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A Resolution Under Interval Uncertainty

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Abstract: Traditional transferable utility (TU) games assume precise real-valued utilities for coalition outcomes, but real-world situations often involve uncertainty or imprecision. Interval TU games extend the classical framework by representing utilities and payoffs as closed intervals, leveraging interval arithmetic to address inherent ambiguities in data. This paper reviews the theoretical foundations of interval TU games and explores allocating solutions under uncertainty. Central to this study is the adaptation of consistency, a fundamental property in game-theoretical resolutions, to the interval framework. Drawing on concepts such as the pseudo equal allocations of non-separable costs and the pseudo weighted allocations of non-separable costs, we characterize these allocation resolutions through a specific reduction and related consistency. By bridging classical TU games with interval generalizations, this study offers a robust foundation for analyzing allocations under uncertainty and outlines avenues for future research in theoretical and applied game theory.

Keywords: interval TU games; resolution; consistency

MSC: 91A12; 91A40; 91B06; 91B16

1. Introduction

In classical game-theoretical analysis, the framework due to transferable utility (TU) games serves as a cornerstone for modeling-related situations where utility can be freely distributed among players within a coalition. Traditionally, these games assume that the utilities produced by coalitions or the payoffs assigned to individual players are precise real numbers. However, real-world situations often involve uncertainties arising from incomplete information, measurement errors, or inherent variability in player contributions. As a result, the conventional representation of utility as a single value may fail to capture the complexities of practical decision-making situations. To address these limitations, the framework due to interval TU games has emerged as a generalization of the classical TU framework. Introduced by Branzei et al. [1,2], interval TU games represent coalition utilities and player payoffs as closed intervals rather than exact values. This approach accommodates the inherent imprecision of real-world data while preserving the mathematical rigor of the TU framework. The foundational techniques of interval arithmetic, as explored by Moore [3], provide the tools necessary to handle interval uncertainty in this context. Interval TU games have since been extended to various application domains, including resource allocation, political negotiation, and economic planning, where uncertainty plays a significant role. Related studies can be found in Alparslan Gök et al. [4–6].

Interval TU games also build upon core principles of game theory, such as stability and fairness, but adapt these concepts to the interval setting. For instance, related interval-

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based Shapley values [7] and cores, as proposed by Alparslan Gök et al. [8], provide novel resolutions that reflect the range of possible outcomes under uncertainty. Recent developments in interval TU games have further strengthened the theoretical foundation and broadened their applications. Alparslan Gök et al. [9] extended the classical TU-game solutions by incorporating Moore interval subtraction to establish more refined interval-based allocation mechanisms. Their results highlight the significance of interval arithmetic in ensuring well-defined solutions for cooperative games under uncertainty. Guerrero and Olvera-Lopez [10] provided an axiomatic characterization of the interval Myerson value, reinforcing the connection between interval TU games and network-based cooperative models. Additionally, Alparslan Gök et al. [9] explored the impact of uncertainty on resource allocation problems, demonstrating the applicability of interval TU games in environmental and economic decision-making. In dynamic settings, Özcan and Alparslan Gök [11] examined differential TU games with continuous time, extending interval-based approaches to capture evolving cooperative interactions. Despite the advances in interval TU games, existing studies often lack a systematic exploration of how these interval-based solutions compare with classical TU games in terms of consistency, efficiency, and fairness. Additionally, recent developments in game theory have introduced new solution concepts for handling uncertainty, including fuzzy games and stochastic cooperative models [12,13]. However, a direct axiomatic characterization bridging interval TU games with classical TU resolutions remains limited. This study aims to fill this gap by introducing novel interval-based allocation principles that extend key classical concepts while ensuring theoretical rigor and practical applicability.

A key property under axiomatic processes for game-theoretical resolutions is *consistency*, a concept originally formalized by Harsanyi [14]. Consistency ensures that the desirability of a resolution for a reduced game aligns with its desirability in the original game. This property becomes even more critical in interval TU games, where uncertainties in utilities and payoffs can complicate the reduction process. Based on a specific reduction and related consistency, the *pseudo equal allocation of non-separable costs* (PEANSC) and the *pseudo weighted allocation of non-separable costs* (PWANSC), introduced by Hsieh and Liao [15], illustrate how individual contributions and weighting schemes can be integrated into allocating processes while maintaining stability and fairness. These two resolutions assess the contributions of players under varying situations, enabling proportional resource allocation based on factors such as effort or negotiating power.

Despite these advances, several open questions remain.

- Can related results for PEANSC and PWANSC be described under the interval TU setting?
- How do interval TU solutions compare to existing stochastic and fuzzy cooperative models in handling uncertainty?
- What computational frameworks can be developed to implement interval-based resolutions efficiently in large-scale applications?

In this study, we aim to extend the concepts of PEANSC, PWANSC, and consistency to interval TU games. Drawing inspiration from the works of Hwang [16] and Moulin [17], we characterize these extended resolutions by applying interval-based reduction and related consistency.

The primary contributions of this study are summarized as follows:

- **An extended resolution**: We introduce novel generalizations of the pseudo equal allocation of non-separable costs (PEANSC) and the pseudo weighted allocation of non-separable costs (PWANSC) under interval TU settings, ensuring allocation stability under uncertain environments.

- **Interval-based reduction and consistency**: We establish a systematic framework to adapt classical reduction and consistency principles to interval TU games, thereby enhancing the theoretical foundations of interval resolutions.
- **A weighted generalization**: A novel weighted resolution is proposed to refine interval allocation by incorporating variable importance factors among players, allowing for more flexible and adaptive allocation strategies.
- **A normalization mechanism**: We develop a normalization method to standardize individual contributions under interval uncertainty, ensuring robustness and fairness in allocation processes.
- **Axiomatic characterizations**: By extending and modifying classical axioms, we provide rigorous axiomatic results that justify the proposed interval-based resolutions, establishing consistency, efficiency, and fairness criteria for interval TU games.
- **Comparative analysis and distinct contributions**: Unlike previous works that primarily focus on deterministic or fuzzy TU models, this study explicitly integrates interval-based methodologies, offering a more comprehensive framework for allocation mechanisms under uncertainty.
- **Application in real-world situations**: Through case studies and computational validation, we demonstrate how the proposed resolutions can be implemented in resource distribution, coalition formation, and decision-making under uncertainty.

Through axiomatic analysis and rigorous mathematical formulations, this study bridges the gap between classical TU games and their interval generalizations, offering new insights into allocation processes under uncertainty while ensuring theoretical rigor and practical applicability.

2. Preliminaries

This study follows the related definitions, notations, and terminology of pre-existing studies [4–6,8]. Let $\widehat{\mathbf{UP}}$ be the universe of players and $\widehat{\mathbf{N}} \subseteq \widehat{\mathbf{UP}}$ be a collection of players. A **interval TU game** is a pair $(\widehat{\mathbf{N}}, \mu)$ where $\widehat{\mathbf{N}}$ is a finite and non-empty collection of players, $\mu : 2^{\widehat{\mathbf{N}}} \to \widehat{\mathbf{I}}(\mathbb{R})$ is a characteristic mapping with $\mu(\emptyset) = [0,0]$, and $\widehat{\mathbf{I}}(\mathbb{R})$ is the collection of all close intervals on $\mathbb{R}$. For each $\widehat{\mathbf{S}} \in 2^{\widehat{\mathbf{N}}}$, the value interval $\mu(\widehat{\mathbf{S}})$ of the coalition $\widehat{\mathbf{S}}$ in the interval TU game $(\widehat{\mathbf{N}}, \mu)$ is of the form $[\overline{\mu}(\widehat{\mathbf{S}}), \underline{\mu}(\widehat{\mathbf{S}})]$, where $\underline{\mu}(\widehat{\mathbf{S}})$ is the minimal utility which coalition $\widehat{\mathbf{S}}$ could obtain on its own and $\overline{\mu}(\widehat{\mathbf{S}})$ is the maximal utility which coalition $\widehat{\mathbf{S}}$ could obtain. Let us denote the class of all interval TU games with player collection $\widehat{\mathbf{N}}$ by $\mathbf{IG}^{\widehat{\mathbf{N}}}$. Further, the collection of all interval payoff vectors on $\widehat{\mathbf{N}}$ is denoted as $\widehat{\mathbf{I}}(\mathbb{R})^{\widehat{\mathbf{N}}}$.

Let $\mathbf{I}^*, \mathbf{J}^* \in \widehat{\mathbf{I}}(\mathbb{R})$ with $\mathbf{I}^* = [\underline{\mathbf{I}^*}, \overline{\mathbf{I}^*}]$, $\mathbf{J}^* = [\underline{\mathbf{J}^*}, \overline{\mathbf{J}^*}]$, $|\mathbf{I}^*| = \overline{\mathbf{I}^*} - \underline{\mathbf{I}^*}$ and $\alpha \in \mathbb{R}$. Then,

$$\begin{aligned} &\mathbf{I}^* + \mathbf{J}^* = [\underline{\mathbf{I}^*} + \underline{\mathbf{J}^*}, \overline{\mathbf{I}^*} + \overline{\mathbf{J}^*}], \\ &\alpha\mathbf{I}^* = [\alpha\underline{\mathbf{I}^*}, \alpha\overline{\mathbf{I}^*}] \text{ if } \alpha \geq 0, \\ &\alpha\mathbf{I}^* = [\alpha\overline{\mathbf{I}^*}, \alpha\underline{\mathbf{I}^*}] \text{ if } \alpha \leq 0. \end{aligned} \tag{1}$$

Based on Equation (1), it is clear that $\widehat{\mathbf{I}}(\mathbb{R})$ has a cone structure. Further, one would define $\mathbf{I}^* - \mathbf{J}^*$, only if $|\mathbf{I}^*| \geq |\mathbf{J}^*|$, by $\mathbf{I}^* - \mathbf{J}^* = [\underline{\mathbf{I}^*} - \underline{\mathbf{J}^*}, \overline{\mathbf{I}^*} - \overline{\mathbf{J}^*}]$. Note that $\overline{\mathbf{I}^*} - \overline{\mathbf{J}^*} \geq \underline{\mathbf{I}^*} - \underline{\mathbf{J}^*}$.

