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Tripled Fixed Points, Obtained by Ran-Reunrings Theorem for Monotone Maps in Partially Ordered Metric Spaces

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Abstract: Using the deep result of Ran & Reunrings, we generalize existing results for tripled fixed points. In contrast to the previously known results for tripled fixed points of maps with or without the mixed monotone property in partially ordered complete metric spaces, we demonstrate that it is possible to obtain results for the existence and uniqueness of such points for arbitrary maps with a type of monotonicity, with the partial ordering in the Cartesian product arising from the maps itself. We prove theorems that ensure the existence and uniqueness for tripled fixed points for maps with different types of monotone properties. We obtain sufficient conditions for the existence and uniqueness of systems of three nonlinear matrix equations. The obtained results are illustrated by solving systems of matrix equations.

Keywords: coupled fixed points; partially ordered metric space; mixed monotone property; matrix equations

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1. Introduction

It is well known that algebraic discrete-type Riccati equations play a central role in modern control theory and signal processing. These equations arise in many important applications, such as in optimal control theory, dynamic programming, stochastic filtering, statistics, and other fields of pure and applied mathematics [1–3]. Nonlinear matrix equations are commonly used in many fields of scientific and engineering computing. Research on the existence and properties of the solution to the matrix equations, as well as the corresponding numerical methods, has important theoretical significance and practical value. In recent years, researchers have shown great interest in the matrix equation

$$X - \sum_{i=1}^{m} \mathcal{A}_i^* X^{-p_i} \mathcal{A}_i = \mathcal{Q},$$

where $0 < p_i \leq 1 (i = 1, 2, \ldots, m)$, with their primary focus being to explore the conditions for the existence of positive definite solutions, perturbation analysis, and developing iterative methods to solve these equations. Notably, there have been several noteworthy studies on the special case for $m = 1, 2$ and $p_i = 1$ [4,5].

In a series of articles [6,7], authors have studied the system of matrix equation $X \pm \mathcal{A}^* X^{-q} \mathcal{A} = \mathcal{Q}$ and have obtained deep necessary conditions for the existence of a solution. More complicated matrix equations are studied in [8]. The studies [9,10] are also related to the presented generalizations in this study. Additionally, in [11], a nonlinear system of matrix equations is investigated through an approach different from ours.

Fixed point results have been recently used across various fields, including those assumed not to be a closely related to them and their applications: fixed points of principal bundles over algebraic curves [12]; fixed points in Higgs bundles over a compact and connected Riemann surface [13,14] and the Hitchin integrable system [13]; and fixed points with applications in physics [15].

Let (X, d) be a metric space, and let $T : X \to X$ be a self-map. The contraction mapping theorem [16] and the abstract monotone iterative technique [17] are widely known and apply in many contexts. The first result on fixed points in partially ordered spaces is found in [18], but [17] laid the groundwork for numerous studies in this area. Recently [19–23], a trend has emerged to relax the contraction condition by requiring it only in a partially ordered metric space (X, d), meaning that instead of requiring $d(Tx, Ty) \leq \alpha d(x, y)$ for all $x, y \in X$, the condition is needed only when $x \preccurlyeq y$, where $\preccurlyeq$ is a partial order on (X, d).

A recent application of fixed points has arisen in solving matrix equations that include the inverted matrix X^{-1} [24]. The authors transform the matrix equation $X + \mathcal{A}^* X^{-1} \mathcal{A} - \mathcal{B}^* X^{-1} \mathcal{B} = Q$, where $\mathcal{A}$ and $\mathcal{B}$ are arbitrary $N \times N$ square matrices and Q is a positive definite $N \times N$ matrix, into a system of two matrix equations:

$$\left| \begin{array}{l} X = \mathcal{Q} - \mathcal{A}^* X^{-1} \mathcal{A} + \mathcal{B}^* Y^{-1} \mathcal{B} \\ Y = \mathcal{Q} - \mathcal{A}^* Y^{-1} \mathcal{A} + \mathcal{B}^* X^{-1} \mathcal{B}, \end{array} \right. \tag{1}$$

and apply the notion of coupled fixed points in metric spaces with a partial ordering from [25].

