{aligned}\tag{2}$$

Firstly, assume that for all $s\in[0,T]$, $f:\Omega\times[0,T]\times H\times H\times L^2(\mathcal{U};H)\to L^2(\mathcal{F}_s;H)$ satisfies the following conditions:

(A_1) There exists a constant $L>0$ such that for all $s\in[0,T]$, $y,y'\in H$, $z,z'\in H$, $q,q'\in L^2(\mathcal{U};H)$, we have

$$\|f(s,y,z,q)-f(s,y',z',q')\|\le L\Big(\|y-y'\|+\|z-z'\|+\|q-q'\|_{L^2(\mathcal{U};H)}\Big);$$

(A_2) $E\int_0^T\|f(s,0,0,0,0)\|^2ds<\infty$.

We first recall the complex Itô formula in [41].

Lemma 1. *Suppose X_t is a one-dimensional complex semi-martingale:*

$$X_t=x_0+\int_0^t b(s,X_s)ds+\int_0^t\sigma(s,X_s)dW_s$$

with

$$\int_0^T|b(s,X_s)|ds+\int_0^T|\sigma(s,X_s)|^2ds<\infty,\ \mathbb{P}-a.s..$$

Then,

$$|X_t|^2=|x_0|^2+2Re\Big(\int_0^t b(s,X_s)\overline{X}_sds\Big)+2Re\Big(\int_0^t\sigma(s,X_s)\overline{X}_sdW_s\Big)+\int_0^t|\sigma(s,X_s)|^2ds.$$

The setting of our problem is as follows: to find $(y,z,q)\in S^2_{\mathcal{F}}([0,T];H)\times L^2_{\mathcal{F}}([0,T];H)\times M^2_{\mathcal{F}}([0,T];H)$ satisfying the BSSEJ (1) by making compactness arguments on the solution $(y^{(n)},z^{(n)},q^{(n)})$ of the approximation Equation (3). The following theorem establishes the existence of the weak solution to the BSSEJ (2).

Theorem 1. *Suppose that f satisfies (A_1) and (A_2); then, for any given terminal condition $\varphi\in L^2(\Omega,\mathcal{F},H)$, the BSSEJ (1) has a unique weak solution $(y,z,q)\in S^2_{\mathcal{F}}([0,T];H)\times L^2_{\mathcal{F}}([0,T];H)\times M^2_{\mathcal{F}}([0,T];H)$. Moreover, we have*

$$\begin{aligned}&E[\sup_{0\le t\le T}\|y_t\|^2]+E\int_0^T\|z_s\|^2ds+E\int_0^T\|q_s\|^2_{L^2(\mathcal{U};H)}ds\\&\qquad\le C\Big(E\|\varphi\|^2+E\int_0^T\|f(s,0,0,0)\|^2ds\Big),\end{aligned}$$

where $C > 0$ is a constant that depends only on d, L, and T.

3. Galerkin's Approximations and a Prior Estimates

By (Theorem 6.5.1, [44]), it follows that $L^2(D,\mathbb{C})$ and $H_0^1(D,\mathbb{C})$ share a common orthogonal basis $\{e_k\}$ such that

$$\begin{aligned} -\Delta e_k &= \lambda_k e_k; \\ 0 \le \lambda_k &\le \lambda_{k+1} \to +\infty. \end{aligned}$$

We can assume that $\{e_k\}$ is orthonormal in $H = L^2(D,\mathbb{C})$. Set $E_n = span\{e_1, e_2, \cdots, e_n\}$. Assume that $\pi_n : H \to E_n$ is the projection operator. For $k = 1, 2, \cdots, n$, consider the following approximation equation:

$$\begin{cases} \mathrm{i}d(y_t^{(n)}, e_k) + (\Delta y_t^{(n)}, e_k)dt = f(t, y_t^{(n)}, z_t^{(n)}, q_t^{(n)}, e_k)dt + (z_t^{(n)}, e_k)dW_t \\ \qquad\qquad + \int_U (q_t^{(n)}(x), e_k)\tilde{N}_p(dt, dx), \quad t \in [0,T], \\ y_T^{(n)} = \varphi_n := \sum_{k=1}^{n} (\varphi, e_k)e_k. \end{cases} \tag{3}$$

Lemma 2. *Assume that f satisfies red(A_1) and red(A_2). Then, for any given terminal condition $\varphi_n \in L^2(\Omega, \mathcal{F}, H)$, the approximation Equation (3) has an adapted solution $(y^{(n)}, z^{(n)}, q^{(n)}) \in S^2_{\mathcal{F}}([0,T];H) \times L^2_{\mathcal{F}}([0,T];H) \times M^2_{\mathcal{F}}([0,T];H)$.*

Proof. Set

$$y_t^{(n)} = \sum_{k=1}^{n} y_k^{(n)}(t)e_k, z_t^{(n)} = \sum_{k=1}^{n} z_k^{(n)}(t)e_k, q_t^{(n)}(x) = \sum_{k=1}^{n} q_k^{(n)}(t,x)e_k;$$

then, we obtain an n-dimensional complex-valued BSDEJ from (3):

$$\begin{cases} \mathrm{i}dy_1^{(n)}(t) - \lambda_1 y_1^{(n)}(t)dt = f\Big(t, \sum_{k=1}^{n} y_k^{(n)}(t)e_k, \sum_{k=1}^{n} z_k^{(n)}(t)e_k, \sum_{k=1}^{n} q_k^{(n)}(t,x)e_k, e_1\Big)dt \\ \qquad\qquad + z_1^{(n)}dW_t + \int_U q_1^{(n)}(t,x)\tilde{N}_p(dt,dx), \\ \mathrm{i}dy_2^{(n)}(t) - \lambda_2 y_2^{(n)}(t)dt = f\Big(t, \sum_{k=1}^{n} y_k^{(n)}(t)e_k, \sum_{k=1}^{n} z_k^{(n)}(t)e_k, \sum_{k=1}^{n} q_k^{(n)}(t,x)e_k, e_2\Big)dt \\ \qquad\qquad + z_2^{(n)}dW_t + \int_U q_2^{(n)}(t,x)\tilde{N}_p(dt,dx), \\ \qquad\qquad \cdots\cdots \\ \mathrm{i}dy_n^{(n)}(t) - \lambda_n y_n^{(n)}(t)dt = f\Big(t, \sum_{k=1}^{n} y_k^{(n)}(t)e_k, \sum_{k=1}^{n} z_k^{(n)}(t)e_k, \sum_{k=1}^{n} q_k^{(n)}(t,x)e_k, e_n\Big)dt \\ \qquad\qquad + z_n^{(n)}dW_t + \int_U q_n^{(n)}(t,x)\tilde{N}_p(dt,dx). \end{cases} \tag{4}$$