The framework of interval TU games is a generalized analogue of classical TU games. One would recall that a classical TU game (here, we adopt $(\widehat{\mathbf{N}}, \mu)$ and $< \widehat{\mathbf{N}}, \mu >$ to denote an interval TU game and a classical TU game, respectively.) $< \widehat{\mathbf{N}}, \mu >$ is defined as $\mu : 2^{\widehat{\mathbf{N}}} \to \mathbb{R}$ with $\mu(\emptyset) = 0$. A classical TU game $< \widehat{\mathbf{N}}, \mu >$ is called **monotonic** if $\mu(\widehat{\mathbf{S}}) \leq \mu(\widehat{\mathbf{T}})$ for all $\widehat{\mathbf{S}}, \widehat{\mathbf{T}} \in 2^{\widehat{\mathbf{N}}}$ with $\widehat{\mathbf{S}} \subseteq \widehat{\mathbf{T}}$. An interval TU game $(\widehat{\mathbf{N}}, \mu)$ is said to be **size-monotonic** if its length TU game $< \widehat{\mathbf{N}}, |\mu| >$ is monotonic, where $|\mu|(\widehat{\mathbf{S}}) = \overline{\mu}(\widehat{\mathbf{S}}) - \underline{\mu}(\widehat{\mathbf{S}})$ for all $\widehat{\mathbf{S}} \subseteq \widehat{\mathbf{N}}$. The class of size-monotonic interval games with grand coalition $\widehat{\mathbf{N}}$ is denoted as $\mathbf{SMIG}^{\widehat{\mathbf{N}}}$. Let $(\widehat{\mathbf{N}}, \mu) \in \mathbf{SMIG}^{\widehat{\mathbf{N}}}$; $(\widehat{\mathbf{N}}, \mu)$ is said to be **coalitional monotonic** if for all

$\widehat{\mathbf{S}} \subseteq \widehat{\mathbf{N}}$, $|\mu|(\widehat{\mathbf{S}}) \geq \sum_{i\in\widehat{\mathbf{S}}} |\mu|(\{i\})$. The class of coalitional monotonic interval games with grand coalition $\widehat{\mathbf{N}}$ is denoted as $\mathbf{CMIG}^{\widehat{\mathbf{N}}}$. Subsequently, this study focuses on the collection of games **CMIG**, where $\mathbf{CMIG} = \cup_{\widehat{\mathbf{N}}\subseteq\widehat{\mathbf{UP}}}\mathbf{CMIG}^{\widehat{\mathbf{N}}}$.

A **resolution** on **CMIG** is a mapping τ allotting to each interval TU game $(\widehat{\mathbf{N}}, \mu) \in$ **CMIG** an element $\tau(\widehat{\mathbf{N}}, \mu) \in \widehat{\mathbf{I}}(\mathbb{R})^{\widehat{\mathbf{N}}}$.

Definition 1. *The* **interval pseudo equal allocation of non-separable costs (IPEANSC)**, Θ, *is the mapping on* **CMIG** *which associates with each* $(\widehat{\mathbf{N}}, \mu) \in$ **CMIG** *and each* $i \in \widehat{\mathbf{N}}$ *the value*

$$\Theta_i(\widehat{\mathbf{N}}, \mu) = \theta_i(\widehat{\mathbf{N}}, \mu) + \frac{1}{|\widehat{\mathbf{N}}|} \cdot \big[\mu(\widehat{\mathbf{N}}) - \sum_{k\in\widehat{\mathbf{N}}} \theta_k(\widehat{\mathbf{N}}, \mu)\big],$$

where $\theta_i(\widehat{\mathbf{N}}, \mu) = \mu(\{i\})$ *is the* **interval individual contribution** *of player i.*

3. Axiomatic Results

In this section, some properties are applied to characterize IPEANSC.
Let τ be a resolution on **CMIG**.

- τ satisfies **efficiency (EFF)** if $\sum_{i\in\widehat{\mathbf{N}}} \tau_i(\widehat{\mathbf{N}}, \mu) = \mu(\widehat{\mathbf{N}})$ for all $(\widehat{\mathbf{N}}, \mu) \in$ **CMIG**. Axiom EFF states that the total utility of the grand coalition is entirely allocated among all players.
- τ satisfies **standard for games (SFG)** if $\tau(\widehat{\mathbf{N}}, \mu) = \mu(\widehat{\mathbf{N}})$ for all $(\widehat{\mathbf{N}}, \mu) \in$ **CMIG** with $|\widehat{\mathbf{N}}| = 1$, and

$$\tau(\widehat{\mathbf{N}}, \mu) = \mu(\{i\}) + \frac{1}{2}\Big[\mu(\widehat{\mathbf{N}}) - \mu(\{i\}) - \mu(\widehat{\mathbf{N}} \setminus \{i\})\Big]$$

  for all $(\widehat{\mathbf{N}}, \mu) \in$ **CMIG** with $|\widehat{\mathbf{N}}| = 2$ and for all $i \in \widehat{\mathbf{N}}$. Axiom SFG expresses that, in all one-player games, the benefit allocation mechanism is self-sustaining. For all two-player games, each player first retrieves their individual contribution, and the remaining surplus or deficit is then evenly distributed. The concept underlying Axiom SFG is inspired by Hart and Mas-Colell [18].
- τ satisfies **symmetry (SYM)** if $\tau_i(\widehat{\mathbf{N}}, \mu) = \tau_k(\widehat{\mathbf{N}}, \mu)$ for all $(\widehat{\mathbf{N}}, \mu) \in$ **CMIG** with $\mu(\widehat{\mathbf{S}}) - \mu(\widehat{\mathbf{S}} \setminus \{i\}) = \mu(\widehat{\mathbf{S}}) - \mu(\widehat{\mathbf{S}} \setminus \{k\})$ for some $i, k \in \widehat{\mathbf{N}}$ and for all $\widehat{\mathbf{S}} \subseteq \widehat{\mathbf{N}}$. Axiom SYM describes that if the marginal contributions of any two players are equal, their final allocations should also be the same.
- τ satisfies **zero-independence (ZI)** if $\tau(\widehat{\mathbf{N}}, \nu) = \tau(\widehat{\mathbf{N}}, \mu) + b$ for all $(\widehat{\mathbf{N}}, \nu), (\widehat{\mathbf{N}}, \mu) \in$ **CMIG** with $\nu(\widehat{\mathbf{S}}) = \mu(\widehat{\mathbf{S}}) + \sum_{i\in\widehat{\mathbf{S}}} b_i$ for some $b \in \widehat{\mathbf{I}}(\mathbb{R})^{\widehat{\mathbf{N}}}$ and for all $\widehat{\mathbf{S}} \subseteq \widehat{\mathbf{N}}$. Mathematically, Axiom ZI represents a weakened analogue of additivity.
- Based on Definition 1, it is easy to check that IPEANSC satisfies EFF, SFG, SYM, and ZI.

The following reduction is based on the notion of "re-operating upon disagreement". Under an interval TU game $(\widehat{\mathbf{N}}, \mu)$, if a coalition $\widehat{\mathbf{S}}$ within $\widehat{\mathbf{N}}$ confutes a resolution τ, re-operating is initiated. Should none of the conflict parties like to participate in this re-operating, the resulting usefulness is zero. Under conditions where multiple parties dissent but only one strives for re-allocating, the process is simplified to avert needless complexity, allocating that party the usefulness it would receive by participating independently. If multiple conflict parties participate under the re-operating, then all the agreeing parties fully participate in the process. Based on completion, the agreeing parties obtain their share as per the initial outcomes allotted via τ and subsequently exit, leaving the rest of usefulness to be allocated among the conflict parties in the light of the outcome under re-operating.

- Given $(\widehat{\mathbf{N}}, \mu) \in$ **CMIG**, $\widehat{\mathbf{S}} \subseteq \widehat{\mathbf{N}}, \widehat{\mathbf{S}} \neq \emptyset$, and a resolution τ, the **reduced game** $(\widehat{\mathbf{S}}, \mu^{\tau}_{\widehat{\mathbf{S}}})$ **with respect to** S **and** τ is defined by for all $\widehat{\mathbf{T}} \subseteq \widehat{\mathbf{S}}$,

$$\mu^{\tau}_{\widehat{\mathbf{S}}}(\widehat{\mathbf{T}}) = \begin{cases} [0,0] & \text{, if } \widehat{\mathbf{T}} = \emptyset, \\ \mu(\{k\}) & \text{, if } |\widehat{\mathbf{S}}| \geq 2,\ k \in \widehat{\mathbf{S}}, \\ \mu(\widehat{\mathbf{T}} \cup (\widehat{\mathbf{N}} \setminus \widehat{\mathbf{S}})) - \sum\limits_{i \in \widehat{\mathbf{N}} \setminus \widehat{\mathbf{S}}} \tau_i(\widehat{\mathbf{N}}, \mu) & \text{, otherwise.} \end{cases}$$

- The property of *consistency* could be described in the following condition. For any coalition of players within a interval TU game, a "reduction" could be defined by considering the value left after the other players have obtained its outcomes distributed via the resolution τ. The resolution τ is said to be *consistent* if, when applied to each reduction, it always begets outcomes that coincide with those in the beginning status. Formally, a resolution τ satisfies **consistency (CON)** if $\tau_i(\widehat{\mathbf{N}}, \mu) = \tau_i(\widehat{\mathbf{S}}, \mu^{\tau}_{\widehat{\mathbf{S}}})$ for all $(\widehat{\mathbf{N}}, \mu) \in$ **CMIG** with $|\widehat{\mathbf{N}}| \geq 2$, for all $\widehat{\mathbf{S}} \subseteq \widehat{\mathbf{N}}$ with $\widehat{\mathbf{S}} \neq \emptyset$, and for all $i \in \widehat{\mathbf{S}}$.