The system of Equation (1) can be viewed as a map $F : \mathcal{X}(N) \times \mathcal{X}(N) \to \mathcal{X}(N)$, where $\mathcal{X}(N)$ represents the space of $N \times N$ matrices, defined by $F(X, Y) = \mathcal{Q} - \mathcal{A}^* X^{-1} \mathcal{A} + \mathcal{B}^* Y^{-1} \mathcal{B}$. We seek the coupled fixed point (X, Y) [26], where $X = F(X, Y)$ and $Y = F(Y, X)$.

It is known that if certain classical assumptions on the map $F : X \times X \to X$ and the underlying normed space $(X, \|\cdot\|)$ hold, then $X = Y$, as in [27]. In [27], two maps $F, G : X \times X \to X$ are introduced, allowing the system of equations $X = F(X, Y)$ and $Y = G(X, Y)$ to have solutions where $X_0 \neq Y_0$. If $G(X, Y) = F(Y, X)$, we obtain the well-known result on coupled fixed points [25].

Following the trend noted above [23,28,29] and the observations in [27], this paper expands the concept of mixed monotone maps to provide a unified framework for a broader class of problems, where existing results become particular cases of the new findings. We establish criteria for guaranteeing the uniqueness and existence of solutions for the matrix equation system:

$$\left| \begin{array}{l} X = \mathcal{Q}_1 + \alpha_1 \mathcal{A}_1^* X^{-k} \mathcal{A}_1 + \alpha_2 \mathcal{B}_1^* Y^{-p} \mathcal{B}_1 + \alpha_3 \mathcal{C}_1^* Y^{-q} \mathcal{C}_1 \\ Y = \mathcal{Q}_2 + \alpha_1 \mathcal{A}_2^* X^{-k} \mathcal{A}_2 + \alpha_2 \mathcal{B}_2^* Y^{-p} \mathcal{B}_2 + \alpha_3 \mathcal{C}_2^* Y^{-q} \mathcal{C}_2 \\ Z = \mathcal{Q}_3 + \alpha_1 \mathcal{A}_3^* X^{-k} \mathcal{A}_3 + \alpha_2 \mathcal{B}_3^* Y^{-p} \mathcal{B}_3 + \alpha_3 \mathcal{C}_3^* Y^{-q} \mathcal{C}_3 \end{array} \right. \tag{2}$$

for $\alpha_i \in \{-1, +1\}$ for $i = 1, 2, 3$ and $k, p, q \in (0, 1]$.

Our work generalizes the results of [17] on the existence and uniqueness of fixed points in partially ordered metric spaces for monotone mappings by removing the assumption of continuity. We adapt the notion of tripled fixed points so that the solution (X, Y, Z), $X = Y = Z$ of Equation (2) represents a solution of

$$X = \mathcal{Q} + \alpha_1 \mathcal{A}^* X^{-k} \mathcal{A} + \alpha_2 \mathcal{B}^* X^{-p} \mathcal{B} + \alpha_3 \mathcal{C}^* X^{-q} \mathcal{C}. \quad (3)$$

The idea of tripled fixed points was introduced in different aspects in [30–32]. Later on, a more sophisticated notion of n-tuples of fixed points was proposed in [33].

Starting with the introduction of coupled fixed points, as well as tripled and n-tuple ones, there has been one drawback: all results lead to solutions $(x_1, x_2, \dots x_k)$ for $k = 2, 3, \dots n$, so that $x_i = x_j$ for $i, j = 1, 2, \dots n$. This reduction in the classes of possible systems of equations for which solutions can be found is overcome by using the concept of generalized coupled fixed points [27] and triples of fixed points [34,35].

It seems that most of the illustrative examples for solving of systems of algebraic equations in partially ordered space naturally satisfy the contractive type conditions for all elements $x, y \in X$ instead of only for the partially ordered ones. Following [17,24,36], it seems that a natural application of the fixed point theorem for monotone maps [17,18] exists in the investigation of matrix equations.

