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Dynamics of a Stochastic Single-Species Kolmogorov System Under Markovian Switching

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Abstract: Understanding the impact of unpredictable environmental fluctuations is crucial in population ecology because such fluctuations are an important feature of natural population systems. In this paper, we consider a stochastic single-species Kolmogorov system under Markovian switching. By using the Lyapunov method, we establish sufficient conditions for stochastic permanence, exponential ergodicity, and extinction of the system. Some examples are presented to illustrate corollaries, showing that our results generalize and improve on some known ones.

Keywords: stochastic single-species Kolmogorov system; Markovian switching; stochastic permanence; stationary distribution; exponential ergodicity; extinction

MSC: 92D25; 60H10; 60J25; 60J70

1. Introduction

In the real world, population dynamics are frequently influenced by unpredictable environmental fluctuations, such as changes in insolation, climate, and the occurrence of disasters. Therefore, modeling population evolution necessitates a stochastic, rather than a deterministic, approach. Stochastic differential equations (SDEs), which model various fixed parameters in deterministic systems as a fluctuation around their mean values according to white noise, are a common and well-established method (see, e.g., [1–4] and the references therein).

Additionally, variability of external circumstances, such as rainfall, temperature, and nutrition, may lead to the environmental regimes of populations switching randomly from one to another. In general, a continuous-time Markov chain is considered a suitable tool for modeling this regime switching. It is worth mentioning that this approach, incorporating Markovian switching into SDEs, has received extensive interest and gained tremendous achievement [5–23]. Especially, these works reveal that Markovian switching can profoundly affect the dynamic behavior of stochastic ecosystems and the robustness of neural networks.

A fundamental issue in mathematical ecology is to determine the permanence or extinction of the population. However, due to environmental noise, the stability of equilibria in the deterministic system may be destroyed in its stochastic counterpart. As we know, under appropriate conditions, the solution of SDEs is a Markov process. Therefore, exploring the dynamics of SDEs is equivalent to studying the ergodic property of the corresponding Markov process (see, e.g., [24,25] for relevant results in the ergodic theory of Markov processes). In particular, our interest lies in ascertaining whether the solution process has a stationary distribution. If so, does its transition probability function converge

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to this stationary distribution in a certain mode? What is the convergence rate? Intuitively, the stationary distribution can be regarded as an analogue to an equilibrium point or an attractor of the deterministic system.

In this paper, we investigate the following stochastic single-species Kolmogorov system under Markovian switching:

$$\mathrm{d}X(t) = X(t)f(X(t), \Lambda(t))\mathrm{d}t + X(t)\sum_{l=1}^{n} g_l(X(t), \Lambda(t))\mathrm{d}B_l(t), \tag{1}$$

where X represents the density of the population. The function $f(\cdot,\cdot)$ denotes the per capita growth rate of the population. $\boldsymbol{B}(t) = (B_1(t), B_2(t), \cdots, B_n(t))$ is an n-dimensional standard Brownian motion, and $g_l(\cdot,\cdot)$, $l = 1, 2, \ldots, n$, describes the white noise intensity. $\Lambda(t)$ is a continuous-time Markov chain with the state space $\mathbb{S} = \{1, 2, \cdots, m\}$. For the fundamental theory of stochastic differential equations under Markovian switching, we refer the reader to [5]. We always make the following fundamental assumption for system (1).

Assumption 1.

(a) For each $i \in \mathbb{S}$, the function $f(\cdot, i) : [0, \infty) \to \mathbb{R}$ is continuously differentiable, and there exist $\Gamma > 0$, $\varrho > 0$ such that $f(x, i) < -\varrho$ for all $x > \Gamma$ and $i \in \mathbb{S}$.

(b) For each $i \in \mathbb{S}$ and $l = 1, 2, \cdots, n$, the function $g_l(\cdot, i) : [0, \infty) \to \mathbb{R}$ is continuously differentiable.

Assumption 1, which guarantees the existence and uniqueness of a global positive solution to (1), is technical, but it is biologically reasonable. In particular, Assumption 1(a) biologically implies that high-density populations are constrained by limited environmental resources, leading to a negative per capita growth rate. Mathematically, this is a constraint of the system at 0. System (1) encompasses several established single-species models previously investigated in the literature. Notable special cases include the logistic model analyzed in [13], the Gilpin–Ayala model examined in [14–16], the Allee effect model explored in [17–21], the Smith model investigated in [22], and the Budworm model studied in [23]. These papers investigated some asymptotic behaviors of the corresponding model, such as stochastic permanence, extinction, and ergodicity. To the best of our knowledge, there is no published paper concerned with the dynamics of system (1).

The aim of this present paper is, by using the Lyapunov method, to establish conditions for stochastic permanence, exponential ergodicity, and extinction of system (1). Specifically, these results will extend and improve on some known ones in [13–23,26].

