mathematics

MDPI

Article

Two-Variable q-General-Appell Polynomials Within the Context of the Monomiality Principle

Noor Alam [1], Waseem Ahmad Khan [2,*], Can Kızılateş [3] and Cheon Seoung Ryoo [4]

[1] Department of Mathematics, College of Science, University of Ha'il, Ha'il 2440, Saudi Arabia; noor.alam@uoh.edu.sa
[2] Department Electrical Engineering, Prince Mohammad Bin Fahd University, P.O. Box 1664, Al Khobar 31952, Saudi Arabia
[3] Department of Mathematics, Faculty of Science, Zonguldak Bülent Ecevit University, Zonguldak 67100, Turkey; cankizilates@gmail.com
[4] Department of Mathematics, Hannam University, Daejeon 34430, Republic of Korea; ryoocs@hnu.kr
* Correspondence: wkhan1@pmu.edu.sa or khan346@gmail.com

Abstract: In this study, we consider the two-variable q-general polynomials and derive some properties. By using these polynomials, we introduce and study the theory of two-variable q-general Appell polynomials (2VqgAP) using q-operators. The effective use of the q-multiplicative operator of the base polynomial produces the generating equation for 2VqgAP involving the q-exponential function. Furthermore, we establish the q-multiplicative and q-derivative operators and the corresponding differential equations. Then, we obtain the operational, explicit and determinant representations for these polynomials. Some examples are constructed in terms of the two-variable q-general Appell polynomials to illustrate the main results. Finally, graphical representations are provided to illustrate the behavior of some special cases of the two-variable q-general Appell polynomials and their potential applications.

Keywords: quantum calculus; two-variable q-general polynomials; two-variable q-general Appell polynomials; q-quasi monomiality principle; dilatation operators

MSC: 05A30; 11B83; 11B68

check for updates

Academic Editor: Francesco Aldo Costabile

Received: 20 January 2025
Revised: 23 February 2025
Accepted: 25 February 2025
Published: 26 February 2025

Citation: Alam, N.; Khan, W.A.; Kızılateş, C.; Ryoo, C.S. Two-Variable q-General-Appell Polynomials Within the Context of the Monomiality Principle. *Mathematics* **2025**, *13*, 765. https://doi.org/10.3390/math13050765

Copyright: © 2025 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https://creativecommons.org/licenses/by/4.0/).

1. Introduction

Appell polynomials, a notable category of special functions, have found widespread utility in mathematical domains. Introduced by Paul Appell in 1880, these polynomials are characterized by distinctive properties that contribute to their significance across various mathematical and scientific fields.

The generating function for two-variable general polynomials (2VgP), represented as $p_n(\zeta,\eta)$, is established as follows in the literature [1,2]:

$$\exp(\zeta t)\phi(\eta,t)=\sum_{n=0}^{\infty}p_n(\zeta,\eta)\frac{t^n}{n!},(p_0(\zeta,\eta)=1), \tag{1}$$

where $\phi(\eta,t)$ has (at least the formal) series expansion

$$\phi(\eta,t)=\sum_{k=0}^{\infty}\phi_k(\eta)\frac{t^k}{k!},(\phi_0(\eta)\neq 0). \tag{2}$$

In the 18th century, the foundation of q-calculus began, establishing a framework integral to the theory of special functions. Recently, q-calculus has garnered significant attention due to its applicability across fields like applied mathematics, mechanical engineering, and physics, bridging classical mathematics and quantum calculus. The development of q-analogues, inspired by classical calculus, has led to foundational results in combinatorics, number theory, and other mathematical disciplines. Key q-special polynomials, such as the q-binomial coefficient $\binom{n}{k}_q$ and the q-Pochhammer symbol $(\gamma;q)_v$, hold central roles in q-analog structures across combinatorics, representation theory, and statistical mechanics. The algebraic and analytic richness of these polynomials offers essential tools for exploring q-analogue phenomena. In this paper, we set $q \in \mathbb{C}$, with $|q| < 1$, and employ q-notations as per Andrews et al. [3]. In recent years, advancements have been made in the study of two-variable q-Hermite polynomials, notably by Zayed et al. [4] and Riyasat and Khan [5], highlighting their importance in extending classical Hermite polynomials into q-calculus frameworks. Additionally, q-special functions have been formulated and explored within the representations of quantum algebras [6–8] underscoring the role of q-calculus in quantum groups and quantum mechanics.

The q-shifted factorial $(\zeta;q)_\nu$ is defined as follows:

$$(\zeta;q)_\nu = \prod_{s=1}^{\nu-1}(1-q^s\zeta), \quad \nu \in \mathbb{N}, \quad \zeta \in \mathbb{C}; \quad 0 < q < 1. \tag{3}$$

In q-calculus, this notation assumes a fundamental significance, embodying the discrete characteristics of q-analogues and enabling a range of identities and transformations crucial to the theoretical framework.

For a complex number $\zeta \in \mathbb{C}$, the q-analogue of ζ is given by

$$[\zeta]_q = \frac{1-q^\zeta}{1-q}, \quad 0 < q < 1, \quad \zeta \in \mathbb{C}. \tag{4}$$

The q-factorial is given by

$$[\zeta]_q! = \begin{cases} \prod_{s=1}^{\zeta}[s]_q, & 0 < q < 1, \quad \zeta \neq 0 \in \mathbb{C} \\ 1, & \zeta = 0 \end{cases}. \tag{5}$$

The Gauss's q-binomial formula is given by [3]

$$(\zeta \pm \gamma)_q^v = \sum_{s=0}^{v} \binom{v}{s}_q q^{\binom{v-s}{2}} \zeta^s (\pm\gamma)^{v-s}, \tag{6}$$

where the Gauss q-binomial coefficient is

$$\binom{v}{s}_q = \frac{[v]_q!}{[s]_q![v-s]_q!}.$$

Two q-exponential functions are given by [9]

$$e_q(t) = \sum_{n=0}^{\infty} \frac{t^n}{[n]_q!}, \quad 0 < q < 1, \quad |\zeta| < \frac{1}{1-q} \tag{7}$$

and

$$E_q(t) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} t^n}{[n]_q!}, \quad 0 < q < 1, \quad \zeta \in \mathbb{C}. \tag{8}$$

The product of both q-exponential functions is given by

$$e_q(\zeta)E_q(\eta) = \sum_{w=0}^{\infty} \frac{(\zeta+\eta)_q^w}{[w]_q!}. \tag{9}$$

Hence

$$e_q(\zeta)E_q(-\zeta) = 1, \quad |\zeta| < \frac{1}{1-q}. \tag{10}$$

The q-derivative of a function $f(\zeta)$ is defined as [10,11]

$$\widehat{D}_{q,\zeta} f(\zeta) = \frac{f(q\zeta) - f(\zeta)}{\zeta(q-1)}, \quad 0 < q < 1, \quad \zeta \neq 0. \tag{11}$$

In particular, we have

$$\widehat{D}_{q,\zeta}\zeta^w = [w]_q \zeta^{w-1}. \tag{12}$$

The q-derivative of the exponential function is given by

$$\widehat{D}_{q,\zeta} e_q(\alpha\zeta) = \alpha e_q(\alpha\zeta), \ \ \alpha \in \mathbb{C} \tag{13}$$

and

$$\widehat{D}^k_{q,\zeta} e_q(\alpha\zeta) = \alpha^k e_q(\alpha\zeta), \ \ k \in \mathbb{N}, \ \alpha \in \mathbb{C}, \tag{14}$$

where $\widehat{D}^k_{q,\zeta}$ denotes the kth order q-derivative with respect to ζ.

The q-derivative of the product of functions $f(\zeta)$ and $g(\zeta)$ has been established in the literature [2,4,12]

$$\widehat{D}_{q,\zeta}(f(\zeta)g(\zeta)) = f(\zeta)\widehat{D}_{q,\zeta}g(\zeta) + g(q\zeta)\widehat{D}_{q,\zeta}f(\zeta). \tag{15}$$

The q-derivative of the division of functions $f(\zeta)$ and $g(\zeta)$ is given by

$$\widehat{D}_{q,\zeta}\left(\frac{f(\zeta)}{g(\zeta)}\right) = \frac{g(q\zeta)\widehat{D}_{q,\zeta}f(\zeta) - f(q\zeta)\widehat{D}_{q,\zeta}g(\zeta)}{g(\zeta)g(q\zeta)}, \quad g(\zeta) \neq 0, \quad g(q\zeta) \neq 0. \tag{16}$$

The q-integral of a function $f(\zeta)$ is defined as [3,4]

$$\int_0^a f(\zeta) d_q\zeta = (1-q)a \sum_{w=0}^{\infty} q^w f(aq^w). \tag{17}$$

By virtue of (17), we can conclude that

$$\int_0^\zeta \lambda^m d_q\lambda = (1-q)\zeta \sum_{w=0}^{\infty} \zeta^m q^{w(m+1)}$$

$$= \frac{\zeta^{m+1}(1-q)}{1-q^{m+1}} = \frac{\zeta^{m+1}}{[m+1]_q}, \ \ m \in \mathbb{N} \sqcup \{0\}. \tag{18}$$

More specifically, $\widehat{D}^{-1}_{q,\zeta}\{1\} = \zeta$, and by using the method of mathematical induction, we have

$$\left(\widehat{D}^{-1}_{q,\zeta}\right)^r\{1\} = \frac{\zeta^r}{[r]_q!}, \quad r \in \mathbb{N} \sqcup \{0\}. \tag{19}$$

For some papers on q-generalizations of the special polynomials, we refer to [9,13–16]. The q-Gould–Hopper polynomials, denoted as $H_{w,q}^{(m)}(\zeta,\eta)$ ($qGHP$), can be defined following generating function

$$e_q(\zeta t)e_q(\eta t^m) = \sum_{w=0}^{\infty} H_{w,q}^{(m)}(\zeta,\eta)\frac{t^w}{[w]_q!} \tag{20}$$

and the series definition

$$H_{w,q}^{(m)}(\zeta,\eta) = [w]_q! \sum_{k=0}^{\left[\frac{w}{m}\right]} \frac{\eta^k \zeta^{w-mk}}{[k]_q![w-mk]_q!}. \tag{21}$$

For $m = 2$, (20) reduces to 2-variable q-Hermite polynomials defined by Raza et al. [11]. The operational identity of q-Gould–Hopper polynomials $H_{w,q}^{(m)}(\zeta,\eta)$ is as follows:

$$H_{w,q}^{(m)}(\zeta,\eta) = e_q\left(\eta D_{q,\zeta}^m\right)\{\zeta^w\}. \tag{22}$$

The q-dilatation operator T_ζ, which acts on any function of the complex variable ζ, in the following manner [8,10]

$$T_\zeta^k f(\zeta) = f(q^k\zeta), \quad k \in \mathbb{R}, \quad 0 < q < 1, \tag{23}$$

satisfies the property

$$T_\zeta^{-1} T_\zeta^1 f(\zeta) = f(\zeta). \tag{24}$$

The q-derivative of the q-exponential function $e_q(\eta t^m)$ is given as [9]

$$\widehat{D}_{q,t} e_q(\eta t^m) = \eta t^{m-1} T_{(\eta;m)} e_q(\eta t^m). \tag{25}$$

where

$$T_{(\eta;m)} = \frac{1-q^m T_\eta^m}{1-qT_\eta} = 1 + qT_\eta + \cdots + q^{m-1}T_\eta^{m-1}.$$

The generating function of q-Appell polynomials $\mathcal{A}_{n,q}(\zeta)$ is elucidated by Al Salam, as delineated in the formula presented in [17,18].

$$\mathcal{A}_q(t)\, e_q(\zeta t) = \sum_{n=0}^{\infty} \mathcal{A}_{n,q}(\zeta)\frac{t^n}{[n]_q!}, \tag{26}$$

$$\mathcal{A}_q(t) = \sum_{n=0}^{\infty} \mathcal{A}_{n,q}\frac{t^n}{[n]_q!}, \qquad \mathcal{A}_q(t) \neq 0, \qquad A_{0,q} = 1. \tag{27}$$

The following list (see Table 1) demonstrates an appropriate option for certain individuals within the class of q-Appell polynomials:

Table 1. Some q-Appell polynomial families.