2. Materials and Methods

2.1. Tripled Fixed Point

In what follows, let (X, d) be a metric space, $\mathbb{N}$ be the set of natural numbers, and $\mathbb{R}$ be the set of real numbers. Whenever we consider a normed space $(X, \|\cdot\|)$, we will assume that the metric in X is the one endowed by the norm, i.e., $d(x, y) = \|x - y\|$.

Starting with the publications [18,26], it is assumed that a partially ordered set is a well-known notion. Just for completeness of the presentation, we will recall its definition, following the presentation from [37].

Definition 1 ([37]). *Let X be a set and $p \subseteq X \times X$ be a relation. The relation p is said to be:*

(d1.i) *reflexive, provided that $(x, x) \in p$ for every $x \in X$*
(d1.ii) *antisymmetric, provided that if $(x, y) \in p$ and $(y, x) \in p$ then $x = y$*
(d1.iii) *transitive, provided that $(x, y) \in p$ and $(y, z) \in p$ then $(x, z) \in p$*

Definition 2 ([37]). *Let X be a set. A partial order on X is a relation $p \subseteq X \times X$ that is reflexive, antisymmetric, and transitive. The ordered pair (X, p) is called a partially ordered set.*

For a partially ordered set (X, p), there are various partial orders with their traditional notations. For example, the relation $\delta \subseteq \mathbb{N} \times \mathbb{N}$ that checks if $m \in \mathbb{N}$ is a divisor of $n \in \mathbb{N}$ is denoted by $m|n$. Therefore, we write $m|n$ to denote that $(m, n) \in \delta$. This is the well-known divisibility relation on $\mathbb{N}$. Whenever practical, for generic partially ordered sets, the partial order relation $\preccurlyeq$ is used. Therefore, we will denote $(x, y) \in p$ by $x \preccurlyeq y$, which is easier to perceive.

We will denote by $(X, d, \preccurlyeq)$ a metric space with a partial ordering.

Definition 3 ([30]). *The ordered triple $(\xi, \eta, \zeta) \in X \times X \times X$ is said to be a tripled fixed points for the map $F : X \times X \times X \to X$ if there hold the equalities $\xi = F(\xi, \eta, \zeta)$, $y = F(\eta, \xi, \eta)$, and $\zeta = F(\zeta, \eta, \xi)$.*

The notion of mixed monotone property for a map of two variables is generalized for a map of three variables $F : X \times X \times X \to X$ in [30].

Definition 4 ([30]). *Let $(X, \preccurlyeq)$ be a partially ordered set and $F : X \times X \times X \to X$. We say that F satisfies the mixed monotone property if $F(x, y, z)$ is monotonely nondecreasing on its first x and third z variables and monotonely non increasing on its second variable y, i.e., for any $x, y, z \in X$, the following holds:*

$$\text{for } x_1, x_2 \in X, \ x_1 \preccurlyeq x_2 \text{ there holds } F(x_1, y, z) \preccurlyeq F(x_2, y, z),$$

$$\text{for } y_1, y_2 \in X, \ y_1 \preccurlyeq y_2 \text{ there holds } F(x, y_1, z) \succcurlyeq F(x, y_2, z)$$

and

$$\text{for } z_1, z_2 \in X, z_1 \preccurlyeq z_2 \text{ there holds } F(x, y, z_1) \preccurlyeq F(x, y, z_2).$$

A sufficient condition for the existence and uniqueness of tripled fixed points is presented in [30], which generalizes the results from [26].

Let (X, ρ) be a metric space. The Cartesian product space $X \times X \times X$ is equipped with the metric $\rho_1((x, y, z), (u, v, w)) = \rho(x, u) + \rho(y, v) + \rho(z, w)$.

Let $(X, \preccurlyeq)$ be a partially ordered set. Following [30], we introduce a partial order in the Cartesian product space $X \times X \times X$ as follows: for any $(x, y, z), (u, v, w) \in X \times X \times X$ we say that $(u, v, w) \preccurlyeq (x, y, z)$ if $x \succcurlyeq u$, $y \preccurlyeq v$, and $z \succcurlyeq w$.