To solve (4), we first consider the following BSDEJ:

$$\begin{cases} dY_t = [AY_t + F(t, Y_t, Z_t, Q_t)]dt + Z_t dW_t + \int_U Q_t(x)\tilde{N}_p(dt,dx), \\ Y_T = \mathrm{i}\varphi_n, \end{cases} \tag{5}$$

where $A = diag\{-\mathrm{i}\lambda_1, -\mathrm{i}\lambda_2, \cdots, -\mathrm{i}\lambda_n\}$,

$$F(t,Y,Z,Q) = \begin{pmatrix} (f(t,-\mathrm{i}Y^Te, Z^Te, Q^Te), e_1) \\ \cdots \\ (f(t,-\mathrm{i}Y^Te, Z^Te, Q^Te), e_n) \end{pmatrix}, \text{ where } e = (e_1, e_2, \cdots, e_n)^T.$$

Because of the theory of BSDEJ [45], we know that (5) has an adapted solution $(Y,Z,Q) \in S^2_{\mathcal{F}}([0,T];\mathbb{C}^n) \times L^2_{\mathcal{F}}([0,T];\mathbb{C}^n) \times M^2_{\mathcal{F}}([0,T];\mathbb{C}^n)$. Set $y = -\mathrm{i}Y, z = Z, q = Q$; it is easy to verify that (y,z,q) is a solution of (4). Therefore, the approximate Equation (3) has a solution. □

The subsequent lemma provides an estimation of the solution to the approximate Equation (3).

Lemma 3. *Suppose that f satisfies (A_1) and (A_2). Then, for any given terminal condition $\varphi_n \in L^2(\Omega,\mathcal{F},H)$, the solution $(y^{(n)}, z^{(n)}, q^{(n)})$ of the approximation Equation (3) satisfies*

$$E\|y_t^{(n)}\|^2 + E\int_0^T \|z_s^{(n)}\|^2 ds + E\int_t^T \int_U \|q_s^{(n)}(x)\|^2 \lambda(dx)ds \\ \leq C\left(E\|\varphi\|^2 + E\int_0^T \|f(s,0,0,0)\|^2 ds\right),$$

where C is a constant depending on d, L, and T.

Proof. Applying the Itô formula to $\|y_t^{(n)}\|^2$ and taking expectation, we have

$$E\|\varphi_n\|^2 - E\|y_t^{(n)}\|^2 = 2\mathrm{Im}\left\{E\int_t^T (f(s,y_s^{(n)}, z_s^{(n)}, q_s^{(n)}), y_s^{(n)})ds\right\} \\ + E\int_t^T \|z_s^{(n)}\|^2 ds + E\int_t^T \int_U \|q_s^{(n)}(x)\|^2 \lambda(dx)ds.$$

Therefore,

$$E\|y_t^{(n)}\|^2 + \frac{1}{2}E\int_t^T \|z_s^{(n)}\|^2 ds + \frac{1}{2}E\int_t^T \int_U \|q_s^{(n)(x)}\|^2 \lambda(dx)ds \\ \leq E\|\varphi_n\|^2 + (4L^2 + 2L + 1)E\int_t^T \|y_s^{(n)}\|^2 ds + E\int_t^T \|f(s,0,0,0)\|^2 ds.$$

According to Gronwall's inequality, it follows that

$$E\|y_t^{(n)}\|^2 + E\int_0^T \|z_s^{(n)}\|^2 ds + E\int_t^T \int_U \|q_s^{(n)}(x)\|^2 \lambda(dx)ds \\ \leq C\left(E\|\varphi\|^2 + E\int_0^T \|f(s,0,0,0)\|^2 ds\right).$$

By the Lipschitz condition (A_1), we have

$$E\int_0^T \|f(s,y_s^{(n)}, z_s^{(n)}, q_s^{(n)})\|^2 ds \leq C\left(E\|\varphi\|^2 + E\int_0^T \|f(s,0,0,0)\|^2 ds\right). \quad (6)$$

□

4. Existence of Weak Solutions to BSSEJ (1)

Proof of Theorem 1 (Existence). The proof is categorized into two distinct cases: D is a bounded C^2 domain or the whole space.

Case 1: D is a bounded C^2 domain. Since $L^2_{\mathcal{F}}([0,T];H), M^2_{\mathcal{F}}([0,T];H)$ is reflexive, there exists $y, z, f^* \in L^2_{\mathcal{F}}([0,T];H), q \in M^2_{\mathcal{F}}([0,T];H)$ such that

$$\begin{aligned} &y^{(n)} \to y, z^{(n)} \to z \ \ weakly \ \ in \ \ L^2_{\mathcal{F}}([0,T];H); \\ &f(\cdot, y^{(n)}_{\cdot}, z^{(n)}_{\cdot}, q^{(n)}_{\cdot}) \to f^* \ \ weakly \ \ in \ \ L^2_{\mathcal{F}}([0,T];H); \\ &q^{(n)} \to q \ \ weakly \ \ in \ \ M^2_{\mathcal{F}}([0,T];H). \end{aligned}$$

We rewrite (3) as

$$\begin{aligned} \mathrm{i}[(\varphi_n, e_k) - (y^{(n)}_t, e_k)] = &\int_t^T (-\Delta y^{(n)}_s, e_k) ds + \int_t^T f(s, y^{(n)}_s, z^{(n)}_s, q^{(n)}_s, e_k) ds \\ &+ \int_t^T (z^{(n)}_s, e_k) dW_s + \int_t^T \int_U (q^{(n)}_s(x), e_k) \tilde{N}_p(ds, dx). \end{aligned} \tag{7}$$