S. No.	q-Appell Polynomials	Generating Function	$\mathcal{A}_q(t)$
I.	The q-Bernoulli Polynomials [16,19]	$\frac{t}{e_q(t)-1}e_q(\zeta t) = \sum_{n=0}^{\infty} \mathcal{B}_{n,q}(\zeta)\frac{t^n}{[n]_q!}$	$\mathcal{A}_q(t) = \frac{t}{e_q(t)-1}$
II.	The q-Euler Polynomials [16,19]	$\frac{[2]_q}{e_q(t)+1}e_q(\zeta t) = \sum_{n=0}^{\infty} \mathcal{E}_{n,q}(\zeta)\frac{t^n}{[n]_q!}$	$\mathcal{A}_q(t) = \frac{[2]_q}{e_q(t)+1}$
III.	The q-Genocchi Polynomials [16,19]	$\frac{[2]_q t}{e_q(t)+1}e_q(\zeta t) = \sum_{n=0}^{\infty} \mathcal{G}_{n,q}(\zeta)\frac{t^n}{[n]_q!}$	$\mathcal{A}_q(t) = \frac{[2]_q t}{e_q(t)+1}$

The elements of the q-Appell polynomials class $\mathcal{A}_{n,q}(\zeta)$ generate the corresponding q-numbers $\mathcal{A}_{n,q}$ when $\zeta = 0$. Table 2 illustrates the initial occurrences of three specific q-numbers: the q-Bernoulli numbers $\mathcal{B}_{n,q}$ [16,19], the q-Euler numbers $\mathcal{E}_{n,q}$ [16,19], and the q-Genocchi numbers $\mathcal{G}_{n,q}$ [16,19].

Table 2. The first five q-numbers of $\mathcal{B}_{n,q}$, $\mathcal{E}_{n,q}$ and $\mathcal{G}_{n,q}$.

n	**0**	**1**	**2**	**3**	**4**
$\mathcal{B}_{n,q}$ [16,19]	1	$-(1+q)^{-1}$	$q^2([3]_q!)^{-1}$	$(1-q)q^3([2]_q)^{-1}([4]_q)^{-1}$	$q^4(1-q^2-2q^3-q^4+q^6)([2]_q^2!)[3]_q[5]_q)^{-1}$
$\mathcal{E}_{n,q}$ [16,19]	1	$-\frac{1}{2}$	$\frac{1}{4}(-1+q)$	$\frac{1}{8}(-1+2q+q^2-q^3)$	$\frac{1}{16}(q-1)[3]_q!(q^2-4q+1)$
$\mathcal{G}_{n,q}$ [16,19]	0	$\frac{2-q}{1+q}$	$\frac{q(q-5)}{(1+q)^2}$	$-\frac{3q^2(q-5)}{(1+q)^3}-\frac{3q(2-q)}{(1+q)^2}-\frac{q}{(1+q)}$	$\frac{-3q}{1+q}\left(\frac{3q^3+10q^2-28q+7}{(1+q)^3}\right)$

The monomiality principle stands as a fundamental tool for investigating specific special polynomials and functions, as well as their properties. Originating from Steffensen's work in the early 19th century, this concept underwent significant refinement and expansion through Dattoli's contributions in 2000. Recent academic research has utilized the monomiality principle to explore innovative hybrid special polynomial sequences and families [14,15]. Notably, Cao et al. [10] extended the application of the monomiality principle to q-special polynomials, opening avenues for the creation of novel q-special polynomial families and shedding light on the quasi monomiality of certain existing q-special polynomials. Through the application of q-operation specific techniques, researchers can derive additional classes of q-generating functions and various generalizations of q-special functions. Among these techniques, the q-operational process demonstrates greater alignment with traditional mathematical approaches and implementations employed in the resolution of q-differential equations. In the context of a q-polynomial set $p_{n,q}(\zeta)(n \in \mathbb{N}, \zeta \in \mathbb{C})$, two q-operators are defined as $\widehat{M}_q$ and $\widehat{P}_q$. These operators, known as the q-multiplicative and q-derivative operators, respectively, are implemented as described in [10]

$$\widehat{M}_q\{p_{n,q}(\zeta)\} = p_{n+1,q}(\zeta) \tag{28}$$

and

$$\widehat{P}_q\{p_{n,q}(\zeta)\} = [n]_q p_{n-1,q}(\zeta). \tag{29}$$

The operators $\widehat{M}_q$ and $\widehat{P}_q$ satisfy the following commutation relation:

$$[\widehat{M}_q, \widehat{P}_q] = \widehat{P}_q\widehat{M}_q - \widehat{M}_q\widehat{P}_q. \tag{30}$$

An analysis of the $\widehat{M}_q$ and $\widehat{P}_q$ operators facilitates the determination of polynomial $p_{n,q}(\zeta)$ properties. When $\widehat{M}_q$ and $\widehat{P}_q$ demonstrate differential realization, the polynomials $p_{n,q}(\zeta)$ conform to a particular differential equation.

$$\widehat{M}_q\widehat{P}_q\{p_{n,q}(\zeta)\} = [n]_q p_{n,q}(\zeta) \tag{31}$$

and

$$\widehat{P}_q\widehat{M}_q\{p_{n,q}(\zeta)\} = [n+1]_q p_{n,q}(\zeta). \tag{32}$$

In view of (28) and (29), we have

$$[\widehat{M}_q, \widehat{P}_q] = [n+1]_q - [n]_q. \tag{33}$$

From (28), we have

$$\widehat{M}_q{}^r\{p_{n,q}\} = p_{n+r,q}(\zeta). \tag{34}$$

More specifically, we have

$$p_{n,q}(\zeta) = \widehat{M}_q{}^n\{p_{0,q}\} = \widehat{M}_q{}^n\{1\}, \tag{35}$$

where $p_{0,q}(\zeta) = 1$ is the q-sequel of polynomial $p_{n,q}(\zeta)$. Furthermore, the generating function of $p_{n,q}(\zeta)$ can be obtained as

$$e_q(\widehat{M}_q t)\{1\} = \sum_{n=0}^{\infty} p_{n,q}(\zeta)\frac{t^n}{[n]_q!}. \tag{36}$$

The subsequent q-multiplicative and q-derivative operators associated with the q-Appell polynomials $A_{n,q}$ are presented as follows [12]

$$\widehat{M}_{qA} = \eta + \frac{A_q'(\widehat{D}_{q,\eta})}{A_q(\widehat{D}_{q,\eta})}T_\eta, \tag{37}$$

or, alternatively,

$$\widehat{M}_{qA} = \eta\frac{A_q(q\widehat{D}_{q,\eta})}{A_q(\widehat{D}_{q,\eta})} + \frac{A_q'(\widehat{D}_{q,\eta})}{A_q(\widehat{D}_{q,\eta})}, \tag{38}$$

and

$$\widehat{P}_{qA} = \widehat{D}_{q,\eta}, \tag{39}$$

respectively.

Inspired by the above papers, in this paper, we examine the specific characteristics of two-variable q-general Appell polynomials through the implementation of the q-analog monomiality principle. Additionally, we present some applications of these newly established two-variable q-general Appell polynomials with graphical representations. Our findings suggest that these polynomials are promising for diverse applications across multiple disciplines.

2. Two-Variable q-General Appell Polynomials

In this section, we introduce the concept of two-variable q-general Appell polynomials, denoted as ${}_pA_{n,q}(\zeta,\eta)$. We elucidate their series definition, q-quasi-monomiality characteristics, operational identities, and associated q-differential equations. Our analysis begins with the formulation of two-variable q-general polynomials, referred to as 2VqgP $p_{n,q}(\zeta,\eta)$.

Here, we consider the two-variable q-general polynomials (2VqgP) $p_{n,q}(\zeta,\eta)$ defined by the following generating function:

$$e_q(\zeta t)\phi_q(\eta,t) = \sum_{n=0}^{\infty} p_{n,q}(\zeta,\eta)\frac{t^n}{[n]_q!}, \quad (p_{0,q}(\zeta,\eta) = 1), \tag{40}$$

where $\phi_q(\eta,t)$ has (at least the formal) series expansion

$$\phi_q(\eta,t) = \sum_{n=0}^{\infty} \phi_{n,q}(\eta)\frac{t^n}{[n]_q!}, \quad (\phi_{0,q}(\eta) \neq 0). \tag{41}$$

With the simplification of the left-hand side of Equation (40) through the application of Equations (7) and (41), the following series definitions of the two-variable q-general polynomials $p_{n,q}(\zeta,\eta)$ are obtained as follows:

$$p_{n,q}(\zeta,\eta) = \sum_{k=0}^{n}\binom{n}{k}_q \zeta^k \phi_{n-k,q}(\eta). \tag{42}$$

We hereby establish the following result concerning the q-quasi-monomial identities of the two-variable q-general polynomials $2VqgP\ p_{n,q}(\zeta,\eta)$.

Theorem 1. *The two-variable q-general polynomials* $2VqgP\ p_{n,q}(\zeta,\eta)$ *demonstrate quasi-monomial characteristics when subjected to the aforementioned q-multiplicative and q-derivative operators as follows:*

$$\widehat{M}_{2VqgP} = \zeta + \frac{\phi_q^{'}(\eta,\widehat{D}_{q,\zeta})}{\phi_q(\eta,\widehat{D}_{q,\zeta})}T_{q,\zeta} \tag{43}$$

and

$$\widehat{P}_{2VqgP} = \widehat{D}_{q,\zeta}, \tag{44}$$

repectively, where $T_{q,\zeta}$ *denote the q-dilation operator given by Equation (23).*

Proof. Taking the q-derivative on both sides of Equation (40), partially with respect to t and by using Equation (15), we have

$$\sum_{n=1}^{\infty} p_{n,q}(\zeta,\eta)\widehat{D}_{q,t}\frac{t^n}{[n]_q!} = e_q(q\zeta t)\widehat{D}_{q,t}\phi_q(\eta,t) + \phi_q(\eta,t)\widehat{D}_{q,t}e_q(\zeta t), \tag{45}$$

which by using Equation (15) by taking $f_q(t) = \phi_q(\eta,t)$ and $g_q(t) = e_q(\zeta t)$, and then simplifying the resultant equation by using Equations (12), (26), and (28) on the right-hand side, we have

$$\left(\zeta + \frac{\phi_q^{'}(\eta,t)}{\phi_q(\eta,t)}T_{q,\zeta}\right)\{e_q(\zeta t)\phi_q(\eta,t)\} = \sum_{n=0}^{\infty} p_{n+1,q}(\zeta,\eta)\frac{t^n}{[n]_q!}. \tag{46}$$