Theorem 1 ([30]). *Let $(X, \rho, \preccurlyeq)$ be a partially ordered complete metric space and $F : X \times X \times X \to X$ be a continuous map with the mixed monotone property in X. Let there be constants $\alpha, \beta, \gamma \in [0, 1)$, such that $\alpha + \beta + \gamma < 1$ and there holds the inequality*

$$\rho(F(x, y, z), F(u, v, w)) \leq \alpha\rho(x, u) + \beta\rho(y, v) + \gamma\rho(z, w) \tag{4}$$

for any $x \succcurlyeq u$, $y \preccurlyeq v$, $z \succcurlyeq w$.

If there exist $x_0, y_0, z_0 \in X$ so that $x_0 \preccurlyeq F(x_0, y_0, z_0)$, $y_0 \succcurlyeq F(y_0, x_0, y_0)$, and $z_0 \preccurlyeq F(z_0, y_0, x_0)$, then there are $\xi, \eta, \zeta \in X$, which are a tripled fixed point for F, i.e., $\xi = F(\xi, \eta, \zeta)$, $\eta = F(\eta, \xi, \eta)$, and $\zeta = F(\zeta, \eta, \xi)$.

If for any ordered triples $(x, y, z), (x_1, y_1, z_1) \in X \times X \times X$ there is an ordered tripled $(u, v, w) \in X \times X \times X$, which is comparable with both (x, y, z) and (x_1, y_1, z_1), then the tripled fixed point (x, y, z) is unique.

Due the technique of the proof, the tripled fixed point introduced in [30] is defined as $\xi = F(\xi, \eta, \zeta)$, $y = F(\eta, \xi, \eta)$, and $\zeta = F(\zeta, \eta, \xi)$, not as $\xi = F(\xi, \eta, \zeta)$, $\eta = F(\eta, \zeta, \xi)$, and $\zeta = F(\zeta, \eta, \xi)$

We will follow the approach introduced in [34,38].

Definition 5 ([34,38]). *Let X be a set and $F : X \times X \times X \to X$, $G : X \times X \times X \to X$, and $H : X \to X$. The ordered triple (F, G, H) is called an ordered triple of maps.*

Definition 6 ([34,38]). *Let X be a set and (F, G, H) be an ordered triple of maps. An element $(x, y, z) \in X \times X \times X$ is called a triple fixed point if*

$$\left| \begin{array}{l} \xi = F(\xi, \eta, \zeta) \\ \eta = G(\xi, \eta, \zeta) \\ \zeta = H(\xi, \eta, \zeta). \end{array} \right.$$

If $G(x, y, z) = F(y, x, y)$ and $H(x, y, z) = F(z, y, x)$ we obtain the tripled fixed points investigated in [30], i.e.,

$$\left| \begin{array}{l} \xi = F(\xi, \eta, \zeta) \\ \eta = F(\eta, \xi, \eta) \\ \zeta = F(\zeta, \eta, \xi). \end{array} \right.$$

2.2. Fixed Points for Monotone Maps in Partially Ordered Complete Metric Spaces

Definition 7 ([17,37]). *Let $(X, \preccurlyeq)$ be a partially ordered set. We say that a map $T : X \to X$ is monotone if it is either order preserving, i.e., $Tx \preccurlyeq Ty$ for all $x \preccurlyeq Y$, or order reversing, i.e., $f(x) \succcurlyeq y$ for all $x \preccurlyeq y$.*

The next result is a slight generalization of the main result from [17].

Theorem 2 ([17,36]). *Let $(X, d, \preccurlyeq)$ be a partially ordered complete metric space and $f : X \to X$ be a monotone contractive map.*

Let one of the following conditions hold:

(th2.i) *f is continuous*

(th2.ii) *for any convergent sequence $\lim_{n\to\infty} z_n = z$*

- *if $z_n \preccurlyeq z_{n+1}$ then $z_n \preccurlyeq z$*
- *if $z_n \succcurlyeq z_{n+1}$ then $z_n \succcurlyeq z$.*

If there exists $z_0 \in X$ such that either $z_0 \preccurlyeq fz_0$ or $z_0 \succcurlyeq fz_0$, then there is a fixed point $\xi \in X$ of f, which is a limit of the sequence $\{z_n\}_{n=1}^{\infty}$, defined by $z_n = fz_{n-1} = f^n z_0$.