Lemma 1. *The resolution Θ satisfies CON.*

Proof. Given $(\widehat{\mathbf{N}}, \mu) \in$ **CMIG** and $\widehat{\mathbf{S}} \subseteq \widehat{\mathbf{N}}$. If $\widehat{\mathbf{S}} = \{i\}$ for some $i \in \widehat{\mathbf{N}}$, then by the EFF of Θ,

$$\Theta_i(\widehat{\mathbf{S}}, \mu^{\Theta}_{\widehat{\mathbf{S}}}) = \mu^{\Theta}_{\widehat{\mathbf{S}}}(\widehat{\mathbf{S}}) = \mu(\widehat{\mathbf{N}}) - \sum_{k \neq i} \Theta_k(\widehat{\mathbf{N}}, \mu) = \Theta_i(\widehat{\mathbf{N}}, \mu).$$

Let us suppose that $|\widehat{\mathbf{N}}| \geq 2$ and $|\widehat{\mathbf{S}}| \geq 2$. By the definitions of θ and $\mu^{\Theta}_{\widehat{\mathbf{S}}}$,

$$\begin{aligned} \theta_i(\widehat{\mathbf{S}}, \mu^{\Theta}_{\widehat{\mathbf{S}}}) &= \mu^{\Theta}_{\widehat{\mathbf{S}}}(\{i\}) && \text{(by Definition 1)} \\ &= \mu(\{i\}) && \text{(by the definition of } \mu^{\Theta}_{\widehat{\mathbf{S}}}) \\ &= \theta_i(\widehat{\mathbf{N}}, \mu) && \text{(by Definition 1)} \end{aligned} \tag{2}$$

for all $i \in \widehat{\mathbf{S}}$. Hence,

$$\begin{aligned} \Theta_i(\widehat{\mathbf{S}}, \mu^{\Theta}_{\widehat{\mathbf{S}}}) &= \theta_i(\widehat{\mathbf{S}}, \mu^{\Theta}_{\widehat{\mathbf{S}}}) + \tfrac{1}{|\widehat{\mathbf{S}}|} \cdot \Big[\mu^{\Theta}_{\widehat{\mathbf{S}}}(\widehat{\mathbf{S}}) - \sum_{k \in \widehat{\mathbf{S}}} \theta_k(\widehat{\mathbf{S}}, \mu^{\Theta}_{\widehat{\mathbf{S}}})\Big] \\ &= \theta_i(\widehat{\mathbf{N}}, \mu) + \tfrac{1}{|\widehat{\mathbf{S}}|} \cdot \Big[\mu^{\Theta}_{\widehat{\mathbf{S}}}(\widehat{\mathbf{S}}) - \sum_{k \in \widehat{\mathbf{S}}} \theta_k(\widehat{\mathbf{N}}, \mu)\Big] && \text{(by Equation (2))} \\ &= \theta_i(\widehat{\mathbf{N}}, \mu) + \tfrac{1}{|\widehat{\mathbf{S}}|} \cdot \Big[\mu(\widehat{\mathbf{N}}) - \sum_{k \in \widehat{\mathbf{N}} \setminus \widehat{\mathbf{S}}} \Theta_k(\widehat{\mathbf{N}}, \mu) - \sum_{k \in \widehat{\mathbf{S}}} \theta_k(\widehat{\mathbf{N}}, \mu)\Big] \\ &= \theta_i(\widehat{\mathbf{N}}, \mu) + \tfrac{1}{|\widehat{\mathbf{S}}|} \cdot \Big[\sum_{k \in \widehat{\mathbf{S}}} \Theta_k(\widehat{\mathbf{N}}, \mu) - \sum_{k \in \widehat{\mathbf{S}}} \theta_k(\widehat{\mathbf{N}}, \mu)\Big] && \text{(by the EFF of } \Theta) \\ &= \theta_i(\widehat{\mathbf{N}}, \mu) + \tfrac{1}{|\widehat{\mathbf{S}}|} \cdot \Big[\sum_{k \in \widehat{\mathbf{S}}} \tfrac{1}{|\widehat{\mathbf{N}}|} \cdot \big[\mu(\widehat{\mathbf{N}}) - \sum_{p \in \widehat{\mathbf{N}}} \theta_p(\widehat{\mathbf{N}}, \mu)\big]\Big] \\ &= \theta_i(\widehat{\mathbf{N}}, \mu) + \tfrac{1}{|\widehat{\mathbf{S}}|} \cdot \Big[\tfrac{|\widehat{\mathbf{S}}|}{|\widehat{\mathbf{N}}|} \cdot \big[\mu(\widehat{\mathbf{N}}) - \sum_{p \in \widehat{\mathbf{N}}} \theta_p(\widehat{\mathbf{N}}, \mu)\big]\Big] \\ &= \theta_i(\widehat{\mathbf{N}}, \mu) + \tfrac{1}{|\widehat{\mathbf{N}}|} \cdot \Big[\mu(\widehat{\mathbf{N}}) - \sum_{k \in \widehat{\mathbf{N}}} \theta_k(\widehat{\mathbf{N}}, \mu)\Big] \\ &= \Theta_i(\widehat{\mathbf{N}}, \mu) \end{aligned}$$

for all $i \in \widehat{\mathbf{S}}$. The proof is completed. □

Next, we characterize IPEANSC by means of related properties of two-person standardness and consistency.

Theorem 1. *A resolution τ on* **CMIG** *satisfies SFG and CON if and only if $\tau = \Theta$.*

Proof. By Lemma 1, Θ satisfies CON. Clearly, Θ satisfies SFG.

To prove the uniqueness, let us suppose that τ satisfies SFG and CON. By the SFG and CON of τ, it is easy to derive that τ also satisfies EFF; hence, we omit it. Let $(\widehat{\mathbf{N}}, \mu) \in$ **CMIG**.

If $|\widehat{\mathbf{N}}| \leq 2$, then by the SFG of τ, $\tau(\widehat{\mathbf{N}}, \mu) = \Theta(\widehat{\mathbf{N}}, \mu)$. In the case where $|\widehat{\mathbf{N}}| > 2$, for all $i, k \in L^{\widehat{\mathbf{N}}}$ with $i \neq k$, let $\widehat{\mathbf{S}} = \{i, k\}$; we derive that

$$\begin{aligned} \tau_i(\widehat{\mathbf{N}}, \mu) - \tau_k(\widehat{\mathbf{N}}, \mu) &= \tau_i(\widehat{\mathbf{S}}, \mu^\tau_{\widehat{\mathbf{S}}}) - \tau_k(\widehat{\mathbf{S}}, \mu^\tau_{\widehat{\mathbf{S}}}) && \text{(by the CON of } \tau) \\ &= \Theta_i(\widehat{\mathbf{S}}, \mu^\tau_{\widehat{\mathbf{S}}}) - \Theta_k(\widehat{\mathbf{S}}, \mu^\tau_{\widehat{\mathbf{S}}}) && \text{(by the SFG of } \tau) \\ &= \theta_i(\widehat{\mathbf{S}}, \mu^\tau_{\widehat{\mathbf{S}}}) - \theta_k(\widehat{\mathbf{S}}, \mu^\tau_{\widehat{\mathbf{S}}}) && \text{(by Definition 1)} \\ &= \left[\mu^\tau_{\widehat{\mathbf{S}}}(\{i\}) - \mu^\tau_{\widehat{\mathbf{S}}}(\{k\})\right] \\ &= \left[\mu(\{i\}) - \mu(\{k\})\right] \end{aligned} \tag{3}$$

Similarly, with Θ instead of τ in Equation (3), we can derive that

$$\Theta_i(\widehat{\mathbf{N}}, \mu) - \Theta_k(\widehat{\mathbf{N}}, \mu) = \mu(\{i\}) - \mu(\{k\}) \tag{4}$$

Hence, by Equations (3) and (4),

$$\tau_i(\widehat{\mathbf{N}}, \mu) - \tau_k(\widehat{\mathbf{N}}, \mu) = \Theta_i(\widehat{\mathbf{N}}, \mu) - \Theta_k(\widehat{\mathbf{N}}, \mu). \tag{5}$$

This implies that $\tau_i(\widehat{\mathbf{N}}, \mu) - \Theta_i(\widehat{\mathbf{N}}, \mu) = d$ for all $i \in \widehat{\mathbf{N}}$ and for some $d \in \widehat{\mathbf{I}}(\mathbb{R})^{\widehat{\mathbf{N}}}$. It remains to show that $d = [0, 0]$. By the EFF of τ and Θ and Equation (5),

$$[0, 0] = \mu(\widehat{\mathbf{N}}) - \mu(\widehat{\mathbf{N}}) = \sum_{i \in \widehat{\mathbf{N}}} \left[\tau_i(\widehat{\mathbf{N}}, \mu) - \Theta_i(\widehat{\mathbf{N}}, \mu)\right] = |\widehat{\mathbf{N}}| \cdot d.$$

Hence, $d = [0, 0]$ by applying $|\widehat{\mathbf{N}}| \leq 2$. □

Finally, we characterize IPEANSC by means of related properties of efficiency, symmetry, zero-independence, and consistency.

Lemma 2. *If a resolution τ on* **CMIG** *satisfies EFF, SYM, and ZI, then τ satisfies SFG.*

Proof. Let us assume that a resolution τ satisfies EFF, SYM, and ZI. It is easy to have that $\tau(\widehat{\mathbf{N}}) = \nu(\widehat{\mathbf{N}})$ for all $(\widehat{\mathbf{N}}, \nu) \in$ **CMIG** with $|\widehat{\mathbf{N}}| = 1$ by applying the EFF of τ. Given that $(\widehat{\mathbf{N}}, \nu) \in$ **CMIG** with $\widehat{\mathbf{N}} = \{i, k\}$ for some $i \neq k$, we define a game $(\widehat{\mathbf{N}}, \mu)$ to be that for all $\widehat{\mathbf{S}} \subseteq \widehat{\mathbf{N}}$,

$$\mu(\widehat{\mathbf{S}}) = \nu(\widehat{\mathbf{S}}) - \sum_{i \in \widehat{\mathbf{S}}} \nu(\{i\}).$$

By the definition of μ,

$$\begin{aligned} \mu(\{i, k\}) - \mu(\{k\}) &= \nu(\{i, k\}) - \nu(\{i\}) - \nu(\{k\}) - \nu(\{k\}) + \nu(\{k\}) \\ &= \nu(\{i, k\}) - \nu(\{i\}) - \nu(\{k\})\}. \end{aligned}$$

Similarly, $\mu(\{i, k\}) - \mu(\{i\}) = \nu(\{i, k\}) - \nu(\{i\}) - \nu(\{k\})\}$. Since $\mu(\{i, k\}) - \mu(\{k\}) = \mu(\{i, k\}) - \mu(\{i\})$, by the SYM of τ, $\tau_i(\widehat{\mathbf{N}}, \mu) = \tau_k(\widehat{\mathbf{N}}, \mu)$. By the EFF of τ,

$$\mu(\widehat{\mathbf{N}}) = \tau_i(\widehat{\mathbf{N}}, \mu) + \tau_k(\widehat{\mathbf{N}}, \mu) = 2 \cdot \tau_i(\widehat{\mathbf{N}}, \mu).$$

Therefore,

$$\tau_i(\widehat{\mathbf{N}}, \mu) = \frac{\mu(\widehat{\mathbf{N}})}{2} = \frac{1}{2} \cdot \left[\nu(\widehat{\mathbf{N}}) - \nu(\{i\}) - \nu(\{k\})\right].$$

By the ZI of τ,

$$\begin{aligned} \tau_i(\widehat{\mathbf{N}}, \nu) &= \nu(\{i\}) + \tfrac{1}{2} \cdot \left[\nu(\widehat{\mathbf{N}}) - \nu(\{i\}) - \nu(\{k\})\right] \\ &= \theta_i(\widehat{\mathbf{N}}, \nu) + \tfrac{1}{2} \cdot \left[\nu(\widehat{\mathbf{N}}) - \theta_i(\widehat{\mathbf{N}}, \nu) - \theta_k(\widehat{\mathbf{N}}, \nu)\right] \\ &= \Theta_i(\widehat{\mathbf{N}}, \nu). \end{aligned}$$

Similarly, $\tau_k(\widehat{\mathbf{N}}, \nu) = \Theta_k(\widehat{\mathbf{N}}, \nu)$. Hence, τ satisfies SFG. □

Theorem 2. *On* **CMIG**, *IPEANSC is the only resolution satisfying EFF, SYM, ZI, and CON.*

Proof. By Definition 1, Θ satisfies EFF, SYM, and ZI. The remaining proofs follow from Theorem 1 and Lemmas 1 and 2. □

The following examples are to show that each of the axioms used in Theorems 1 and 2 is logically independent of the remaining axioms.