Since

$$\widehat{D}_{q,\zeta}\{e_q(\zeta t)\phi_q(\eta,t)\} = te_q(\zeta t)\phi_q(\eta,t), \tag{47}$$

and $\frac{\phi_q^{'}(\eta,t)}{\phi_q(\eta,t)}$ has a q-power series expansion in t, as $\phi_q(\eta,t)$ is an invertible series of t. Therefore, by (40) and (46), we have

$$\left(\zeta + \frac{\phi_q^{'}(\eta,\widehat{D}_{q,\zeta})}{\phi_q(\eta,\widehat{D}_{q,\zeta})}T_{q,\zeta}\right)\left\{\sum_{n=0}^{\infty} p_{n,q}(\zeta,\eta)\frac{t^n}{[n]_q!}\right\} = \sum_{n=0}^{\infty} p_{n+1,q}(\zeta,\eta)\frac{t^n}{[n]_q!}. \tag{48}$$

Comparing the coefficients of like powers of t on both sides, gives

$$\left(\zeta + \frac{\phi_q^{'}(\eta,\widehat{D}_{q,\zeta})}{\phi_q(\eta,\widehat{D}_{q,\zeta})}T_{q,\zeta}\right)\{p_{n,q}(\zeta,\eta)\} = p_{n+1,q}(\zeta,\eta). \tag{49}$$

In accordance with the monomiality principle Equation (28), the aforementioned equation substantiates assertion (43) of Theorem 1. Furthermore, with the application of identity (13) to Equation (40), we have

$$\widehat{D}_{q,\zeta}\left\{\sum_{n=0}^{\infty} p_{n,q}(\zeta,\eta)\frac{t^n}{[n]_q!}\right\} = \sum_{n=1}^{\infty} p_{n-1,q}(\zeta,\eta)\frac{t^n}{[n-1]_q!}. \tag{50}$$

Upon comparing the coefficients of equivalent powers of t on both sides of Equation (50), we derive

$$\widehat{D}_{q,\zeta}\{p_{n,q}(\zeta,\eta)\} = [n]_q p_{n-1,q}(\zeta,\eta), \quad (n \geqslant 1). \tag{51}$$

Upon examining the monomiality principle Equation (29), we can conclude that assertion (44) of Theorem 1 is validated. □

Theorem 2. *The following q-differential equation for two-variable q-general polynomials* $2VqgP$ $p_{n,q}(\zeta,\eta)$ *holds:*

$$\left(\zeta\widehat{D}_{q,\zeta}+\frac{\phi_q^{'}(\eta,\widehat{D}_{q,\zeta})}{\phi_q(\eta,\widehat{D}_{q,\zeta})}T_{q,\zeta}\widehat{D}_{q,\zeta}-[n]_q\right)p_{n,q}(\zeta,\eta)=0. \tag{52}$$

Proof. In view of Equations (31), (43), and (44), we obtain the assertion (52) of Theorem 2. □

Remark 1. *Since* $p_{0,q}(\zeta,\eta)=1$, *in view of monomiality principle Equation (35), we have*

$$p_{n,q}(\zeta,\eta)=\left(\zeta+\frac{\phi_q^{'}(\eta,\widehat{D}_{q,\zeta})}{\phi_q(\eta,\widehat{D}_{q,\zeta})}T_{q,\zeta}\right)^n\{1\}\quad(p_{0,q}(\zeta,\eta)=1). \tag{53}$$

Furthermore, in view of Equations (35), (40) and (43), we have

$$e_q\left(\widehat{M}_{2VqgP}\right)\{1\}=e_q(\zeta t)\phi_q(\eta,t)=\sum_{n=0}^{\infty}p_{n,q}(\zeta,\eta)\frac{t^n}{[n]_q!}. \tag{54}$$

Subsequently, we introduce the two-variable q-general-Appell polynomials ($2VqgAP$), in order to derive the generating functions for the two-variable q-general-Appell polynomials by means of exponentially generating the function of q-Appell polynomials. Thus, replacing ζ on the left-hand side of (43) by the q-multiplicative operator of $2VqgP$ $p_{n,q}(\zeta,\eta)$, given by (40) and denoting the resultant of two-variable q-general-Appell polynomials $2VqgAP$ ${}_pA_{n,q}(\zeta,\eta)$, we obtain

$$A_q(t)e_q\left(\widehat{M}_{2VqgP}\right)\{1\}=\sum_{n=0}^{\infty}{}_pA_{n,q}(\zeta,\eta)\frac{t^n}{[n]_q!}, \tag{55}$$

which, by using Equation (43), we obtain the following two equivalent forms of ${}_pA_{n,q}(\zeta,\eta)$:

$$A_q(t)e_q\left(\zeta+\frac{\phi_q^{'}(\eta,\widehat{D}_{q,\zeta})}{\phi_q(\eta,\widehat{D}_{q,\zeta})}T_{q,\zeta}\right)\{1\}=\sum_{n=0}^{\infty}{}_pA_{n,q}(\zeta,\eta)\frac{t^n}{[n]_q!}. \tag{56}$$

The generating function for the two-variable q-general Appell polynomials ${}_pA_{n,q}(\zeta,\eta)$ can be formulated by applying the relation (54) to the left-hand side of Equation (55), resulting in the following representation:

$$A_q(t)e_q(\zeta t)\phi_q(\eta,t)=\sum_{n=0}^{\infty}{}_pA_{n,q}(\zeta,\eta)\frac{t^n}{[n]_q!}. \tag{57}$$

Theorem 3. *The two-variable q-general-Appell polynomials* ${}_pA_{n,q}(\zeta,\eta)$ *are defined by the following series:*

$${}_pA_{n,q}(\zeta,\eta)=\sum_{k=0}^{n}\binom{n}{k}_q A_{k,q}p_{n-k,q}(\zeta,\eta), \tag{58}$$

where $A_{k,q}$ *is given by Equation (27).*

Proof. In view of Equations (27) and (40), we can write

$$\sum_{n=0}^{\infty}{}_pA_{n,q}(\zeta,\eta)\frac{t^n}{[n]_q!}=A_q(t)\sum_{n=0}^{\infty}p_{n,q}(\zeta,\eta)\frac{t^n}{[n]_q!}. \tag{59}$$

By employing the expansion (27) of $A_q(t)$ on the left side of Equation (59), performing simplification, and subsequently comparing the coefficients of equivalent t powers on both sides of the resulting expression, we arrive at assertion (58). □

By using a similar approach given in [20,21], and in view of Equation (57), the following determinant form for ${}_pA_{n,q}(\zeta,\eta)$ is obtained.

Theorem 4. *The determinant representation of two-variable q-general Appell polynomials* ${}_pA_{n,q}(\zeta,\eta)$ *of degree n is*

$$ {}_pA_{0,q}(\zeta,\eta) = \frac{1}{\beta_{0,q}},$$

$$ {}_pA_{n,q}(\zeta,\eta) = \frac{(-1)^n}{(\beta_{0,q})^{n+1}} \begin{vmatrix} 1 & p_{1,q}(\zeta,\eta) & p_{2,q}(\zeta,\eta) & \dots & p_{n-1,q}(\zeta,\eta) & p_{n,q}(\zeta,\eta) \\ \beta_{0,q} & \beta_{1,q} & \beta_{2,q} & \dots & \beta_{n-1,q} & \beta_{n,q} \\ 0 & \beta_{0,q} & \binom{2}{1}_q\beta_{1,q} & \dots & \binom{n-1}{1}_q\beta_{n-2,q} & \binom{n}{1}_q\beta_{n-1,q} \\ 0 & 0 & \beta_{0,q} & \dots & \binom{n-1}{1}_q\beta_{n-3,q} & \binom{n}{2}_q\beta_{n-2,q} \\ \vdots & \vdots & \vdots & \dots & \vdots & \\ 0 & 0 & 0 & \dots & \beta_{0,q} & \binom{n}{n-1}_q\beta_{1,q} \end{vmatrix}, \tag{60}$$

$$ \beta_{n,q} = -\frac{1}{A_{0,q}}\left(\sum_{k=1}^{n}\binom{n}{k}_q A_{k,q}\beta_{n-k,q}\right), \quad n = 0,1,2,\dots,$$

where $\beta_{0,q} \neq 0$, $\beta_{0,q} = \frac{1}{A_{0,q}}$ *and* $p_{n,q}(\zeta,\eta)$, $n = 0,1,2,\dots$ *are the two-variable q-general polynomials defined by Equation (40).*

Theorem 5. *The two-variable q-general Appell polynomials* ${}_pA_{n,q}(\zeta,\eta)$ *are quasi-monomials under the following q-multiplicative and q-derivative operators:*

$$ \widehat{M}_{2VqgAP} = \left(\zeta + \frac{\phi_q'(\eta,\widehat{D}_{q,\zeta})}{\phi_q(\eta,\widehat{D}_{q,\zeta})}T_{q,\zeta}\right)\frac{A_q(q\widehat{D}_{q,\zeta})}{A_q(\widehat{D}_{q,\zeta})} + \frac{A_q'(\widehat{D}_{q,\zeta})}{A_q(\widehat{D}_{q,\zeta})} \tag{61}$$

and

$$ \widehat{P}_{2VqgAP} = \widehat{D}_{q,\zeta}, \tag{62}$$

respectively, where $T_{q,\zeta}$ *denote the q-dilatation operators given by Equation (23).*

Proof. Taking the q-derivative on both sides of Equation (57) with respect to t by using Equation (15), we obtain

$$\sum_{n=1}^{\infty} {}_pA_{n,q}(\zeta,\eta)\widehat{D}_{q,t}\frac{t^n}{[n]_q!}$$

$$= A_q(qt)\widehat{D}_{q,t}\left(e_q(\zeta t)\phi_q(\eta,t)\right) + e_q(\zeta t)\phi_q(\eta,t)A_q'(t), \tag{63}$$

which by using Equation (15) by taking $f_q(t) = e_q(\eta t)\phi_q(\eta,t)$ and $g(t) = A_q(t)$, and then simplifying the resultant equation by using Equations (14), (23) and (28) on the left-hand side, we have

$$\left(\left(\zeta + \frac{\phi_q'(\eta,t)}{\phi_q(\eta,t)}T_{q,\zeta}\right)\frac{A_q(qt)}{A_q(t)} + \frac{A_q'(t)}{A_q(t)}\right)A_q(t)e_q(\zeta t)\phi_q(\eta,t)$$

$$= \sum_{n=1}^{\infty} {}_pA_{n,q}(\zeta,\eta)\frac{t^{n-1}}{[n-1]_q!}. \tag{64}$$

Let $A_q(t)$ and $\phi_q(\eta,t)$ be invertible series of t, $\frac{A_q'(t)}{A_q(t)}$ and $\frac{\phi_q'(\eta,t)}{\phi_q(\eta,t)}$ have a q-power series expansion in t. Since

$$\widehat{D}_{q,\eta}A_q(t)e_q(\zeta t)\phi_q(\eta,t) = tA_q(t)e_q(\zeta t)\phi_q(\eta,t), \tag{65}$$

Equation (64) becomes

$$\left(\left(\zeta+\frac{\phi_q'(\eta,\widehat{D}_{q,\zeta})}{\phi_q(\eta,\widehat{D}_{q,\zeta})}T_{q,\zeta}\right)\frac{A_q(q\widehat{D}_{q,\eta})}{A_q(\widehat{D}_{q,\eta})}+\frac{A_q'(\widehat{D}_{q,\eta})}{A_q(\widehat{D}_{q,\eta})}\right)A_q(t)e_q(\zeta t)\phi_q(\eta,t)$$

$$=\sum_{n=1}^{\infty}{}_pA_{n,q}(\zeta,\eta)\frac{t^{n-1}}{[n-1]_q!}, \tag{66}$$

which, by using (57), gives

$$\left(\left(\zeta+\frac{\phi_q'(\eta,\widehat{D}_{q,\zeta})}{\phi_q(\eta,\widehat{D}_{q,\zeta})}T_{q,\zeta}\right)\frac{A_q(q\widehat{D}_{q,\eta})}{A_q(\widehat{D}_{q,\eta})}+\frac{A_q'(\widehat{D}_{q,\eta})}{A_q(\widehat{D}_{q,\eta})}\right)\sum_{n=0}^{\infty}{}_pA_{n,q}(\zeta,\eta)\frac{t^n}{[n]_q!}$$

$$=\sum_{n=1}^{\infty}{}_pA_{n,q}(\zeta,\eta)\frac{t^{n-1}}{[n-1]_q!}. \tag{67}$$