The error estimations are listed below:

(th2.I) *A priori error estimate $d(z_n, \xi) \leq \frac{k^n}{1-k} d(z_0, z_1)$*

(th2.II) *A posteriori error estimate $d(z_{n+1}, \xi) \leq \frac{k}{1-k} d(z_n, z_{n+1})$*

(th2.III) *The rate of convergence $d(z_{n+1}, \xi) \leq \frac{k}{1-k} d(z_n, z_{n+1})$.*

The fixed point ξ is unique if each pair of elements $x, y \in X$ additionally has a lower bound and an upper bound.

Proposition 1 ([36]). *Let $(X, d, \preccurlyeq)$ be a partially ordered metric space and $f : X \to X$ be a monotone contractive map. Then inequality (4) is also satisfied for any $x, y \in X$ that are comparable.*

Proposition 2 ([36]). *Let $(X, d, \preccurlyeq)$ be a partially ordered metric space and $f : X \to X$ be a monotone contractive map. Let ξ be a fixed point for f and $x_0 \in X$ be comparable with ξ. Then each element of the iterated sequence $x_n = fx_{n-1} = f^n x_0$, for $n \in \mathbb{N}$ is comparable with ξ.*

Proposition 3 ([39]). *Let $(X, \preccurlyeq)$ be a partially ordered set. Every pair of elements $x, y \in X$ has either a lower or an upper bound if and only if there is $z \in X$, comparable with both of them.*

A different approach is used in [33]. Instead of considering a partial order, the notion of an F-invariant M set is proposed. We will recall the basic notions from [33] for maps $F : X^3 \to X$ instead of $F : X^n \to X$. An element $(x_1, x_2, x_3) \in X^3$ is said to be a fixed point of order 3 if

$$\left| \begin{array}{l} x_1 = F(x_1, x_2, x_3) \\ x_2 = F(x_2, x_3, x_1) \\ x_3 = F(x_3, x_1, x_2). \end{array} \right.$$

Let $F : X^3 \to X$. A non-empty subset $M \subset X^6$ is said to be an F–invariant one if for any $x_1, x_2, x_3, y_1, y_2, y_3 \in X$ there holds

$$(x_1, x_2, x_3, y_1, y_2, y_3) \in M \Leftrightarrow \left\{ \begin{array}{l} (x_2, x_3, y_1, y_2, y_3, x_1) \in M \\ (x_3, y_1, y_2, y_3, x_1, x_2) \in M \\ (y_1, y_2, y_3, x_1, x_2, x_3) \in M \\ (y_2, y_3, x_1, x_2, x_3, y_1) \in M \\ (y_3, x_1, x_2, x_3, y_1, y_2) \in M \end{array} \right.$$

and if $(x_1, x_2, x_3, y_1, y_2, y_3) \in M$, then

$$(F(x_1,x_2,x_3),F(x_2,x_3,x_1),F(x_3,x_1,x_2),F(y_1,y_2,y_3),F(y_2,y_3,y_1),F(y_3,y_1,y_2))\in M$$

Theorem 3. *Let (X,d) be a complete metric space, $F:X^3\to X$, be a continuous mapping, and $M\subset X^6$ be a non empty F–invariant set. There is $(x_1,x_2,x_3)\in X^3$ so that*

$$(F(x_1,x_2,x_3),F(x_2,x_3,x_1),F(x_3,x_1,x_2),x_1,x_2,x_3)\in M.$$

For any $(x,y,z,u,v,w)\in M$ there holds the inequality

$$d(F(x,y,z),F(u,v,w))\leq\frac{k}{3}(d(x,u)+d(y,v)+d(z,w)).$$

Then, F has a fixed point of order 3*. If, in addition, for any $(x_1,x_2,x_3),(y_1,y_2,y_3)\in X^3$ there is $(z_1,z_2,z_3)\in X^3$ so that $(x_1,x_2,x_3,z_1,z_2,z_3),(y_1,y_2,y_3,z_1,z_2,z_3)\in M$, then the fixed point is unique and its components are equal.*