Step 1: In what follows, we are going to prove that for any $k \geq 1$,

$$\begin{aligned} \mathrm{i}[(\varphi, e_k) - (y_t, e_k)] = &\int_t^T (y_s, -\Delta e_k) ds + \int_t^T (f^*(s), e_k) ds \\ &+ \int_t^T (z_s, e_k) dW_s + \int_t^T \int_U (q_s(x), e_k) \tilde{N}_p(ds, dx). \end{aligned} \tag{8}$$

Assume that ξ is a bounded complex random variable defined on the probability space $(\Omega, \mathcal{F}_T)$, and let ψ be a bounded complex Borel-measurable function on the interval $[0,T]$. As $n \to \infty$, we claim that

$$E \int_0^T \xi \psi(t) (\varphi_n, e_k) dt \to E \int_0^T \xi \psi(t) (\varphi, e_k) dt, \tag{9}$$

$$E \int_0^T \xi \psi(t) (y^{(n)}_t, e_k) dt \to E \int_0^T \xi \psi(t) (y_t, e_k) dt, \tag{10}$$

$$E \int_0^T \xi \psi(t) \{ \int_t^T (y^{(n)}_s, e_k) ds \} dt \to E \int_0^T \xi \psi(t) \{ \int_t^T (y_s, e_k) ds \} dt, \tag{11}$$

$$E \int_0^T \xi \psi(t) \{ \int_t^T (f(s, y^{(n)}_s, z^{(n)}_s, q^{(n)}_s), e_k) ds \} dt \to E \int_0^T \xi \psi(t) \{ \int_t^T (f^*(s), e_k) ds \} dt, \tag{12}$$

$$E \int_0^T \xi \psi(t) \{ \int_t^T (z^{(n)}_s dW_s, e_k) \} dt \to E \int_0^T \xi \psi(t) \{ \int_t^T (z_s dW_s, e_k) \} dt, \tag{13}$$

and

$$E \int_0^T \xi \psi(t) \{ \int_t^T (q^{(n)}_s(x) \tilde{N}_p(ds, dx), e_k) \} dt \to E \int_0^T \xi \psi(t) \{ \int_t^T (q_s(x) \tilde{N}_p(ds, dx), e_k) \} dt. \tag{14}$$

Upon establishing the proofs for Equations (9)–(14),we can then proceed by multiplying $\xi \psi(t)$ to both sides of Equation (7) and subsequently performing integration and taking expectations. By carefully taking the limits, we can readily arrive at

$$
\begin{aligned}
E\int_0^T \psi(t)\xi \mathrm{i}[(\varphi, e_k) - (y_t, e_k)]dt = E\int_0^T \psi(t)\xi \int_t^T (y_s, -\Delta e_k)dsdt \\
+ E\int_0^T \psi(t)\xi \int_t^T (f^*(s), e_k)dsdt \\
+ E\int_0^T \psi(t)\xi \int_t^T (z_s, e_k)dW_s dt \\
+ E\int_0^T \psi(t)\xi \int_t^T \int_U (q_s(x), e_k)\tilde{N}_p(ds, dx)dt.
\end{aligned}
$$

By the arbitrariness of ξ and $\psi(t)$, we obtain (8) immediately.

The subsequent step in the process is to give the proof of (9)–(14) one by one. Since $\varphi_n \to \varphi$ strongly in $L^2_{\mathcal{F}}([0,T];H)$, (9) obviously holds. Since $\xi\psi(t)(y_t^{(n)} - y_t, e_k) \in L^1(\Omega \times [0,T])$, we have

$$
\begin{aligned}
E\int_0^T \xi\psi(t)(y_t^{(n)} - y_t, e_k)dt &= \int_0^T E[\xi\psi(t)(y_t^{(n)} - y_t, e_k)]dt \\
&= \int_0^T E\big[E[\xi\psi(t)(y_t^{(n)} - y_t, e_k)|\mathcal{F}_t]\big]dt \\
&= \int_0^T E\big[\psi(t)(y_t^{(n)} - y_t, e_k)E[\xi|\mathcal{F}_t]\big]dt \\
&= E\int_0^T (y_t^{(n)} - y_t, \overline{\psi(t)E[\xi|\mathcal{F}_t]}e_k)dt \to 0,
\end{aligned}
$$

which yields (10). Similarly, we have

$$
E\int_t^T \xi(y_s^{(n)} - y_s, e_k)ds \to 0.
$$

By using the Fubini theorem and the Lebesgue convergence theorem, we obtain

$$
E\int_0^T \psi(t)\xi\int_t^T (y_s^{(n)} - y_s, e_k)dsdt \to 0,
$$

which leads to (11) at once. In a similar way, we can prove (12). To establish (13), we firstly show that, for any fixed t, the following holds:

$$
E\left[\xi\int_t^T (z_s^{(n)}dW_s, e_k)\right] \to E\left[\xi\int_t^T (z_s dW_s, e_k)\right]. \tag{15}
$$

According to the martingale representation theorem, it is known that there exists

$$
\rho \in L^2_{\mathcal{F}}([0,T];\mathbb{C})
$$

such that

$$
\xi = E\xi + \int_0^T \rho_s dW_s.
$$

Then,

$$
\begin{aligned}
E\left[\xi\int_t^T (z_s^{(n)}, e_k)dW_s\right] &= E\left[\int_t^T (z_s^{(n)}dW_s, e_k)\int_0^T \rho_s dW_s\right] \\
&= E\int_t^T (z_s^{(n)}, \overline{\rho_s}e_k)ds \\
&= E\int_0^T (z_s^{(n)}, \overline{\rho_s}1_{[t,T]}e_k)ds.
\end{aligned} \tag{16}
$$