An examination of the coefficients of t on both sides of Equation (67), in conjunction with a consideration of Equation (28), leads to the derivation of assertion (61). Furthermore, upon applying identity (13) to Equation (57), we have

$$\widehat{D}_{q,\zeta}\left\{\sum_{n=0}^{\infty}{}_pA_{n,q}(\zeta,\eta)\frac{t^n}{[n]_q!}\right\}=\sum_{n=1}^{\infty}{}_pA_{n-1,q}(\zeta,\eta)\frac{t^n}{[n-1]_q!}. \tag{68}$$

Upon comparing the coefficients of equivalent powers of t on both sides of Equation (68), we derive

$$\widehat{D}_{q,\zeta}\{{}_pA_{n,q}(\zeta,\eta)\}=[n]_q{}_pA_{n-1,q}(\zeta,\eta),\quad(n\geqslant 1). \tag{69}$$

Upon examining the monomiality principle Equation (29), we can conclude that assertion (62) of Theorem 5 is validated. □

Theorem 6. *The following q-differential equation for* ${}_pA_{n,q}(\zeta,\eta)$ *holds true:*

$$\left(\zeta\frac{A_q(q\widehat{D}_{q,\eta})}{A_q(\widehat{D}_{q,\eta})}\widehat{D}_{q,\zeta}+\frac{\phi_q'(\eta,\widehat{D}_{q,\zeta})}{\phi_q(\eta,\widehat{D}_{q,\zeta})}T_{q,\zeta}\frac{A_q(q\widehat{D}_{q,\eta})}{A_q(\widehat{D}_{q,\eta})}\widehat{D}_{q,\zeta}+\frac{A_q'(\widehat{D}_{q,\eta})}{A_q(\widehat{D}_{q,\eta})}\widehat{D}_{q,\eta}-[n]_q\right){}_pA_{n,q}(\zeta,\eta)=0. \tag{70}$$

Proof. Using (61) and (62) in (31), we obtain the assertion (70). □

3. Applications

Specifically, when considering the case where $\phi_q(\eta,t)=e_q(\eta t^m)$ in generating function (57), which results in the reduction of 2VqgAP ${}_pA_{n,q}(\zeta,\eta)$ to the q-Gould–Hopper-based Appell polynomials (q-GHbAP) ${}_HA_{n,q}^{(m)}(\zeta,\eta)$ can be characterized by a specific generating function as follows:

$$A_q(t)e_q(\zeta t)e_q(\eta t^m)=\sum_{n=0}^{\infty}{}_HA_{n,q}^{(m)}(\zeta,\eta)\frac{t^n}{[n]_q!}. \tag{71}$$

In other words, we have

$${}_HA_{n,q}^{(m)}(\zeta,\eta)=e_q(\eta\widehat{D}_{q,\eta}^{(m)})\{A_{n,q}(\zeta)\}. \tag{72}$$

Through the application of Equations (20) and (26), we can expand the left-hand side of Equation (71) as follows

$$ {}_{H}A_{n,q}^{(m)}(\zeta,\eta) = \sum_{k=0}^{n} \binom{n}{k}_{q} A_{k,q} H_{n-k,q}^{(m)}(\zeta,\eta). \tag{73} $$

By using a similar approach given in [20,21] and in view of Equation (71), the following determinant form for ${}_{H}A_{n,q}^{(m)}(\zeta,\eta)$ is obtained.

Theorem 7. *The determinant representation of q-Gould–Hopper-based Appell polynomials* ${}_{H}A_{n,q}^{(m)}(\zeta,\eta)$ *of degree n is*

$$ {}_{H}A_{0,q}^{(m)}(\zeta,\eta) = \frac{1}{\beta_{0,q}}, $$

$$ {}_{H}A_{n,q}^{(m)}(\zeta,\eta) = \frac{(-1)^n}{(\beta_{0,q})^{n+1}} \begin{vmatrix} 1 & H_{1,q}^{(m)}(\zeta,\eta) & H_{2,q}^{(m)}(\zeta,\eta) & \dots & H_{n-1,q}^{(m)}(\zeta,\eta) & H_{n,q}^{(m)}(\zeta,\eta) \\ \beta_{0,q} & \beta_{1,q} & \beta_{2,q} & \dots & \beta_{n-1,q} & \beta_{n,q} \\ 0 & \beta_{0,q} & \binom{2}{1}_{q}\beta_{1,q} & \dots & \binom{n-1}{1}_{q}\beta_{n-2,q} & \binom{n}{1}_{q}\beta_{n-1,q} \\ 0 & 0 & \beta_{0,q} & \dots & \binom{n-1}{1}_{q}\beta_{n-3,q} & \binom{n}{2}_{q}\beta_{n-2,q} \\ \vdots & \vdots & \vdots & \dots & \vdots & \\ 0 & 0 & 0 & \dots & \beta_{0,q} & \binom{n}{n-1}_{q}\beta_{1,q} \end{vmatrix}, \tag{74} $$

$$ \beta_{n,q} = -\frac{1}{A_{0,q}} \left(\sum_{k=1}^{n} \binom{n}{k}_{q} A_{k,q}\beta_{n-k,q} \right), \quad n = 0,1,2,\dots, $$

where $\beta_{0,q} \neq 0$, $\beta_{0,q} = \frac{1}{A_{0,q}}$ *and* $H_{n,q}^{(m)}(\zeta,\eta)$, $n = 0,1,2,\dots$ *are the q-Gould–Hopper polynomials defined by Equation (20).*

We shall now demonstrate the q-multiplicative and q-derivative operators of ${}_{H}A_{n,q}^{(m)}(\zeta,\eta)$. The following theorem is presented:

Theorem 8. *The q-Gould–Hopper-based Appell polynomials* ${}_{H}A_{n,q}^{(m)}(\zeta,\eta)$ *are quasi-monomial under the following q-multiplicative and q-derivative operators:*

$$ \widehat{M}_{2VqGHAP} = \left(\zeta T_{\eta} + \eta \widehat{D}_{q,\zeta}^{m-1} T_{(\eta;m)} \right) \frac{A_q(q\widehat{D}_{q,\eta})}{A_q(\widehat{D}_{q,\eta})} + \frac{A_q'(\widehat{D}_{q,\eta})}{A_q(\widehat{D}_{q,\eta})}, \tag{75} $$

or, equivalently,

$$ \widehat{M}_{2VqGHAP} = \left(\zeta + \eta \widehat{D}_{q,\zeta}^{m-1} T_{(\eta;m)} T_{\zeta} \right) \frac{A_q(q\widehat{D}_{q,\eta})}{A_q(\widehat{D}_{q,\eta})} + \frac{A_q'(\widehat{D}_{q,\eta})}{A_q(\widehat{D}_{q,\eta})} \tag{76} $$

and

$$ \widehat{P}_{2VqGHAP} = \widehat{D}_{q,\zeta}, \tag{77} $$

respectively, where T_{ζ} *and* T_{η} *denote the q-dilatation operators given by Equation (23), and* $D_{q,\zeta}^{-1}$ *is defined by Equation (19).*

Proof. By applying the q-derivative to both sides of Equation (71) with respect to t, utilizing Equation (15), we obtain

$$ \sum_{n=1}^{\infty} {}_{H}A_{n,q}^{(m)}(\zeta,\eta)\widehat{D}_{q,t}\frac{t^n}{[n]_q!} $$

$$= A_q(qt)e_q\widehat{D}_{q,t}\left(e_q(\zeta t)e_q(\eta t^m)\right) + e_q(\zeta t)e_q(\eta t^m)A_q^{'}(t), \tag{78}$$

which, by using Equation (15) by taking $f(t) = e_q(\zeta t)e_q(\eta t^m)$ and $g(t) = A_q(t)$, and then simplifying the resultant equation by using Equations (14), (26), and (27) on the left-hand side, we have

$$\left(\left(\zeta T_\eta + \eta t^{m-1}T_{(\eta;m)}\right)\frac{A_q(qt)}{A_q(t)} + \frac{A_q^{'}(t)}{A_q(t)}\right)A_q(t)e_q(\zeta t)e_q(\eta t^m)$$

$$= \sum_{n=1}^{\infty} {}_HA_{n,q}^{(m)}(\zeta,\eta)\frac{t^{n-1}}{[n-1]_q!}. \tag{79}$$

Since

$$\widehat{D}_{q,\zeta}A_q(t)e_q(\zeta t)e_q(\eta t^m) = tA_q(t)e_q(\zeta t)e_q(\eta t^m) \tag{80}$$

and $\frac{A_q^{'}(t)}{A_q(t)}$ has a q-power series expansion in t, as $A_q(t)$ is an invertible series of t.

Therefore, by using Equation (71), we obtain

$$\left(\left(\eta T_\eta + \eta\widehat{D}_{q,\zeta}^{m-1}T_{(\eta;m)}\right)\frac{A_q(q\widehat{D}_{q,\eta})}{A_q(\widehat{D}_{q,\eta})} + \frac{A_q^{'}(\widehat{D}_{q,\eta})}{A_q(\widehat{D}_{q,\eta})}\right)$$

$$\times\sum_{n=0}^{\infty} {}_HA_{n,q}^{(m)}(\zeta,\eta)\frac{t^n}{[n]_q!} = \sum_{n=1}^{\infty} {}_HA_{n,q}^{(m)}(\zeta,\eta)\frac{t^{n-1}}{[n-1]_q!}. \tag{81}$$

Comparing the coefficients of t on both sides of Equation (81), and then, in view of Equation (28), the resultant equation gives assertion (75).

Employing Equation (78) with $f_q(t) = e_q(\eta t^m)$ and $g_q(t) = e_q(\zeta t)$, and applying the methodology utilized in the proof of assertion (75), we obtain assertion (76).

Upon examining Equations (29) and (80), we establish assertion (77). □

Theorem 9. *The following q-differential equation for q-GAbP* ${}_HA_{n,q}^{(m)}(\zeta,\eta)$ *holds true:*

$$\left(\zeta\widehat{D}_{q,\zeta}\frac{A_q(q\widehat{D}_{q,\zeta})}{A_q(\widehat{D}_{q,\zeta})}T_\eta + \eta\widehat{D}_{q,\zeta}^{m}T_{(\eta;m)}\frac{A_q(q\widehat{D}_{q,\zeta})}{A_q(\widehat{D}_{q,\zeta})} + \frac{A_q^{'}(\widehat{D}_{q,\eta})}{A_q(\widehat{D}_{q,\zeta})}\widehat{D}_{q,\zeta} - [n]_q\right){}_HA_{n,q}^{(m)}(\zeta,\eta) = 0, \tag{82}$$

or, equivalently

$$\left(\zeta\widehat{D}_{q,\zeta}\frac{A_q(q\widehat{D}_{q,\zeta})}{A_q(\widehat{D}_{q,\zeta})} + \eta\widehat{D}_{q,\zeta}^{m}T_{(\eta;m)}\frac{A_q(q\widehat{D}_{q,\zeta})}{A_q(\widehat{D}_{q,\zeta})}T_\zeta + \frac{A_q^{'}(\widehat{D}_{q,\eta})}{A_q(\widehat{D}_{q,\zeta})}\widehat{D}_{q,\zeta} - [n]_q\right){}_HA_{n,q}^{(m)}(\zeta,\eta) = 0. \tag{83}$$

Proof. Using (75), (76), and (77) in (31), we obtain the assertions (82) and (83). □

4. Examples

In this section, we consider certain members of the family of two-variable q-Gould–Hopper-based Appell polynomials (71).