2.3. Matrix Equations

Throughout this section, we will denote by $\mathcal{H}(\mathcal{N})$ the set of all $N\times N$ Hermitian matrices. From [40], we will denote by $\mathcal{A}^*$ the conjugated A matrix. $\mathcal{A}$ is called Hermitian if $\mathcal{A}=\mathcal{A}^*$, skew-Hermitian if $\mathcal{A}=-\mathcal{A}^*$, unitary if $\mathcal{A}\mathcal{A}^*=I$, where I is identity matrix, and the matrix is normal if $\mathcal{A}\mathcal{A}^*=\mathcal{A}^*\mathcal{A}$. A matrix $\mathcal{A}$ is said to be positive semidefinite if $\langle x,\mathcal{A}x\rangle\geq 0$ for all nonzero vectors x. The notation $\mathcal{A}\geq 0$ will be used to express the fact that $\mathcal{A}$ is a positive matrix. If $\langle x,\mathcal{A}x\rangle>0$ for all nonzero x, we will say $\mathcal{A}$ is a positive definite. We will then write $\mathcal{A}>0$. A positive matrix is strictly positive if and only if it is invertible. If $\mathcal{A}$ and $\mathcal{B}$ are Hermitian matrices, then we say $\mathcal{A}\geq\mathcal{B}(\mathcal{A}>\mathcal{B})$ if the matrix $\mathcal{A}-\mathcal{B}$ is positive semidefinite (positive definite), i.e., $\mathcal{A}-\mathcal{B}\geq 0(\mathcal{A}-\mathcal{B}>0)$. For any matrix $\mathcal{A}$ the matrix $\mathcal{A}\mathcal{A}^*$ is always positive, and its unique positive definite square root is denoted by $|\mathcal{A}|$. The eigenvalues of $|\mathcal{A}|$ counted with multiplicities are called the singular values of $\mathcal{A}$. We will always enumerate these in decreasing order, and use for them the notation $\sigma_1(\mathcal{A})\geq\sigma_2(\mathcal{A})\geq\ldots\geq\sigma_n(\mathcal{A})$. If rank $\mathcal{A}=k$, then $\sigma_k(\mathcal{A})>0$, but $\sigma_{k+1}(\mathcal{A})=\ldots=\sigma_n(\mathcal{A})=0$. Every matrix $\mathcal{A}$ is unitarily equivalent (or unitarily similar) to an upper triangular matrix T, i.e., $\mathcal{A}=QTQ^*$, where Q is unitary.

We will endow $\mathcal{H}(\mathcal{N})$ with a partial ordering $\preccurlyeq$, i.e., $\mathcal{A}=(a_{i,j})\preccurlyeq\mathcal{B}=(b_{i,j})$ if $a_{i,j}\leq b_{i,j}$ for all $i,j=1,2,\ldots N$. If $\mathcal{A},\mathcal{B}\in\mathcal{H}(\mathcal{N})$, we will write $\mathcal{A}\succcurlyeq\mathcal{B}$ if $\mathcal{B}\preccurlyeq\mathcal{A}$. If $\mathcal{A}\preccurlyeq\mathcal{B}$ and $\mathcal{A}\neq\mathcal{B}$, we will denote it by $\mathcal{A}\prec\mathcal{B}$. By $\mathcal{A}=0$ we will denote the $N\times N$ matrix, that satisfies $a_{ij}=0$ for all $i,j=1,\ldots N$. If $\mathcal{A}\in\mathcal{H}(\mathcal{N})$ by $\mathcal{A}\succcurlyeq 0$ ($\mathcal{A}\succ 0$) we will denote that $\mathcal{A}$ is a positive semi-definite (positive definite). For $\mathcal{A},\mathcal{B}\in\mathcal{H}(\mathcal{N})$ by $\mathcal{A}\succcurlyeq\mathcal{B}$ ($\mathcal{A}\succ\mathcal{B}$) we will assume that $\mathcal{A}-\mathcal{B}$ is a positive semi-definite (positive definite) matrix. If there hold the inequalities $\mathcal{A}\preccurlyeq X\preccurlyeq\mathcal{B}$ we will use the notation $X\in[\mathcal{A},\mathcal{B}]$. We will denote by $\mathcal{A}^*$ the conjugated transpose, and by $r(\mathcal{A})$ we will denote the spectral radius of $\mathcal{A}$. We will denote by $\|.\|$ the spectral norm, i.e., $\|\mathcal{A}\|=\sqrt{\mathcal{A}^+(\mathcal{A}^*\mathcal{A})}$, where $\mathcal{A}^+(\mathcal{A}^*\mathcal{A})$ is the largest eigenvalue of $\mathcal{A}^*\mathcal{A}$. The $N\times N$ identity matrix will be written as I. We will endow the space $\mathcal{H}(\mathcal{N})$ with the norm $\|\cdot\|_{\mathrm{tr}}$. Let it be recalled, just for completeness, that $\|\mathcal{A}\|_{\mathrm{tr}}=\sum_{j=1}^{N}\sigma_j(\mathcal{A})$, where $\sigma_j(\mathcal{A})$, $j=1,2,\ldots,N$ are the singular values of $\mathcal{A}$.