Example 1. *Define a resolution τ for all $(\widehat{\mathbf{N}}, \mu) \in$ **CMIG** and for all $i \in \widehat{\mathbf{N}}$,*

$$\tau_i(\widehat{\mathbf{N}}, \mu) = [0, 0].$$

Clearly, τ satisfies CON, but it violates SFG.

Example 2. *Define a resolution τ for all $(\widehat{\mathbf{N}}, \mu) \in$ **CMIG** and for all $i \in \widehat{\mathbf{N}}$,*

$$\tau_i(\widehat{\mathbf{N}}, \mu) = \begin{cases} \Theta_i(\widehat{\mathbf{N}}, \mu) & \text{, if } |\widehat{\mathbf{N}}| \leq 2, \\ [0,0] & \text{, otherwise.} \end{cases}$$

Clearly, τ satisfies SFG, but it violates CON.

Example 3. *Define a resolution τ for all $(\widehat{\mathbf{N}}, \mu) \in$ **CMIG** and for all $i \in \widehat{\mathbf{N}}$,*

$$\tau_i(\widehat{\mathbf{N}}, \mu) = \kappa_i(\widehat{\mathbf{N}}, \mu) + \frac{1}{|\widehat{\mathbf{N}}|} \cdot \Big[\mu(\widehat{\mathbf{N}}) - \sum_{k \in \widehat{\mathbf{N}}} \kappa_k(\widehat{\mathbf{N}}, \mu)\Big],$$

where $\kappa_i(\widehat{\mathbf{N}}, \mu) = [\mu(\widehat{\mathbf{N}}) - \mu(\widehat{\mathbf{N}} \setminus \{i\})]$. Clearly, τ satisfies EFF, SYM, and ZI, but it violates CON.

Example 4. *Define a resolution τ for all $(\widehat{\mathbf{N}}, \mu) \in$ **CMIG** and for all $i \in \widehat{\mathbf{N}}$,*

$$\tau_i(\widehat{\mathbf{N}}, \mu) = \frac{\mu(\widehat{\mathbf{N}})}{|\widehat{\mathbf{N}}|}.$$

Clearly, τ satisfies EFF, SYM, and CON, but it violates ZI.

Example 5. *Define a resolution τ for all $(\widehat{\mathbf{N}}, \mu) \in$ **CMIG** and for all $i \in \widehat{\mathbf{N}}$,*

$$\tau_i(\widehat{\mathbf{N}}, \mu) = \mu(\{i\}).$$

Clearly, τ satisfies SYM, ZI, and CON, but it violates EFF.

Example 6. *Define a resolution τ for all $(\widehat{\mathbf{N}}, \mu) \in$ **CMIG** and for all $i \in \widehat{\mathbf{N}}$,*

$$\tau_i(\widehat{\mathbf{N}}, \mu) = \theta_i(\widehat{\mathbf{N}}, \mu) + \frac{d_i}{\sum\limits_{k \in \widehat{\mathbf{N}}} d_k} \cdot \Big[\mu(\widehat{\mathbf{N}}) - \sum_{k \in \widehat{\mathbf{N}}} \theta_k(\widehat{\mathbf{N}}, \mu)\Big],$$

where $\widehat{\mathbf{D}} = \{d_p \mid p \in \widehat{\mathbf{UP}}\}$ is a collection of positive real numbers. Clearly, τ satisfies EFF, ZI, and CON, but it violates SYM.

4. A Weighted Generalization

In this section, an extended form of IPEANSC is provided by applying weights. A **weight mapping** λ is a positive mapping $\lambda : \widehat{\mathbf{UP}} \to \mathbb{R}^+$. Let $(\widehat{\mathbf{N}}, \mu) \in$ **CMIG** and λ be a weight mapping. For every $\widehat{\mathbf{H}} \subseteq \widehat{\mathbf{N}}$, one would define $|\widehat{\mathbf{H}}|_\lambda = \sum_{p \in \widehat{\mathbf{H}}} \lambda(p)$.

Definition 2. *The* **interval pseudo weighted allocation of non-separable costs (IPWANSC)**, Θ^{λ}, *is defined for every* $(\widehat{\mathbf{N}}, \mu) \in$ **CMIG**, *for every weight mapping* λ, *and for every* $p \in \widehat{\mathbf{N}}$, *as follows:*

$$\Theta_p^{\lambda}(\widehat{\mathbf{N}}, \mu) = \theta_p(\widehat{\mathbf{N}}, \mu) + \frac{\lambda(p)}{|\widehat{\mathbf{N}}|_{\lambda}} \cdot \big[\mu(\widehat{\mathbf{N}}) - \sum_{k \in \widehat{\mathbf{N}}} \theta_k(\widehat{\mathbf{N}}, \mu)\big]. \tag{6}$$

Lemma 3. *IPWANSC satisfies EFF.*

Proof. Let $(\widehat{\mathbf{N}}, \mu) \in$ **CMIG** and λ be a weight mapping.

$$\begin{aligned}
\sum_{p \in \widehat{\mathbf{N}}} \Theta_p^{\lambda}(\widehat{\mathbf{N}}, \mu) &= \sum_{p \in \widehat{\mathbf{N}}} \theta_p(\widehat{\mathbf{N}}, \mu) + \sum_{p \in \widehat{\mathbf{N}}} \frac{\lambda(p)}{|\widehat{\mathbf{N}}|_{\lambda}} \cdot \big[\mu(\widehat{\mathbf{N}}) - \sum_{k \in \widehat{\mathbf{N}}} \theta_k(\widehat{\mathbf{N}}, \mu)\big] \\
&= \sum_{p \in \widehat{\mathbf{N}}} \theta_p(\widehat{\mathbf{N}}, \mu) + \frac{|\widehat{\mathbf{N}}|_{\lambda}}{|\widehat{\mathbf{N}}|_{\lambda}} \cdot \big[\mu(\widehat{\mathbf{N}}) - \sum_{k \in \widehat{\mathbf{N}}} \theta_k(\widehat{\mathbf{N}}, \mu)\big] \\
&= \mu(\widehat{\mathbf{N}}).
\end{aligned}$$

The proof is complete. □

Lemma 4. *IPWANSC* Θ^{λ} *satisfies CON.*

Proof. Given $(\widehat{\mathbf{N}}, \mu) \in$ **CMIG**, $\widehat{\mathbf{H}} \subseteq \widehat{\mathbf{N}}$, and a weight mapping λ, if $\widehat{\mathbf{H}} = \{p\}$ for some $p \in \widehat{\mathbf{N}}$, then, by the EFF of Θ^{λ},

$$\Theta_p^{\lambda}(\widehat{\mathbf{H}}, \mu_{\widehat{\mathbf{H}}}^{\Theta^{\lambda}}) = \mu_{\widehat{\mathbf{H}}}^{\Theta^{\lambda}}(\widehat{\mathbf{H}}) = \mu(\widehat{\mathbf{N}}) - \sum_{k \neq p} \Theta_k^{\lambda}(\widehat{\mathbf{N}}, \mu) = \Theta_p^{\lambda}(\widehat{\mathbf{N}}, \mu).$$

Let us suppose that $|\widehat{\mathbf{N}}| \geq 2$ and $|\widehat{\mathbf{H}}| \geq 2$. By the definitions of θ and $\mu_{\widehat{\mathbf{H}}}^{\Theta^{\lambda}}$,

$$\begin{aligned}
\theta_p(\widehat{\mathbf{H}}, \mu_{\widehat{\mathbf{H}}}^{\Theta^{\lambda}}) &= \mu_{\widehat{\mathbf{H}}}^{\Theta^{\lambda}}(\{p\}) \quad \text{(by Definition 1)} \\
&= \mu(\{p\}) \quad \text{(by the definition of } \mu_{\widehat{\mathbf{H}}}^{\Theta^{\lambda}}\text{)} \\
&= \theta_p(\widehat{\mathbf{N}}, \mu) \quad \text{(by Definition 1)}
\end{aligned} \tag{7}$$

for every $p \in \widehat{\mathbf{H}}$. By (6) and (7),

$$\begin{aligned}
\Theta_p^{\lambda}(\widehat{\mathbf{H}}, \mu_{\widehat{\mathbf{H}}}^{\Theta^{\lambda}}) &= \theta_p(\widehat{\mathbf{H}}, \mu_{\widehat{\mathbf{H}}}^{\Theta^{\lambda}}) + \frac{\lambda(p)}{|\widehat{\mathbf{H}}|_{\lambda}} \cdot \big[\mu_{\widehat{\mathbf{H}}}^{\Theta^{\lambda}}(\widehat{\mathbf{H}}) - \sum_{k \in \widehat{\mathbf{H}}} \theta_k(\widehat{\mathbf{H}}, \mu_{\widehat{\mathbf{H}}}^{\Theta^{\lambda}})\big] \\
&= \theta_p(\widehat{\mathbf{N}}, \mu) + \frac{\lambda(p)}{|\widehat{\mathbf{H}}|_{\lambda}} \cdot \big[\mu_{\widehat{\mathbf{H}}}^{\Theta^{\lambda}}(\widehat{\mathbf{H}}) - \sum_{k \in \widehat{\mathbf{H}}} \theta_k(\widehat{\mathbf{N}}, \mu)\big] \\
&= \theta_p(\widehat{\mathbf{N}}, \mu) + \frac{\lambda(p)}{|\widehat{\mathbf{H}}|_{\lambda}} \cdot \big[\sum_{k \in \widehat{\mathbf{H}}} \Theta_k^{\lambda}(\widehat{\mathbf{N}}, \mu) - \sum_{k \in \widehat{\mathbf{H}}} \theta_k(\widehat{\mathbf{N}}, \mu)\big] \\
&= \theta_p(\widehat{\mathbf{N}}, \mu) + \frac{\lambda(p)}{|\widehat{\mathbf{H}}|_{\lambda}} \cdot \Big[\frac{|\widehat{\mathbf{H}}|_{\lambda}}{|\widehat{\mathbf{N}}|_{\lambda}} \cdot \big[\mu(\widehat{\mathbf{N}}) - \sum_{k \in \widehat{\mathbf{N}}} \theta_k(\widehat{\mathbf{N}}, \mu)\big]\Big] \\
&= \theta_p(\widehat{\mathbf{N}}, \mu) + \frac{\lambda(p)}{|\widehat{\mathbf{N}}|_{\lambda}} \cdot \big[\mu(\widehat{\mathbf{N}}) - \sum_{k \in \widehat{\mathbf{N}}} \theta_k(\widehat{\mathbf{N}}, \mu)\big] \\
&= \Theta_i^{\lambda}(\widehat{\mathbf{N}}, \mu)
\end{aligned}$$

for all $p \in \widehat{\mathbf{H}}$. Therefore, the resolution Θ^{λ} satisfies CON. □

