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Möbius Transformations in the Second Symmetric Product of $\mathbb{C}$

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Abstract: Let $F_2(\mathbb{C})$ denote the second symmetric product of the complex plane $\mathbb{C}$ endowed with the Hausdorff topology, i.e., $F_2(\mathbb{C}) = \{A \subset \mathbb{C} : |A| \leq 2, A \neq \emptyset\}$. In this paper, we extended the concept of Möbius transformations to $F_2(\mathbb{C})$. More precisely, given a Möbius transformation T of $\mathbb{C}$, we define the map $\widetilde{T}(\{z,w\}) = \{T(z), T(w)\}$ within $F_2(\mathbb{C})$. We describe some general properties of these maps, including the structure of their generators, characteristics related to transitivity, and the geometry of the conjugacy classes.

Keywords: second symmetric product; Möbius transformations; transitivity; conjugacy classes

MSC: 30G35; 54H15

1. Introduction

The second symmetric product of the complex plane $\mathbb{C}$ is defined as the set $F_2(\mathbb{C}) = \{A \subset \mathbb{C} : |A| \leq 2, A \neq \emptyset\}$; that is, $F_2(\mathbb{C})$ consists of points of the form $\{z,w\}$, with $z,w \in \mathbb{C}$ and $z \neq w$, and points of the form $\{z\}$, called singletons when $z = w$, endowed with the Hausdorff metric (see [1–3] for more about symmetric products of topological spaces). In this paper, we translated the properties of the Möbius transformations of the Riemann sphere to the space $F_2(\mathbb{C})$.

Consider a Möbius transformation $T : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, where $\widehat{\mathbb{C}}$ denotes the one-point compactification of $\mathbb{C}$, i.e., $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the Riemann sphere. We extend the notion of Möbius transformation to $F_2(\mathbb{C})$ as follows. Given a point $\{z,w\} \in F_2(\mathbb{C})$, we define $\widetilde{T} : F_2(\mathbb{C}) \to F_2(\mathbb{C})$ as $\widetilde{T}(\{z,w\}) = \{T(z), T(w)\}$, whenever T is defined in z and w; note that $\widetilde{T}(\{z,w\}) = \{T(z)\}$, if $z = w$.

The purpose of this paper is to extend the study of the geometry of the Möbius transformations in the second symmetric product of the complex plane $\mathbb{C}$. To do that, we introduced a model for $F_2(\mathbb{C})$. That is, there is a homeomorphism from $F_2(\mathbb{C})$ to a more suitable space in which we can have a better understanding of the geometry induced by $\{z,w\} \mapsto \{T(z), T(w)\}$. The homeomorphic model of $F_2(\mathbb{C})$ is the space

$$\mathbb{M}_2 = (\mathbb{R}^3_+ \times \mathbb{S}^1)/s,$$

where $\mathbb{R}^3_+ = \{(x,y,z) \in \mathbb{R}^3 : z \geq 0\}$ and s is a relation on elements of the form $(x,y,0,t) \in \mathbb{R}^3 \times \mathbb{S}^1$; see Section 2.3 below.

Given $T(z) = (az+b)/(cz+d)$, a Möbius transformation in the Riemann sphere, recall that for $z = -d/c$, $T(-d/c) = \infty$, so we need to change the definition of $\widetilde{T}$ when z or w is equal to $-d/c$; this change will produce discontinuities at some points, but on the

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other hand, the change will be compatible with having some results similar to properties inherent in the set of Möbius transformations.

We start with the basic results of the classical theory of Möbius transformations in Section 2. In addition, we describe the topology of the second symmetric product of $\mathbb{C}$ and the model $\mathbb{M}_2$ for $F_2(\mathbb{C})$.

In Section 3, we define the set $M(F_2(\mathbb{C})) = \{\widetilde{T} : T \in \mathrm{Aut}(\widehat{\mathbb{C}})\}$, where each $\widetilde{T}$ is taken with its corresponding domain and image, and $\mathrm{Aut}(\widehat{\mathbb{C}})$ is the set of complex automorphisms of $\widehat{\mathbb{C}}$, that is, the set of all automorphisms of the Riemann sphere; see Section 2.1. That is, we properly defined the extension of a Möbius transformation to the space $F_2(\mathbb{C})$. We look closely at how the domains of these maps change depending on T; for instance, we describe the maximum domain where any map $\widetilde{T}$ is continuous, for any Möbius transformation T. We also describe the action of these maps $\widetilde{T}$, via the extension to $F_2(\mathbb{C})$ of the usual generators of the group of Möbius transformations; that is, we prove in Proposition 1 that any map in $M(F_2(\mathbb{C}))$ is a composition of the extensions to $F_2(\mathbb{C})$ of the maps described in Theorem 1. We study in Propositions 2–5 the action of the generators of $M(F_2(\mathbb{C}))$ in the space $\mathbb{M}_2$.

Some transitivity properties of the usual Möbius transformations can be translated on transitivity features of the set $M(F_2(\mathbb{C}))$ in $F_2(\mathbb{C})$, which we described in Section 4. In Proposition 8, we prove that $M(F_2(\mathbb{C}))$ is 2-transitive in $F_2(\mathbb{C})$ if the corresponding points have the same cross ratio.

Now, considering the set of Euclidean circles and the family of lines in $\mathbb{C}$, the corresponding objects in $\mathbb{M}_2$ are Möbius strips and semi-planes, respectively. Proving first that $M(F_2(\mathbb{C}))$ sends the set of these Möbius strips and semi-planes into itself, we show in Theorem 9 that $M(F_2(\mathbb{C}))$ acts transitively in this set. We also define maps that preserve the Möbius strips in $F_2(\mathbb{C})$ generated by Euclidean circles in $\mathbb{C}$. These maps are the extensions to $F_2(\mathbb{C})$ of inversions in Euclidean circles in the Riemann sphere, and we finish Section 4 proving some properties for these maps.

As any Möbius transformation T, different from the identity, is conjugated to a map of the form $U_\lambda(z) = \lambda z$ with $\lambda \in \mathbb{C} \setminus \{0, 1\}$ or to the map $U_1(z) = z + 1$, in Section 5, we extend this result for maps in the set $M(F_2(\mathbb{C}))$ in Theorems 11–13, depending on whether T is parabolic, hyperbolic, or elliptic, respectively. Finally, we show how the corresponding maps $\widetilde{U}_\lambda$ act in $\mathbb{M}_2$.

In summary, in this paper, we explain how the classical theory of Möbius transformations in the Riemann sphere can be extended not just to $\mathbb{R}^n$ or $\mathbb{C}^n$ but to different topological spaces, as we have done in the second symmetric product of the complex plane. This extension includes the description of the generators of $M(F_2(\mathbb{C}))$ and how they behave geometrically in the model $\mathbb{M}_2$, which allows us to observe some similarities with the geometry of the orbits of points under classical Möbius transformations. We also prove that $M(F_2(\mathbb{C}))$ has some transitivity properties; for instance, we prove transitivity in a large set of Möbius strips contained in $F_2(\mathbb{C})$, and they are characterized using cross ratio; moreover, we prove that the concept of inversion in a Euclidean circle can be extended to an inversion in a Möbius strip. Finally, we extend the geometric classification of conjugacy classes in $\mathrm{Aut}(\widehat{\mathbb{C}})$ to our setting, describing geometrically the representatives of every conjugacy class.

We would like to note that extending the Möbius transformations to the second symmetric space $F_2(\mathbb{C})$ allows us to explore this space from the perspective of conformal geometry. This approach makes $F_2(\mathbb{C})$ even more intriguing, as it retains all the properties we have proven.

Furthermore, we highlight that in recent years, several open problems in plane geometry have been resolved by examining the non-embeddability of the second symmetric product of certain spaces into $\mathbb{R}^3$ and $\mathbb{R}^4$ (see [4–6]). In light of these findings, we believe it

is crucial to study, in general, the second symmetric product of $\mathbb{C}$. We consider it possible that certain geometric problems in the plane can be translated to different geometric spaces, as it is demonstrated by the results mentioned above.

2. Preliminaries

In this section, we will briefly present the definitions and results of Möbius transformations and the second symmetric product of $\mathbb{C}$, which we will need in the rest of the paper.

2.1. Möbius Transformations

First, let us describe some basic facts, properties, and results about Möbius transformations in the Riemann sphere; for more details, see [7,8], where all the definitions and proofs of the results stated here can be found. Let $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere. We will denote by $\mathrm{Aut}(\widehat{\mathbb{C}})$ the set of all automorphisms of $\widehat{\mathbb{C}}$, that is, functions of the form

$$T(z) = \frac{az + b}{cz + d},$$

with a, b, c, d complex numbers such that $ad - bc \neq 0$. Transformations $w = T(z)$ are known as linear fractional or Möbius transformations. These transformations form a group under composition, where the inverse map of T is given by

$$T^{-1}(z) = \frac{dz - b}{-cz + a}.$$

There is a group homomorphism from the group of matrices of 2 by 2 with complex coefficients, $GL(2, \mathbb{C})$, to the group $\mathrm{Aut}(\widehat{\mathbb{C}})$; its kernel is the set K of all matrices of the form λI, $(\lambda \in \mathbb{C} \setminus \{0\})$, where I is the identity matrix. Hence, $\mathrm{Aut}(\widehat{\mathbb{C}})$ is isomorphic to the group $GL(2, \mathbb{C})/K = PGL(2, \mathbb{C})$, the projective general linear group.

Equivalently, as T does not determine the coefficients a, b, c, d uniquely, since λa, λb, λc, λd correspond to the same transformation T, for $\lambda \in \mathbb{C} \setminus \{0\}$, we see again that the group $\mathrm{Aut}(\widehat{\mathbb{C}})$ is isomorphic to the projective general linear group. It can be proven that $\mathrm{Aut}(\widehat{\mathbb{C}})$ is also isomorphic to the projective special linear group, that is, the group $PSL(2, \mathbb{C})$, where each element has determinant equal to 1; see Theorem 2.1.3 in [7], which states that $\mathrm{Aut}(\widehat{\mathbb{C}}) \cong PGL(2, \mathbb{C}) = PSL(2, \mathbb{C})$. Thus, from now on, we can assume that $ad - bc = 1$.

The group $\mathrm{Aut}(\widehat{\mathbb{C}})$ is generated by four special types of Möbius transformations, as is stated in the following theorem.

Theorem 1. *Any element in $Aut(\widehat{\mathbb{C}})$ can be written as a composition of maps of the following form:*

(i) The map $R_\theta(z) = e^{i\theta}z$ $(\theta \in \mathbb{R})$ is a rotation of the Riemann sphere $\widehat{\mathbb{C}}$ by an angle θ.

(ii) The transformation $J(z) = 1/z$, that interchange zero and ∞.

(iii) The map $S_r(z) = rz$ $(r \in \mathbb{R}, r > 0)$ fixes 0 and ∞, and acts in the plane $\mathbb{C}$ as a similarity transformation.

(iv) The transformation $T_t(z) = z + t$ $(t \in \mathbb{C})$ fixes ∞ and acts as a translation in the complex plane.

One important property of the automorphisms of $\widehat{\mathbb{C}}$ is that they send circles in $\widehat{\mathbb{C}}$ to circles in $\widehat{\mathbb{C}}$. To be more precise, the circles in $\widehat{\mathbb{C}}$ are the usual Euclidean circles, and the straight lines in $\mathbb{C}$ (which can be thought of as circles through infinity).

Theorem 2. *If C is a circle in $\widehat{\mathbb{C}}$ and $T \in Aut(\widehat{\mathbb{C}})$, then $T(C)$ is a circle in $\widehat{\mathbb{C}}$.*

The group $\mathrm{Aut}(\widehat{\mathbb{C}})$ also has several properties about transitivity; the following are the ones we will use in this paper.