In the next section, we devote to the main results and their proofs. In Section 3, we list several examples. Finally, we give the conclusion in the last section.

2. Main Results and Proofs

Throughout this paper, we work on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions. Let $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$, $\|\cdot\|_{\mathrm{var}}$ be the total variation norm of signed measures and $C^2(\mathbb{R})$ be the family of all twice differentiable functions on $\mathbb{R}$. $\Lambda(t)$ is independent of $\boldsymbol{B}(t)$ with the generator $Q = (q_{ij})_{m\times m}$ satisfying

$$q_{ij} \geq 0 \ \text{ for } \ i \neq j \ \text{ and } \ \sum_{j=1}^{m} q_{ij} = 0 \ \text{ for each } \ i \in \mathbb{S}. \tag{2}$$

Suppose that Q is irreducible. Then Q admits a unique stationary distribution $\pi = (\pi_1, \pi_2, \ldots, \pi_m)$ that constitutes a unique solution to linear equations, as follows:

$$\pi Q = 0 \text{ and } \sum_{i=1}^{m} \pi_i = 1 \tag{3}$$

satisfying $\pi_i > 0$ for each $i \in \mathbb{S}$.

For a function $V : \mathbb{R} \times \mathbb{S} \to \mathbb{R}$ and $V(\cdot, i) \in C^2(\mathbb{R})$, we define an operator $\mathscr{L}$ associated with system (1) by

$$\mathscr{L}V(x,i) = xf(x,i)\frac{\partial V(x,i)}{\partial x} + \frac{x^2}{2}\sum_{l=1}^{n} g_l^2(x,i)\frac{\partial^2 V}{\partial x^2} + \sum_{j=1}^{m} q_{ij}V(x,j),\ i \in \mathbb{S}.$$

For simplicity of presentation, we denote

$$h(x,i) = \sum_{l=1}^{n} g_l^2(x,i),\ \ \alpha(i) = f(0,i) - \frac{1}{2}h(0,i),\ \ i \in \mathbb{S},$$

and $\alpha = (\alpha(1), \alpha(2), \ldots, \alpha(m))^T$. The solution process of system (1) will be denoted by $(X(t), \Lambda(t))$.

From the perspective of mathematical ecology, our focus is exclusively on the positive solution. Therefore, establishing the existence and uniqueness of a globally positive solution constitutes a fundamental prerequisite for the systematic analysis of system (1).

Proposition 1. *For any given $(x,i) \in \mathbb{R}_+ \times \mathbb{S}$, there exists a unique solution $X(t)$ to system (1) for all $t \geq 0$ with $X(0) = x$, $\Lambda(0) = i$, and the solution will stay forever in $\mathbb{R}_+$ with probability 1.*

Proof. Choose $p \in (0,1)$ and define a non-negative C^2–function by

$$V_1(x) = \frac{x^p}{p} - \ln x - \frac{1}{p}.$$

It is obvious that

$$\lim_{x\to\infty} V_1(x) = \infty. \tag{4}$$

Then we find that

$$\mathscr{L}V_1 = (x^p - 1)f(x,i) + \frac{1}{2}[(p-1)x^p + 1]h(x,i) =: \Phi_1(x,i).$$

One can easily find a $\Gamma_1 > \Gamma$ such that $(p-1)x^p + 1 < 0$ for all $x > \Gamma_1$. Then by (A_1), we have that for all $x > \Gamma_1$ and $i \in \mathbb{S}$,

$$\Phi_1(x,i) < -\varrho x^p + \varrho,$$

which yields that for each $i \in \mathbb{S}$,

$$\limsup_{x\to\infty} \Phi_1(x,i) = -\infty. \tag{5}$$

Moreover, we have that for each $i \in \mathbb{S}$,

$$\lim_{x\to 0^+} \Phi_1(x,i) = f(0,i) + \frac{1}{2}h(0,i).$$

This, together with (5) and the continuity of function $\Phi_1(\cdot, i)$, leads that

$$c_1 = \sup_{(x,i)\in(\mathbb{R}_+\times\mathbb{S})} \Phi_1(x,i) < \infty.$$

Noticing the non-negativity of V_1, we have that for all $(x,i) \in \mathbb{R}_+ \times \mathbb{S}$,

$$\mathscr{L}V_1(x) \le \max\{c_1, 1\} + V_1(x). \tag{6}$$

As the coefficients of Equation (1) are local Lipschitz, utilizing (4) and (6), Proposition 1 follows from Theorem 3.19 of [5]. □

Moreover, applying Theorem 5.1 in [7], we can deduce from (4) and (6) that $(X(t), \Lambda(t))$ is a time-homogeneous Markov process on $\mathbb{R}_+ \times \mathbb{S}$.