The next lemmas will be useful later.

Lemma 1 ([17])**.** *Let $\mathcal{A}\succcurlyeq 0$ and $\mathcal{B}\succcurlyeq 0$ be $N\times N$ matrices, then $0\leq tr(\mathcal{A}\mathcal{B})\leq\|\mathcal{A}\|tr(\mathcal{B})$.*

Lemma 2 ([40])**.** *If $0<\theta\leq 1$, and P and Q are positive definite matrices of the same order with $P,Q\geq bI>0$, then for every unitarily invariant norm $|||\cdot|||$ there hold the inequalities $|||P^\theta-Q^\theta|||\leq\theta b^{\theta-1}|||P-Q|||$ and $|||P^{-\theta}-Q^{-\theta}|||\leq\theta b^{-(\theta+1)}|||P-Q|||$.*

Lemma 3 ([40]). *Let $\mathcal{A} \in \mathcal{H}(\mathcal{N})$ satisfying $-I \prec \mathcal{A} \prec I$, then $\|\mathcal{A}\| < 1$.*

Theorem 4 (e.g., [41]). *Let S be a non-empty, compact, convex subset of a normed vector space. Every continuous function $f : S \to S$ mapping S into itself has a fixed point.*

3. Results

3.1. Tripled Fixed Points for Maps with the Mixed Monotone Property in Partially Ordered Metric Spaces

Proposition 4. *Let $(X, d, \preccurlyeq)$ be a metric space with a partial ordering, $F_k : X \times X \times X \to X$, $k = 1, 2, 3$ be such that (F_1, F_2, F_3) has the mixed monotone property. If $(X \times X \times X, \preccurlyeq)$ is endowed with the partial ordering $w = (x, y, z) \preccurlyeq (p, q, s) = u$, provided that $x \preccurlyeq p$, $y \succcurlyeq q$, and $z \preccurlyeq s$, then $Tz = (F_1(w), F_2(w), F_3(w))$ is monotone increasing map.*

Proof. From the mixed monotone property of (F_1, F_2, F_3) for any $w = (x, y, z) \preccurlyeq (p, q, s) = u$, i.e., $x \preccurlyeq p$, $y \succcurlyeq q$, and $z \preccurlyeq s$, it follows

$$F_k(x, y, z) \preccurlyeq F(p, y, z) \preccurlyeq F_k(p, q, z) \preccurlyeq F_k(p, q, s)$$

for $k = 1, 2, 3$. Thus

$$\begin{aligned} Tw = T(x, y, z) &= (F_1(x, y, z), F_2(x, y, z), F_3(x, y, z)) \\ &\preccurlyeq (F_1(p, q, s), F_2(p, q, s), F_3(p, q, s)) = T(p, q, s) = Tu. \end{aligned}$$

Consequently, T is a monotone increasing with respect to the ordering $\preccurlyeq$ in the Cartesian product space $(X \times X \times X, \preccurlyeq)$. □

Till the end of this subsection we will endow the Cartesian product set $X \times X \times X$ with the partial order $w = (x, y, z) \preccurlyeq (p, q, s) = u$, whenever $x \preccurlyeq p$, $y \succcurlyeq q$, provided that $(X, \preccurlyeq)$ be a partially ordered set.