Theorem 3. *If (z_1, z_2, z_3) and (w_1, w_2, w_3) are triples of distinct points in $\widehat{\mathbb{C}}$, then there is a unique $T \in Aut(\widehat{\mathbb{C}})$ such that $T(z_j) = w_j$, for $j = 1, 2, 3$.*

Corollary 1. *If $T \in Aut(\widehat{\mathbb{C}})$ and T fixes three distinct points of $\widehat{\mathbb{C}}$, then T is the identity map.*

Theorem 4. *If C and C' are circles in $\widehat{\mathbb{C}}$, then there exists some $T \in Aut(\widehat{\mathbb{C}})$ such that $T(C) = C'$.*

In general, $\mathrm{Aut}(\widehat{\mathbb{C}})$ is not 4-transitive, but if two 4-tuple of distinct points have the same cross-ratio, there is some Möbius transformation that sends one 4-tuple into the other. Recall that the cross-ratio of four complex numbers is defined as $\lambda = (z_0, z_1; z_2, z_3) = \frac{(z_0 - z_1)(z_2 - z_3)}{(z_1 - z_3)(z_3 - z_0)}$ with the convention of taking limits if some $z_j = \infty$.

Theorem 5. *Let (z_0, z_1, z_2, z_3) and (w_0, w_1, w_2, w_3) be 4-tuples of distinct elements of $\widehat{\mathbb{C}}$. Then, there exists some $T \in Aut(\widehat{\mathbb{C}})$ with $T(z_j) = w_j$, $j = 0, 1, 2, 3$ if and only if the two 4-tuples have the same cross-ratio.*

Consider a circle C in $\widehat{\mathbb{C}}$ given by the equation $az\bar{z} + bz + \bar{b}\bar{z} + c = 0$, with $a, c \in \mathbb{R}$, $b \in \mathbb{C}$. If $a \neq 0$, then C is an Euclidean circle in the complex plane, and then a transformation exists in the complex plane that fixes C. This transformation is given by

$$I_C(z) = -\frac{\bar{b}\bar{z} + c}{a\bar{z} + b},$$

which is called the inversion in C. Moreover, if $T \in \mathrm{Aut}(\widehat{\mathbb{C}})$, then $T(C) = C'$ is another circle, and we have that $I_{C'} = T I_C T^{-1}$.

To study the geometry of the Möbius transformations, we would like to recall that there is a classification of the automorphisms of $\widehat{\mathbb{C}}$ in conjugacy classes, according to the number of fixed points, and the corresponding trace of the matrix associated in $PSL(2, \mathbb{C})$. The next results summarize this classification.

Theorem 6. *Let $T(z) = (az + b)/(cz + d)$, with $ad - bc = 1$. If $(a + d)^2 \neq 4$, then T has two fixed points in $\widehat{\mathbb{C}}$; if $(a + d)^2 = 4$ and T is not the identity map, T has one fixed point in $\widehat{\mathbb{C}}$.*

For $\lambda \in \mathbb{C} \setminus \{0\}$, consider the maps $U_\lambda(z) = \lambda z$ if $\lambda \neq 1$ and $U_1(z) = z + 1$. We will say that two maps T and S are conjugated if there exists another transformation V such that $T = V^{-1} \circ S \circ V$.

Theorem 7. *Let T be a non-identity element in $Aut(\widehat{\mathbb{C}})$; then, there exists some $\lambda \in \mathbb{C} \setminus \{0\}$ such that T is conjugate to U_λ in $Aut(\widehat{\mathbb{C}})$.*

Remark 1. *When $\lambda = 1$, the map T has only one fixed point z_0, and it is conjugated to $U_1(z)$ by a Möbius transformation S that sends z_0 to ∞. Since $\lim_{n\to\infty} U_1^n(z) = \infty$, any $z \in \mathbb{C}$ is moved by T^n towards z_0 as n goes to infinity. In this case, T is called* parabolic.

Remark 2. *If T is not parabolic, then it has two fixed points z_1 and z_2 and is conjugated to U_λ, with $\lambda \in \mathbb{C} \setminus \{0, 1\}$, that fixes 0 and ∞, by means of a Möbius transformation S such that $S(z_1) = 0$ and $S(z_2) = \infty$. If $|\lambda| < 1$, $\lim_{n\to\infty} U_\lambda^n(z) = 0$ for all $z \neq \infty$ and hence $\lim_{n\to\infty} T^n(z) = z_1$ for all $z \neq z_2$. In the same way, if $|\lambda| > 1$, then $\lim_{n\to\infty} T^n(z) = z_2$ for all $z \neq z_1$ (the two cases for λ*

are the same since we can replace λ by $1/\lambda$). We conclude that if $|\lambda| \neq 1$, all points $z \neq z_1, z_2$ are moved by T away from one of these fixed points towards the other. If $\lambda > 0$, T is called hyperbolic *and* loxodromic *otherwise. If $|\lambda| = 1$, with $\lambda \neq 1$, then U_λ is a rotation R_θ, so $U_\lambda^n(z)$ has no limit for $z \neq 0, \infty$; hence neither $T^n(z)$ for $z \neq z_1, z_2$. In this case, T is called elliptic.*

2.2. Second Symmetric Product of $\mathbb{C}$

The *second symmetric product* of $\mathbb{C}$, denoted by $F_2(\mathbb{C})$, is the set

$$F_2(\mathbb{C}) = \{A \subset \mathbb{C} \ : \ A \text{ has at most 2 elements and } A \text{ is not empty}\}.$$

The space $F_2(\mathbb{C})$ has the topology induced by the following metric:

$$\mathcal{H}(A, B) = \inf\{\varepsilon > 0 \ : \ A \subset \mathcal{V}_\varepsilon(B) \text{ and } B \subset \mathcal{V}_\varepsilon(A)\},$$

where $\mathcal{V}_\varepsilon(A) = \{x \in \mathbb{C} \ : d(x, A) < \varepsilon\}$, $d(\,)$ is the usual metric in $\mathbb{C}$, and A and B are subsets of $\mathbb{C}$. Given X a compact subset of $\mathbb{C}$, the space $F_2(X)$ can also be topologized through the Vietoris topology: if $U_1, \ldots, U_m$ are nonempty subsets of $\mathbb{C}$ and $m \in \mathbb{N}$, then define

$$\langle U_1, \ldots, U_m \rangle = \Big\{ A \subset X \ : \ A \neq \emptyset, \ |A| \leq 2, \ A \subset \bigcup_{j=1}^{m} U_j \text{ and } A \cap U_j \neq \emptyset, \text{ for all } j \in \{1, \ldots, m\} \Big\};$$

a base for the Vietoris topology is given by the family of the sets $\langle U_1, \ldots, U_m \rangle$, where $m \in \mathbb{N}$ and $U_1, \ldots, U_m$ are open subsets of $\mathbb{C}$. The Vietoris topology and the topology induced by the Hausdorff metric coincide in $F_2(\mathbb{C})$.

Let X be a connected and compact subspace of $\mathbb{C}$. It is known that $F_2(X)$ is a continuum itself [9] (Corollary 1.8.8). In [2], it is proven that, for $I = [0, 1]$, $F_2(I)$ is homeomorphic to a 2-cell. In [3], it is proven that for the 1-sphere $\mathbb{S}^1$, $F_2(\mathbb{S}^1)$ is homeomorphic to a Möbius strip; see also [1].

2.3. A Model for $F_2(\mathbb{C})$

In order to have a better understanding of the space $F_2(\mathbb{C})$, we will introduce a model $\mathbb{M}_2$ of $F_2(\mathbb{C})$, that is, a continuous and bijective copy of $F_2(\mathbb{C})$. Let $\mathbb{M}_2$ be the space $(\mathbb{R}^3_+ \times \mathbb{S}^1)/s$, where $\mathbb{R}^3_+ = \{(x, y, z) \in \mathbb{R}^3 \ : \ z \geq 0\}$, $\mathbb{S}^1$ is the unit circle, and such that s is a relation defined by $(x, y, 0, t) \sim (x, y, 0, t')$, for all $t, t' \in \mathbb{S}^1$.

Definition 1. *Let Φ be the function $\Phi : F_2(\mathbb{C}) \to \mathbb{M}_2$ given by*

$$\Phi(\{a, b\}) = \begin{cases} \left(\frac{a+b}{2}, \|a - b\|, e^{2i(\arg(a-b)(mod\,\pi))}\right), & \text{if } a \neq b; \\ (\text{class } [a, 0, t]), & \text{if } a = b. \end{cases}$$

We observe that Φ is a well-defined, bijective, and bicontinuous function with the corresponding topologies. We will call $\mathbb{M}_2$ the model of $F_2(\mathbb{C})$. This function is inspired in the map defined by Vaughan in [10]; also see [11,12].

Remark 3. *Observe that given a point $(u, a, t) \in \mathbb{M}_2$, with $u \in \mathbb{R}^2$, $a \geq 0$, and $t \in \mathbb{S}^1$, we can obtain its pre-image under Φ as follows: u must be the midpoint of two points z_u and w_u in the complex plane such that $\|z_u - w_u\| = a$ and $e^{2i\theta} = t$, where $\theta = \arg(z_u - w_u)$; then, z_u and w_u are points in the circle with center u and radius $a/2$, such that the segment $\overline{z_u w_u}$ is a diameter of the circle. Hence, $z_u = u + (a/2)e^{i\pi\theta}$ and $w_u = u - (a/2)e^{i\pi\theta}$.*

In Figure 1, we observe a representation of the model $\mathbb{M}_2$; for instance, over any point $t = \frac{a+b}{2}$, the midpoint of a, $b \in \mathbb{C}$, there is a cone V with vertex at t, so any two points $z, w \in \mathbb{C}$ with midpoint t have a representation in V at height $\|z-w\|$ and angle $e^{2i(\arg(z-w)(\mathrm{mod}\pi))}$, for example see the red point in the figure.

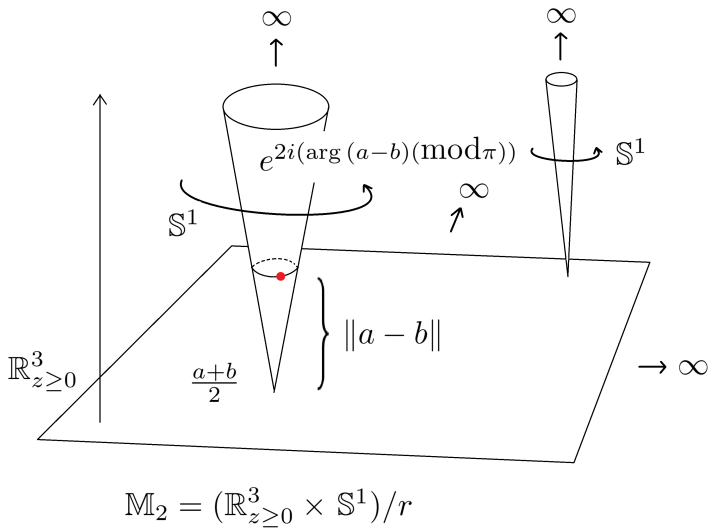

Figure 1. The model $\mathbb{M}_2$ for $F_2(\mathbb{C})$.

Let $\{x, y\} \in F_2(\mathbb{C})$; observe that there exists a closed disk D that contains x and y in its interior; then, $\langle D \rangle \cap F_2(\mathbb{C})$ is a neighborhood of $\{x, y\}$ in $F_2(\mathbb{C})$. Given that $F_2(D)$ is a compact set, it follows that $F_2(\mathbb{C})$ is a Hausdorff and a locally compact topological space; then, it is possible to consider the Alexandroff's compactification, denoted by $F_2(\mathbb{C})^*$. The point added is denoted by ∞ (observe that this point will correspond to the pair of points $\{z, \infty\}$ in $F_2(\mathbb{C})$, for each $z \in \mathbb{C}$). Note that in $F_2(\mathbb{C})$, the sets $\{x, y\}$ such that $(x+y)/2 = constant$ are mapped by Φ to an open topological disk. Hence, the Alexandroff's compactification of such a set will be homeomorphic to $\mathbb{S}^2$. Moreover, observe that the set of singletons together with the point ∞ in $F_2(\mathbb{C})^*$ is homeomorphic to $\mathbb{S}^2$.