Definition 1. *Let*

$$P(t,(x,i),B) = \mathbb{P}\{(X(t),\Lambda(t)) \in B \,|(X(0),\Lambda(0)) = (x,i)\}, \quad B \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{S})$$

be the transition probability function of $(X(t), \Lambda(t))$. *A probability measure* μ *on* $\mathbb{R}_+ \times \mathbb{S}$ *is said to be a stationary distribution of* $(X(t), \Lambda(t))$ *if*

$$\sum_{i=1}^{m} \int_{\mathbb{R}_+} P(t,(x,i),B)\mu(\mathrm{d}x) = \mu(B) \text{ for any } t \ge 0 \text{ and } B \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{S}).$$

Definition 2. *The process* $(X(t), \Lambda(t))$ *is said to be exponentially ergodic if it processes a unique stationary distribution* μ *on* $\mathbb{R}_+ \times \mathbb{S}$, *and there exists a constant* $c > 0$ *such that for any* $(x,i) \in \mathbb{R}_+ \times \mathbb{S}$,

$$\lim_{t\to\infty} \mathrm{e}^{ct} \|P(t,(x,i),\cdot) - \mu(\cdot)\|_{\mathrm{var}} = 0.$$

Definition 3. *System* (1) *is said to be stochastically permanent if, for arbitrary* $\varepsilon \in (0,1)$, *there exist* $K_1 = K_1(\varepsilon) > 0$, $K_2 = K_2(\varepsilon) > 0$ *such that for any initial value* $(x,i) \in \mathbb{R}_+ \times \mathbb{S}$,

$$\liminf_{t\to\infty} \mathbb{P}\{K_1 \le X(t) \le K_2\} \ge 1 - \varepsilon.$$

From the definitions of exponential ergodicity and stochastic permanence, one can easily see that exponential ergodicity implies stochastic permanence. The following condition ensures stochastic permanence.

Assumption 2. $\pi\alpha > 0$.

Remark 1. *The quantity* $\pi\alpha$ *can be interpreted as the stochastic per capita intrinsic growth rate. Mathematically, Assumption 2 is a constraint of the system at 0.*

Now, we present a lemma that is crucial for establishing the stochastic permanence and exponential ergodicity.

Lemma 1. *Suppose that Assumptions 1 and 2 hold. Then there exist a function* $V_2 : \mathbb{R}_+ \times \mathbb{S} \to \mathbb{R}_+$ *with* $V_2(\cdot, i) \in C^2(\mathbb{R}_+)$ *and two positive constants* κ, c_2, *satisfying that*

(a) *for each* $i \in \mathbb{S}$, $\lim_{k\to\infty} \inf\{V_2(x,i) : x \vee x^{-1} > k\} = \infty$;

(b) *for any* $(x,i) \in \mathbb{R}_+ \times \mathbb{S}$, $\mathscr{L}V_2(x,i) \le -\kappa V_2(x,i) + c_2$.

Proof. By (3), we have

$$\pi[-\alpha + \pi\alpha\mathbb{I}_m] = 0,$$

where $\mathbb{I}_m = (1, 1, \ldots, 1)^T \in \mathbb{R}^m$. Owing to Lemma 2.3 of [27], the equation

$$Q\xi = -\alpha + \pi\alpha\mathbb{I}_m$$

has a solution $\xi = (\xi_1, \xi_2, \ldots, \xi_m)^T \in \mathbb{R}^m$. This, together with Assumption 2, implies that for each $i \in \mathbb{S}$,

$$\sum_{j=1}^{m} q_{ij}\xi_j + \alpha(i) = \pi\alpha > 0. \tag{7}$$

Then, one can choose $q \in (0, 1)$ that is small enough such that for each $i \in \mathbb{S}$,

$$1 - \xi_i q > 0 \quad \text{and} \quad \pi\alpha + q\left(\frac{\xi_i}{1-\xi_i q}\sum_{j=1}^{m} q_{ij}\xi_j - \frac{1}{2}h(0,i)\right) =: \chi_i > 0.$$

Define

$$V_2(x,i) = (1-\xi_i q)\frac{x^{-q}}{q} + \frac{x^p}{p}, \quad (x,i) \in \mathbb{R}_+ \times \mathbb{S}. \tag{8}$$

It is obvious that assertion (a) holds.