Through the appropriate selection of the function $A_q(t)$ in Table 1 in Equation (71), it is possible to establish the following generating functions for the q-Gould–Hopper-based Bernoulli ${}_H\mathbb{B}_{n,q}^{(m)}(\zeta,\eta)$, Euler ${}_H\mathbb{E}_{n,q}^{(m)}(\zeta,\eta)$, and Genocchi ${}_H\mathbb{G}_{n,q}^{(m)}(\zeta,\eta)$ polynomials:

$$\frac{t}{e_q(t)-1}e_q(\zeta t)e_q(\eta t^m) = \sum_{n=0}^{\infty} {}_H\mathbb{B}_{n,q}^{(m)}(\zeta,\eta)\frac{t^n}{[n]_q!}, \tag{84}$$

$$\frac{2}{e_q(t)+1}e_q(\zeta t)e_q(\eta t^m) = \sum_{n=0}^{\infty} {}_H\mathbb{E}_{n,q}^{(m)}(\zeta,\eta)\frac{t^n}{[n]_q!} \tag{85}$$

and

$$\frac{2t}{e_q(t)+1}e_q(\zeta t)e_q(\eta t^m)=\sum_{n=0}^{\infty}{}_H\mathbb{G}_{n,q}^{(m)}(\zeta,\eta)\frac{t^n}{[n]_q!}. \tag{86}$$

Furthermore, in view of expression (73), the polynomials ${}_H\mathbb{B}_{n,q}^{(m)}(\zeta,\eta)$, ${}_H\mathbb{E}_{n,q}^{(m)}(\zeta,\eta)$, and ${}_H\mathbb{G}_{n,q}^{(m)}(\zeta,\eta)$ satisfy the following explicit form:

$${}_H\mathbb{B}_{n,q}^{(m)}(\zeta,\eta)=\sum_{k=0}^{n}\binom{n}{k}_q\mathbb{B}_{k,q}\,H_{n-k,q}^{(m)}(\zeta,\eta), \tag{87}$$

$${}_H\mathbb{E}_{n,q}^{(m)}(\zeta,\eta)=\sum_{k=0}^{n}\binom{n}{k}_q\mathbb{E}_{k,q}\,H_{n-k,q}^{(m)}(\zeta,\eta) \tag{88}$$

and

$${}_H\mathbb{G}_{n,q}^{(m)}(\zeta,\eta)=\sum_{k=0}^{n}\binom{n}{k}_q\mathbb{G}_{k,q}\,H_{n-k,q}^{(m)}(\zeta,\eta). \tag{89}$$

Furthermore, in view of expressions (74), the polynomials ${}_H\mathbb{B}_{n,q}^{(m)}(\zeta,\eta)$, ${}_H\mathbb{E}_{n,q}^{(m)}(\zeta,\eta)$ and ${}_H\mathbb{G}_{n,q}^{(m)}(\zeta,\eta)$ satisfy the following determinant representations:

$${}_H\mathbb{B}_{0,q}^{(m)}(\zeta,\eta)=1,$$

$${}_H\mathbb{B}_{n,q}^{(m)}(\zeta,\eta)=(-1)^n\begin{vmatrix} 1 & H_{1,q}^{(m)}(\zeta,\eta) & H_{2,q}^{(m)}(\zeta,\eta) & \cdots & H_{n-1,q}^{(m)}(\zeta,\eta,) & H_{n,q}^{(m)}(\zeta,\eta)\\ 1 & \frac{1}{[2]_q} & \frac{1}{[3]_q} & \cdots & \frac{1}{[n]_q} & \frac{1}{[n+1]_q}\\ 0 & 1 & \binom{2}{1}_q\frac{1}{[2]_q} & \cdots & \binom{n-1}{1}_q\frac{1}{[n-1]_q} & \binom{n}{1}_q\frac{1}{[n]_q}\\ 0 & 0 & 1 & \cdots & \binom{n-1}{2}_q\frac{1}{[n-2]_q} & \binom{n}{2}_q\frac{1}{[n-1]_q}\\ . & . & . & \cdots & . & .\\ . & . & . & \cdots & . & .\\ 0 & 0 & 0 & \cdots & 1 & \binom{n}{n-1}_q\frac{1}{[2]_q} \end{vmatrix}, \tag{90}$$

$${}_H\mathbb{E}_{0,q}^{(m)}(\zeta,\eta)=1,$$

$${}_H\mathbb{E}_{n,q}^{(m)}(\zeta,\eta)=(-1)^n\begin{vmatrix} 1 & H_{1,q}^{(m)}(\zeta,\eta) & H_{2,q}^{(m)}(\zeta,\eta) & \cdots & H_{n-1,q}^{(m)}(\zeta,\eta) & H_{n,q}^{(m)}(\zeta,\eta)\\ 1 & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2}\\ 0 & 1 & \binom{2}{1}_q\frac{1}{2} & \cdots & \binom{n-1}{1}_q\frac{1}{2} & \binom{n}{1}_q\frac{1}{2}\\ 0 & 0 & 1 & \cdots & \binom{n-1}{2}_q\frac{1}{2} & \binom{n}{2}_q\frac{1}{2}\\ . & . & . & \cdots & . & .\\ . & . & . & \cdots & . & .\\ 0 & 0 & 0 & \cdots & 1 & \binom{n}{n-1}_q\frac{1}{2} \end{vmatrix} \tag{91}$$

and

$${}_H\mathbb{G}_{0,q}^{(m)}(\zeta,\eta)=1,$$

$$
{}_H\mathbb{G}_{n,q}^{(m)}(\zeta,\eta) = (-1)^n \begin{vmatrix} 1 & H_{1,q}^{(m)}(\zeta,\eta) & H_{2,q}^{(m)}(\zeta,\eta) & \cdots & H_{n-1,q}^{(m)}(\zeta,\eta) & H_{n,q}^{(m)}(\zeta,\eta) \\ 1 & \frac{1}{2[2]_q} & \frac{1}{2[3]_q} & \cdots & \frac{1}{2[n]_q} & \frac{1}{2[n+1]_q} \\ 0 & 1 & \binom{2}{1}_q \frac{1}{2[2]_q} & \cdots & \binom{n-1}{1}_q \frac{1}{2[n-1]_q} & \binom{n}{1}_q \frac{1}{2[n]_q} \\ 0 & 0 & 1 & \cdots & \binom{n-1}{2}_q \frac{1}{2[n-2]_q} & \binom{n}{2}_q \frac{1}{2[n-1]_q} \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & 1 & \binom{n}{n-1}_q \frac{1}{2[2]_q} \end{vmatrix}, \tag{92}
$$

where $H_{n,q}^{(m)}(\zeta,\eta), \quad (n = 0, 1, 2\cdots)$ are the q-Gould–Hopper polynomials of degree n.

5. Distribution of Zeros and Graphical Representation

In this section, we aim to provide the graphical representations and zeros for the q-Gould–Hopper-based Bernoulli ${}_H\mathbb{B}_{n,q}^{(m)}(\zeta,\eta)$ and q-Gould–Hopper-based Euler ${}_H\mathbb{E}_{n,q}^{(m)}(\zeta,\eta)$ polynomials. By appropriately choosing the function $A_q(t)$ from Table 1 in Equation (57), we can establish the following generating functions for the q-Gould–Hopper-based Bernoulli ${}_H\mathbb{B}_{n,q}^{(m)}(\zeta,\eta)$ polynomial:

$$
\frac{t}{e_q(t)-1} e_q(\zeta t) e_q(\eta t^m) = \sum_{n=0}^{\infty} {}_H\mathbb{B}_{n,q}^{(m)}(\zeta,\eta) \frac{t^n}{[n]_q!}. \tag{93}
$$

A few of them are

$$
\begin{aligned}
&{}_H\mathbb{B}_{0,q}^{(3)}(\zeta,\eta)) = 1,\\
&{}_H\mathbb{B}_{1,q}^{(3)}(\zeta,\eta) = \zeta - \frac{1}{[2]_q!},\\
&{}_H\mathbb{B}_{2,q}^{(3)}(\zeta,\eta) = -\zeta + \zeta^2 + \frac{1}{[2]_q!} - \frac{[2]_q!}{[3]_q!},\\
&{}_H\mathbb{B}_{3,q}^{(3)}(\zeta,\eta) = -\zeta + \zeta^3 + \frac{2}{[2]_q!} + \eta[3]_q! - \frac{[3]_q!}{[2]_q^3} + \frac{\zeta[3]_q!}{[2]_q!^2} - \frac{\zeta^2[3]_q!}{[2]_q!^2} - \frac{[3]_q!}{[4]_q!},\\
&{}_H\mathbb{B}_{4,q}^{(3)}(\zeta,\eta) = -\zeta + \zeta^4 + \frac{2}{[2]_q!} + \zeta\eta[4]_q! + \frac{[4]_q!}{[2]_q!^4} - \frac{\zeta[4]_q!}{[2]_q!^3} + \frac{\zeta^2[4]_q!}{[2]_q!^3}\\
&\qquad - \frac{\eta[4]_q!}{[2]_q!} + \frac{[4]_q!}{[3]_q!^2} - \frac{3[4]_q!}{[2]_q!^2[3]_q!} + \frac{2\zeta[4]_q!}{[2]_q![3]_q!} - \frac{\zeta^2[4]_q!}{[2]_q![3]_q!} - \frac{\zeta^3[4]_q!}{[2]_q![3]_q!} - \frac{[4]_q!}{[5]_q!},\\
&{}_H\mathbb{B}_{5,q}^{(3)}(\zeta,\eta) = -\zeta + \zeta^5 + \frac{2}{[2]_q!} - \frac{[5]_q!}{[2]_q!^5} + \frac{\zeta[5]_q!}{[2]_q!^4} - \frac{\zeta^2[5]_q!}{[2]_q!^4} + \frac{\eta[5]_q!}{[2]_q!^2} - \frac{\zeta\eta[5]_q!}{[2]_q!}\\
&\qquad + \frac{\zeta^2\eta[5]_q!}{[2]_q!} + \frac{\zeta[5]_q!}{[3]_q!^2} - \frac{\zeta^3[5]_q!}{[3]_q!^2} - \frac{3[5]_q!}{[2]_q![3]_q!^2} - \frac{\eta[5]_q!}{[3]_q!} + \frac{4[5]_q!}{[2]_q!^3[3]_q!}\\
&\qquad - \frac{3\zeta[5]_q!}{[2]_q!^2[3]_q!} + \frac{2\zeta^2[5]_q!}{[2]_q!^2[3]_q!} + \frac{\zeta^3[5]_q!}{[2]_q!^2[3]_q!} - \frac{3[5]_q!}{[2]_q!^2[4,]_q!} + \frac{2\zeta[5]_q!}{[2]_q![4]_q!}\\
&\qquad - \frac{\zeta^2[5]_!}{[2]_q![4]_q!} - \frac{\zeta^4[5]_q!}{[2]_q![4]_q!} + \frac{2[5]_q!}{[3]_q![4]_q!} - \frac{[5]_q!}{[6]_q!}.
\end{aligned}
$$

We investigate the beautiful zeros of the q-Gould–Hopper-based Bernoulli ${}_H\mathbb{B}_{n,q}^{(m)}(\zeta,\eta) = 0$ by using a computer. We plot the zeros of q-Gould–Hopper-based Bernoulli ${}_H\mathbb{B}_{n,q}^{(m)}(\zeta,\eta) = 0$ for $n = 24$ (Figure 1).