3. Extension of the Möbius Transformations to the Space $F_2(\mathbb{C})$

In this section, we will define how a Möbius transformation extends to the space $F_2(\mathbb{C})$. We will examine how the domains of these extended maps vary based on the specific Möbius transformation being considered. Additionally, we will describe the extensions to $F_2(\mathbb{C})$ of the maps outlined in Theorem 1, which will serve as a set of generators for all the extensions of Möbius transformations.

Let $T(z) = (az+b)/(cz+d)$ be a Möbius transformation in the Riemann sphere, define in the second symmetric space $F_2(\mathbb{C}) = \{a = \{z, w\} \ : \ z, w \in \mathbb{C}\}$, the function $\widetilde{T}$ given by

$$\widetilde{T}(a) = \widetilde{T}(\{z, w\}) = \{T(z), T(w)\}, \ z, w \neq -d/c. \tag{1}$$

In particular, observe that if $z = w$, then $\widetilde{T}(a) = \widetilde{T}(\{z\}) = \{T(z)\}$; hence the geometry of T in $\mathbb{C}$ will be reflected in $F_2(\mathbb{C})$. Since the transformation T has an inverse map $T^{-1}(z) = (dz-b)/(-cz+a)$, it is easy to see that in some appropriate domains, $\widetilde{T}^{-1} \circ \widetilde{T}$ and $\widetilde{T} \circ \widetilde{T}^{-1}$ are the identity maps.

Observe that we can use the homeomorphism $\Phi : F_2(\mathbb{C}) \to \mathbb{M}_2$ to translate the definition of $\widetilde{T}$ to $\mathbb{M}_2$; that is, we can conjugate the map $\widetilde{T}$ in some appropriate domain via Φ to obtain a map $\widehat{T}$ in $\mathbb{M}_2$. So, from now on, by convention, for any object X in $\mathbb{C}$, we will use $\widetilde{X}$ for the object in $F_2(\mathbb{C})$ generated by X and $\widehat{X}$ for the corresponding object in the model $\mathbb{M}_2$.

Recall that a Möbius transformation T has at most two fixed points, and let us assume that T does not fix the point at infinity in the Riemann sphere. First, suppose that T has only one fixed point z_0; then, the map $\widetilde{T}$ also has z_0 as its only fixed point; meanwhile, if T fixes two distinct points, z_0 and z_1, then $\widetilde{T}$ has three fixed points: $\{z_0\}, \{z_1\}, \{z_0, z_1\}$.

As the map T is defined in $\widehat{\mathbb{C}}$, we need to consider the image and pre-image of the point at infinity, that is, $T(\infty) = a/c$ and $T(-d/c) = \infty$. Let us define the sets $D_T = F_2(\mathbb{C}) \setminus \{\{z, -d/c\} : z \in \mathbb{C}\}$ and $R_T = F_2(\mathbb{C}) \setminus \{\{z, a/c\} : z \in \mathbb{C}\}$; then, we have our first result for the map $\widetilde{T}$.

Lemma 1. *For any $T \in Aut(\widehat{\mathbb{C}})$, the map $\widetilde{T} : D_T \to R_T$ is a homeomorphism.*

Proof. Assume that $T(z) = (az + b)/(cz + d)$. First, let us prove that $\widetilde{T}$ is a bijection. Let $\{z_1, w_1\}$ and $\{z_2, w_2\}$ be two points in D_T, such that $\widetilde{T}(\{z_1, w_1\}) = \widetilde{T}(\{z_2, w_2\})$; then, it follows that $\{T(z_1), T(w_1)\} = \{T(z_2), T(w_2)\}$. If $z_1 = w_1$, then $\{T(z_1), T(w_1)\} = \{T(z_1)\} = \{T(z_2), T(w_2)\}$ for which $T(z_1) = T(z_2) = T(w_2)$; therefore $z_1 = z_2 = w_2$; if now $z_1 \neq w_1$, then $z_2 \neq w_2$, and then $T(z_1) = T(z_2)$ or $T(z_1) = T(w_2)$; in the former case, $T(w_1) = T(w_2)$, and in the latter case, $T(z_2) = T(w_1)$. In any case, we have that $\{z_1, w_1\} = \{z_2, w_2\}$ since T is a one-to-one map, for which it follows the injectivity of $\widetilde{T}$.

It is clear that for any pair of point $z, w \in \mathbb{C}$, neither equal to a/c, there are points $u, v \in \widehat{\mathbb{C}}$ such that $T(u) = z$ and $T(v) = w$, by the surjectivity of T, and therefore $\widetilde{T}$ is onto. Now, observe that $\widetilde{T}^{-1} : R_T \to D_T$ is the inverse map of $\widetilde{T}$.

Finally, to establish the continuity of the map $\widetilde{T}$, observe that $\widetilde{T}(\{z\}) = \{T(z)\}$ and $\widetilde{T}(\{z, w\}) = \{T(z), T(w)\}$, so by the continuity of T and the characterization of the open sets in the Hausdorff topology on $F_2(\mathbb{C})$, we have the result. □

Observe that if $c = 0$, then the map $\widetilde{T}$ can be defined in all $F_2(\mathbb{C})$ as in relation (1), and it is a homeomorphism there. For a general map $T(z) = (az + b)/(cz + d)$, we can think of the action of $\widetilde{T}$ in $F_2(\mathbb{C})$ as follows. For any $w \in \mathbb{C}$, we define the cone of vertex at w as the set $V_w = \{\{z, w\} : z \in \mathbb{C}\} \subset F_2(\mathbb{C})$. Let $V_w^T = V_w \setminus \{-d/c, w\}$ and $V_w^{T*} = V_w \setminus \{a/c, w\}$. Then, $\widetilde{T}$ acts sending the cone V_w^T with vertex at $w \neq -d/c$ one-to-one to the cone $V_{T(w)}^{T*}$ with vertex at $T(w)$, since $\widetilde{T}(\{z, w\}) = (\{T(z), T(w)\}) \in V_{T(w)}$, for any $\{z, w\} \in V_w^T$. We have the following result using the same arguments in the proof of Lemma 1.

Lemma 2. *Let $T(z) = (az + b)/(cz + d)$ be an element in $Aut(\widehat{\mathbb{C}})$; then, the map $\widetilde{T} : V_w^T \to V_{T(w)}^{T*}$ is a homeomorphism, for any $w \neq -d/c$.*

Some special cones need to be considered in the definition of $\widetilde{T}$. Suppose that z_0 is a fixed point of T; then, the cone $V_{z_0}^T$ is invariant under $\widetilde{T}$, that is, $\widetilde{T}$ is a homeomorphism from $V_{z_0}^T$ to $V_{z_0}^{T*}$; when T has two fixed points z_1 and z_2, the two cones V_{z_1} and V_{z_2} intersect each other in the other fixed point $\{z_1, z_2\}$ of $\widetilde{T}$.

So far, we have defined $\widetilde{T}$ only in D_T (and then $\widehat{T}$ only in $\Phi(D_T)$), so we need to extend the definition of $\widetilde{T}$. Observe that the set where $\widetilde{T}$ is not defined yet, is precisely the cone $V_T := V_{-d/c} = \{\{z, -d/c\} : z \in \mathbb{C}\}$, which will be called the singular cone for T, and the other cone $V_T' = \{\{z, a/c\} : z \in \mathbb{C}\}$ will be called the singular value cone for T. For $\{z, -d/c\} \in V_T$, we define the function $\widetilde{T}$ as follows:

$$\widetilde{T}(\{z, -d/c\}) = \{T(z), a/c\} \in V_T'. \tag{2}$$

Remark 4. *Since T is a bijective map in $\mathbb{C}$, we have that $\widetilde{T}$ is a bijection from $V_T \setminus \{-d/c\}$ to $V_T' \setminus \{a/c\}$. Also, observe that in the cone $\Phi(V_T)$, the map $\widehat{T}$ continuously sends circles at some particular height to topological circles in $\Phi(V_T')$. Moreover, $\widehat{T}$ sends points in the cone $\Phi(V_T)$ close*

to the vertex $-d/c$ to points in the cone $\Phi(V_T')$ close to infinity, and points in V_T close to infinity to points in V_T' close to the vertex a/c.

In this way, we have defined $\widetilde{T}$ in $V_T \setminus \{-d/c\}$ and therefore in all $F_2(\mathbb{C}) \setminus \{-d/c\}$, since $\widetilde{T}$ was already defined in D_T. Moreover, $\widetilde{T}(V_T \setminus \{-d/c\}) = V_T' \setminus \{a/c\}$. Thus, we have extended the definition of $\widetilde{T}$ to $F_2(\mathbb{C}) \setminus \{-d/c\}$ with image $F_2(\mathbb{C}) \setminus \{a/c\}$, so naturally, we can extend the definition of $\widetilde{T}$ to $F_2(\mathbb{C})^*$, sending $\{-d/c\} \to \infty$ and $\infty \to a/c$. Using the notation established thus far, we have the following result.

Theorem 8. *Let $T(z) = (az + b)/(cz + d)$ be a Möbius transformation in the Riemann sphere. Then, the map $\widetilde{T} : F_2(\mathbb{C})^* \to F_2(\mathbb{C})^*$ is a bijective map, continuous in D_T and continuous in V_T.*

Proof. By Lemma 1, the map $\widetilde{T}$ is a homeomorphism in D_T. As $\widetilde{T}(V_T \setminus \{-d/c\}) = V_T' \setminus \{a/c\}$ in a bijective way by Equation (2), and $\{-d/c\} \to \infty$ and $\infty \to a/c$, we conclude that $\widetilde{T}$ is a bijection. By Remark 4, we see that $\widetilde{T}$ is continuous within V_T. □

Remark 5. *Since by Lemma 1, the map $\widetilde{T} : D_T \to R_T$ is a homeomorphism, any extension of the map in V_T must have image V_T'. If we consider a sequence of points $s_n = \{z_n, w_n\} \in D_T$ that converges to a point $\{z, -d/c\}$ in V_T and consider the open set $\mathcal{V}_\varepsilon(\{z, -d/c\})$ in $F_2(\mathbb{C})$ that contains the point $\{z, -d/c\}$, for some $\epsilon > 0$, then there exists $N \in \mathbb{N}$ such that if $n \geq N$, it follows that $s_n = \{z_n, w_n\} \in \mathcal{V}_\varepsilon(\{z, -d/c\})$. Hence, for all $n \geq N$, $|z_n - z| < \epsilon$ and $|w_n - (-d/c)| < \epsilon$ or $|w_n - z| < \epsilon$ and $|z_n - (-d/c)| < \epsilon$; then, there are sequences of complex points $\{a_n\}$, $\{b_n\}$ such that $a_n \to z$, $b_n \to -d/c$, as $n \to \infty$ and $\{a_n, b_n\} = \{z_n, w_n\}$ for $n \geq N$. As T is a continuous map, it follows that $T(b_n) \to T(-d/c) = \infty$; therefore, we can not have continuity for the map $\widetilde{T}$ when we approach V_T from points in D_T.*

Remark 6. *It seems that we can use another compactification of $F_2(\mathbb{C})$, different from the compactification of Aleksandrov, so that the map $\widetilde{T}$ is a homeomorphism in this new space; we add a cone with vertex at infinity compatible with the topology of $F_2(\mathbb{C})$; however, we will lose the advantages of having the model for $F_2(\mathbb{C})$ to have a geometric description of the maps $\widehat{T}$. Another possible direction is to work in the second symmetric product of the Riemann sphere $F_2(\widehat{\mathbb{C}})$, but again we lose the possible model to describe the geometry of the maps $\widetilde{T}$.*

However, the map $\widetilde{T} : F_2(\mathbb{C})^* \to F_2(\mathbb{C})^*$ is bijective, so we can define the set of transformations $M(F_2(\mathbb{C})) = \{\widetilde{T} : F_2(\mathbb{C})^* \to F_2(\mathbb{C})^* : T \in \mathrm{Aut}(\widehat{\mathbb{C}})\}$, where $\widetilde{T}$ is defined as before. Hence, the set $M(F_2(\mathbb{C}))$ is a group with the composition of maps as its group operation. If $T(z) = (az + b)/(cz + d)$ and $S(z) = (a'z + b')/(c'z + d')$ are two Möbius transformations, then we see that $\widetilde{S} \circ \widetilde{T}(\{z, w\}) = \{S(T(z)), S(T(w))\}$ is well-defined in all $F_2(\mathbb{C})$. We will explore the structure of this group in a future manuscript.