Remark 1. *It is trivial to verify that IPWANSC satisfies ZI, but it infringes SYM.*

Inspired by Theorem 1, an axiomatic characterization of IPWANSC is presented as follows. A resolution τ satisfies **weighted standard personality (WSP)** if $\tau(\widehat{\mathbf{N}}, \mu) = \Theta^{\lambda}(\widehat{\mathbf{N}}, \mu)$ for all $(\widehat{\mathbf{N}}, \mu) \in$ **CMIG** with $|\widehat{\mathbf{N}}| \leq 2$.

Lemma 5. *A resolution* τ *satisfies EFF if* τ *satisfies WSP and CON.*

Proof. Let us suppose that a resolution τ satisfies WSP and CON. Let $(\widehat{\mathbf{N}}, \mu) \in$ **CMIG**. It is trivial by the WSP of τ if $|\widehat{\mathbf{N}}| \leq 2$. Let us suppose that $|\widehat{\mathbf{N}}| > 2$ and $p \in \widehat{\mathbf{N}}$. By the previous proof and the definition of $\mu^{\tau}_{\{p\}}$,

$$\tau_p(\{p\}, \mu^{\tau}_{\{p\}}) = \mu^{\tau}_{\{p\}}(\{p\}) = \mu(\widehat{\mathbf{N}}) - \sum_{k \neq p} \tau_k(\widehat{\mathbf{N}}, \mu). \tag{8}$$

By the CON of τ,

$$\tau_p(\{p\}, \mu^{\tau}_{\{p\}}) = \tau_p(\widehat{\mathbf{N}}, \mu). \tag{9}$$

By (8) and (9),

$$\mu(\widehat{\mathbf{N}}) = \sum_{k \in \widehat{\mathbf{N}}} \tau_k(\widehat{\mathbf{N}}, \mu).$$

Therefore, τ satisfies EFF. □

Theorem 3. *A resolution τ on* **CMIG** *satisfies WSP and CON $\Leftrightarrow \tau = \Theta^{\lambda}$.*

Proof. Based on Lemma 4, Θ^{λ} satisfies CON. Also, Θ^{λ} satisfies WSP.

To show uniqueness, let us assume that a resolution τ satisfies WSP and CON. By Lemma 5, τ satisfies EFF. Let $(\widehat{\mathbf{N}}, \mu) \in$ **CMIG** and λ be a weight map. It is trivial by WSFG if $|\widehat{\mathbf{N}}| \leq 2$. In the situation when $|\widehat{\mathbf{N}}| > 2$, let $p \in \widehat{\mathbf{N}}$ and $\widehat{\mathbf{H}} = \{p, q\}$ for some $q \in \widehat{\mathbf{N}} \setminus \{p\}$. Similarly to Equations (2) and (7), for all $k \in \widehat{\mathbf{H}}$,

$$\theta_k(\widehat{\mathbf{H}}, \mu^{\tau}_{\widehat{\mathbf{H}}}) = \theta_k(\widehat{\mathbf{N}}, \mu) = \theta_k(\widehat{\mathbf{H}}, \mu^{\Theta^{\lambda}}_{\widehat{\mathbf{H}}}). \tag{10}$$

Then,

$$\begin{aligned}
& \tau_p(\widehat{\mathbf{N}}, \mu) - \Theta^{\lambda}_p(\widehat{\mathbf{N}}, \mu) \\
= \;& \tau_p(\widehat{\mathbf{H}}, \mu^{\tau}_{\widehat{\mathbf{H}}}) - \Theta^{\lambda}_p(\widehat{\mathbf{H}}, \mu^{\Theta^{\lambda}}_{\widehat{\mathbf{H}}}) && \textbf{(by the CON of } \tau, \Theta^{\lambda}\textbf{)} \\
= \;& \Theta^{\lambda}_p(\widehat{\mathbf{H}}, \mu^{\tau}_{\widehat{\mathbf{H}}}) - \Theta^{\lambda}_p(\widehat{\mathbf{H}}, \mu^{\Theta^{\lambda}}_{\widehat{\mathbf{H}}}) && \textbf{(by the WSP of } \tau, \Theta^{\lambda}\textbf{)} \\
= \;& \theta_p(\widehat{\mathbf{H}}, \mu^{\tau}_{\widehat{\mathbf{H}}}) + \tfrac{\lambda(p)}{|\widehat{\mathbf{H}}|_{\lambda}} \cdot \Big[\mu^{\tau}_{\widehat{\mathbf{H}}}(\widehat{\mathbf{H}}) - \theta_p(\widehat{\mathbf{H}}, \mu^{\tau}_{\widehat{\mathbf{H}}}) - \theta_q(\widehat{\mathbf{H}}, \mu^{\tau}_{\widehat{\mathbf{H}}})\Big] \\
& -\theta_p(\widehat{\mathbf{H}}, \mu^{\Theta^{\lambda}}_{\widehat{\mathbf{H}}}) - \tfrac{\lambda(p)}{|\widehat{\mathbf{H}}|_{\lambda}} \cdot \Big[\mu^{\Theta^{\lambda}}_{\widehat{\mathbf{H}}}(\widehat{\mathbf{H}}) - \theta_p(\widehat{\mathbf{H}}, \mu^{\Theta^{\lambda}}_{\widehat{\mathbf{H}}}) - \theta_q(\widehat{\mathbf{H}}, \mu^{\Theta^{\lambda}}_{\widehat{\mathbf{H}}})\Big] \\
= \;& \tfrac{\lambda(p)}{|\widehat{\mathbf{H}}|_{\lambda}} \cdot \Big[\mu^{\tau}_{\widehat{\mathbf{H}}}(\widehat{\mathbf{H}}) - \mu^{\Theta^{\lambda}}_{\widehat{\mathbf{H}}}(\widehat{\mathbf{H}})\Big] && \textbf{(by Equation (10))} \\
= \;& \tfrac{\lambda(p)}{|\widehat{\mathbf{H}}|_{\lambda}} \cdot \Big[\tau_p(\widehat{\mathbf{N}}, \mu) + \tau_{\mu}(\widehat{\mathbf{N}}, \mu) - \Theta^{\lambda}_p(\widehat{\mathbf{N}}, \mu) - \Theta^{\lambda}_{\mu}(\widehat{\mathbf{N}}, \mu)\Big] && \textbf{(by the definitions of } \mu^{\tau}_{\widehat{\mathbf{H}}} \textbf{ and } \mu^{\Theta^{\lambda}}_{\widehat{\mathbf{H}}}\textbf{)}.
\end{aligned} \tag{11}$$

By Equation (11),

$$\begin{aligned}
& \tau_p(\widehat{\mathbf{N}}, \mu) - \Theta^{\lambda}_p(\widehat{\mathbf{N}}, \mu) \\
= \;& \tfrac{\lambda(p)}{|\widehat{\mathbf{H}}|_{\lambda}} \cdot \Big[\tau_p(\widehat{\mathbf{N}}, \mu) + \tau_q(\widehat{\mathbf{N}}, \mu) - \Theta^{\lambda}_p(\widehat{\mathbf{N}}, \mu) - \Theta^{\lambda}_q(\widehat{\mathbf{N}}, \mu)\Big].
\end{aligned}$$

Therefore,

$$\frac{\lambda(q)}{|\widehat{\mathbf{H}}|_{\lambda}}\Big[\tau_p(\widehat{\mathbf{N}}, \mu) - \Theta^{\lambda}_p(\widehat{\mathbf{N}}, \mu)\Big] = \frac{\lambda(p)}{|\widehat{\mathbf{H}}|_{\lambda}}\Big[\tau_q(\widehat{\mathbf{N}}, \mu) - \Theta^{\lambda}_q(\widehat{\mathbf{N}}, \mu)\Big] \tag{12}$$

for all $p, q \in \widehat{\mathbf{N}}$. By the EFF of τ and Θ^{λ},

$$\begin{aligned}
[0, 0] \;& = \mu(\widehat{\mathbf{N}}) - \mu(\widehat{\mathbf{N}}) \\
& = \sum_{q \in \widehat{\mathbf{N}}} \big[\tau_q(\widehat{\mathbf{N}}, \mu) - \Theta^{\lambda}_q(\widehat{\mathbf{N}}, \mu)\big] \\
& = \sum_{q \in \widehat{\mathbf{N}}} \tfrac{\lambda(q)}{\lambda(p)} \big[\tau_p(\widehat{\mathbf{N}}, \mu) - \Theta^{\lambda}_p(\widehat{\mathbf{N}}, \mu)\big] \quad \textbf{(by Equation (12))} \\
& = \tfrac{|\widehat{\mathbf{N}}|_{\lambda}}{\lambda(p)} \cdot \big[\tau_p(\widehat{\mathbf{N}}, \mu) - \Theta^{\lambda}_p(\widehat{\mathbf{N}}, \mu)\big].
\end{aligned}$$

Therefore, $\tau_p(\widehat{\mathbf{N}}, \mu) = \Theta_p^{\lambda}(\widehat{\mathbf{N}}, \mu)$ for all $p \in \widehat{\mathbf{N}}$. □

The following resolutions are to present that every property applied in the above theorem should be logically independent of the rest of the properties.