Next, we verify assertion (b). It follows from (2) that

$$\begin{aligned} -\frac{1}{q(1-\xi_i q)}\sum_{j=1}^{m} q_{ij}(1-\xi_j q) &= \frac{1}{1-\xi_i q}\sum_{j=1}^{m} q_{ij}\xi_j = \frac{1-\xi_i q+\xi_i q}{1-\xi_i q}\sum_{j=1}^{m} q_{ij}\xi_j \\ &= \sum_{j=1}^{m} q_{ij}\xi_j + \frac{\xi_i q}{1-\xi_i q}\sum_{j=1}^{m} q_{ij}\xi_j. \end{aligned} \tag{9}$$

Choose $\kappa \in (0, 1)$ that is small enough such that for each $i \in \mathbb{S}$,

$$\chi_i - \frac{\kappa}{q} > 0 \quad \text{and} \quad -\frac{1}{2}\varrho + \frac{\kappa}{p} < 0. \tag{10}$$

Denote

$$\Psi(x,i) = f(x,i) - \frac{1}{2}h(x,i) + \sum_{j=1}^{m} q_{ij}\xi_j + q\left(\frac{\xi_i}{1-\xi_i q}\sum_{j=1}^{m} q_{ij}\xi_j - \frac{1}{2}h(x,i)\right) - \frac{\kappa}{q}.$$

Then using (9), we obtain that

$$\begin{aligned} \mathscr{L}V_2(x,i) + \kappa V_2(x,i) =& -(1-\xi_i q)x^{-q}\left[f(x,i) - \frac{q+1}{2}h(x,i) - \frac{\kappa}{q}\right] \\ &+ \sum_{j=1}^{m} q_{ij}(1-\xi_j q)\frac{x^{-q}}{q} + x^p\left[f(x,i) + \frac{p-1}{2}h(x,i) + \frac{\kappa}{p}\right] \\ =& -\Psi(x,i)(1-\xi_i q)x^{-q} + x^p\left[f(x,i) + \frac{p-1}{2}h(x,i) + \frac{\kappa}{p}\right] \\ =:& \Phi_2(x,i). \end{aligned} \tag{11}$$

Since, for each $i \in \mathbb{S}$,

$$\lim_{x\to 0^+}\Psi(x,i) = \chi_i - \frac{\kappa}{q} > 0,$$

then there exists a $\delta_1 > 0$ such that for all $x \in (0, \delta_1)$ and $i \in \mathbb{S}$,

$$\Psi(x,i) \geq \frac{1}{2}(\chi_i - \frac{\kappa}{q}),$$

which leads to that for all $x \in (0, \delta_1)$ and $i \in \mathbb{S}$,

$$\Phi_2(x,i) \leq -\frac{1}{2}\left(\chi_i - \frac{\kappa}{q}\right)(1-\xi_i q)x^{-q} + x^p\left[f(x,i) + \frac{p-1}{2}h(x,i) + \frac{\kappa}{p}\right].$$

Hence, we deduce that for each $i \in \mathbb{S}$,

$$\lim_{x\to 0^+} \Phi_2(x,i) = -\infty. \tag{12}$$

On the other hand, $\Phi_2(x,i)$ can be rewritten as

$$\begin{aligned}\Phi_2(x,i) = &\left[\frac{1}{2}x^{p+q} - (1-\xi_i q)\right]x^{-q}f(x,i) + \frac{1}{2}\left[(p-1)x^{p+q} + (q+1)(1-\xi_i q)\right]x^{-q}h(x,i)\\ &+ \left[\sum_{j=1}^{m} q_{ij}(1-\xi_j q) + \kappa(1-\xi_i q)\right]\frac{x^{-q}}{q} + x^p\left[\frac{1}{2}f(x,i) + \frac{\kappa}{p}\right].\end{aligned} \tag{13}$$

It is easy to find a $\Gamma_2 > \Gamma$ such that for all $x > \Gamma_2$ and $i \in \mathbb{S}$,

$$\left[\frac{1}{2}x^{p+q} - (1-\xi_i q)\right]f(x,i) < 0, \quad (p-1)x^{p+q} + (q+1)(1-\xi_i q) < 0. \tag{14}$$

It follows from (10), (13), and (14) that for all $x > \Gamma_2$ and $i \in \mathbb{S}$,

$$\Phi_2(x,i) \leq \left[\sum_{j=1}^{m} q_{ij}(1-\xi_j q) + \kappa(1-\xi_i q)\right]\frac{x^{-q}}{q},$$

which yields that for each $i \in \mathbb{S}$,

$$\limsup_{x\to\infty} \Phi_2(x,i) \leq 0. \tag{15}$$

This, together with (12) and the continuity of function $\Phi_2(\cdot, i)$, gives that

$$c_2 = \sup_{(x,i)\in(\mathbb{R}_+\times\mathbb{S})} \Phi_2(x,i) < \infty,$$

which, together with (11), gives assertion (b). □

Now we present the result on the stochastic permanence of system (1).