Since $\overline{\rho_s}1_{[t,T]}e_k \in L^2_{\mathcal{F}}([0,T];H)$ and $z^{(n)} \to z$ weakly in $L^2_{\mathcal{F}}([0,T];H)$, we obtain that (16) tends to

$$\begin{aligned} E\int_0^T (z_s, \overline{\rho_s}1_{[t,T]}e_k)ds &= E[\int_0^T \rho_s dW_s \int_t^T (z_s dW_s, e_k)] \\ &= E[(\int_t^T (z_s dW_s, e_k))\xi], \end{aligned}$$

which yields (15). By applying the Burkholder–Davis–Gundy inequality (see, e.g., Theorem 3.28 [46]), we obtain

$$E\left|\xi(\int_t^T (z_s^{(n)} dW_s, e_k))\right| \leq C < \infty. \tag{17}$$

Similar to (15) and (17), we can prove (13) by the dominated convergence theorem. By the martingale representation theorem of jump processes, there exists $\phi \in M^2_{\mathcal{F}}([0,T];\mathbb{C})$ such that

$$\xi = E\xi + \int_0^T \int_U \phi_s(x)\tilde{N}_p(ds,dx).$$

Thus, (14) can be obtained in a similar way as to derive (13).
Step 2: In what follows, we show that

$$f(t, y_t, z_t, q_t(x)) = f^*(t),\ a.e.\ (t,\omega,x) \in [0,T]\times\Omega\times U.$$

Firstly, we project y_s, z_s, q_s onto $E_n = span\{e_1, e_2, \cdots, e_n\}$:

$$\begin{aligned} \tilde{y}_s^{(n)} &:= \pi_n y_s = \sum_{k=1}^n (y_s, e_k)e_k; \\ \tilde{z}_s^{(n)} &:= \pi_n z_s = \sum_{k=1}^n (z_s, e_k)e_k; \\ \tilde{q}_s^{(n)}(x) &:= \pi_n q_s(x) = \sum_{k=1}^n (q_s(x), e_k)e_k. \end{aligned}$$

Through linear combinations of Equation (8), we arrive at

$$\begin{aligned} \mathrm{i}[(\varphi_n, e_k) - (\tilde{y}_t^{(n)}, e_k)] = &\int_t^T (-\Delta\tilde{y}_s^{(n)}, e_k)ds + \int_t^T (\pi_n f^*(s), e_k)ds \\ &+ \int_t^T (\tilde{z}_s^{(n)}, e_k)dW_s + \int_t^T \int_U (\tilde{q}_s^{(n)}(x), e_k)\tilde{N}_p(ds,dx). \end{aligned}$$

Taking into account (7), we have

$$\begin{aligned} \mathrm{i}(y_t^{(n)} - \tilde{y}_t^{(n)}, e_k) = &\int_t^T (\Delta(y_s^{(n)} - \tilde{y}_s^{(n)}), e_k)ds \\ &+ \int_t^T (\pi_n f^*(s) - \pi_n f(s, y_s^{(n)}, z_s^{(n)}, q_s^{(n)}), e_k)ds \\ &+ \int_t^T (\tilde{z}_s^{(n)} - z_s^{(n)}, e_k)dW_s + \int_t^T \int_U (\tilde{q}_s^{(n)}(x) - q_s^{(n)}(x), e_k)\tilde{N}_p(ds,dx). \end{aligned}$$

By using the Itô formula to $e^{\lambda t}\|y_t^{(n)} - \tilde{y}_t^{(n)}\|^2,\ \lambda \in \mathbb{R}$, we have

$$
\begin{aligned}
&Ee^{\lambda t}\|y_t^{(n)}-\tilde{y}_t^{(n)}\|^2+\lambda E\int_t^T e^{\lambda s}\|y_s^{(n)}-\tilde{y}_s^{(n)}\|^2ds\\
&\quad+E\int_t^T e^{\lambda s}\|z_s^{(n)}-\tilde{z}_s^{(n)}\|^2ds+E\int_t^T\int_U e^{\lambda s}\|q_s^{(n)}(x)-\tilde{q}_s^{(n)}(x)\|^2\lambda(dx)ds\\
&=-2\mathrm{Im}\Big(E\int_t^T e^{\lambda s}(\pi_n f(s,y_s^{(n)},z_s^{(n)},q_s^{(n)})-\pi_n f^*(s),y_s^{(n)}-\tilde{y}_s^{(n)})ds\Big)\\
&=-2\mathrm{Im}\Big(E\int_t^T e^{\lambda s}(f(s,y_s^{(n)},z_s^{(n)},q_s^{(n)})-f^*(s),y_s^{(n)}-\tilde{y}_s^{(n)})ds\Big).
\end{aligned}
$$

Then, with (A_1) and the Young inequality, we deduce that

$$
\begin{aligned}
&Ee^{\lambda t}\|y_t^{(n)}-\tilde{y}_t^{(n)}\|^2+\lambda E\int_t^T e^{\lambda s}\|y_s^{(n)}-\tilde{y}_s^{(n)}\|^2ds\\
&\quad+E\int_t^T e^{\lambda s}\|z_s^{(n)}-\tilde{z}_s^{(n)}\|^2ds+E\int_t^T e^{\lambda s}\|q_s^{(n)}-\tilde{q}_s^{(n)}\|^2_{L^2(\mathcal{U},H)}ds\\
&\le 2E\int_t^T e^{\lambda s}\|f(s,y_s^{(n)},z_s^{(n)},q_s^{(n)})-f^*(s)\|\|y_s^{(n)}-\tilde{y}_s^{(n)}\|ds\\
&\le (4L^2+5L)E\int_t^T e^{\lambda s}\|y_s^{(n)}-\tilde{y}_s^{(n)}\|^2ds\\
&\quad+\frac{1}{2}E\int_t^T e^{\lambda s}\|z_s^{(n)}-\tilde{z}_s^{(n)}\|^2ds+\frac{1}{2}E\int_t^T e^{\lambda s}\|q_s^{(n)}-\tilde{q}_s^{(n)}\|^2_{L^2(\mathcal{U},H)}ds\\
&\quad+LE\int_t^T e^{\lambda s}\|\tilde{y}_s^{(n)}-y_s\|^2ds+LE\int_t^T e^{\lambda s}\|\tilde{z}_s^{(n)}-z_s\|^2ds\\
&\quad+LE\int_t^T e^{\lambda s}\|\tilde{q}_s^{(n)}-q_s\|^2_{L^2(\mathcal{U},H)}ds\\
&\quad+2\Big|E\int_t^T e^{\lambda s}(f(s,y_s,z_s,q_s)-f^*(s),y_s^{(n)}-\tilde{y}_s^{(n)})ds\Big|.
\end{aligned}
$$