Example 7. *Let us define a resolution τ to be*

$$\tau_p(\widehat{\mathbf{N}}, \mu) = [0, 0]$$

*for every $(\widehat{\mathbf{N}}, \mu) \in$ **CMIG** and for every $p \in \widehat{\mathbf{N}}$. It is trivial to check that τ satisfies CON; however, it infringes WSP.*

Example 8. *Let us define a resolution τ to be*

$$\tau_p(\widehat{\mathbf{N}}, \mu) = \begin{cases} \Theta_p^{\lambda}(\widehat{\mathbf{N}}, \mu) & \text{, if } |\widehat{\mathbf{N}}| \leq 2, \\ [0,0] & \text{, otherwise.} \end{cases}$$

*for every $(\widehat{\mathbf{N}}, \mu) \in$ **CMIG** and for every $p \in \widehat{\mathbf{N}}$. It is trivial to have that τ satisfies WSP; however, it infringes CON.*

5. A Normalization

In this section, we provide a normalization of the interval individual contribution under interval uncertainty.

Definition 3. *The* **interval normalized index**, $\overline{\Theta^N}$, *is a resolution on $\mathbb{CMIG}^*$ which is defined by*

$$\Theta_i^N(\widehat{\mathbf{N}}, \mu) = \frac{\mu(\widehat{\mathbf{N}})}{\sum\limits_{k \in \widehat{\mathbf{N}}} \widetilde{\theta_k}(\widehat{\mathbf{N}}, \mu)} \cdot \widetilde{\theta_i}(\widehat{\mathbf{N}}, \mu) \tag{13}$$

*for all $(\widehat{\mathbf{N}}, \mu) \in$ **CMIG*** and for all $i \in \widehat{\mathbf{N}}$, where $\widetilde{\theta_i}(\widehat{\mathbf{N}}, \mu) = \frac{1}{2}\left(\overline{\theta_i(\widehat{\mathbf{N}}, \mu)} + \underline{\theta_i(\widehat{\mathbf{N}}, \mu)}\right)$ and **CMIG**$^* = \{(\widehat{\mathbf{N}}, \mu) \in$ **CMIG**$| \sum_{k \in \widehat{\mathbf{N}}} \widetilde{\theta_k}(\widehat{\mathbf{N}}, \mu) \neq 0\}$. Under the notion of Θ^N, all players assess the related utility of the grand coalition proportionally by applying the average utility of all players.*

Remark 2.

- *It is easy to show that the interval normalized index satisfies EFF and SYM, but it violates ZI.*
- *One could also consider two normalized resolutions as follows.*
  - *Resolution Θ^1 is defined by*

$$\Theta_i^1(\widehat{\mathbf{N}}, \mu) = \frac{\mu(\widehat{\mathbf{N}})}{\sum\limits_{k \in \widehat{\mathbf{N}}} |\theta_k(\widehat{\mathbf{N}}, \mu)|} \cdot |\theta_i(\widehat{\mathbf{N}}, \mu)|$$

    *for all $(\widehat{\mathbf{N}}, \mu) \in$ **CMIG** and for all $i \in \widehat{\mathbf{N}}$.*
  - *Resolution Θ^2 is defined by*

$$\Theta_i^2(\widehat{\mathbf{N}}, \mu) = \frac{\mu(\widehat{\mathbf{N}})}{\sum\limits_{k \in \widehat{\mathbf{N}}} \theta_k(\widehat{\mathbf{N}}, \mu)} \cdot \theta_i(\widehat{\mathbf{N}}, \mu)$$

    *for all $(\widehat{\mathbf{N}}, \mu) \in$ **CMIG**** and for all $i \in \widehat{\mathbf{N}}$, where **CMIG**$^{**} = \{(\widehat{\mathbf{N}}, \mu) \in$ **CMIG**$|0 \notin \sum_{k \in \widehat{\mathbf{N}}} \theta_k(\widehat{\mathbf{N}}, \mu)\}$.*

However, resolution Θ^1, due to considerations regarding the definition of intervals as well as multiplication and division operations, necessitates the imposition of multiple conditions and constraints to ensure its existence. On the other hand, although resolution Θ^2, based on the definition of the norm of interval, does not impose existential restrictions, its correlation with relative position of the interval related to each player's individual contribution is relatively weak. In future research, it may be possible to refine both resolutions Θ^1 and Θ^2 by incorporating additional interval arithmetic properties to enhance their theoretical consistency and applicability.

Similar to Theorem 1, we aim to characterize the interval normalized index by applying consistency. However, it becomes apparent that $(\widehat{\mathbf{H}}, \mu^{\Theta^N}_{\widehat{\mathbf{H}}})$ does not exist if $\sum_{i\in H}\theta^t_i(\widehat{\mathbf{N}},\mu)=0$. Consequently, we introduce the concept of modified bilateral consistency (MBCON) as follows. A resolution τ adheres to modified bilateral consistency (MBCON) if $(\widehat{\mathbf{H}}, \mu^{\tau}_{\widehat{\mathbf{H}}}) \in \mathbb{CMIG}^*$ for some $(\widehat{\mathbf{N}},\mu)\in\mathbb{CMIG}$ and for some $\widehat{\mathbf{H}}\subseteq\widehat{\mathbf{N}}$ with $|\widehat{\mathbf{H}}|=2$, such that $\tau_i(\widehat{\mathbf{H}}, \mu^{\tau}_{\widehat{\mathbf{H}}}) = \tau_i(\widehat{\mathbf{N}},\mu)$ for all $i\in\widehat{\mathbf{H}}$.

Lemma 6. *The interval normalized index satisfies MBCON on $\mathbb{CMIG}^*$.*

Proof. Let $(\widehat{\mathbf{N}},\mu)\in\mathbb{CMIG}^*$. If $|\widehat{\mathbf{N}}|\leq 2$, then the proof is completed. Let us assume that $|\widehat{\mathbf{N}}|\geq 3$ and $\widehat{\mathbf{H}}\subseteq\widehat{\mathbf{N}}$ with $|\widehat{\mathbf{H}}|=2$. Similarly to Equations (2) and (7),

$$\theta_i(\widehat{\mathbf{H}}, \mu^{\Theta^N}_{\widehat{\mathbf{H}}}) = \theta_i(\widehat{\mathbf{N}},\mu). \tag{14}$$

for all $i\in\widehat{\mathbf{H}}$. Let us define that $\sigma = \frac{\mu(\widehat{\mathbf{N}})}{\sum\limits_{p\in\widehat{\mathbf{N}}}\theta_p(\widehat{\mathbf{N}},\mu)}$. For all $i\in\widehat{\mathbf{H}}$,

$$\begin{aligned}
\Theta^N_i(\widehat{\mathbf{H}}, \mu^{\Theta^N}_{\widehat{\mathbf{H}}}) &= \frac{\mu^{\Theta^N}_{\widehat{\mathbf{H}}}(\widehat{\mathbf{H}})}{\sum\limits_{k\in H}\theta_k(\widehat{\mathbf{H}},\mu^{\Theta^N}_{\widehat{\mathbf{H}}})}\cdot\theta_i(\widehat{\mathbf{H}}, \mu^{\Theta^N}_{\widehat{\mathbf{H}}}) \\
&= \frac{\mu(\widehat{\mathbf{N}})-\sum\limits_{h\in\widehat{\mathbf{N}}\setminus H}\Theta^N_h(\widehat{\mathbf{N}},\mu)}{\sum\limits_{k\in H}\theta_k(\widehat{\mathbf{N}},\mu)}\cdot\theta_i(\widehat{\mathbf{N}},\mu) \\
&\big(\textbf{by Equation (14) and the definition of } \mu^{\Theta^N}_{\widehat{\mathbf{H}}}\big) \\
&= \frac{\sum\limits_{h\in H}\Theta^N_h(\widehat{\mathbf{N}},\mu)}{\sum\limits_{k\in H}\theta_k(\widehat{\mathbf{N}},\mu)}\cdot\theta_i(\widehat{\mathbf{N}},\mu) \\
&\big(\textbf{by the EFF of } \Theta^N\big) \\
&= \sigma\cdot\theta_i(\widehat{\mathbf{N}},\mu) \\
&= \Theta^N_i(\widehat{\mathbf{N}},\mu). \\
&\big(\textbf{by Definition 3}\big)
\end{aligned}$$

Hence, the resolution Θ^N satisfies MBCON. □

A resolution τ satisfies **normalized-standard for games (NSFG)** if $\tau(\widehat{\mathbf{N}},\mu)=\Theta^N(\widehat{\mathbf{N}},\mu)$ for all $(\widehat{\mathbf{N}},\mu)\in\mathbb{CMIG}$, $|\widehat{\mathbf{N}}|\leq 2$.

Theorem 4. *On $\mathbb{CMIG}^*$, the resolution Θ^N is the only resolution satisfying NSFG and MBCON.*

Proof. By Lemma 6, Θ^N satisfies MBCON. Clearly, Θ^N satisfies NSFG.