Theorem 1. *Suppose that Assumptions 1 and 2 hold. Then system* (1) *is stochastically permanent.*

Proof. One can easily obtain from assertion (b) of Lemma 1 that

$$\mathscr{L}\left[\mathrm{e}^{\kappa t}V_2(X(t),\Lambda(t))\right] \leq c_2\mathrm{e}^{\kappa t}.$$

Utilizing Itô's formula, integrating over $[0,t]$, and taking expectations, we obtain that

$$\mathrm{e}^{\kappa t}\mathbb{E}[V_2(X(t),\Lambda(t))] \leq V_2(x,i) + \frac{c_2}{\kappa}(\mathrm{e}^{\kappa t} - 1),$$

which results in

$$\limsup_{t\to\infty} \mathbb{E}[V_2(X(t),\Lambda(t))] \le \frac{c_2}{\kappa}. \tag{16}$$

For any $\varepsilon \in (0,1)$, let

$$K_1(\varepsilon) = \left(\frac{\varepsilon\kappa\xi_*}{c_2}\right)^{1/q} \text{ and } K_2(\varepsilon) = \left(\frac{pc_2}{\kappa\varepsilon}\right)^{1/p},$$

where $\xi_* = (1-\max\{\xi_i\}q)/q$. Then, an application of Chebyshev's inequality combined with (8) and (16) yields that

$$\begin{aligned}\limsup_{t\to\infty} \mathbb{P}\{X(t) < K_1\} &= \limsup_{t\to\infty} \mathbb{P}\{X^{-q}(t) > K_1^{-q}\} \le K_1^q \limsup_{t\to\infty} \mathbb{E}[X^{-q}(t)] \\ &\le K_1^q \xi_*^{-1} \limsup_{t\to\infty} \mathbb{E}[V_2(X(t),\Lambda(t))] \le \varepsilon,\end{aligned}$$

$$\limsup_{t\to\infty} \mathbb{P}\{X(t) > K_2\} \le K_2^{-p} \limsup_{t\to\infty} \mathbb{E}[X^p(t)] \le K_2^{-p} p \limsup_{t\to\infty} \mathbb{E}[V_2(X(t),\Lambda(t))] \le \varepsilon.$$

Hence,

$$\liminf_{t\to\infty} \mathbb{P}\{K_1 \le X(t) \le K_2\} \ge 1-\varepsilon.$$

The proof is complete. □

In order to obtain the ergodic result, we further need the following condition to ensure that the solution to (1) is a nondegenerate diffusion.

Assumption 3. *For any $(x,i) \in \mathbb{R}_+ \times \mathbb{S}$, $h(x,i) > 0$.*

Lemma 2. *Suppose that Assumptions 1 and 3 hold. Then for any $T > 0$, $i \in \mathbb{S}$ and compact set $D \subset \mathbb{R}_+$, $D \times \{i\}$ is a petite set for the Markov chain $\{X(kT), \Lambda(kT)\}_{k\in\mathbb{N}}$.*

Proof. The proof follows a similar approach to that of Lemma 3.6 of [12], so we omit it here. □

We note that the concept and properties of petite sets are essential for the stability theory of Markov chains. We refer the reader to [24] for the definition and properties of petite sets. Then, we are in condition to state our result on the exponential ergodicity of system (1).

Theorem 2. *Suppose that Assumptions 1–3 hold. Then process $(X(t),\Lambda(t))$ is exponentially ergodic.*

Proof. Applying Theorem 6.1 in [25], we can deduce Theorem 2 from Lemmas 1 and 2. □

Next, we turn to explore the extinction of system (1). To this end, we need to introduce the following assumption.

Assumption 4.

(a) *For any $(x,i) \in \mathbb{R}_+ \times \mathbb{S}$, there exist $\beta(i) > 0$ and $\sigma(i) > 0$ such that $f(x,i) \le \beta(i)$ and $h(x,i) \ge \sigma^2(i)$.*

(b) $\displaystyle\sum_{i=1}^{m} \pi_i[\beta(i) - \frac{1}{2}\sigma^2(i)] < 0.$

Remark 2. *Assumption 4(a) necessitates a positive upper bound for the per capita growth rate and a lower bound for the white noise intensity. This is not a heavy burden for many ecological models (see Section 3). Assumption 4(b) implies that the maximum stochastic per capita growth rate is negative.*

Theorem 3. *Suppose that Assumptions 1 and 4 hold. Then for any initial value* $(x, i) \in \mathbb{R}_+ \times \mathbb{S}$, $\lim_{t\to\infty} X(t) = 0$ *a.s., i.e., the population in system* (1) *will be driven to extinction.*

Proof. By an argument similar to that of (7), we can find that there exists a vector $\zeta = (\zeta_1, \zeta_2, \cdots, \zeta_m) \in \mathbb{R}^m$ such that for each $i \in \mathbb{S}$,

$$\sum_{j=1}^{m} q_{ij}\zeta_j + \beta(i) - \frac{1}{2}\sigma^2(i) = \sum_{j=1}^{m} \pi_j[\beta(j) - \frac{1}{2}\sigma^2(j)].$$