Set $\lambda=4L^2+5L+1$; then, we have

$$
\begin{aligned}
&Ee^{\lambda t}\|y_t^{(n)}-\tilde{y}_t^{(n)}\|^2+\lambda E\int_t^T e^{\lambda s}\|y_s^{(n)}-\tilde{y}_s^{(n)}\|^2ds\\
&\quad+\frac{1}{2}E\int_t^T e^{\lambda s}\|z_s^{(n)}-\tilde{z}_s^{(n)}\|^2ds+\frac{1}{2}E\int_t^T e^{\lambda s}\|q_s^{(n)}-\tilde{q}_s^{(n)}\|^2_{L^2(\mathcal{U},H)}ds\\
&\le LE\int_t^T e^{\lambda s}\|\tilde{y}_s^{(n)}-y_s\|^2ds+LE\int_t^T e^{\lambda s}\|\tilde{z}_s^{(n)}-z_s\|^2ds\\
&\quad+LE\int_t^T e^{\lambda s}\|\tilde{q}_s^{(n)}-q_s\|^2_{L^2(\mathcal{U},H)}ds\\
&\quad+2\Big|E\int_t^T e^{\lambda s}(f(s,y_s,z_s,q_s)-f^*(s),y_s^{(n)}-\tilde{y}_s^{(n)})ds\Big|.
\end{aligned}
\tag{18}
$$

Note that

$$
\tilde{y}^{(n)}\to y,\tilde{z}^{(n)}\to z\ \textit{strongly in}\ L^2_{\mathcal{F}}([0,T];H),
$$

$$
\tilde{q}^{(n)}\to q\ \textit{strongly in}\ M^2_{\mathcal{F}}([0,T];H)
$$

and

$$
y^{(n)}-\tilde{y}^{(n)}\to 0\ \textit{weakly in}\ L^2_{\mathcal{F}}([0,T];H).
$$

Setting $t=0$ and letting $n\to\infty$ in (18), we have

$$
\lim_{n\to\infty}E\int_0^T\big(\|y_t^{(n)}-\tilde{y}_t^{(n)}\|^2+\|z_t^{(n)}-\tilde{z}_t^{(n)}\|^2+\|q_t^{(n)}-\tilde{q}_t^{(n)}\|^2_{L^2(\mathcal{U},H)}\big)dt=0.
$$

Combining with

$$\lim_{n\to\infty} E\int_0^T (\|y_t-\tilde{y}_t^{(n)}\|^2+\|z_t-\tilde{z}_t^{(n)}\|^2+\|q_t-\tilde{q}_t^{(n)}\|^2_{L^2(\mathcal{U},H)})dt=0,$$

yields

$$\lim_{n\to\infty} E\int_0^T (\|y_t-y_t^{(n)}\|^2+\|z_t-z_t^{(n)}\|^2dt+\|q_t-q_t^{(n)}\|^2_{L^2(\mathcal{U},H)})dt=0. \tag{19}$$

Then, by (A_1), we arrive at

$$f(\cdot,y_\cdot^{(n)},z_\cdot^{(n)},q_\cdot^{(n)})\to f(\cdot,y_\cdot,z_\cdot,q_\cdot) \text{ strongly in } L^2_{\mathcal{F}}([0,T];H).$$

Note that

$$f(\cdot,y_\cdot^{(n)},z_\cdot^{(n)},q_\cdot^{(n)})\to f^*(\cdot) \text{ weakly in } L^2_{\mathcal{F}}([0,T];H).$$

Through the uniqueness of the limit, it follows that

$$f(t,y_t,z_t,q_t(x))=f^*(t),\ a.e.(t,\omega,x)\in[0,T]\times\Omega\times U. \tag{20}$$

Combining (8) and (20), for any $\eta\in H_0^1(D)\cap H^2(D)$, we have

$$\begin{aligned}\mathrm{i}[(\varphi,\eta)-(y_t,\eta)]=&\int_t^T(y_s,-\Delta\eta)ds+\int_t^T(f(s,y_s,z_s,q_s),\eta)ds\\&+\int_t^T(z_s,\eta)dW_s+\int_t^T\int_U(q_s(x),\eta)\tilde{N}_p(ds,dx),\end{aligned} \tag{21}$$

which implies that BSSEJ (1) has a weak solution $(y,z,q)\in L^2_{\mathcal{F}}([0,T];H)\times L^2_{\mathcal{F}}([0,T];H)\times M^2_{\mathcal{F}}([0,T];H)$.

Step 3: Next, we estimate $E\sup\limits_{0\le t\le T}\|y_t\|^2$ to prove that $(y,z,q)\in S^2_{\mathcal{F}}([0,T];H)\times L^2_{\mathcal{F}}([0,T];H)\times M^2_{\mathcal{F}}([0,T];H)$. By (21) and the Itô formula, we have

$$\begin{aligned}&\|y_t\|^2+\int_t^T\|z_s\|^2ds+\int_t^T\|q_s\|^2_{L^2(\mathcal{U},H)}ds\\&\quad=\|\varphi\|^2-2\mathrm{Im}\left(\int_t^T(f(s,y_s,z_s,q_s),y_s)ds\right)\\&\qquad-2\mathrm{Im}\left(\int_t^T(z_s,y_s)dW_s\right)-2\mathrm{Im}\left(\int_t^T\int_U(q_s(x),y_s)\tilde{N}_p(ds,dx)\right).\end{aligned}$$

Thus,

$$\begin{aligned}&\sup_{t\le s\le T}\|y_s\|^2+\int_t^T\|z_s\|^2ds+\int_t^T\|q_s\|^2_{L^2(\mathcal{U},H)}ds\\&\quad\le\|\varphi\|^2+2L\int_t^T(\|y_s\|+\|z_s\|+\|q_s\|_{L^2(\mathcal{U},H)})\|y_s\|ds\\&\qquad+\int_t^T\|f(s,0,0,0)\|^2ds+\int_t^T\|y_s\|^2ds+2\sup_{t\le s\le T}\left|\int_s^T(y_u,z_u)dW_u\right|\\&\qquad+2\sup_{t\le s\le T}\left|\int_s^T\int_U(y_u,q_u(x))\tilde{N}_p(du,dx)\right|.\end{aligned}$$