Generators of $M(F_2(\mathbb{C}))$

We will show now that all the maps in $M(F_2(\mathbb{C}))$ are compositions of the following four maps:

(i) $\widetilde{R}_\theta(\{z, w\}) = \{e^{i\theta}z, e^{i\theta}w\}$, $\theta \in \mathbb{R}$;
(ii) $\widetilde{J}(\{z, w\}) = \{1/z, 1/w\}$, for $zw \neq 0$;
(iii) $\widetilde{S}_r(\{z, w\}) = \{rz, rw\}$, $r \in \mathbb{R}, r > 0$;
(iv) $\widetilde{T}_t(\{z, w\}) = \{z + t, w + t\}$, $t \in \mathbb{C}$.

Observe that $\widetilde{R}_\theta$, $\widetilde{S}_r$, and $\widetilde{T}_t$ are homeomorphisms defined in all $F_2(\mathbb{C})$. Meanwhile, $\widetilde{J}$ is defined at all points $\{z, w\} \in F_2(\mathbb{C})$, with $zw \neq 0$, but we can extend the definition of $\widetilde{J}$ in its singular cone $V_J = \{\{z, 0\} : z \in \mathbb{C}\}$ as in relation (2), that is, $\widetilde{J}(\{z, 0\}) = \{J(z), 0\}$, for $z \neq 0$, and observe that for J its singular cone coincides with its singular value cone.

Proposition 1. *Let S be a map in $M(F_2(\mathbb{C}))$; then, S can be expressed as a composition in some order of the maps $\widetilde{R}_\theta$, $\widetilde{S}_r$, $\widetilde{T}_t$, and $\widetilde{J}$.*

Proof. Let $T \in \mathrm{Aut}(\widehat{\mathbb{C}})$ such that $\widetilde{T} = S$, and assume that $T(z) = (az+b)/(cz+d)$. If $c = 0$, we know that $T = T_t \circ S_r \circ R_\theta$, where $b/d = t$ y $a/d = re^{i\theta}$, hence it is straightforward to see that $S = \widetilde{T}_t \circ \widetilde{S}_r \circ \widetilde{R}_\theta$.

Now, when $c \neq 0$, $T(z) = (T_t \circ J)(-c^2 z - cd)$, where $t = a/c$. By the first part of the proof, $-c^2 z - cd = V(z) = T_{t'} \circ S_r \circ R_\theta$, for some $t' \in \mathbb{C}$, $r > 0$ and $\theta \in \mathbb{R}$. Therefore $S = \widetilde{T}_t \circ \widetilde{J} \circ \widetilde{V}$. Note that the previous decomposition of $S = \widetilde{T}$ even works for the singular cone V_T, take $\{z, -d/c\} \in V_T$, then $\widetilde{T}_t \circ \widetilde{J} \circ \widetilde{V}(\{z, -d/c\}) = \widetilde{T}_t(\widetilde{J}(\{V(z), V(-d/c)\})) = \widetilde{T}_t(\widetilde{J}(\{V(z), 0\})) = \widetilde{T}_t(\{J(V(z)), 0\}) = \{J(V(z)) + a/c, a/c\} = \{T(z), a/c\}$. □

Let us analyze the geometry of these generators maps in the space $F_2(\mathbb{C})$. In order to do that, let us work on the model $\mathbb{M}_2$ of $F_2(\mathbb{C})$. Since $\Phi : F_2(\mathbb{C}) \to \mathbb{M}_2$ is a homeomorphism, we can conjugate any map $\widetilde{F} : F_2(\mathbb{C}) \to F_2(\mathbb{C})$ to a map $\widehat{F} : \mathbb{M}_2 \to \mathbb{M}_2$, that is, $\Phi \circ \widetilde{F} = \widehat{F} \circ \Phi$, extending the definition to infinity in a natural way. In particular, the elements of $M(F_2(\mathbb{C}))$ can be thought of as acting in $\mathbb{M}_2$, so in some cases, we will not make a distinction if the context is clear.

Let us start with the map $\widetilde{R}_\theta(\{z, w\}) = \{e^{i\theta} z, e^{i\theta} w\}$, $\theta \in \mathbb{R}$, and the analysis for the other maps will be similar. In this case, the conjugation gives a map $\widehat{R}_\theta$ such that $\Phi \circ \widetilde{R}_\theta = \widehat{R}_\theta \circ \Phi$; the left side composition satisfies that

$$\Phi(\widetilde{R}_\theta(\{z, w\}) = \Phi(\{e^{i\theta} z, e^{i\theta} w\}) = \Big(e^{i\theta}(z+w)/2, \|z-w\|, e^{2i(\arg e^{i\theta}(z-w)(\mathrm{mod}\pi))}\Big),$$

and the right side composition is equal to

$$\widehat{R}_\theta(\Phi(\{z, w\})) = \widehat{R}_\theta\Big((z+w)/2, \|z-w\|, e^{2i(\arg(z-w)(\mathrm{mod}\pi))}\Big),$$

then the next result follows directly.

Proposition 2. *The map $\widehat{R}_\theta : \mathbb{M}_2 \to \mathbb{M}_2$ acts as follows: $\widehat{R}_\theta(u, l, t) = (e^{i\theta} u, l, e^{2i\theta} t)$, for $u \in \mathbb{R}^2$, $l \geq 0$ and $t \in \mathbb{S}^1$.*

As a result, we can determine the geometry of the map $\widetilde{R}_\theta$ in $F_2(\mathbb{C})$, stated as follows.

Corollary 2. *The map $\widetilde{R}_\theta$ acts conjugated as a double rotation with the same angle; this double rotation moves a point around a topological torus.*

Proof. Just observe that since $\widetilde{R}_\theta$ is conjugated to $\widehat{R}_\theta$, and by Proposition 2, $\widehat{R}_\theta(u, l, t) = (e^{i\theta} u, l, e^{2i\theta} t)$, the orbit of the point (u, r, t) remains at the same height and the first and third coordinates are rotated by the same angle, so the result follows. □

Similarly, we can determine the action of the corresponding maps $\widetilde{S}_r$ and $\widetilde{T}_t$ in the space $\mathbb{M}_2$.

Proposition 3. *The map $\widehat{S}_r : \mathbb{M}_2 \to \mathbb{M}_2$ acts as follows: $\widehat{S}_r(u, l, t) = (ru, rl, t)$, for $u \in \mathbb{R}^2$, $l \geq 0$ and $t \in \mathbb{S}^1$.*

Proof. From the conjugation $\Phi \circ \widetilde{S}_r = \widehat{S}_r \circ \Phi$, we obtain that

$$\begin{aligned}\Phi(\widetilde{S}_r(\{z,w\})) &= \Phi(\{rz,rw\}) = (r(z+w)/2, \|rz-rw\|, e^{2i(\arg(rz-rw)(\mathrm{mod}\pi))}) \\ &= (r(z+w)/2, r\|z-w\|, e^{2i(\arg r+\arg(z-w))(\mathrm{mod}\pi))}) \\ &= \widehat{S}_r(\Phi(\{z,w\}) = \widehat{S}_r((z+w)/2, \|z-w\|, e^{2i(\arg(z-w)(\mathrm{mod}\pi))}),\end{aligned}$$

from where it follows the claim, observing that $\arg r = 0$. □

Using the definition in [13] of a topological attractor, we have the following result.

Corollary 3. *The point $O \in \mathbb{M}_2$ with coordinates $(0,0,0,1)$ is a fixed point of $\widehat{S}_r$, which is a global topological attractor for the dynamics of $\widehat{S}_r$ when $r < 1$.*

Proof. Recall that s is the relation defined by $(x,y,0,t) \sim (x,y,0,t')$ for all t, t', so all points $(0,0,0,t)$ can be identified to the point $(0,0,0,1)$. It is clear that $O \in \mathbb{M}_2$ is a fixed point of $\widehat{S}_r$. By Proposition 3, the map $\widehat{S}_r$ is defined as $\widehat{S}_r(u,l,t) = (ru, rl, t)$; then, iterating this map, we obtain that $\widehat{S}_r^n(u,l,t) = (r^n u, r^n l, t)$, and since $r < 1$, it follows that $\widehat{S}_r^n(u,l,t) \to (0,0,0,1)$, as $n \to \infty$. □

Proposition 4. *The map $\widehat{T}_t : \mathbb{M}_2 \to \mathbb{M}_2$ acts in the following way $\widehat{T}_t(u,l,\theta) = (u+t, l, \theta)$, for $u \in \mathbb{R}^2$, $l \geq 0$ and $\theta \in \mathbb{S}^1$.*

Proof. From the conjugation $\Phi \circ \widehat{T}_t = \widehat{T}_t \circ \Phi$, we obtain that

$$\begin{aligned}\Phi(\widehat{T}_t(\{z,w\})) &= \Phi(\{z+t, w+t\}) = (((z+w)/2) + t, \|z-w\|, e^{2i(\arg(z-w)(\mathrm{mod}\pi))}) \\ &= \widehat{T}_t(\Phi(\{z,w\}) = \widehat{T}_t((z+w)/2, \|z-w\|, e^{2i(\arg(z-w)(\mathrm{mod}\pi))}),\end{aligned}$$

from where it follows the claim. □

The next result follows directly from Proposition 4.

Corollary 4. *The orbit of every point in $\mathbb{M}_2$ under the map $\widehat{T}_t$ goes to infinity.*

Finally, let us analyze the action of the map $\widehat{J}$ on $\mathbb{M}_2$. Using the conjugation $\Phi \circ \widetilde{J} = \widehat{J} \circ \Phi$, we get, first of all for $zw \neq 0$, that

$$\begin{aligned}\Phi \circ \widetilde{J}(\{z,w\}) &= \Phi(\{1/z, 1/w\}) \\ &= (1/2(1/z + 1/w), \|1/z - 1/w\|, e^{2i(\arg(1/z-1/w)(\mathrm{mod}\pi))}) \\ &= ((z+w)/2) \cdot (1/(zw)), \|z-w\|/\|zw\|, e^{2i(\arg((w-z)/zw)(\mathrm{mod}\pi))}).\end{aligned}$$

On the other hand, $\widehat{J} \circ \Phi(\{z,w\}) = \widehat{J}(((z+w)/2), \|z-w\|, e^{2i(\arg(z-w)(\mathrm{mod}\pi))})$; hence, $\widehat{J}(u,l,t) = (u/(z_u w_u), l/(\|z_u w_u\|), e^{-2i\arg(z_u w_u)} t)$, where $u \in \mathbb{R}^2$, $l \geq 0$, $t \in \mathbb{S}^1$ and z_u, w_u are the complex numbers that depend of u as in Remark 3.