From Assumption 4(b), one can choose $r \in (0,1)$ that is small enough such that for each $i \in \mathbb{S}$,

$$1 + \zeta_i r > 0 \text{ and } \sum_{j=1}^{m} \pi_j[\beta(j) - \frac{1}{2}\sigma^2(j)] + r\left(\frac{1}{2}\sigma^2(i) - \frac{\zeta_i}{1+\zeta_i r}\sum_{j=1}^{m} q_{ij}\zeta_j\right) < 0. \tag{17}$$

Similar to (9), we have that

$$\frac{1}{r(1+\zeta_i r)}\sum_{j=1}^{m} q_{ij}(1+\zeta_j r) = \sum_{j=1}^{m} q_{ij}\zeta_j - \frac{\zeta_i r}{1+\zeta_i r}\sum_{j=1}^{m} q_{ij}\zeta_j. \tag{18}$$

Define

$$V_3(x,i) = (1+\zeta_i r)x^r, \quad (x,i) \in \mathbb{R}_+ \times \mathbb{S}.$$

Then by Assumption 4(a), (17), and (18), we obtain that

$$\begin{aligned}
\mathscr{L}V_3(x,i) &= r(1+\zeta_i r)x^r\left[f(x,i) + \frac{r-1}{2}h(x,i) + \frac{1}{r(1+\zeta_i r)}\sum_{j=1}^{m} q_{ij}(1+\zeta_j r)\right] \\
&= r(1+\zeta_i r)x^r\left[f(x,i) + \frac{r-1}{2}h(x,i) + \sum_{j=1}^{m} q_{ij}\zeta_j - \frac{\zeta_i r}{1+\zeta_i r}\sum_{j=1}^{m} q_{ij}\zeta_j\right] \\
&\le r(1+\zeta_i r)x^r\left[\sum_{j=1}^{m} q_{ij}\zeta_j + \beta(i) + \frac{r-1}{2}\sigma^2(i) - \frac{\zeta_i r}{1+\zeta_i r}\sum_{j=1}^{m} q_{ij}\zeta_j\right] \\
&= r(1+\zeta_i r)\left[\sum_{j=1}^{m} \pi_j[\beta(j) - \frac{1}{2}\sigma^2(j)] + r\left(\frac{1}{2}\sigma^2(i) - \frac{\zeta_i}{1+\zeta_i r}\sum_{j=1}^{m} q_{ij}\zeta_j\right)\right]x^r \\
&\le -\varphi x^r,
\end{aligned} \tag{19}$$

where

$$\varphi = \min_{i\in\mathbb{S}}\left\{-r(1+\zeta_i r)\left[\sum_{j=1}^{m} \pi_j[\beta(j) - \frac{1}{2}\sigma^2(j)] + r\left(\frac{1}{2}\sigma^2(i) - \frac{\zeta_i}{1+\zeta_i r}\sum_{j=1}^{m} q_{ij}\zeta_j\right)\right]\right\} > 0.$$

Obviously,

$$\min_{i\in\mathbb{S}}\{1+\zeta_i r\}x^r \le V_3(x,i) \le \max_{i\in\mathbb{S}}\{1+\zeta_i r\}x^r. \tag{20}$$

By virtue of (19) and (20), Theorem 3 follows from Theorem 5.37 in [5]. □

It is well known that the threshold dynamics are an important issue in mathematical ecology. In order to obtain the threshold dynamics of system (1), it is necessary to impose some further restriction on $f(\cdot,\cdot)$ and $h(\cdot,\cdot)$.

Assumption 5. *For each $i \in \mathbb{S}$,*

(a) $\max_{x\in\mathbb{R}_+} f(x,i) = f(0,i)$;

(b) $\min_{x\in\mathbb{R}_+} h(x,i) = h(0,i)$.

Remark 3. *We point out that Assumption 5 is not a severe restriction. All decreasing growth functions in the literature, such as logistic growth, Gilpin-Ayala growth, and Smith growth, satisfy Assumption 5(a). For the case that $g_l(x,i)$ is independent of x, Assumption 5(b) clearly holds. In addition, several systems with multi-parameter stochastic perturbations also satisfy Assumption 5(b) (see Examples 2 and 4).*

According to Theorems 1 and 3, we obtain the following result on the threshold dynamics of system (1).

Theorem 4. *Suppose that Assumptions 1, 2, and 5 hold. Then the stochastic permanence and extinction of system (1) can be determined by the sign of $\pi\alpha$. That is, if $\pi\alpha > 0$, then system (1) is stochastically permanent; if $\pi\alpha < 0$, then the population in system (1) will be driven to extinction. Additionally, if Assumption 3 further holds, then system (1) exhibits exponential ergodicity when $\pi\alpha > 0$.*

Remark 4. *We note that the threshold is a weighted combination, with the weights determined by $\Lambda(t)$. In fact, it is this structure that makes Markovian switching essential for adapting to environmental change and achieving robust ecosystem management. In addition, Theorems 1–4 remain valid in the absence of regime switching (i.e., $\mathbb{S} = \{1\}$).*

3. Example

In this section, as an application of our main results, we shall present several examples.