By using the Burkholder–Davis–Gundy inequality, we obtain

$$2E\left[\sup_{t\le s\le T}\left|\int_s^T (y_u, z_u)dW_u\right|\right] \le c_1 E\sqrt{\int_t^T \|y_u\|^2\|z_u\|^2 du},$$

and

$$2E\left[\sup_{t\le s\le T}\left|\int_s^T \int_U (y_u, q_u(x))\tilde{N}_p(du,dx)\right|\right] \le c_2 E\sqrt{\int_t^T \|y_u\|^2\|q_u\|^2_{L^2(\mathcal{U},H)} du}.$$

Then, we arrive at

$$\begin{aligned} E\sup_{t\le s\le T}\|y_s\|^2 + E\int_t^T \|z_s\|^2 ds + E\int_t^T \|q_s\|^2_{L^2(\mathcal{U},H)} ds \\ \le E\|\varphi\|^2 + 2LE\int_t^T (\|y_s\| + \|z_s\| + \|q_s\|_{L^2(\mathcal{U},H)})\|y_s\| ds \\ + E\int_t^T \|f(s,0,0,0)\|^2 ds + E\int_t^T \|y_s\|^2 ds \\ + c_1 E\sqrt{\int_t^T \|y_u\|^2\|z_u\|^2 du} + c_2 E\sqrt{\int_t^T \|y_u\|^2\|q_u\|^2_{L^2(\mathcal{U},H)} du}, \end{aligned} \tag{22}$$

where $c_1, c_2 > 0$ are constants. By (19) and (6), we obtain

$$\begin{aligned} E\int_0^T \|y_s\|^2 ds + E\int_0^T \|z_s\|^2 ds + E\int_0^T \|q_s\|^2_{L^2(\mathcal{U},H)} ds \\ \le C\left(E\|\varphi\|^2 + E\int_0^T \|f(s,0,0,0)\|^2 ds\right). \end{aligned} \tag{23}$$

Then, we deduce from (22) and (23) that

$$\begin{aligned} E[\sup_{0\le t\le T}\|y_t\|^2] + E\int_0^T \|z_s\|^2 ds + E\int_0^T \|q_s\|^2_{L^2(\mathcal{U},H)} ds \\ \le C\left(E[\|\varphi\|^2 + \int_0^T \|f(s,0,0,0)\|^2 ds]\right), \end{aligned}$$

where $C > 0$ is a constant depending on d, L, and T.

Case 2: D is the whole space, i.e., $D = \mathbb{R}^d$. Choose the sequence $\varphi_m \to \varphi$ in $L^2(\Omega, \mathcal{F}_T, H)$, with supp $\varphi^m \subset D_m = B(0,m)$. Here, supp $\varphi^m = \overline{\{x : \varphi^m(x) \neq 0\}}$, and $B(0,m)$ denotes the ball centered at 0 with radius m in $\mathbb{R}^d$. Take $m = j, l, j < l$ and consider BSSEJ (1) in D_l. Let (y^j, z^j, q^j), and (y^l, z^l, q^l) be two solutions under the terminal value φ^j and φ^l, respectively. By applying the Itô formula to $|(y_t^j - y_t^l, \eta)|^2$, we have

$$\begin{aligned} |(\varphi^j - \varphi^l, \eta)|^2 - |(y_t^j - y_t^l, \eta)|^2 = -2\mathrm{Im}\left(\int_t^T (y_s^j - y_s^l, \Delta\eta)\overline{(y_s^j - y_s^l, \eta)} ds\right) \\ + 2\mathrm{Im}\left(\int_t^T (f(s, y_s^j, z_s^j, q_s^j) - f(s, y_s^l, z_s^l, q_s^l), \eta)\overline{(y_s^j - y_s^l, \eta)} ds\right) \\ + 2\mathrm{Im}\left(\int_t^T (z_s^j - z_s^l, \eta)\overline{(y_s^j - y_s^l, \eta)} dW_s\right) \\ + 2\mathrm{Im}\left(\int_t^T \int_U (q_s^j(x) - q_s^l(x), \eta)\overline{(y_s^j - y_s^l, \eta)}\tilde{N}_p(ds,dx)\right) \\ + \int_t^T |(z_s^j - z_s^l, \eta)|^2 ds + \int_t^T \int_U |(q_s^j(x) - q_s^l(x), \eta)|^2 \lambda(dx) ds. \end{aligned}$$

Suppose $\{e_k, k = 1, 2, \cdots, \infty\}$ are eigenvectors of Laplacian in D_l. Take $\eta = e_k$, and note that

$$(y_s^j - y_s^l, \Delta e_k)\overline{(y_s^j - y_s^l, e_k)} = -\lambda_k (y_s^j - y_s^l, e_k)\overline{(y_s^j - y_s^l, e_k)} \in \mathbb{R}.$$

We arrive at

$$\begin{aligned} &E\|y_t^j - y_t^l\|^2 + E\int_t^T \|z_s^j - z_s^l\|^2 ds + E\int_t^T \|q_s^j(x) - q_s^l(x)\|^2_{L^2(\mathcal{U}, L^2(\mathcal{D}_l))} ds \\ &= E\|(\varphi^j - \varphi^l, \eta)\|^2 - 2\mathrm{Im}\Big\{E\int_t^T (f(s, y_s^j, z_s^j, q_s^j) - f(s, y_s^l, z_s^l, q_s^l), y_s^j - y_s^l) ds\Big\}. \end{aligned}$$

By (A_1), we have

$$\begin{aligned} &E\|y_t^j - y_t^l\|^2 + E\int_t^T \|z_s^j - z_s^l\|^2 ds + E\int_t^T \|q_s^j(x) - q_s^l(x)\|^2_{L^2(\mathcal{U}, L^2(D_l))} ds \\ &\le E\|\varphi^j - \varphi^l\|^2 + 2LE\int_t^T \|y_s^j - y_s^l\|^2 ds \\ &\quad + 2LE\int_t^T (\|z_s^j - z_s^l\| + \|q_s^j(x) - q_s^l(x)\|_{L^2(\mathcal{U}, L^2(D_l))})\|y_s^j - y_s^l\| ds \\ &\le E\|\varphi^j - \varphi^l\|^2 + (4L^2 + 2L)E\int_t^T \|y_s^j - y_s^l\|^2 ds \\ &\quad + \frac{1}{2}E\int_t^T \|z_s^j - z_s^l\|^2 ds + \frac{1}{2}E\int_t^T \|q_s^j - q_s^l\|^2_{L^2(\mathcal{U}, L^2(D_l))} ds, \end{aligned}$$

which leads to

$$\begin{aligned} &E\|y_t^j - y_t^l\|^2 + \frac{1}{2}E\int_t^T \|z_s^j - z_s^l\|^2 ds \\ &\qquad + \frac{1}{2}E\int_t^T \|q_s^j - q_s^l\|^2_{L^2(\mathcal{U}, L^2(D_l))} ds \le CE\|\varphi^j - \varphi^l\|^2. \end{aligned} \tag{24}$$