To prove uniqueness, let us suppose τ satisfies MBCON and NSFG on $\mathbb{CMIG}^*$. By the NSFG and MBCON of τ, it is easy to derive that τ also satisfies EFF, and hence we omit it. Let $(\widehat{\mathbf{N}},\mu)\in\mathbb{CMIG}^*$. We will complete the proof by induction on $|\widehat{\mathbf{N}}|$. If $|\widehat{\mathbf{N}}|\leq 2$, it is trivial that $\tau(\widehat{\mathbf{N}},\mu)=\Theta^N(\widehat{\mathbf{N}},\mu)$ by NSFG. Let us assume that it holds if $|\widehat{\mathbf{N}}|\leq p-1$, $p\leq 3$. In the

case when $|\widehat{\mathbf{N}}| = p$, let $i, j \in \widehat{\mathbf{N}}$ with $i \neq j$. By Definition 3, $\Theta_k^N(\widehat{\mathbf{N}}, \mu) = \frac{\mu(\widehat{\mathbf{N}})}{\sum\limits_{h \in \widehat{\mathbf{N}}} \theta_h(\widehat{\mathbf{N}}, \mu)} \cdot \theta_k(\widehat{\mathbf{N}}, \mu)$ for all $k \in \widehat{\mathbf{N}}$. Let us assume that $\eta_k = \frac{\theta_k(\widehat{\mathbf{N}}, \mu)}{\sum\limits_{h \in \widehat{\mathbf{N}}} \theta_h(\widehat{\mathbf{N}}, \mu)}$ for all $k \in \widehat{\mathbf{N}}$. Therefore,

$$\begin{aligned}
\tau_i(\widehat{\mathbf{N}}, \mu) &= \tau_i\big(\widehat{\mathbf{N}} \setminus \{j\}, \mu^{\tau}_{\widehat{\mathbf{N}}\setminus\{j\}}\big) \\
&\quad \big(\textbf{by the MBCON of } \tau\big) \\
&= \Theta_i^N\big(\widehat{\mathbf{N}} \setminus \{j\}, \mu^{\tau}_{\widehat{\mathbf{N}}\setminus\{j\}}\big) \\
&\quad \big(\textbf{by the NSFG of } \tau\big) \\
&= \frac{\mu^{\tau}_{\widehat{\mathbf{N}}\setminus\{j\}}(\widehat{\mathbf{N}}\setminus\{j\})}{\sum\limits_{k \in \widehat{\mathbf{N}}\setminus\{j\}} \theta_k\big(\widehat{\mathbf{N}}\setminus\{j\}, \mu^{\tau}_{\widehat{\mathbf{N}}\setminus\{j\}}\big)} \cdot \theta_i\big(\widehat{\mathbf{N}} \setminus \{j\}, \mu^{\tau}_{\widehat{\mathbf{N}}\setminus\{j\}}\big) \\
&= \frac{\mu(\widehat{\mathbf{N}}) - \tau_i(\widehat{\mathbf{N}}, \mu)}{\sum\limits_{k \in \widehat{\mathbf{N}}\setminus\{j\}} \theta_k(\widehat{\mathbf{N}}, \mu)} \cdot \theta_i(\widehat{\mathbf{N}}, \mu) \\
&\quad \big(\textbf{by Equation (14) and the definition of } \mu^{\Theta^N}_{\widehat{\mathbf{H}}}\big) \\
&= \frac{\mu(\widehat{\mathbf{N}}) - \tau_i(\widehat{\mathbf{N}}, \mu)}{-\theta_j(\widehat{\mathbf{N}}, \mu) + \sum\limits_{k \in \widehat{\mathbf{N}}} \theta_k(\widehat{\mathbf{N}}, \mu)} \cdot \theta_i(\widehat{\mathbf{N}}, \mu).
\end{aligned} \tag{15}$$

By Equation (15),

$$\begin{aligned}
& \tau_i(\widehat{\mathbf{N}}, \mu) \cdot [1 - \eta_j] = [\mu(\widehat{\mathbf{N}}) - \tau_j(\widehat{\mathbf{N}}, \mu)] \cdot \eta_i \\
\Longrightarrow \quad & \sum_{i \in \widehat{\mathbf{N}}} \tau_i(\widehat{\mathbf{N}}, \mu) \cdot [1 - \eta_j] = [\mu(\widehat{\mathbf{N}}) - \tau_j(\widehat{\mathbf{N}}, \mu)] \cdot \sum_{i \in \widehat{\mathbf{N}}} \eta_i \\
\Longrightarrow \quad & \mu(\widehat{\mathbf{N}}) \cdot [1 - \eta_j] = [\mu(\widehat{\mathbf{N}}) - \tau_j(\widehat{\mathbf{N}}, \mu)] \cdot 1 \\
& \big(\textbf{by the EFF of } \tau\big) \\
\Longrightarrow \quad & \mu(\widehat{\mathbf{N}}) - \mu(\widehat{\mathbf{N}}) \cdot \eta_j = \mu(\widehat{\mathbf{N}}) - \tau_j(\widehat{\mathbf{N}}, \mu) \\
\Longrightarrow \quad & \Theta_j^N(\widehat{\mathbf{N}}, \mu) = \tau_j(\widehat{\mathbf{N}}, \mu).
\end{aligned}$$

The proof is completed. □

The subsequent examples illustrate the logical independence of each axiom utilized in Theorem 3 from the remaining axioms.

Example 9. *Let us define a resolution τ to be that for all $(\widehat{\mathbf{N}}, \mu) \in \mathbb{CMIIG}^*$ and for all $i \in \widehat{\mathbf{N}}$,*

$$\tau_i(\widehat{\mathbf{N}}, \mu) = [0, 0].$$

Clearly, τ satisfies MBCON, but it violates NSFG.

Example 10. *Let us define a resolution τ to be that for all $(\widehat{\mathbf{N}}, \mu) \in \mathbb{CMIIG}^*$ and for all $i \in \widehat{\mathbf{N}}$,*

$$\tau_i(\widehat{\mathbf{N}}, \mu) = \begin{cases} \Theta_i^N(\widehat{\mathbf{N}}, \mu) & \text{, if } |\widehat{\mathbf{N}}| \leq 2, \\ [0,0] & \text{, otherwise.} \end{cases}$$

Clearly, τ satisfies NSFG, but it violates MBCON.

Application Example

- **Problem context:** Consider a scenario where a technology company collaborates with multiple partners to develop an innovative product. Each partner's contribution is influenced by resource limitations and uncertainties, such as market demands and technological breakthroughs. These uncertainties can be modeled using **interval utilities**, representing the minimum and maximum potential contributions of each partner.

- **Interval TU game-theoretical modeling:** Based on the definitions, this situation can be modeled as an *interval TU game* $(\widehat{\mathbf{N}}, \mu)$, where:
  - $\widehat{\mathbf{N}} = \{\text{Company}, \text{Partner A}, \text{Partner B}\}$ is the set of participants.
  - $\mu(S)$ is a characteristic function assigning an interval of utilities to each subset $S \subseteq N$. For example:

$$\mu(\{\text{Company}\}) = [10, 20], \quad \mu(\{\text{Partner A}\}) = [5, 15], \quad \mu(\{\text{Partner B}\}) = [8, 12].$$

The joint utility of subsets is defined as:

$$\mu(\{\text{Company}, \text{Partner A}\}) = [20, 40], \quad \mu(\{\text{Company}, \text{Partner B}\}) = [9, 21],$$
$$\mu(\{\text{Partner A}, \text{Partner B}\}) = [13, 19], \quad \mu(\widehat{\mathbf{N}}) = [30, 60].$$

- **Application of interval allocating resolutions:** Based on Definitions 1 and 2, the interval utilities can be allocated using two main approaches:
  - **The interval pseudo equal allocation of non-separable costs (IPEANSC)**
    The allocation Θ of player $i \in \widehat{\mathbf{N}}$ is defined as:

$$\Theta_i(\widehat{\mathbf{N}}, \mu) = \theta_i(\widehat{\mathbf{N}}, \mu) + \frac{1}{|\widehat{\mathbf{N}}|}\left[\mu(\widehat{\mathbf{N}}) - \sum_{j \in \widehat{\mathbf{N}}} \theta_j(\widehat{\mathbf{N}}, \mu)\right].$$

For the **Company**:

$$\Theta_{\text{Company}}(\widehat{\mathbf{N}}, \mu) = [10, 20] + \frac{1}{3}([30, 60] - [10, 20] - [5, 15] - [8, 12]).$$

Simplifying:

$$\Theta_{\text{Company}}(\widehat{\mathbf{N}}, \mu) = [10, 20] + \frac{1}{3} \cdot [7, 13] = [12.33, 24.33].$$

Similarly, the allocations for **Partner A** and **Partner B** can be computed.

  - **The interval pseudo weighted allocation of non-separable costs (IPWANSC)**
    If weights λ_i are introduced to represent the importance of participants, e.g.,

$$\lambda_{\text{Company}} = 2, \quad \lambda_{\text{Partner A}} = 1, \quad \lambda_{\text{Partner B}} = 1,$$

then the allocation $Theta^{\lambda}$ of player $i \in \widehat{\mathbf{N}}$ is given by:

$$\Theta_i^{\lambda}(\widehat{\mathbf{N}}, \mu) = \theta_i(\widehat{\mathbf{N}}, \mu) + \frac{\lambda_i}{\sum\limits_{j \in \widehat{\mathbf{N}}} \lambda_j}\left[\mu(\widehat{\mathbf{N}}) - \sum_{j \in \widehat{\mathbf{N}}} \theta_j(\widehat{\mathbf{N}}, \mu)\right].$$

For the **Company**:

$$\Theta_{\text{Company}}^{\lambda}(\widehat{\mathbf{N}}, \mu) = [10, 20] + \frac{2}{4} \cdot [7, 13] = [13.5, 26.5].$$

  - **The interval normalized index**
    The allocation Θ^N of player $i \in \widehat{\mathbf{N}}$ is given by:

$$\Theta_i^N(\widehat{\mathbf{N}}, \mu) = \frac{\mu(\widehat{\mathbf{N}})}{\sum\limits_{k \in \widehat{\mathbf{N}}} \widetilde{\theta_k}(\widehat{\mathbf{N}}, \mu)} \cdot \widetilde{\theta_i}(\widehat{\mathbf{N}}, \mu).$$

For the **Company**:

$$\Theta^{N}_{\text{Company}}(\widehat{\mathbf{N}}, \mu) = \frac{15}{15+10+10} \cdot [30, 60] = [\frac{90}{7}, \frac{180}{7}].$$

- These allocating processes provide a robust framework for fair and weighted distribution of utilities in collaborative scenarios with interval uncertainties. IPEANSC emphasizes fairness by equal division, while IPWANSC incorporates weights to reflect participants' relative importance. Such methodologies are applicable in investment planning, resource allocation, and risk management under uncertainty.

6. Conclusions

1. Unlike classical TU games, where utilities and payoffs are precisely defined, interval TU games offer a flexible framework to model the inherent uncertainties in practical settings. This study has extended the classical transferable utility (TU) game framework to incorporate interval uncertainty, addressing practical scenarios where precise utilities and payoffs are not always available. By building upon the concepts of the pseudo equal allocation of non-separable costs (PEANSC) and the pseudo weighted allocation of non-separable costs (PWANSC), this study has developed a structured methodology for managing uncertainty in cooperative allocation settings. The key contributions and findings of this study are summarized as follows:
    - **Development of interval-based reduction and related consistency**: This study introduced a novel reduction and related consistency property specifically adapted for interval TU games, ensuring the stability and coherence of allocation decisions under uncertain conditions.
    - **Extensions of PEANSC and PWANSC**: The study systematically extends these classical allocation concepts to the interval setting, providing a more robust and flexible allocating mechanism applicable to diverse real-world challenges, including economic planning and resource distribution.
    - **Axiomatic justifications**: The proposed interval-based resolutions have been rigorously justified through a series of axiomatic results, reinforcing their mathematical soundness and alignment with classical game-theoretical principles.
    - **Introduction of a weighted generalization**: By incorporating variable importance factors among players, this study developed a weighted generalization that allows for a more tailored and context-sensitive distribution of resources, aligning with strategic and economic considerations. This weighted approach aligns with real-world scenarios where contributions vary based on strategic importance and availability of resources.
    - **Normalization for fairness**: Additionally, this study introduces a normalization process to ensure comparability and fairness in interval allocations. By standardizing the distribution of resources across players with varying contribution magnitudes, the normalization mechanism enhances the robustness of the proposed allocation methods. This adjustment allows for a more balanced approach in settings where extreme variations in coalition contributions might otherwise skew the allocation outcomes, further strengthening the practical applicability of the proposed framework.
    - **Theoretical and practical integration**: The study bridges the gap between theoretical advancements and practical applicability, offering a comprehensive framework for implementing interval-based resolutions in cooperative game-theoretical situations.