In the cone $V_J = \{\{z,0\} : z \in \mathbb{C}\}$, we get that

$$\begin{aligned}\Phi \circ \widetilde{J}(\{z,0\}) = \Phi(\{1/z, 0\} &= (1/(2z), \|1/z\|, e^{2i\arg(1/z)(\mathrm{mod}\pi)}) \\ &= \widehat{J} \circ \Phi(\{z,0\}) = \widehat{J}(z/2, \|z\|, e^{2i(\arg(z)(\mathrm{mod}\pi))}).\end{aligned}$$

That is, $\widehat{J}(u, 2\|u\|, t) = (1/4u, 1/(2\|u\|), -t)$, for $u \in \mathbb{R}^2$ and $t = e^{2i(\arg(2u))}$. In this way, we can prove the following.

Proposition 5. *The map $\widehat{J}:\mathbb{M}_2\to\mathbb{M}_2$ satisfies that $\widehat{J}\circ\widehat{J}=Id_{\mathbb{M}_2}$, the identity map in the model of $F_2(\mathbb{C})$.*

Proof. For $zw\neq 0$, we have that $\widetilde{J}\circ\widetilde{J}(\{z,w\})=(\{J^2(z),J^2(w)\})=(\{z,w\})$; then, as $\widehat{J}\circ\widehat{J}\circ\Phi=\Phi\circ\widetilde{J}\circ\widetilde{J}$, the result follows. In the cone V_J, just notice that $\widetilde{J}\circ\widetilde{J}(\{z,0\})=\widetilde{J}(\{J(z),0\})=\widetilde{J}(\{1/z,0\})=\{z,0\}$; conjugating with the map, Φ we have the result. □

4. Transitivity of $M(F_2(\mathbb{C}))$

In this section, we will present several results regarding transitivity in the space $M(F_2(\mathbb{C}))$. Let us start with two triples of distinct points in $\mathbb{C}$, that is, (z_1,z_2,z_3) and (w_1,w_2,w_3); then, we can consider the triples of distinct points in $F_2(\mathbb{C})$: $(\{z_1,z_2\},\{z_2,z_3\},\{z_3,z_1\})$, and $(\{w_1,w_2\},\{w_2,w_3\},\{w_3,w_1\})$. The first instance of transitivity is the following.

Proposition 6. *If $(\{z_1,z_2\},\{z_2,z_3\},\{z_3,z_1\})$ and $(\{w_1,w_2\},\{w_2,w_3\},\{w_3,w_1\})$ are triples of distinct points in $F_2(\mathbb{C})$, then there is a unique $\widetilde{T}\in M(F_2(\mathbb{C}))$ such that $\widetilde{T}(\{z_i,z_j\})=\{w_i,w_j\}$, for all $i,j\in\{1,2,3\}$, with $i\neq j$.*

Proof. By Proposition 3, there is $T\in\mathrm{Aut}(\widehat{\mathbb{C}})$ such that $T(z_i)=w_i$, for $i=1,2,3$; then, $\widetilde{T}(\{z_i,z_j\})=\{w_i,w_j\}$, for all $i,j\in\{1,2,3\}$, with $i\neq j$.

Suppose there is another element $\widetilde{S}$ in $M(F_2(\mathbb{C}))$ such that $\widetilde{S}(\{z_i,z_j\})=\{w_i,w_j\}$. Consider the image of the first point, $\widetilde{S}(\{z_1,z_2\})=\{S(z_1),S(z_2)\}=\{w_1,w_2\}$ then, there are two cases. If $S(z_1)=w_1$, then $S(z_2)=w_2$, and taking one of the other two points in $F_2(\mathbb{C})$, we see that $S(z_3)=w_3$. By Proposition 3, we have that $S=T$ and therefore $\widetilde{S}=\widetilde{T}$. In case $S(z_1)=w_2$, then $S(z_2)=w_1$, but we have that $\widetilde{S}(\{z_2,z_3\})=\{S(z_2),S(z_3)\}=\{w_2,w_3\}$, which is a contradiction since $S(z_2)=w_1$; this finishes the proof of the uniqueness of the map $\widetilde{T}$. □

Using the same argument as in the proof of the uniqueness in the previous result, we obtain the next Corollary.

Corollary 5. *If $\widetilde{T}\in M(F_2(\mathbb{C}))$ fixes three distinct point of the form $\{z_1,z_2\},\{z_2,z_3\},\{z_3,z_1\}$, then $\widetilde{T}$ is the identity map.*

We can again use the 3-transitivity of $\mathrm{Aut}(\widehat{\mathbb{C}})$ and the arguments of the proof of Proposition 6 to prove the following result.

Proposition 7. *Consider two pairs of points $\{z_0\},\{z_1,z_2\}$ and $\{w_0\},\{w_1,w_2\}$ in $F_2(\mathbb{C})$ with $z_1\neq z_2$ and $w_1\neq w_2$; then, there is a unique $\widetilde{T}\in M(F_2(\mathbb{C}))$ such that $\widetilde{T}(\{z_0\})=\{w_0\}$ and $\widetilde{T}(\{z_1,z_2\})=\{w_1,w_2\}$.*

As a corollary, we obtain that the Möbius transformations in $F_2(\mathbb{C})$ act transitively in the set of cones $\mathcal{V}=\{V_a : V_a=\{\{z,a\} : z\in\mathbb{C}\}\}$.

Corollary 6. *Let $\{z_0\}$ and $\{w_0\}$ be two singletons in $F_2(\mathbb{C})$. Then, there exists $\widetilde{T}\in M(F_2(\mathbb{C}))$ such that $\widetilde{T}(V_{z_0})=V_{w_0}$; that is, $M(F_2(\mathbb{C}))$ is transitive in $\mathcal{V}$.*

Proof. Consider different points $\{z_0,z_1\},\{z_0,z_2\}\in V_{z_0}$ and $\{w_0,w_1\},\{w_0,w_2\}\in V_{w_0}$; by the 3-transitivity of $\mathrm{Aut}(\widehat{\mathbb{C}})$, there exists a transformation $T(z)=(az+b)/(cz+d)$ such that $T(z_i)=w_i$ for $i=0,1,2$. Hence, $\widetilde{T}(\{z_0\})=\{w_0\}$, $\widetilde{T}(\{z_0,z_1\})=\{w_0,w_1\}$, and $\widetilde{T}(\{z_0,z_2\})=\{w_0,w_2\}$. Let $\{z_0,z\}$ be a point in V_{z_0}, with $z\neq -d/c$; then, $\widetilde{T}(\{z_0,z\})=$

$\{T(z_0), T(z)\} = \{w_0, T(z)\} \in V_{w_0}$; for points $\{z_0, -d/c\}$, we get $\widetilde{T}(\{z_0, -d/c\}) = \{T(z_0), a/c\} = \{w_0, a/c\} \in V_{w_0}$. □

For general points in $F_2(\mathbb{C})$, we can prove the 2-transitivity of the set $M(F_2(\mathbb{C}))$ if these points combined have the same cross ratio.

Proposition 8. *If $(\{z_1, z_2\}, \{z_3, z_4\})$ and $(\{w_1, w_2\}, \{w_2, w_3\})$ are pairs of distinct points in $F_2(\mathbb{C})$, such that the cross ratio of (z_1, z_2, z_3, z_4) is equal to the cross ratio of (w_1, w_2, w_2, w_3), then there is $\widetilde{T} \in M(F_2(\mathbb{C}))$ such that $\widetilde{T}(\{z_1, z_2\}) = \{w_1, w_2\}$ and $\widetilde{T}(\{z_3, z_4\}) = \{w_3, w_4\}$.*

Proof. By Proposition 5, there exists $T \in \mathrm{Aut}(\widehat{\mathbb{C}})$ such that $T(z_i) = w_i$, for $i = 1, 2, 3, 4$; then, $\widetilde{T}(\{z_1, z_2\}) = \{w_1, w_2\}$ and $\widetilde{T}(\{z_3, z_4\}) = \{w_3, w_4\}$. □

4.1. Transitivity of Möbius Bands

Let us consider $\mathcal{C}$ the family of Euclidean circles in $\mathbb{C}$ and $\mathcal{L}$ the family of lines in $\mathbb{C}$; remember that $\mathrm{Aut}(\widehat{\mathbb{C}})$ sends $\mathcal{C} \cup \mathcal{L}$ into itself; in fact, the action is transitive there.

We have observed that $F_2(\mathbb{S}^1)$ is homeomorphic to a Möbius strip; then, $F_2(C)$ is a Möbius band $\widetilde{C}$ for any $C \in \mathcal{C}$. Moreover, passing to the model $\mathbb{M}_2$, we can see that $\Phi(\widetilde{C}) = \widehat{C}$ is a Möbius band that intersects the subset $\{(x, y, 0, 0) : (x, y, 0, 0) \in \mathbb{M}_2\}$ of $\mathbb{M}_2$ exactly in C.

It is not difficult to see that $F_2(L)$ is homeomorphic to a semi-plane $\widehat{L}$ in the model $\mathbb{M}_2$, for any $L \in \mathcal{L}$; in fact, $\widehat{L} \cap \{(x, y, 0, 0) : (x, y, 0, 0) \in \mathbb{M}_2\} = L$.

Lemma 3. *Let K be an element in $\mathcal{C} \cup \mathcal{L}$; then, for any map S in $M(F_2(\mathbb{C}))$, the set $S(\widetilde{K})$ is homeomorphic to a Möbius strip or homeomorphic to a semi-plane in the model $\mathbb{M}_2$.*

Proof. Let S be an element of $M(F_2(\mathbb{C}))$; then, the corresponding map $T \in \mathrm{Aut}(\widehat{\mathbb{C}})$ (that is, $\widetilde{T} = S$) satisfies that $T(K)$ is a Euclidean circle or a line in $\mathbb{C}$. Assume that $K = C$ is an element of $\mathcal{C}$; the proof for the other case is similar. When passing to the model $\mathbb{M}_2$, consider the set $\widehat{T}(\widehat{C})$.

First, if $T(C)$ is a Euclidean circle, then $\widehat{T(C)}$ is a Möbius band. Since $\Phi \circ \widetilde{T} = \widehat{T} \circ \Phi$ and $\widetilde{T}(\{z, w\}) = \{T(z), T(w)\} \in \widetilde{T}(\widetilde{C}) = \widetilde{T(C)}$, for any $z, w \in C$, it follows that $\widehat{T}(\widehat{C}) = \widehat{T}(\Phi(\widetilde{C})) = \Phi(\widetilde{T}(\widetilde{C})) = \Phi(\widetilde{T(C)}) = \widehat{T(C)}$.

Now, assume that $T(C)$ is a line in $\mathbb{C}$; this happens if $T(z) = (az + b)/(cz + d)$ and the point $-d/c$ is a point on C. Remember that in this case $\widetilde{T} : D_T \to R_T$, and $\widetilde{T}(V_T \setminus \{-d/c\}) = V_T' \setminus \{a/c\}$; then, we only consider the image of $C' = C \setminus \{-d/c\}$; that is, $T(C')$ is a complete line since $T(-d/c) = \infty$. It follows that $\widehat{T(C')}$ is still a whole semi-plane, and once again, using that $\widetilde{T}(\widetilde{C'}) = \widetilde{T(C')}$, we get that $\widehat{T}(\widehat{C'}) = \widehat{T(C')}$, which concludes the proof. □

Now we will prove transitivity for a family of Möbius strips in $\mathbb{M}_2$. Consider the set $M_{\mathcal{C}\cup\mathcal{L}} = \{\Phi(F_2(K)) : K \in \mathcal{C} \cup \mathcal{L}\}$; that is, $M_{\mathcal{C}\cup\mathcal{L}}$ consists of Möbius bands and semi-planes generated by Euclidean circles and lines in $\mathbb{C}$, respectively.