We mention that the norm in (24) can be regarded as the norm of $L^2(\mathbb{R}^d)$ if we define $y^j, y^l, z^j, z^l, q^j, q^l$ to be 0 outside D_l. By (24), one can check that (y^j, z^j, q^j) is a Cauchy sequence, which has a limit $(y, z, q) \in L^2_{\mathcal{F}}([0, T]; L^2(\mathbb{R}^d)) \times L^2_{\mathcal{F}}([0, T]; L^2(\mathbb{R}^d)) \times M^2_{\mathcal{F}}([0, T]; L^2(\mathbb{R}^d))$. For any fixed $\eta \in C_c^\infty(D_m)$, when $j \ge m$, we have

$$\begin{aligned} \mathrm{i}(\varphi^j, \eta) - \mathrm{i}(y_t^j, \eta) + \int_t^T (y_s^j, \Delta\eta) ds &= \int_t^T (f(s, y_s^j, z_s^j, q_s^j), \eta) ds \\ &\quad + \int_t^T (z_s^j, \eta) dW_s + \int_t^T \int_U (q_s^j(x), \eta) \tilde{N}_p(ds, dx). \end{aligned}$$

Let $j \to \infty$, we obtain

$$\begin{aligned} \mathrm{i}(\varphi, \eta) - \mathrm{i}(y_t, \eta) + \int_t^T (y_s, \Delta\eta) ds &= \int_t^T (f(s, y_s, z_s, q_s), \eta) ds \\ &\quad + \int_t^T (z_s, \eta) dW_s + \int_t^T \int_U (q_s(x), \eta) \tilde{N}_p(ds, dx). \end{aligned} \tag{25}$$

Since (25) holds for any $\eta \in C_c^\infty(D_m)$, it holds for any $\eta \in C_c^\infty(\mathbb{R}^d)$. Hence, (y, z, q) is a weak solution of BSSEJ (1). By the same method in Case 1, we obtain

$$\begin{aligned} &E[\sup_{0 \le t \le T} \|y_t\|^2] + E\int_0^T \|z_s\|^2 ds + E\int_0^T \|q_s\|^2_{L^2(\mathcal{U}, L^2(\mathbb{R}^d))} ds \\ &\qquad \le C\Big(E\|\varphi\|^2 + E\int_0^T \|f(s, 0, 0, 0)\|^2 ds\Big), \end{aligned}$$

which implies that

$$(y,z,q) \in S^2_{\mathcal{F}}([0,T];H) \times L^2_{\mathcal{F}}([0,T];H) \times M^2_{\mathcal{F}}([0,T];H).$$

Therefore, we establish the case when $D = \mathbb{R}^d$ and finalize the proof of existence. □

5. Uniqueness of Adapted Solutions to BSSEJ (1)

Proof of Theorem 1 (Uniqueness). Like the proof of existence, we can prove uniqueness using two cases.

Case 1: D is a bounded C^2 domain. Suppose there exist two weak solutions of BSSEJ (1):

$$(y^i,z^i,q^i) \in S^2_{\mathcal{F}}([0,T];H) \times L^2_{\mathcal{F}}([0,T];H) \times M^2_{\mathcal{F}}([0,T];H), i = 1,2.$$

Then, according to Definition 1, for any $\eta \in H^1_0(D) \cap H^2(D)$, we have

$$\begin{aligned} -\mathrm{i}(y^1_t - y^2_t, \eta) &+ \int_t^T (y^1_s - y^2_s, \Delta\eta)ds \\ &= \int_t^T (f(s,y^1_s,z^1_s,q^1_s) - f(s,y^2_s,z^2_s,q^2_s), \eta)ds \\ &\quad + \int_t^T (z^1_s - z^2_s, \eta)dW_s + \int_t^T \int_U (q^1_s(x) - q^2_s(x), \eta)\tilde{N}_p(ds,dx). \end{aligned}$$

By applying the complex Itô formula to $|(y^1_t - y^2_t, \eta)|^2$, we deduce

$$\begin{aligned} |(y^1_t - y^2_t, \eta)|^2 = &-2\mathrm{Im}\left(\int_t^T -(y^1_s - y^2_s, \Delta\eta)\overline{(y^1_s - y^2_s, \eta)}ds\right) \\ &- 2\mathrm{Im}\left(\int_t^T (f(s,y^1_s,z^1_s,q^1_s) - f(s,y^2_s,z^2_s,q^2_s), \eta)\overline{(y^1_s - y^2_s, \eta)}ds\right) \\ &- 2\mathrm{Im}\left(\int_t^T (z^1_s - z^2_s, \eta)\overline{(y^1_s - y^2_s, \eta)}dW_s - \int_t^T |(z^1_s - z^2_s, \eta)|^2 ds\right) \\ &- 2\mathrm{Im}\left(\int_t^T \int_U (q^1_s(x) - q^2_s(x), \eta)\overline{(y^1_s - y^2_s, \eta)}\tilde{N}_p(ds,dx)\right. \\ &\left. - \int_t^T \int_U |(q^1_s(x) - q^2_s(x), \eta)|^2 \lambda(dx)ds\right). \end{aligned} \tag{26}$$

Observing the first term on the right-hand side of (26), we note that the Laplacian operator Δ cannot be moved in front of $(y^1_s - y^2_s)$ because of the insufficient spatial regularity of $(y^1_s - y^2_s)$. Consequently, this term is generally difficult to handle. However, by selecting $\eta = e_k \ \ (k = 1,2,\cdots,\infty)$, we note that

$$(y^1_s - y^2_s, \Delta e_k)\overline{(y^1_s - y^2_s, e_k)} = -\lambda_k (y^1_s - y^2_s, e_k)\overline{(y^1_s - y^2_s, e_k)} \in \mathbb{R}.$$