Example 1. *Consider a stochastic logistic model under Markovian switching,*

$$\mathrm{d}X(t) = X(t)\Big[a(\Lambda(t)) - b(\Lambda(t))X(t)\Big]\mathrm{d}t + \sigma_1(\Lambda(t))\mathrm{d}B_1(t). \tag{21}$$

which has been studied by [13]. Here, for each $i \in \mathbb{S}$, $b(i), \sigma_1(i) > 0$ and $a(i) \geq 0$. For the biological significance of these parameters, one is referred to [13]. Denote

$$\lambda_1 = \sum_{i=1}^{m} \pi_i[a(i) - \frac{1}{2}\sigma_1^2(i)].$$

According to Theorem 4, we obtain the following result.

Corollary 1.

(1) *If $\lambda_1 > 0$, then the solution to system (21) with an initial value on $\mathbb{R}_+ \times \mathbb{S}$ is exponentially ergodic.*

(2) *If $\lambda_1 < 0$, then the population in system (21) will be driven to extinction.*

Remark 5. *In their Theorem 3.2, Li et al. [13] showed that system (21) is stochastically persistent under the following conditions:*

$$\lambda_1 > 0 \text{ and for some } j \in \mathbb{S},\ q_{ij} > 0 (i \neq j).$$

Clearly, the second condition is superfluous. Hence, our Corollary 1 improves on their Theorem 3.2.

Example 2. *Consider a stochastic Gilpin–Ayala model under Markovian switching,*

$$\begin{aligned} \mathrm{d}X(t) = X(t)\Big[a(\Lambda(t)) - b(\Lambda(t))X^{\theta(\Lambda(t))}(t)\Big]\mathrm{d}t + \sigma_1(\Lambda(t))X(t)\mathrm{d}B_1(t) \\ + \sigma_2(\Lambda(t))X^{1+\eta(\Lambda(t))}(t)\mathrm{d}B_2(t), \end{aligned} \tag{22}$$

which has been studied in [14–16]. Here, for each $i \in \mathbb{S}$, $b(i), \theta(i), \sigma_1(i) > 0$, and other parameters are considered to be non-negative. For the biological significance of these parameters, one is referred to [14]. Denote

$$\lambda_2 = \sum_{i=1}^{m} \pi_i [a(i) - \frac{1}{2}\sigma_1^2(i)].$$

According to Theorem 4, we obtain the following result.

Corollary 2.

(1) If $\lambda_2 > 0$, then the solution to system (22) with an initial value on $\mathbb{R}_+ \times \mathbb{S}$ is exponentially ergodic.

(2) If $\lambda_2 < 0$, then the population in system (22) will be driven to extinction.

Remark 6. *In their Theorem 5.1, Wang et al. [14] showed that system (22) admits a unique stationary distribution under the following conditions:*

$$\lambda_2 > 0 \text{ and } \ \theta(i) + 1 \geq 2\eta(i) \text{ for each } i \in \mathbb{S}.$$

We observe that the second condition is superfluous, and the convergence rate to this stationary distribution was not investigated in [14]. Hence, our Corollary 2 improves this result. Similarly, our Theorem 2 also improves Theorem 5 of [15] and Theorem 4.2 of [16].

Example 3. *Consider a stochastic Allee effect model under Markovian switching,*

$$dX(t) = X(t)\left[a(\Lambda(t))\left(1 - \frac{X(t)}{K(\Lambda(t))}\right) - \frac{b(\Lambda(t))}{d(\Lambda(t))b(\Lambda(t))X(t) + 1}\right]dt + \sigma_1(\Lambda(t))X(t)dB_1(t), \tag{23}$$

which has been studied in [17]. Here, for each $i \in \mathbb{S}$, all parameters are positive. For the biological significance of these parameters, one is referred to [17]. Let

$$f(x,i) = \left[a(i)\left(1 - \frac{x}{K(i)}\right) - \frac{b(i)}{d(i)b(i)x + 1}\right].$$

A simple calculation gives that

$$f(x,i) \leq a(i) - b(i) \text{ as } \min_{i\in\mathbb{S}} \gamma(i) > 0,$$

$$f(x,i) \leq \beta_2(i) = a(i) + \frac{a(i) - 2b(i)\sqrt{a(i)d(i)k(i)}}{b(i)d(i)k(i)} \text{ as } \max_{i\in\mathbb{S}} \gamma(i) \leq 0,$$

where $\gamma(i) = \frac{a(i)}{K(i)} - d(i)b^2(i)$. *Denote*

$$\lambda_3 = \lambda_1 - \sum_{i=1}^{m} \pi_i b(i) \text{ and } \lambda_4 = \sum_{i=1}^{m} \left[\beta_2(i) - \frac{1}{2}\sigma_1^2(i) \right].$$

An application of Theorem 2 and 3 yields the following result.