In Figure 1 (top-left), we choose $m = 3, \eta = 6$, and $q = \frac{3}{10}$. In Figure 1 (top-right), we choose $m = 3, \eta = 6$, and $q = \frac{5}{10}$. In Figure 1 (bottom-left), we choose $m = 3, \eta = 6$, and $q = \frac{7}{10}$. In Figure 1 (bottom-right), we choose $m = 3, \eta = 6$, and $q = \frac{9}{10}$.

Stacks of zeros of the q-Gould–Hopper-based Bernoulli ${}_H\mathbb{B}_{n,q}^{(m)}(\zeta, \eta) = 0$ for $1 \leq n \leq 24$, forming a 3D structure, are presented (Figure 2).

In Figure 2 (top-left), we choose $m = 3, \eta = 6$, and $q = \frac{3}{10}$. In Figure 2 (top-right), we choose $m = 3, \eta = 6$, and $q = \frac{5}{10}$. In Figure 2 (bottom-left), we choose $m = 3, \eta = 6$, and $q = \frac{7}{10}$. In Figure 2 (bottom-right), we choose $m = 3, \eta = 6$, and $q = \frac{9}{10}$.

Plots of the real zeros of the q-Gould–Hopper-based Bernoulli ${}_H\mathbb{B}_{n,q}^{(m)}(\zeta, \eta) = 0$ for $1 \leq n \leq 24$ are presented (Figure 3).

In Figure 3 (top-left), we choose $m = 3, \eta = 6$, and $q = \frac{3}{10}$. In Figure 3 (top-right), we choose $m = 3, \eta = 6$, and $q = \frac{5}{10}$. In Figure 3 (bottom-left), we choose $m = 3, \eta = 6$, and $q = \frac{7}{10}$. In Figure 3 (bottom-right), we choose $m = 3, \eta = 6$, and $q = \frac{9}{10}$.

Next, we calculate an approximate solution satisfying the q-Gould–Hopper-based Bernoulli ${}_H\mathbb{B}_{n,q}^{(m)}(\zeta, \eta) = 0$ for $m = 3, \eta = 6$, and $q = \frac{9}{10}$. The results are given in Table 3.

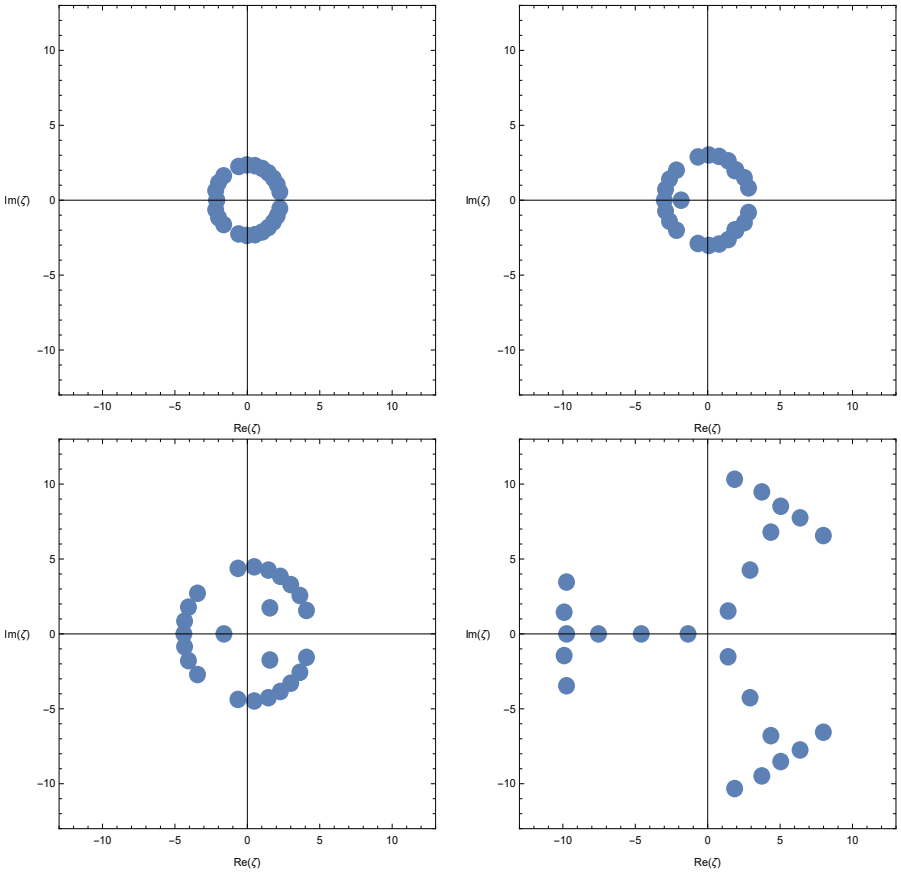

Figure 1. Zeros of ${}_H\mathbb{B}_{n,q}^{(m)}(\zeta, \eta) = 0$.

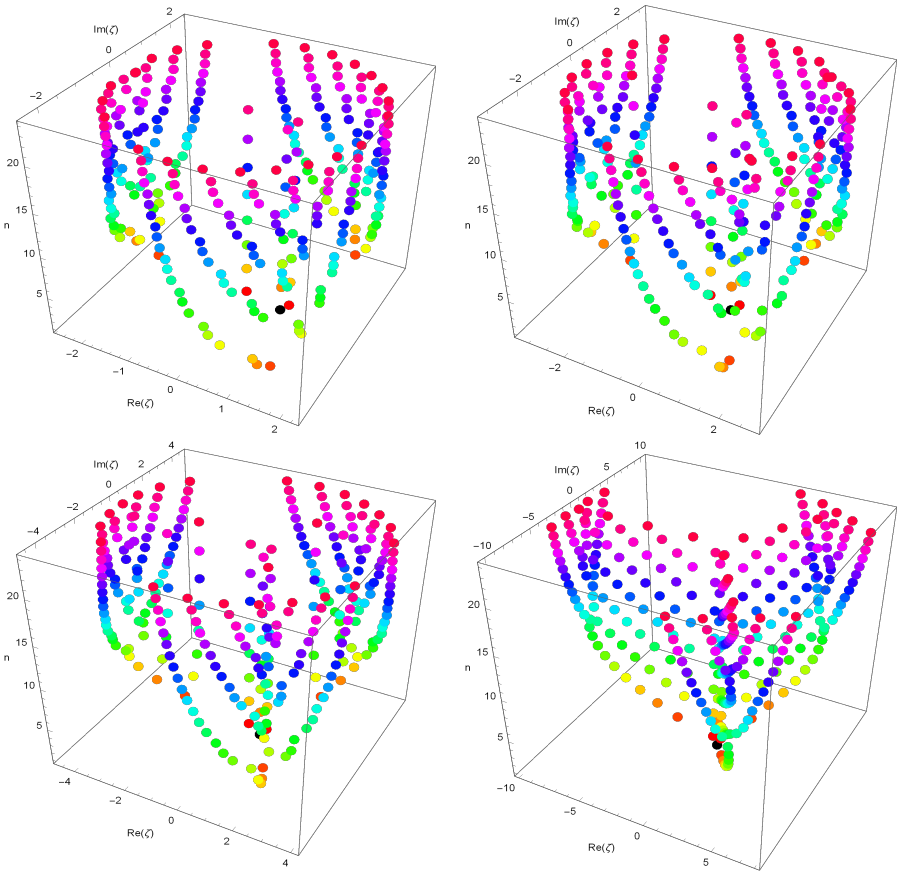

Figure 2. Zeros of ${}_H\mathbb{B}_{n,q}^{(m)}(\zeta,\eta)=0$.

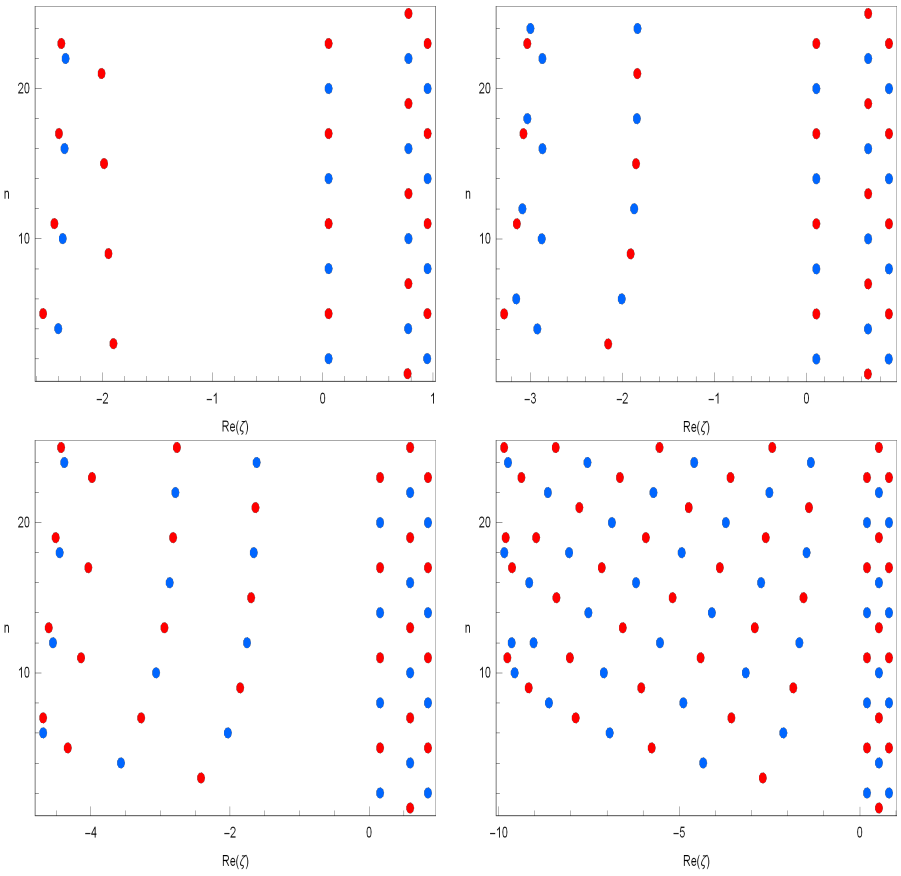

Figure 3. Real zeros of ${}_H\mathbb{B}_{n,q}^{(m)}(\zeta,\eta)=0$.

Table 3. Approximate solutions of ${}_H\mathbb{B}_{n,q}^{(m)}(\zeta,\eta)=0$.