Therefore,

$$\begin{aligned} &|(y^1_t - y^2_t, e_k)|^2 + \int_t^T |(z^1_s - z^2_s, e_k)|^2 ds + \int_t^T \int_U |(q^1_s(x) - q^2_s(x), e_k)|^2 \lambda(dx)ds \\ &= -2\mathrm{Im}\left(\int_t^T (z^1_s - z^2_s, e_k)\overline{(y^1_s - y^2_s, e_k)}dW_s\right) \\ &\quad - 2\mathrm{Im}\left(\int_t^T \int_U (q^1_s(x) - q^2_s(x), e_k)\overline{(y^1_s - y^2_s, \eta)}\tilde{N}_p(ds,dx)\right) \\ &\quad - 2\mathrm{Im}\left(\int_t^T (f(s,y^1_s,z^1_s,q^1_s) - f(s,y^2_s,z^2_s,q^2_s), e_k)\overline{(y^1_s - y^2_s, e_k)}ds\right). \end{aligned}$$

By the Burkholder–Davis–Gundy inequality, we have

$$E|(y_t^1 - y_t^2, e_k)|^2 + E\int_t^T |(z_s^1 - z_s^2, e_k)|^2 ds + E\int_t^T \int_U |(q_s^1(x) - q_s^2(x), e_k)|^2 \lambda(dx)ds$$
$$= -2\mathrm{Im}\left(E\int_t^T (f(s, y_s^1, z_s^1, q_s^1) - f(s, y_s^2, z_s^2, q_s^2, e_k)\overline{(y_s^1 - y_s^2, e_k)}ds\right), k \geq 1. \tag{27}$$

By (A_1) and (27), we obtain

$$\begin{aligned} &E\|y_t^1 - y_t^2\|^2 + E\int_t^T \|z_s^1 - z_s^2\|^2 ds + E\int_t^T \int_U \|q_s^1(x) - q_s^2(x)\|^2 \lambda(dx)ds \\ &= -2\mathrm{Im}\left(E\int_t^T (f(s, y_s^1, z_s^1, q_s^1) - f(s, y_s^2, z_s^2, q_s^2), y_s^1 - y_s^2)ds\right) \\ &\leq 2LE\int_t^T \left(\|y_s^1 - y_s^2\| + \|z_s^1 - z_s^2\| + \|q_s^1 - q_s^2\|_{L^2(\mathcal{U},H)}\right)\|y_s^1 - y_s^2\|ds \\ &\leq (4L^2 + 2L)E\int_t^T \|y_s^1 - y_s^2\|^2 ds + \frac{1}{2}E\int_t^T \|z_s^1 - z_s^2\|^2 ds \\ &\qquad + \frac{1}{2}E\int_t^T \int_U \|q_s^1(dx) - q_s^2(x)\|^2 \lambda(dx)ds. \end{aligned} \tag{28}$$

Then,

$$E\|y_t^1 - y_t^2\|^2 \leq (4L^2 + 2L)E\int_t^T \|y_s^1 - y_s^2\|^2 ds.$$

Applying Gronwall's inequality, we have

$$E\|y_t^1 - y_t^2\|^2 = 0, \ \ \forall t \in [0, T], \tag{29}$$

which means $y_t^1 = y_t^2, a.s.$. Then, according to the continuity of the path of y^1, y^2, we have

$$P(y_t^1 = y_t^2, \forall t \in [0, T]) = 1.$$

Substituting (29) into (28), we obtain

$$z_t^1 = z_t^2, a.e.(t, \omega) \in [0, T] \times \Omega,$$

$$q_t^1(x) = q_t^2(x), a.e.(t, \omega, x) \in [0, T] \times \Omega \times U,$$

which completes the proof of Case 1 .

Case 2: D is the whole space, i.e. $D = \mathbb{R}^d$. Let $D_n = \{x \in \mathbb{R}^d | |x| \leq n\}$, then $\partial D_n \in C^2$. By making similar arguments as in Case 1, we obtain

$$E\|y_t^1 - y_t^2\|^2_{L^2(D_n)} = 0, \ E\int_0^T \|z_t^1 - z_t^2\|^2_{L^2(D_n)} dt = 0$$

and

$$E\int_0^T \|q_t^1 - q_t^2\|^2_{L^2(\mathcal{U}, L^2(D_n))} dt = 0.$$

Hence,

$$\begin{aligned} &y_t^1 = y_t^2, \quad \forall (t, x) \in [0, T] \times D_n, \ \ a.s., \\ &z_t^1 = z_t^2, \quad a.e. \ \ (t, \omega, x) \in [0, T] \times \Omega \times D_n, \\ &q_t^1 = q_t^2, \quad a.e. \ \ (t, \omega, u, x) \in [0, T] \times \Omega \times U \times D_n. \end{aligned}$$

Since $\bigcup_{n=1}^{\infty} = \mathbb{R}^d$, it is easy to obtain

$$\begin{aligned} y_t^1 &= y_t^2, \quad \forall (t,x) \in [0,T] \times \mathbb{R}^d, \ \ a.s., \\ z_t^1 &= z_t^2, \quad a.e. \ \ (t,\omega,x) \in [0,T] \times \Omega \times \mathbb{R}^d, \\ q_t^1 &= q_t^2, \quad a.e. \ \ (t,\omega,u,x) \in [0,T] \times \Omega \times U \times \mathbb{R}^d. \end{aligned}$$

The proof is complete. □

6. Conclusions

In this research, we consider a class of backward stochastic semi-linear Schrödinger equations with Poisson jumps, defined either in $\mathbb{R}^d$ or within a bounded domain with a C^2 boundary. These equations are closely linked to the stochastic control problem of nonlinear Schrödinger equations driven by Lévy noise. The methods are mainly based on the complex Itô formula, Galerkin's approximation method, and the martingale representation theorem. The main contribution of this study lies in the analytical insights and mathematical rigor, which provide a foundation for future numerical analysis. Meanwhile, the results presented in this manuscript provide a solid foundation for control design, which we plan to explore in subsequent research, potentially leveraging methods from [36,47–49].

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