Theorem 9. *The set $M(F_2(\mathbb{C}))$ acts transitively on $M_{\mathcal{C}\cup\mathcal{L}}$; that is, if $\widehat{K}_1, \widehat{K}_2 \in M_{\mathcal{C}\cup\mathcal{L}}$, then there exists $\widetilde{S} \in M(F_2(\mathbb{C}))$ such that $\widehat{S}(\widehat{K}_1) = \widehat{K}_2$.*

Proof. Let $\widehat{K}_1$ and $\widehat{K}_2$ be two elements in $M_{\mathcal{C}\cup\mathcal{L}}$. Let $\{z_1, w_1\}$, $\{z_2, w_2\}$ be two different points in $\widetilde{K}_1$, and let $\{u_1, v_1\}$, $\{u_2, v_2\}$ be two different points in $\widetilde{K}_2$; then, z_1, w_1, z_2, w_2 are in the same Euclidean circle or in the same line K_1 in $\mathbb{C}$ that generates the Möbius strip or the semi-plane $\widehat{K}_1$. The same holds for u_1, v_1, u_2, v_2; i.e., they are in the same Euclidean circle or in the same line K_2 in $\mathbb{C}$ that generates the Möbius strip or the semi-plane $\widehat{K}_2$.

Notice that since $\{z_1, w_1\}$, $\{z_2, w_2\}$ are different points, then there are at least three different complex numbers in the set $A = \{z_1, w_1, z_2, w_2\}$, and the same happens in the set $B = \{u_1, v_1, u_2, v_2\}$. According to Proposition 3, there is a unique Möbius transformation S that sends the three different points in A into the three different points in B. Since three points are sufficient to determine a circle or a line, then $S(K_1) = K_2$, and then $\widetilde{S}(\widetilde{K}_1) = \widetilde{K}_2$. □

The next result characterizes the sets in $M_{\mathcal{C}\cup\mathcal{L}}$ using the cross-ratio.

Corollary 7. *Let $\widehat{K}$ be an element in $M_{\mathcal{C}\cup\mathcal{L}}$, and let $\{z_0, w_0\}$ be a point in $\widetilde{K}$. Then, $\widetilde{K} = \{\{z, w\} : (z_0, w_0; z, w) \in \mathbb{R} \cup \{\infty\}\}$.*

Proof. First, observe that $\mathbb{R}$ is a line in $\mathbb{C}$. The set $\widetilde{K}$ is generated by a Euclidean circle or a line K in $\mathbb{C}$. Let T be the Möbius transformation such that $T(K) = \mathbb{R} \cup \{\infty\}$; then, if $\{z, w\} \in \widetilde{K}$, it follows that $T(z_0), T(w_0), T(z), T(w) \in \mathbb{R} \cup \{\infty\}$. By Theorem 5, $(z_0, w_0; z, w) = (T(z_0), T(w_0); T(z), T(w))$, and the result follows. □

4.2. Inversion in Möbius Strips

Let C be a circle in $\widehat{\mathbb{C}}$ given by the equation $az\bar{z} + bz + \bar{b}\bar{z} + c = 0$, with $a, c \in \mathbb{R}$, $b \in \mathbb{C}$. If $a \neq 0$, then C is a Euclidean circle in the complex plane; hence, there exists a transformation I_C in the complex plane that fixes C given by

$$I_C(z) = -\frac{\bar{b}\bar{z} + c}{a\bar{z} + b}, \tag{3}$$

which is called the inversion in C. This transformation fixes point-wise the set C, sends the center of C to infinity and vice versa, and $I_C \circ I_C$ is the identity map. Moreover, if $T \in \mathrm{Aut}(\widehat{\mathbb{C}})$, then $T(C) = C'$ is another circle, and we have $I_{C'} = TI_CT^{-1}$.

Given a Euclidean circle C in $\mathbb{C}$, we have that $F_2(C)$ is homeomorphic to a Möbius strip, for which we can define its inversion as follows. Let $\widehat{C}$ be the corresponding Möbius band in the model $\mathbb{M}_2$, and let $\widehat{I}_C : \mathbb{M}_2 \to \mathbb{M}_2$ such that $\Phi \circ \widetilde{I}_C = \widehat{I}_C \circ \Phi$, where $\widetilde{I}_C : F_2(\mathbb{C}) \to F_2(\mathbb{C})$ is given by $\widetilde{I}_C(\{z, w\}) = \{I_C(z), I_C(w)\}$, for $z, w \neq -\bar{b}/a$, and $\widetilde{I}_C(\{z, -\bar{b}/a\}) = \{I_C(z), -\bar{b}/a\}$, where $-\bar{b}/a$ is the center of C. We call the map $\widehat{I}_C$ the inversion in the Möbius band $\widehat{C}$. Then, we have the following properties for the map $\widehat{I}_C$, taking I_C as in (3) from now on.

Proposition 9. *The map $\widehat{I}_C$ fixes point-wise the Möbius strip $\widehat{C}$, and $\widehat{I}_C \circ \widehat{I}_C$ is the identity map in $\mathbb{M}_2$.*

Proof. As I_C fixes the set C point-wise, it follows that $\widetilde{I}_C(\{z, w\}) = \{I_C(z), I_C(w)\} = \{z, w\}$ if $z, w \in C$. Thus, $\Phi(\{z, w\}) = \Phi \circ \widetilde{I}_C(\{z, w\}) = \widehat{I}_C \circ \Phi(\{z, w\})$, we conclude that $\widehat{I}_C$ fixes $\widehat{C}$ point-wise.

The second statement follows from the fact that $\widetilde{I}_C \circ \widetilde{I}_C(\{z, w\}) = \{I_C \circ I_C(z), I_C \circ I_C(w)\} = \{z, w\}$, for any $z, w \in \mathbb{C} \setminus \{-\bar{b}/a\}$; and $\widetilde{I}_C \circ \widetilde{I}_C(\{z, -\bar{b}/a\}) = \widetilde{I}_C(\{I_C(z), -\bar{b}/a\}) = \{I_C(I_C(z)), -\bar{b}/a\} = \{z, -\bar{b}/a\}$. □

Remark 7. *For any complex number z, we know that $I_C \circ I_C(z) = z$; then, it follows that $\{z, I_C(z)\}$ is a fixed point for $\widetilde{I}_C$; that is, $\widetilde{I}_C$ not only fixes the Möbius strip $\widetilde{C}$, but it has infinitely many other fixed points. Observe that these points correspond to infinite rays coming out from the manifold boundary of the fixed Möbius strip, and these rays do not intersect. Therefore, the fixed set is homeomorphic to a real projective plane minus a point. Moreover, every point $\{z, w\}$ in $F_2(\mathbb{C})$ is a fixed point or a periodic point of period 2 under $\widetilde{I}_C$.*

Now, let us consider two Möbius bands $\widehat{C}$, $\widehat{C}'$ in $M_{\mathcal{C}\cup\mathcal{L}}$, so we know that there is a Möbius transformation T such that $T(C) = C'$; then, the next result follows.

Proposition 10. *Let $\widehat{I}_C$ and $\widehat{I}_{C'}$ be two inversions in the Möbius bands $\widehat{C}$ and $\widehat{C}'$ in $M_{\mathcal{C}\cup\mathcal{L}}$, respectively. Then, they are conjugated in the subset $\mathbb{M}_2 \setminus \Phi(V_T \cup V_T')$ of $\mathbb{M}_2$.*

Proof. Just observe that there is a Möbius transformation $T(z) = (az+b)/(cz+d)$ such that $T(C) = C'$. Since $I_{C'} = TI_CT^{-1}$, it follows that $\widetilde{I}_{C'} = \widetilde{T} \circ \widetilde{I}_C \circ \widetilde{T}^{-1}$ in $F_2(\mathbb{C}) \setminus (V_T \cup V_T')$, and then after conjugating with the map Φ, we obtain $\widehat{I}_{C'} = \widehat{T} \circ \widehat{I}_C \circ \widehat{T}^{-1}$ in $\mathbb{M}_2 \setminus \Phi(V_T \cup V_T')$. □

Note that we can extend the conjugation to V_T', since we must have that

$$\begin{aligned} \widetilde{T} \circ \widetilde{I}_C \circ \widetilde{T}^{-1}(\{z, a/c\}) &= \widetilde{T}(\widetilde{I}_C(\{T^{-1}(z), -d/c\})) = \widetilde{T}(\{I_C(T^{-1}(z)), I_C(-d/c)\}) \\ &= \{T(I_C(T^{-1}(z))), T(I_C(-d/c))\} = \{I_{C'}(z), T(I_C(-d/c))\}, \end{aligned}$$

and then use the map Φ. Notice that when C' is a line, we have $-d/c \in C'$; then, $\widetilde{T} \circ \widetilde{I}_C \circ \widetilde{T}^{-1}(\{z, a/c\}) = \{I_{C'}(z), a/c\}$.

In particular, consider the real line $\mathbb{R}$; then, for any Euclidean circle C, we can send C to $\mathbb{R} \cup \{\infty\}$ by a Möbius transformation $T(z) = (az+b)/(cz+d)$, then the point $-d/c \in C$ since $T(-d/c) = \infty$. Thus, $I_{\mathbb{R}} = TI_CT^{-1}$, where $I_{\mathbb{R}}(z) = \bar{z}$. In $F_2(\mathbb{C}) \setminus V_T'$, we get that $\widetilde{T} \circ \widetilde{I}_C \circ \widetilde{T}^{-1} = \widetilde{I}_{\mathbb{R}}$, and $\widetilde{T} \circ \widetilde{I}_C \circ \widetilde{T}^{-1} : V_T' \to V_T'$, since $\widetilde{T} \circ \widetilde{I}_C \circ \widetilde{T}^{-1}(\{z, a/c\}) = \widetilde{T}(\widetilde{I}_C(\{T^{-1}(z), -d/c\})) = \widetilde{T}(\{I_C(T^{-1}(z)), I_C(-d/c)\}) = \widetilde{T}(\{I_C(T^{-1}(z)), -d/c\}) = \{T(I_C(T^{-1}(z))), a/c\} = \{\bar{z}, a/c\}$, where $I_C(-d/c) = -d/c$ as $-d/c \in C$. In this way, we have defined the conjugation in all $F_2(\mathbb{C})$, and then we can pass to $\mathbb{M}_2$.

Theorem 10. *For any $\widehat{C}$ in $M_{\mathcal{C}\cup\mathcal{L}}$, the inversion in $\widehat{C}$ is conjugated to the map $\widehat{I}_{\mathbb{R}} : \mathbb{M}_2 \to \mathbb{M}_2$, given by $\widehat{I}_{\mathbb{R}}(u, r, \theta) = (\bar{u}, r, -\theta)$, for $u \in \mathbb{R}^2$, $r \geq 0$ and $\theta \in \mathbb{S}^1$.*

Proof. Let T be the Möbius transformation such that $T(C) = \mathbb{R}$, and we assume that C is a Euclidean circle; the case when C is a line is similar. Since $I_{\mathbb{R}} = TI_CT^{-1}$, it follows that the map $\widetilde{I}_{\mathbb{R}} = \widetilde{T} \circ \widetilde{I}_C \circ \widetilde{T}^{-1}$ is defined in $F_2(\mathbb{C}) \setminus V_T'$ as $\widetilde{I}_{\mathbb{R}}(\{z, w\}) = \{I_{\mathbb{R}}(z), I_{\mathbb{R}}(w)\} = \{\bar{z}, \bar{w}\}$. Thus, $\widehat{I}_{\mathbb{R}} = \widehat{T} \circ \widehat{I}_C \circ \widehat{T}^{-1}$ is defined in $\mathbb{M}_2 \setminus \Phi(V_T')$; using the conjugation $\Phi \circ \widetilde{I}_{\mathbb{R}} = \widehat{I}_{\mathbb{R}} \circ \Phi$, we obtain that

$$\begin{aligned} \Phi \circ \widetilde{I}_{\mathbb{R}}(\{z, w\}) = \Phi(\{\bar{z}, \bar{w}\}) &= ((\bar{z} + \bar{w})/2, \|\bar{z} - \bar{w}\|, e^{2i(\arg(\bar{z}-\bar{w})(\mathrm{mod}\pi))}) \\ &= ((\bar{z} + \bar{w})/2, \|z - w\|, e^{2i(-\arg(z-w)(\mathrm{mod}\pi))}), \end{aligned}$$

is equal to $\widehat{I}_{\mathbb{R}} \circ \Phi(\{z, w\}) = \widehat{I}_{\mathbb{R}}((z+w)/2, \|z-w\|, e^{2i(\arg(z-w)(\mathrm{mod}\pi))})$, and the result follows in $\mathbb{M}_2 \setminus \Phi(V_T')$.