Degree n	ζ
1	0.52632
2	0.19555, 0.80445
3	-2.6891, $2.0577-2.6942i$, $2.0577+2.6942i$
4	-4.3398, 0.52609, $2.8119-4.0737i$, $2.8119+4.0737i$
5	-5.7645, 0.19543, 0.80436, $3.4600-5.2700i$, $3.4600+5.2700i$
6	-6.9272, -2.1200, $1.7821-2.2002i$, $1.7821+2.2002i$, $3.9746-6.2386i$, $3.9746+6.2386i$
7	-7.8678, -3.5588, 0.52589, $2.4441-3.4039i$, $2.4441+3.4039i$, $4.3791-7.0171i$, $4.3791+7.0171i$
8	-8.6091, -4.8888, 0.19532, 0.80428, $3.0642-4.5295i$, $3.0642+4.5295i$, $4.6837-7.6242i$, $4.6837+7.6242i$
9	-9.1693, -6.0540, -1.8456, $1.6480-1.9596i$, $1.6480+1.9596i$, $3.6030-5.5115i$, $3.6030+5.5115i$, $4.8955-8.0749i$, $4.8955+8.0749i$
10	-9.5569, -7.0891, -3.1609, 0.52570, $2.2537-3.0594i$, $2.2537+3.0594i$, $4.0884-6.3858i$, $4.0884+6.3858i$, $5.0125-8.3748i$, $5.0125+8.3748i$

Similarly, by appropriately choosing the function $A_q(t)$ from Table 1 in Equation (57), we can obtain the following generating functions for the q-Gould–Hopper-based Euler ${}_H\mathbb{E}_{n,q}^{(m)}(\zeta,\eta)$ polynomial:

$$\frac{2}{e_q(t)+1}e_q(\zeta t)e_q(\eta t^m)=\sum_{n=0}^{\infty}{}_H\mathbb{E}_{n,q}^{(m)}(\zeta,\eta)\frac{t^n}{[n]_q!}. \tag{94}$$

A few of them are

$$\begin{aligned}
{}_H\mathbb{E}_{0,q}^{(2)}(\zeta,\eta))&=1,\\
{}_H\mathbb{E}_{1,q}^{(2)}(\zeta,\eta)&=-\frac{1}{2}+\zeta,\\
{}_H\mathbb{E}_{2,q}^{(2)}(\zeta,\eta)&=-\frac{1}{2}+\zeta^2+\frac{1}{4}[2]_q!-\frac{1}{2}\zeta[2]_q!+\eta[2]_q!,\\
{}_H\mathbb{E}_{3,q}^{(2)}(\zeta,\eta)&=-\frac{1}{2}+\zeta^3-\frac{1}{8}[3]_q!+\frac{1}{4}\zeta[3]_q!-\frac{1}{2}\eta[3]_q!+\zeta\eta[3]_q!\\
&\quad+\frac{[3]_q!}{2[2]_q!}-\frac{\zeta[3]_q!}{2[2]_q!}-\frac{\zeta^2[3]_q!}{2[2]_q!},\\
{}_H\mathbb{E}_{4,q}^{(2)}(\zeta,\eta)&=-\frac{1}{2}+\zeta^4+\frac{1}{16}[4]_q!-\frac{1}{8}\zeta[4]_q!+\frac{1}{4}\eta[4]_q!-\frac{1}{2}\zeta\eta[4]_q!+\frac{[4]_q!}{4[2]_q!^2}\\
&\quad-\frac{\zeta^2[4]_q!}{2[2]_q!^2}-\frac{3[4]_q!}{8[2]_q!}+\frac{\zeta[4]_q!}{2[2]_q!}+\frac{\zeta^2[4]_q!}{4[2]_q!}-\frac{\eta[4]_q!}{2[2]_q!}+\frac{\zeta^2\eta[4]_q!}{[2]_q!}\\
&\quad+\frac{\eta^2[4]_q!}{[2]_q!}+\frac{[4]_q!}{2[3]_q!}-\frac{\zeta[4]_q!}{2[3]_q!}-\frac{\zeta^3[4]_q!}{2[3]_q!},\\
{}_H\mathbb{E}_{5,q}^{(2)}(\zeta,\eta)&=-\frac{1}{2}+\zeta^5-\frac{1}{32}[5]_q!+\frac{1}{16}\zeta[5]_q!-\frac{1}{8}\eta[5]_q!+\frac{1}{4}\zeta\eta[5]_q!\\
&\quad-\frac{3[5]_q!}{8[2]_q!^2}+\frac{\zeta[5]_q!}{4[2]_q!^2}+\frac{\zeta^2[5]_q!}{2[2]_q!^2}+\frac{[5]_q!}{4[2]_q!}-\frac{3\zeta[5]_q!}{8[2]_q!}-\frac{\zeta^2[5]_q!}{8[2]_q!}\\
&\quad+\frac{\eta[5]_q!}{2[2]_q!}-\frac{\zeta\eta[5]_q!}{2[2]_q!}-\frac{\zeta^2\eta[5]_q!}{2[2]_q!}-\frac{\eta^2[5]_q!}{2[2]_q!}+\frac{\zeta\eta^2[5]_q!}{[2]_q!}\\
&\quad-\frac{3[5]_q!}{8[3]_q!}+\frac{\zeta[5]_q!}{2[3]_q!}+\frac{\zeta^3[5]_q!}{4[3]_q!}-\frac{\eta[5]_q!}{2[3]_q!}+\frac{\zeta^3\eta[5]_q!}{[3]_q!}+\frac{[5]_q!}{2[2]_q![3]_q!}\\
&\quad-\frac{\zeta^2[5]_q!}{2[2]_q![3]_q!}-\frac{\zeta^3[5]_q!}{2[2]_q![3]_q!}+\frac{[5]_q!}{2[4]_q!}-\frac{\zeta[5]_q!}{2[4]_q!}-\frac{\zeta^4[5]_q!}{2[4]_q!}.
\end{aligned}$$

We investigate the beautiful zeros of the q-Gould–Hopper-based Euler ${}_H\mathbb{E}_{n,q}^{(m)}(\zeta,\eta)=0$ by using a computer. We plot the zeros of q-Gould–Hopper-based Euler ${}_H\mathbb{E}_{n,q}^{(m)}(\zeta,\eta)=0$ for $n=24$ (Figure 4).

In Figure 4 (top-left), we choose $m=2,\eta=6$, and $q=\frac{3}{10}$. In Figure 4 (top-right), we choose $m=2,\eta=6$, and $q=\frac{5}{10}$. In Figure 4 (bottom-left), we choose $m=2,\eta=6$, and $q=\frac{7}{10}$. In Figure 4 (bottom-right), we choose $m=2,\eta=6$, and $q=\frac{9}{10}$.

The stacks of zeros of the q-Gould–Hopper-based Euler ${}_H\mathbb{E}_{n,q}^{(m)}(\zeta,\eta)=0$ for $1\leq n\leq 24$, forming a 3D structure, are presented (Figure 5).

In Figure 5 (top-left), we choose $m=2,\eta=6$, and $q=\frac{3}{10}$. In Figure 5 (top-right), we choose $m=2,\eta=6$, and $q=\frac{5}{10}$. In Figure 5 (bottom-left), we choose $m=2,\eta=6$, and $q=\frac{7}{10}$. In Figure 5 (bottom-right), we choose $m=2,\eta=6$, and $q=\frac{9}{10}$.

Plots of the real zeros of the q-Gould–Hopper-based Euler ${}_H\mathbb{E}_{n,q}^{(m)}(\zeta,\eta)=0$ for $1\leq n\leq 24$ are presented (Figure 6).

In Figure 6 (top-left), we choose $m=2,\eta=6$, and $q=\frac{3}{10}$. In Figure 6 (top-right), we choose $m=2,\eta=6$, and $q=\frac{5}{10}$. In Figure 6 (bottom-left), we choose $m=2,\eta=6$, and $q=\frac{7}{10}$. In Figure 6 (bottom-right), we choose $m=2,\eta=6$, and $q=\frac{9}{10}$.

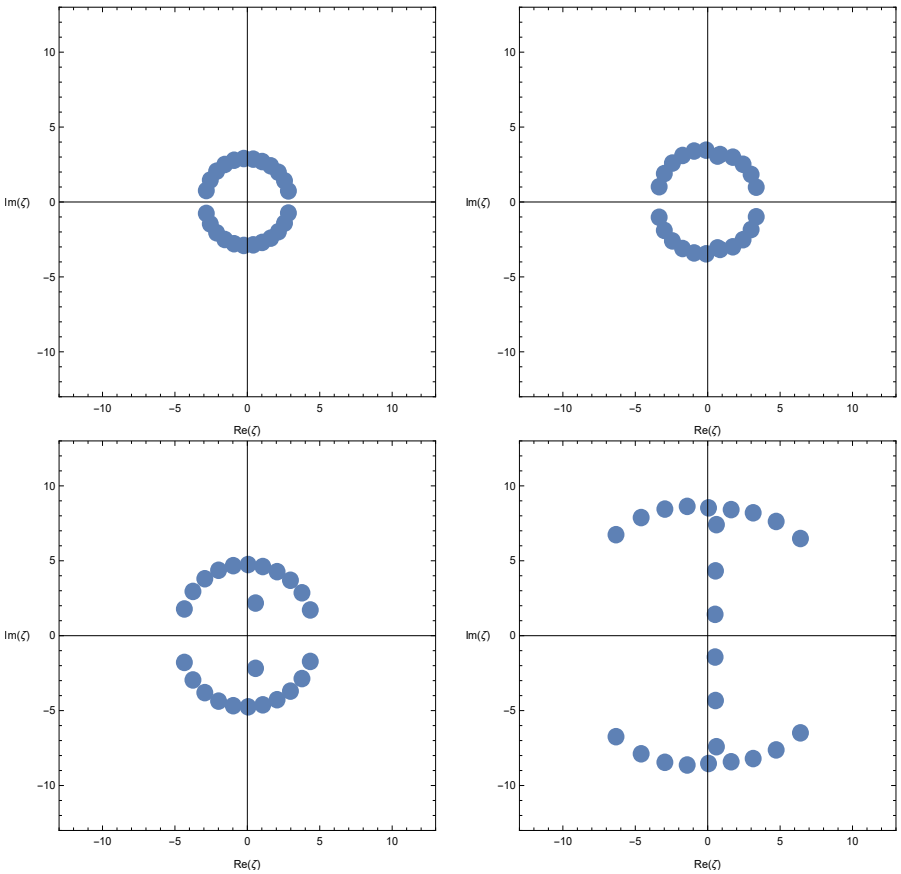

Figure 4. Zeros of ${}_H\mathbb{E}_{n,q}^{(m)}(\zeta,\eta)=0$.

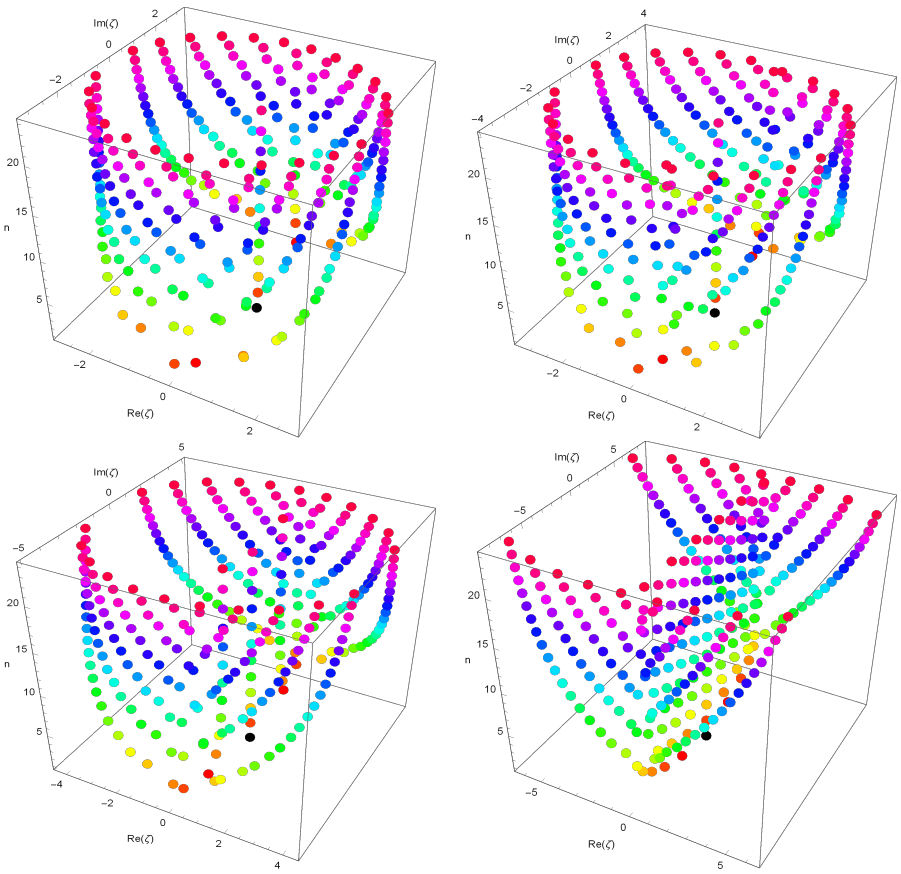

Figure 5. Zeros of ${}_H\mathbb{E}_{n,q}^{(m)}(\zeta,\eta)=0$.