Corollary 3.

(1) *If* $\lambda_3 > 0$, *then the solution to system* (23) *with an initial value on* $\mathbb{R}_+ \times \mathbb{S}$ *is exponentially ergodic.*

(2) *Suppose* $\min_{i\in\mathbb{S}} \gamma(i) > 0$. *If* $\lambda_3 < 0$, *then the population in system* (23) *will be driven to extinction.*
Suppose $\max_{i\in\mathbb{S}} \gamma(i) \leq 0$. *If* $\lambda_4 < 0$, *then the population in system* (23) *will be driven to extinction.*

Remark 7. *In their Theorem 3.1, Yu et al. [17] showed that when* $\max_{i\in\mathbb{S}} \gamma(i) \leq 0$, *the population of system* (23) *goes extinct under the following condition:*

$$\sum_{i=1}^{m} \pi_i \left[a(i) - \sqrt{\frac{a(i)}{d(i)K(i)}} - \frac{1}{2}\sigma_1^2(i) \right] < 0. \tag{24}$$

An easy verification shows that inequation (24) *implies* $\lambda_4 < 0$. *On the other hand, while their Theorem 4.1 demonstrates the existence of a unique stationary distribution under the condition* $\lambda_3 > 0$, *it fails to specify its convergence rate. Hence, our Corollary 3 improves on* (b_1) *of Theorems 3.1 and 4.1 in [17]. Similarly, our Theorem 2 also generalizes and improves the results presented in Theorem 3 of [18,19], Theorem 2 of [20], and Theorem 1 of [23], while also confirming the first assertion of Theorem 1 of [21,22].*

Example 4. *Consider a stochastic Gilpin–Ayala model without Markovian switching,*

$$\mathrm{d}X(t) = X(t)\left[a - bX^{\theta}(t)\right]\mathrm{d}t + \sum_{l=1}^{n} X(t)\sigma_{1l}\mathrm{d}B_l(t) + \sum_{l=1}^{n} \sigma_{2l}X^{1+\theta}(t)dB_l(t), \tag{25}$$

which has been studied in [26]; here, for every $l = 1, 2, \cdots, n$, $\sigma_{2l} \geq 0$ *is non-negative, and* a, b, θ, *and* $\sum_{l=1}^{n} \sigma_{1l}^2$ *are considered to be positive. For the biological significance of these parameters, the reader is referred to [26]. Denote*

$$\lambda_5 = a - \frac{1}{2}\sum_{l=1}^{n} \sigma_{1l}^2.$$

From Theorem 4 and Remark 4, we obtain the following result.

Corollary 4.

(1) *If* $\lambda_5 > 0$, *then the solution to system* (25) *with an initial value on* $\mathbb{R}_+ \times \mathbb{S}$ *is exponentially ergodic.*

(2) *If* $\lambda_5 < 0$, *then the population in system* (25) *will be driven to extinction.*

Remark 8. *Lv et al. [26] only showed that system* (25) *exhibits exponential ergodicity when* $l = 1$. *Clearly, our Corollary 4 generalizes Lemma 3 of [26].*

4. Conclusions

This paper is devoted to a stochastic single-species Kolmogorov system under Markovian switching. The sufficient conditions for stochastic permanence, exponential ergodicity,

and extinction of system (1) have been established. Notably, we have given an explicit threshold (see Theorem 4) that delineates extinction from stochastic permanence. The threshold dynamics result has been successfully applied to typical models in Section 3, demonstrating the practical relevance of our main results. Meanwhile, these examples clearly illustrate that our results extend and improve on some known ones in [13–23,26]. Our results reinforce the critical role of unpredictable environmental fluctuations in determining species richness, providing mathematical foundations for biodiversity conservation strategies and sustainable resource management policies.

Moreover, recognizing that discontinuous stochastic events such as volcanic eruptions, earthquakes, forest fires, and sudden disease outbreaks can significantly impact population dynamics, we acknowledge that it is necessary to incorporate Lévy jumps into SDEs under Markovian switching. Therefore, the study of a stochastic single-species Kolmogorov system driven by Lévy jumps under Markovian switching would enable systematic exploration of synergistic noise interactions, potentially yielding enhanced ecological realism and dynamical complexity through the superposition of multiple stochastic processes. We will pursue this line of research in our future work, anticipating the need for advanced mathematical tools and analytical skills.

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