To complete the proof, observe that for points $\{z, a/c\} \in V_T'$, we get that

$$\widehat{I}_{\mathbb{R}}((z + a/c)/2, \|z - a/c\|, e^{2i(\arg(z-a/c)(\mathrm{mod}\pi))})$$

$$= ((\bar{z} + a/c)/2, \|\bar{z} - a/c\|, e^{2i(\arg(\bar{z}-a/c)(\mathrm{mod}\pi))}),$$

but since $a, c \in \mathbb{R}$, then $\|z - a/c\| = \|\bar{z} - a/c\|$, $\overline{(z + a/c)/2} = (\bar{z} + a/c)/2$ and $\arg(z - a/c) = -\arg(\bar{z} - a/c)$, so we can conclude that $\widehat{I}_{\mathbb{R}}(u, r, \theta) = (\bar{u}, r, -\theta)$ for $(u, r, \theta) \in \Phi(V_T')$, as well. □

5. Conjugacy Classes in $M(F_2(\mathbb{C}))$

Given $T \in \mathrm{Aut}(\widehat{\mathbb{C}})$, it is well-known that if T is different from the identity, then T is conjugated to U_λ for some $\lambda \in \mathbb{C} \setminus \{0\}$, where $U_\lambda(z) = \lambda z$ if $\lambda \neq 1$ and $U_1(z) = z + 1$, otherwise. In this section, we will extend this result for maps in $M(F_2(\mathbb{C}))$ and show how the corresponding maps $\widetilde{U}_\lambda$ act in $\mathbb{M}_2$.

5.1. Parabolic Maps

Let $T(z) = (az + b)/(cz + d)$ be a Möbius transformation with only one fixed point at z_0; then, T is called a parabolic transformation, and it is conjugated to the map $U_1(z) = z + 1 = T_1(z)$. Let S be the Möbius transformation that conjugates T and T_1, such that $S(z_0) = \infty$, so $S(z) = t/(z - z_0)$ for some $t \in \mathbb{C} \setminus 0$. In order to see the conjugation in $F_2(\mathbb{C})$, we need to consider the singular cones of T and S.

First, consider the singular cone of S, that is, V_S coincides with the cone V_{z_0}; then, $\widetilde{S} : V_S \to V_0$ is given by $\widetilde{S}(\{z, z_0\}) = \{S(z), 0\}$. So the conjugation map $\widetilde{S} \circ \widetilde{T} = \widetilde{T}_1 \circ \widetilde{S}$ in V_{z_0} is given by $\widetilde{S} \circ \widetilde{T}(\{z, z_0\}) = \widetilde{S}(\{T(z), z_0\}) = \{S(T(z)), 0\} = \widetilde{T}_1(\{S(z), 0\})$; then, we define $\widetilde{T}_1$ in V_0 as $\widetilde{T}_1(\{z, 0\}) = \{z + 1, 0\}$. Similarly, in the singular cone V_T of T, we have $\widetilde{S} \circ \widetilde{T}(\{z, -d/c\}) = \widetilde{S}(\{T(z), a/c\}) = \{S(T(z)), S(a/c)\}$, and we would like the last quantity to be equal to $\widetilde{T}_1(\{S(z), S(-d/c)\})$; then, we define $\widetilde{T}_1(\{z, S(-d/c)\}) = \{z + 1, S(a/c)\}$ in $V_{S(-d/c)}$.

In any other cone $V_z \subset F_2(\mathbb{C})$, with $z \neq z_0, -d/c$, we have that $\widetilde{S} \circ \widetilde{T} = \widetilde{T}_1 \circ \widetilde{S}$, in $V_z \setminus \{\{z, z_0\}, \{z, -d/c\}\}$, that is, $\widetilde{T}_1(\{z, w\}) = \{z + 1, w + 1\}$. Using the relation $\Phi \circ \widetilde{T}_1 = \widehat{T}_1 \circ \Phi$, we can conjugate the action of $\widetilde{T}$ in $F_2(\mathbb{C})$ to the action of $\widehat{T}_1$ in $\mathbb{M}_2$.

Remark 8. *In V_0, we obtain that*

$$\begin{aligned} \Phi \circ \widetilde{T}_1(\{z, 0\}) &= \Phi((\{z + 1, 0\}) = ((z + 1)/2, \|z + 1\|, e^{2i(\arg(z+1)(mod\pi))}) \\ &= \widehat{T}_1 \circ \Phi(\{z, 0\}) = \widehat{T}_1((z/2, \|z\|, e^{2i(\arg(z)(mod\pi))}). \end{aligned}$$

Remark 9. *Meanwhile, in $V_{S(-d/c)}$, we get, for $w = S(-d/c)$ and $v = S(a/c)$*

$$\begin{aligned} \Phi \circ \widetilde{T}_1(\{z, w\}) &= \Phi((\{z + 1, v\}) = ((z + 1 + v)/2, \|z + 1 - v\|, e^{2i(\arg(z+1-v)(mod\pi))}) \\ &= \widehat{T}_1 \circ \Phi(\{z, w\}) = \widehat{T}_1((z + w)/2, \|z - w\|, e^{2i(\arg(z-w)(mod\pi))}). \end{aligned}$$

Setting $\mathbb{M}_2^p = \mathbb{M}_2 \setminus \Phi(V_0 \cup V_{S(-d/c)})$, we get the following result.

Theorem 11. *Let $W \in M(F_2(\mathbb{C}))$ be a map with only one fixed point; then, W is conjugated to the map $\widehat{T}_1 : \mathbb{M}_2^p \to \mathbb{M}_2$ given by $\widehat{T}_1(u, r, \theta) = (u + 1, r, \theta)$, where $u \in \mathbb{R}^2$, $r \geq 0$ and $\theta \in \mathbb{S}^1$.*

Proof. As W has only one fixed point in $F_2(\mathbb{C})$, then $W = \widetilde{T}$ for some parabolic map $T(z) = (az + b)/(cz + d)$. Since T is conjugated to the map $U_1(z) = z + 1 = T_1(z)$, then the result follows by Proposition 4, taking $t = 1$, since for any $\{z, w\} \notin V_0 \cup V_{S(-d/c)}$, the map $\widetilde{T}_1(\{z, w\}) = \{z + 1, w + 1\}$. □

Corollary 8. *The orbit of every point in $F_2(\mathbb{C})$ under a parabolic map in $M(F_2(\mathbb{C}))$ tends to the fixed point of the map.*

Proof. Let us start in V_0; since $\Phi \circ \widetilde{T}_1^n(\{z, 0\}) = \widehat{T}_1^n \circ \Phi(\{z, 0\})$ by 8, it follows that

$$\begin{aligned} \widehat{T}_1^n \circ \Phi(\{z, 0\}) &= \widehat{T}_1^n((z/2, \|z\|, e^{2i(\arg(z)(\mathrm{mod}\pi))}) \\ &= ((z + n)/2, \|z + n\|, e^{2i(\arg(z+n)(\mathrm{mod}\pi))}) \\ &\to \infty, \text{ as } n \to \infty, \end{aligned}$$

then $\widetilde{T}_1^n(\{z,0\}) \to \infty$, as n goes to infinity. Since $\widetilde{S} \circ \widetilde{T} = \widetilde{T}_1 \circ \widetilde{S}$, then $\lim_{n\to\infty} \widetilde{T}^n(\{z,0\}) = \lim_{n\to\infty} \widetilde{S}^{-1}(\widetilde{T}_1^n(\widetilde{S}(\{z,0\}))) = \widetilde{S}^{-1}(\infty) = z_0$. The argument is the same for points in $V_{S(-d/c)}$ and $F_2(\mathbb{C}) \setminus V_0 \cup V_{S(-d/c)}$, by Remark 9 and Theorem 11. □

5.2. Hyperbolic, Loxodromic and Elliptic Maps

Now, let $T(z) = (az+b)/(cz+d)$ be a Möbius transformation with two fixed points z_1 and z_2. It is conjugated to the map $U_\lambda(z) = \lambda z$ with $\lambda \neq 1$, using a Möbius transformation S such that $S(z_1) = 0$ and $S(z_2) = \infty$; that is, we can take $S(z) = (z - z_1)/(z - z_2)$. If $|\lambda| \neq 1$ and $\lambda > 0$, then T is called hyperbolic; otherwise, T is called loxodromic. If $|\lambda| = 1$, the map T is called elliptic. As in the parabolic case, to find the map $\widetilde{U}_\lambda$ that is conjugated to $\widetilde{T}$, we need to consider some special subsets of $F_2(\mathbb{C})$ and some generalities about the conjugation in this setting before analyzing the different cases.

Let us consider three special cones: the singular cone V_T of T, and the cones V_{z_1} and V_{z_2}, where the singular cone of S coincides with V_{z_2}. Observe that $\widetilde{T} : V_T \to V_T'$, $\widetilde{T} : V_{z_i} \to V_{z_i}$, for $i = 1,2$, $\widetilde{S} : V_{z_1} \to V_0$ and $\widetilde{S} : V_{z_2} \to V_S' = V_1$. For points in V_{z_1}, we would like to have $\widetilde{S} \circ \widetilde{T}(\{z,z_1\}) = \widetilde{S}(\{T(z),z_1\}) = \{S(T(z)),0\}) = \widetilde{U}_\lambda \circ \widetilde{S}(\{z,z_1\}) = \widetilde{U}_\lambda(\{S(z),0\})$, so we define $\widetilde{U}_\lambda(\{z,0\}) = \{\lambda z,0\})$ in V_0. In the same way, in V_{z_2} we need to happen that $\widetilde{S} \circ \widetilde{T}(\{z,z_2\}) = \widetilde{S}(\{T(z),z_2\}) = \{S(T(z)),1\}) = \widetilde{U}_\lambda \circ \widetilde{S}(\{z,z_2\}) = \widetilde{U}_\lambda(\{S(z),1\})$, so we define $\widetilde{U}_\lambda(\{z,1\}) = \{\lambda z,1\})$ in V_1. Finally, if we set $w = S(-d/c)$ and $v = S(a/c)$, then in V_T we must have $\widetilde{S} \circ \widetilde{T}(\{z,-d/c\}) = \widetilde{S}(\{T(z),a/c\}) = \{S(T(z)),v\}) = \widetilde{U}_\lambda \circ \widetilde{S}(\{z,-d/c\}) = \widetilde{U}_\lambda(\{S(z),w\})$, so we define $\widetilde{U}_\lambda(\{z,w\}) = \{\lambda z,v\})$ in $V_{S(-d/c)}$.

For $z \neq z_1, z_2, -d/c$, we have that $\widetilde{S} \circ \widetilde{T} = \widetilde{U}_\lambda \circ \widetilde{S}$; that is, $\widetilde{U}_\lambda(\{z,w\}) = \{\lambda z, \lambda w\}$ in any cone $V_z \setminus \{\{z,z_1\},\{z,z_2\},\{z,-d/c\}\} \subset F_2(\mathbb{C})$. Using the relation $\Phi \circ \widetilde{U}_\lambda = \widehat{U}_\lambda \circ \Phi$, we can conjugate the action of $\widetilde{T}$ in $F_2(\mathbb{C})$ to the action of $\widehat{U}_\lambda$ in $\mathbb{M}_2$.