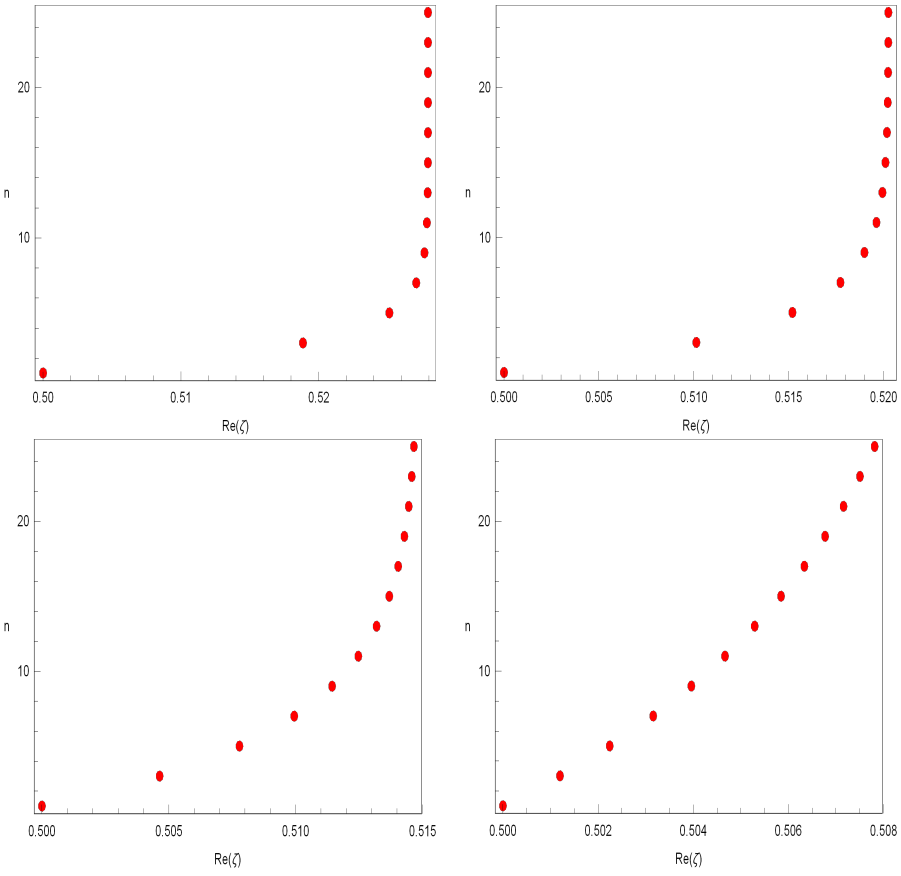

Figure 6. Real zeros of ${}_H\mathbb{E}_{n,q}^{(m)}(\zeta,\eta)=0$.

Next, we calculated an approximate solution satisfying the q-Gould–Hopper-based Euler ${}_H\mathbb{E}_{n,q}^{(m)}(\zeta,\eta)=0$ for $m=2,\eta=6$, and $q=\frac{9}{10}$. The results are given in Table 4.

Table 4. Approximate solutions of ${}_H\mathbb{E}_{n,q}^{(m)}(\zeta,\eta)=0$.

Degree n	ζ
1	0.50000
2	$0.4750-3.3391i,\quad 0.4750+3.3391i$
3	$0.4269-5.4969i,\quad 0.4269+5.4969i,\quad 0.50120$
4	$0.3686-6.9276i,\quad 0.3686+6.9276i,\quad 0.4912-2.5888i,\quad 0.4912+2.5888i$
5	$0.2992-7.8166i,\quad 0.2992+7.8166i,\quad 0.4734-4.6431i,\quad 0.4734+4.6431i,\quad 0.50224$
6	$0.1944-8.2120i,\quad 0.1944+8.2120i,\quad 0.4795-6.3458i,\quad 0.4795+6.3458i,\quad 0.4975-2.2225i,\quad 0.4975+2.2225i$
7	$-0.3810-8.1652i,\quad -0.3810+8.1652i,\quad 0.4901-4.1360i,\quad 0.4901+4.1360i,\quad 0.50316,$ $0.9436-7.7827i,\quad 0.9436+7.7827i$
8	$-0.9757-8.4614i,\quad -0.9757+8.4614i,\quad 0.4911-5.8222i,\quad 0.4911+5.8222i,\quad 0.5012-1.9994i,$ $0.5012+1.9994i,\quad 1.4072-8.2319i,\quad 1.4072+8.2319i$
9	$-1.4546-8.5648i,\quad -1.4546+8.5648i,\quad 0.4537-7.3398i,\quad 0.4537+7.3398i,\quad 0.4993-3.7990i,$ $0.4993+3.7990i,\quad 0.50396,\quad 1.7811-8.2699i,\quad 1.7811+8.2699i$
10	$-1.9729-8.6021i,\quad -1.9729+8.6021i,\quad 0.2771-8.2181i,\quad 0.2771+8.2181i,\quad 0.5036-1.8475i,$ $0.5036+1.8475i,\quad 0.5055-5.4479i,\quad 0.5055+5.4479i,\quad 2.3151-8.2932i,\quad 2.3151+8.2932i$

6. Conclusions

In this paper, we have presented precise formulas and illuminated the fundamental characteristics of these polynomials, enhancing our comprehension of two-variable q-general-Appell polynomials and their relationships with established polynomial categories. Such advancements enrich the mathematical landscape and pave the way for novel research. The wide-ranging potential applications of these polynomials span quantum mechanics, mathematical physics, statistical mechanics, information theory, and computational science, warranting extensive investigation. Subsequent research may probe the structural properties and algebraic facets of these polynomials, potentially revealing profound insights and practical applications. Moreover, interdisciplinary collaboration can amplify the real-world impact of these polynomials across diverse fields. In summary, the introduction and examination of q-hybrid polynomials constitute a pivotal development in mathematics and science, catalyzing new research trajectories and applications across various disciplines.

Author Contributions: Conceptualization, W.A.K. and C.K.; Formal analysis, W.A.K. and C.K.; Funding acquisition, N.A. and W.A.K.; Investigation, W.A.K.; Methodology, W.A.K.; Project administration, N.A., W.A.K., C.K. and C.S.R.; Software, C.S.R.; Writing—original draft, W.A.K. and C.K.; Writing—review and editing, W.A.K., C.K. and C.S.R. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: No data were used to support this study.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Khan, S.; Raza, N. General Appell polynomials within the context of monomiality principle. *Int. J. Anal.* **2013**, *2013*, 328032. [CrossRef]
2. Riyasat, M.; Khan, S.; Haneef, M. Generalized 2D extension of q-Bessel polynomials. *Rep. Math. Phys.* **2024**, *90*, 63–80. [CrossRef]

3. Andrews, G.E.; Askey, R.; Roy, R. *Special Functions*; Cambridge University Press: Cambridge, UK, 1999.
4. Zayed, M.; Wani, S.A.; Ramírez, W.; Cesarano, C. Advancements in q-Hermite-Appell polynomials: A three-dimensional exploration. *AIMS Math.* **2024**, *9*, 26799–26824. [CrossRef]
5. Riyasat, M.; Khan, S. Some results on q-Hermite based hybrid polynomials. *Glas. Mat.* **2018**, *53*, 9–31. [CrossRef]
6. Florenini, R.; Vinet, L. Quantum algebras and q-special functions. *Ann. Phys.* **1993**, *221*, 53–70. [CrossRef]
7. Fadel, M.; Alatawi, M.S.; Khan, W.A. Two variable q-Hermite-based Appell polynomials and their applications. *Mathematics* **2024**, *12*, 1358. [CrossRef]
8. Wani, S.A.; Riyasat, M.; Khan, S.; Ramirez, W. Certain advancements in multidimensional q-Hermite polynomials. *Rep. Math. Phys.* **2024**, *94*, 117–141. [CrossRef]
9. Alatawi, M.S.; Khan, W.A.; Ryoo, C.S. Explicit properties of q-Cosine and q-Sine Array-type polynomials containing symmetric structures. *Symmetry* **2022**, *14*, 1675. [CrossRef]
10. Cao, J.; Raza, N.; Fadel, M. Two-variable q-Laguerre polynomials from the context of quasi-monomiality. *J. Anal. Appl.* **2024**, *535*, 128126. [CrossRef]
11. Raza, N.; Fadel, M.; Nisar, K.S.; Zakarya, M. On 2-variable q-Hermite polynomials. *AIMS Math.* **2021**, *6*, 8705–8727. [CrossRef]
12. Srivastava, H.M. Some characterization of Appell and q-Appell polynomials. *Ann. Mat. Pura Appl.* **1982**, *4*, 321–329. [CrossRef]
13. Askey, R. Limits of some q-Laguerre polynomials. *J. Approx. Theory* **1986**, *46*, 213–216. [CrossRef]
14. Dattoli, G. Hermite-Bessel and Laguerre-Bessel functions: A by-product of the monomiality principle. *Adv. Spec. Funct. Appl.* **2000**, *1*, 147–164
15. Dattoli, G.; Torre, A. Exponential operators, quasi-monomials and generalized polynomials. *Radiat. Phys. Chem.* **2000**, *57*, 21–26. [CrossRef]
16. Ernst, T. q-Bernoulli and q-Euler polynomials, an umbral calculus approach. *Int. J. Differ. Equ.* **2006**, *1*, 31–80.
17. Al-Salam, W.A. q-Bernoulli numbers and polynomials. *Math. Nachr.* **1958**, *17*, 239–260. [CrossRef]
18. Al-Salam, W.A. q-Appell polynomials. *Ann. Mat. Pura Appl.* **1967**, *77*, 31–45. [CrossRef]
19. Ernst, T. *A Comprehensive Treatment of q-Calculus;* Springer: Basel, Switzerland, 2012.
20. Costabile, F.A.; Gualtieri, M.T.; Napoli, A. General bivariate Appell polynomials via matrix calculus and related interpolation hints. *Mathematics* **2022**, *9*, 964. [CrossRef]
21. Costabile, F.A.; Gualtieri, M.T.; Napoli, A. Bivariate general Appell interpolation problem. *Numer. Algorithms* **2022**, *91*, 531–556. [CrossRef]

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.