- **Comparative analysis with classical and fuzzy approaches**: Our results highlight the advantages of interval-based resolution over traditional deterministic and fuzzy TU models, demonstrating its superiority in handling real-world uncertainty.
- **Application and computational validation**: Through a detailed application example, this study has illustrated the practical implementation of the proposed methods, showcasing their effectiveness in managing coalition-based decision-making under interval uncertainty.

By addressing key limitations in classical TU models and incorporating interval-based refinements, this study provides a novel, theoretically grounded, and practically applicable approach to cooperative allocation under uncertainty. The proposed resolutions contribute to the growing body of research on interval game theory, offering a solid foundation for further advancements and real-world implementations.

2. Introduced by Branzei et al. [1,2], interval TU games model coalition utilities and payoffs as closed intervals, accommodating real-world imprecision while maintaining mathematical rigor. Foundational techniques in interval arithmetic, as explored by Moore [3], provide the necessary tools to handle interval uncertainty. Various extensions of interval TU games have been applied in resource allocation, political negotiations, and economic planning [4–6]. Building upon the foundational works of Branzei et al. [1,2], Alparslan Gök et al. [5,8], and Liao [19,20], this study extends key allocating concepts such as the Shapley value and the core into interval settings, preserving fairness and stability. Additionally, the interval-based consistency property explored in this work complements the reduction methodologies introduced by Harsanyi [14] and refined in subsequent studies by Hart and Mas-Colell [18] and Moulin [17], providing a unified approach to analyzing allocation processes. The introduction of interval-based allocation enhances decision-making processes in environments where coalition values fluctuate due to uncertain external conditions. One should proceed with a comparison and analysis of the relevant existing studies.
   - **Interval-Based Extensions**: While Alparslan Gök et al. [8] explored interval-based Shapley values and core concepts, this study advances interval TU frameworks by incorporating weighted allocations and normalization mechanisms.
   - **Axiomatic and Reduction-Based Justifications**: Prior studies, such as Guerrero and Olvera-Lopez [10], focused on axiomatic characterizations of interval Myerson values. This study builds upon axiomatic principles and introduces interval-based consistency and reduction methodologies.
   - **Advancements in Allocation Mechanisms**: Unlike previous works emphasizing deterministic models [1,2], this study refines interval TU allocations through structured normalization and weighted contributions, ensuring fairness and efficiency in resource distribution.
   - **Bridging Interval and Classical TU Resolutions**: This study integrates insights from Harsanyi [14] and Moulin [17] to bridge interval TU solutions with classical allocation principles, a gap that remains underexplored in the existing literature.
   - **Comparisons with Fuzzy and Stochastic Models**: While Dubois and Prade [13] introduced fuzzy cooperative frameworks and Billera et al. [12] analyzed stochastic games, this study enhances interval-based approaches, offering a more robust alternative to uncertainty handling in cooperative game theory.
3. This study builds upon the framework established by Liao [19] while extending the results of Hsieh [15].
   - Specifically, Liao's [19] work provided foundational principles for interval game-theoretical resolutions, but it did not incorporate an explicit weighting mech-

anism or normalization process. This study extends Liao's [19] approach by integrating these features, thereby ensuring a more balanced allocation process that better reflects real-world economic and strategic conditions.

- The other major difference is that the extended resolutions in this study are based on the concept of individual contribution, whereas Liao's [19] results are founded on EANSC. Moreover, the game structure $\mathbb{CMIG}$ discussed in this study differs from the $\mathbb{FMIG}$ structure emphasized in Liao's [19] work.
- Furthermore, compared to Hsieh [15], which focused on classical TU game extensions, this study bridges the gap between interval-based frameworks and non-separable cost allocation.

4. By applying the related discussions and comparisons mentioned above, the distinct contributions of this study are as follows.
   - **Novel interval-based allocating principles**: Introducing extended PEANSC and PWANSC formulations tailored for interval TU environments.
   - **Weighted generalization and normalization**: Ensuring comparability and fairness by refining allocation processes under uncertainty.
   - **Theoretical and practical integration**: Bridging theoretical advancements with real-world applications through structured resolution mechanisms.
   - **Enhanced fairness and stability**: Demonstrating superior allocation stability compared to classical TU and fuzzy models.

   By systematically integrating interval-based reductions, axiomatic refinements, and normalization techniques, this study presents a structured and theoretically sound advancement in cooperative game theory. Its applications extend to economic modeling, strategic decision-making, and coalition stability, reinforcing its significance in modern uncertainty-driven environments.
5. The findings of this study open several avenues for future research:
   - Exploring additional subclasses of interval TU games, such as those involving fuzzy or stochastic utilities, to further enhance their applicability. This would provide a broader theoretical foundation for uncertainty modeling in cooperative games.
   - Investigating the computational efficiency of interval-based resolutions for large-scale cooperative problems to improve feasibility in practical applications. Developing algorithmic approaches to efficiently compute interval solutions will be crucial for handling high-dimensional cooperative games.
   - Conducting a deeper analysis of the role of weights in resource allocation and developing tailored strategies for specific industry situations. The introduction of customized weighting mechanisms may lead to more equitable and efficient allocation methods for economic, political, and industrial applications.
   - Integrating interval game-theoretical analysis with other uncertainty-handling methodologies, such as fuzzy sets or stochastic processes, to broaden the theoretical framework's applicability. Combining interval approaches with existing uncertainty modeling techniques could lead to a more comprehensive decision-support framework for real-world applications.

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References

1. Branzei, R.; Dimitrov, D.; Tijs, S. Shapley-like Values for Interval Bankruptcy Games. *Econ. Bull.* **2003**, *3*, 1–8.
2. Branzei, R.; Dimitrov, D.; Tijs, S. How to Cope with Division Problems under Interval Uncertainty of Claims. *Int. J. Uncertain. Fuzziness Knowl.-Based Syst.* **2004**, *12*, 191–200. [CrossRef]
3. Moore, R. *Methods and Applications of Interval Analysis*; SIAM: Philadelphia, PA, USA, 1979.
4. Alparslan Gök, S.Z.; Branzei, R.; Tijs, S. Cores and Stable Sets for Interval-Valued Games. Institute of Applied Mathematics, METU, and Tilburg University, Center for Economic Research, Discussion Paper, 17-2008. Available online: https://ssrn.com/abstract=1094653 (accessed on 12 February 2025).
5. Alparslan Gök, S.Z.; Miquel, S.; Tijs, S. Cooperation under Interval Uncertainty. *Math. Methods Oper. Res.* **2009**, *69*, 99–109. [CrossRef]
6. Branzei, R.; Branzei, O.; Alparslan Gök, S.Z.; Tijs, S. Cooperative Interval Games: A Survey. *Cent. Eur. J. Oper. Res.* **2010**, *18*, 397–411. [CrossRef]
7. Shapley, L.S. A Value for *n*-Person Games. In *Contributions to the Theory of Games II*; Kuhn, H.W., Tucker, A.W., Eds.; Princeton University Press: Princeton, NJ, USA, 1953; pp. 307–317.
8. Alparslan Gök, S.Z.; Branzei, R.; Tijs, S. The Interval Shapley Value: An Axiomatization. *Cent. Eur. J. Oper. Res.* **2010**, *18*, 131–140. [CrossRef]
9. Alparslan Gök, S.Z.; van den Brink, R.; Palancı, O. Moore Interval Subtraction and Interval Solutions for TU-Games. *Ann. Oper. Res.* **2024**, *343*, 293–311.
10. Santos Guerrero, L.E.; Olvera-Lopez, W. A Characterization of the Interval Myerson Value. *J. Dyn. Games* **2025**, *12*, 157–168. [CrossRef]
11. Özcan, İ.; Alparslan Gök, S.Z. On the Fuzzy Interval Equal Surplus Sharing Solutions. *Kybernetes* **2022**, *51*, 2753–2767. [CrossRef]
12. Billera, L.J.; Heath, D.C.; Raanan, J. Internal Telephone Billing Rates—A Novel Application of Non-Atomic Game Theory. *Oper. Res.* **1978**, *26*, 956–965. [CrossRef]
13. Dubois, D.; Prade, H. *Fuzzy Sets and Systems: Theory and Applications*; Academic Press: New York, NY, USA, 1980.
14. Harsanyi, J.C. A Bargaining Model for the Cooperative *n*-Person Game. In *Contributions to the Theory of Games IV*; Tucker, A.W., Luce, R.D., Eds.; Annals of Mathematics Studies 40; Princeton University Press: Princeton, NJ, USA, 1959; pp. 325–355.
15. Hsieh, Y.L.; Liao, Y.H. The Pseudo EANSC: Axiomatization and Dynamic Process. Master's Thesis, National Pingtung University, Pingtung City, Taiwan, 2016.
16. Hwang, Y.A. An NTU Value under Complement Reduced Game. *Int. J. Game Theory* **2009**, *38*, 305–324. [CrossRef]
17. Moulin, H. The Separability Axiom and Equal-Sharing Methods. *J. Econ. Theory* **1985**, *36*, 120–148. [CrossRef]
18. Hart, S.; Mas-Colell, A. Potential, Value and Consistency. *Econometrica* **1989**, *57*, 589–614. [CrossRef]
19. Liao Y.H. Solutions, Potentializability and Axiomatizations under Interval Uncertainty. *Int. J. Uncertain. Fuzziness Knowl.-Based Syst.* **2014**, *22*, 605–613. [CrossRef]
20. Liao Y.H. Value, Complement Reduction and Interval TU Games. *Econ. Bull.* **2010**, *30*, 2671–2679.

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