Proceeding as in Remarks 8 and 9, we obtain the conjugation in the corresponding domains. In V_0, we obtain that

$$\begin{aligned} \Phi \circ \widetilde{U}_\lambda(\{z,0\}) &= \Phi((\{\lambda z,0\}) = (\lambda z/2, \|\lambda z\|, e^{2i(\arg(\lambda z)(\mathrm{mod}\pi))}) \\ &= \widehat{U}_\lambda \circ \Phi(\{z,0\}) = \widehat{U}_\lambda(z/2, \|z\|, e^{2i(\arg(z)(\mathrm{mod}\pi))}), \end{aligned} \tag{4}$$

and, in V_1 we get that

$$\begin{aligned} \Phi \circ \widetilde{U}_\lambda(\{z,1\}) &= \Phi((\{\lambda z,1\}) = ((\lambda z+1)/2, \|\lambda z-1\|, e^{2i(\arg(\lambda z-1)(\mathrm{mod}\pi))}) \\ &= \widehat{U}_\lambda \circ \Phi(\{z,1\}) = \widehat{U}_\lambda((z+1)/2, \|z-1\|, e^{2i(\arg(z-1)(\mathrm{mod}\pi))}). \end{aligned} \tag{5}$$

Meanwhile, in $V_{S(-d/c)}$, we get, setting $w = S(-d/c)$ and $v = S(a/c)$

$$\begin{aligned} \Phi \circ \widetilde{U}_\lambda(\{z,w\}) &= \Phi((\{\lambda z,v\}) = ((\lambda z+v)/2, \|\lambda z-v\|, e^{2i(\arg(\lambda z-v)(\mathrm{mod}\pi))}) \\ &= \widehat{U}_\lambda \circ \Phi(\{z,w\}) = \widehat{T}_1((z+w)/2, \|z-w\|, e^{2i(\arg(z-w)(\mathrm{mod}\pi))}). \end{aligned} \tag{6}$$

5.2.1. Hyperbolic and Loxodromic Maps

Let $T(z) = (az+b)/(cz+d)$ be a Möbius transformation conjugated to $U_\lambda(z) = \lambda z$ with $|\lambda| \neq 1$. Let $\mathbb{M}_2^h = \mathbb{M}_2 \setminus \Phi(V_0 \cup V_1 \cup V_{S(-d/c)})$. Then, we get the following result.

Theorem 12. *Let $\widetilde{T} \in M(F_2(\mathbb{C}))$ be a hyperbolic map. Then, $\widetilde{T}$ is conjugated to the map $\widehat{U}_\lambda : \mathbb{M}_2^h \to \mathbb{M}_2$ given by $\widehat{U}_\lambda(z,a,t) = (\lambda z, |\lambda| a, e^{2i\arg\lambda} t)$, for $z \in \mathbb{R}^2$, $a \geq 0$ and $t \in \mathbb{S}^1$.*

Proof. From the conjugation $\Phi \circ \widetilde{U}_\lambda = \widehat{U}_\lambda \circ \Phi$, we obtain that

$$\begin{aligned}\Phi(\widetilde{U}_\lambda(\{z,w\})) &= \Phi(\{\lambda z, \lambda w\}) = (\lambda(z+w)/2, \|\lambda z - \lambda w\|, e^{2i(\arg(\lambda z - \lambda w)(\mathrm{mod}\pi))}) \\ &= (\lambda(z+w)/2, |\lambda|\|z-w\|, e^{2i(\arg\lambda + \arg(z-w))(\mathrm{mod}\pi))}) \\ &= \widehat{U}_\lambda(\Phi(\{z,w\}) = \widehat{U}_\lambda((z+w)/2, \|z-w\|, e^{2i(\arg(z-w)(\mathrm{mod}\pi))}),\end{aligned}$$

from where it follows the claim. □

By the action of $\widehat{U}_\lambda$ in $\mathbb{M}_2$, that is, by Equations (4)–(6) and by Theorem 12, as well as Remark 2, we conclude the following result.

Corollary 9. *The orbit of every point in $F_2(\mathbb{C})$ under a hyperbolic or loxodromic map in $M(F_2(\mathbb{C}))$ tends to one of the fixed points of the map and away from the other fixed point.*

As in the classical theory of Möbius transformation, we can make a geometric distinction between hyperbolic and loxodromic elements in $M(F_2(\mathbb{C}))$. Remember that a hyperbolic Möbius transformation T always has an invariant disc in the complex plane, that is, it leaves its boundary invariant, so the corresponding map $\widetilde{T}$ must leave a Möbius strip invariant; meanwhile, a loxodromic element can not leave any Möbius band invariant.

5.2.2. Elliptic Maps

Let $T(z) = (az+b)/(cz+d)$ be a Möbius transformation with two fixed points z_1 and z_2 conjugated to the map $U_\lambda(z) = \lambda z$ with $\lambda \neq 1$ but $|\lambda| = 1$. Consider $\mathbb{M}_2^e = \mathbb{M}_2 \setminus \Phi(V_0 \cup V_1 \cup V_{S(-d/c)})$. We get the following.

Theorem 13. *Let $\widetilde{T} \in M(F_2(\mathbb{C}))$ be a map such that T is an elliptic map. Then $\widetilde{T}$ is conjugated to the map $\widehat{U}_\lambda : \mathbb{M}_2^e \to \mathbb{M}_2$ given by $\widehat{U}_\lambda(z,a,t) = (\lambda z, a, e^{2i\arg\lambda}t)$, for $z \in \mathbb{R}^2$, $a \geq 0$ and $t \in \mathbb{S}^1$.*

Proof. The proof follows the same lines as before; from the relation $\Phi \circ \widetilde{U}_\lambda = \widehat{U}_\lambda \circ \Phi$, we obtain that

$$\begin{aligned}\Phi(\widetilde{U}_\lambda(\{z,w\})) &= \Phi(\{\lambda z, \lambda w\}) = (\lambda(z+w)/2, \|\lambda z - \lambda w\|, e^{2i(\arg(\lambda z - \lambda w)(\mathrm{mod}\pi))}) \\ &= (\lambda(z+w)/2, \|z-w\|, e^{2i(\arg\lambda + \arg(z-w))(\mathrm{mod}\pi))}) \\ &= \widehat{U}_\lambda(\Phi(\{z,w\}) = \widehat{U}_\lambda((z+w)/2, \|z-w\|, e^{2i(\arg(z-w)(\mathrm{mod}\pi))}),\end{aligned}$$

from where it follows the claim. □

By the final part of the Remark 4 and the previous Theorem, for an elliptic map T, the set $\widetilde{T}^n(\{z,w\})$ has no limit, for any point $\{z,w\} \in F_2(\mathbb{C})$, with $z, w \notin \{z_1, z_2\}$.

We recall that the *period or order* of a Möbius transformation T is the least positive integer m such that $T^m = I$ is the identity map if such an integer exists. So, we have the next consequence of Theorem 13.

Corollary 10. *If T is a non-identity Möbius map with finite period n, then the map $\widehat{U}_\lambda : \mathbb{M}_2^e \to \mathbb{M}_2$ conjugated to $\widetilde{T}$ satisfies that $\widehat{U}_\lambda^n$ is the identity map in $\mathbb{M}_2^e$.*

Proof. Since T has a finite period, then it is an elliptic map conjugated to the map $U_\lambda(z) = \lambda z$, with $\lambda \neq 1$, $|\lambda| = 1$, and $\lambda^n = 1$. By Theorem 13, $\widetilde{T}$ is conjugated to the map $\widehat{U}_\lambda : \mathbb{M}_2^e \to \mathbb{M}_2$ given by $\widehat{U}_\lambda(z,a,t) = (\lambda z, a, e^{2i\arg\lambda}t)$, for $z \in \mathbb{R}^2$, $a \geq 0$ and $t \in \mathbb{S}^1$. Then,

$$\widehat{U}_\lambda^n(z,a,t) = (\lambda^n z, a, e^{2in\arg\lambda}t) = (z,a,t),$$

which proves the claim. □

We can say a little more about elliptic maps $T(z) = (az + b)/(cz + d)$ with finite period. Since T^n is the identity map, we have that $\widetilde{T}^n(\{z, w\}) = \{T^n(z), T^n(w)\} = \{z, w\}$, for $\{z, w\} \notin V_T$. Recall that $T(-d/c) = \infty$, $T(\infty) = a/c$ and then $T^{n-2}(a/c) = -d/c$ since T^n is the identity. Hence, for $\{z, -d/c\} \in V_T$, we have that $\widetilde{T}(\{z, -d/c\}) = \{T(z), a/c\}$, and then $\widetilde{T}^{n-1}(\{z, -d/c\}) = \{T^{n-1}(z), -d/c\}$; therefore, $\widetilde{T}^n(\{z, -d/c\}) = \{z, a/c\}$. To get the identity, we need to iterate the map $n(n-1)$ times, that is, $\widetilde{T}^{n(n-1)}(\{z, -d/c\}) = \{z, -d/c\}$. Thus, for any elliptic Möbius map with a finite period, we get a map in $F_2(\mathbb{C})$ that also has a finite period, which gives us an example of a finite subgroup in $M(F_2(\mathbb{C}))$.

6. Conclusions and Future Work

For any Möbius transformation $T(z) = (az + b)/(cz + d)$ in the Riemann sphere, we have defined a map $\widetilde{T} : F_2(\mathbb{C})^* \to F_2(\mathbb{C})^*$ in two steps. For the singular cone of T, $V_T = \{z, -d/c\}$, we have defined $\widetilde{T} : V_T \to V_T'$ as $\widetilde{T}(\{z, -d/c\}) = \{T(z), a/c\}$, and for $F_2(\mathbb{C}) \setminus V_T$, we have $\widetilde{T}(\{z, w\}) = \{T(z), T(w)\}$, and then we extended it to ∞ sending $\infty \to a/c$ and $-d/c \to \infty$. In this way, $\widetilde{T}$ is a bijection, and it is a homeomorphism if we restrict the map to $F_2(\mathbb{C}) \setminus V_T$ and V_T, separately. The lack of continuity in between is not an impediment to extend several classical results of the set of Möbius transformations in the complex plane to the set of maps $M(F_2(\mathbb{C})) = \{\widetilde{T} : F_2(\mathbb{C})^* \to F_2(\mathbb{C})^* : T \in \mathrm{Aut}(\widehat{\mathbb{C}})\}$ such as properties of transitivity, decomposition in generators, and conjugation to simple maps.

We describe the generators of $M(F_2(\mathbb{C}))$, proving how they behave geometrically in the model $\mathbb{M}_2$, which gives a better understanding of the orbits under these generators, so we observe several similarities with the geometry of the orbits of points under classical Möbius transformations.

Another important feature that we can extend are the transitivity properties of the set $M(F_2(\mathbb{C}))$, not only at the points in $F_2(\mathbb{C})$ but also in the so-called cones. Moreover, we prove transitivity in a large set of Möbius strips contained in $F_2(\mathbb{C})$, we give a characterization of these Möbius bands using cross-ratio, and we prove that the concept of inversion in a Euclidean circle can be extended to this set of Möbius strips, given a geometric description in $\mathbb{M}_2$ of this new inversion map.

The conjugacy classes in $\mathrm{Aut}(\widehat{\mathbb{C}})$ are important since they give a geometric classification of the Möbius transformations in the Riemann sphere. We show that we can extend this geometric classification in our setting, describing geometrically the representatives of every conjugacy class.

In summary, we demonstrate in this paper that the classical theory of Möbius transformation can be extended not just to $\mathbb{R}^n$ or $\mathbb{C}^n$ but to other topological spaces, as we have done for the second symmetric product of the complex plane.

In future work, we would like to explore the group properties of the set $M(F_2(\mathbb{C}))$ and extend the action of the special linear group with real coefficients, $\mathrm{PSL}(2, \mathbb{R})$, in $\mathbb{H}$ to $F_2(\mathbb{H})$.

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