Theorem 5. *Let $(X, \rho, \preccurlyeq)$ be a partially ordered complete metric space, and (F_1, F_2, F_3) be an ordered triple of continuous maps with the mixed monotone property. Let there exist $\alpha \in [0, 1)$, so that the inequality*

$$\sum_{k=1}^{3} \rho(F_k(x, y, z), F_k(u, v, w)) \leq \alpha(\rho(x, u) + \rho(y, v) + \rho(z, w)) \tag{5}$$

holds for all $x \succcurlyeq u$, $y \preccurlyeq v$ and $z \succcurlyeq w$.

Let one of the following hold:

(th5.i) *F_k, $k = 1, 2, 3$ are continuous maps*

(th5.ii) *for any convergent sequence $\lim_{n\to\infty}(x_n, y_n, z_n) = (x, y, z)$, $(x_n, y_n, z_n) \in X \times X \times X$*

- *if $(x_n, y_n, z_n) \preccurlyeq (x_{n+1}, y_{n+1}, z_{n+1})$ then $(x_n, y_n, z_n) \preccurlyeq (x, y, z)$*
- *if $(x_n, y_n, z_n) \succcurlyeq (x_{n+1}, y_{n+1}, z_{n+1})$ then $(x_n, y_n, z_n) \succcurlyeq (x, y, z)$.*

If there are $x_0, y_0, z_0 \in X$ so that one of the following holds

- $x_1^{(0)} \preccurlyeq F_1\left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right)$, $x_2^{(0)} \succcurlyeq F_2\left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right)$, *and* $x_3^{(0)} \preccurlyeq F_3\left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right)$
- $x_1^{(0)} \succcurlyeq F_1\left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right)$, $x_2^{(0)} \preccurlyeq F_2\left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right)$, *and* $x_3^{(0)} \succcurlyeq F_3\left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right)$

then, a tripled fixed point $(\xi_1, \xi_2, \xi_3) \in X \times X \times X$ exists, which is a limit of the sequence $\left\{\left(x_1^{(n)}, x_2^{(n)}, x_3^{(n)}\right)\right\}_{n=0}^{\infty}$, defined by $x_k^{(n)} = F_k\left(x_1^{(n)}, x_2^{(n)}, x_3^{(n)}\right)$ for $k = 1, 2, 3$ and $n \in \mathbb{N}$.

The following error estimates hold:

(th5.I) *a priori error estimate*

$$\max_{k=1,2,3}\left\{d\left(x_k^{(n)},\xi_k\right)\right\} \leq \frac{k^n}{1-k}\sum_{k=1}^{3} d\left(x_k^{(0)},x_k^{(1)}\right)$$

(th5.II) *a posteriori error estimate*

$$\max_{k=1,2,3}\left\{d\left(x_k^{(n+1)},\xi_k\right)\right\} \leq \frac{k}{1-k}\sum_{k=1}^{3} d\left(x_k^{(n)},x_k^{(n+1)}\right)$$

(th5.III) *the rate of convergence*

$$\max_{k=1,2,3}\left\{d\left(x_k^{(n+1)},\xi_k\right)\right\} \leq \frac{k}{1-k}\sum_{k=1}^{3} d\left(x_k^{(n)},\xi_k\right).$$

If each pair of components $(x,y,z),(u,v,w) \in (X\times X\times X,\preccurlyeq)$, *where* $(x_1,x_2,x_3) \preccurlyeq (y_1,y_2,y_3)$ *if* $x_1 \preccurlyeq y_1$, $x_2 \succcurlyeq y_2$, *and* $x_3 \preccurlyeq y_3$, *has either a lower bound or an upper bound, then* (ξ_1,ξ_2,ξ_3) *is a unique coupled fixed point.*

Remark 1. *It should be pointed out that a particular case of Theorem 5 is proven in [38], where a variational technique is used, and only the uniqueness and existence result is obtained without the convergence of the iterated sequence or error estimates. It is also assumed that it holds that* F_k, $k=1,2,3$ *are continuous maps and* $x_0 \preccurlyeq F_1(x_0,y_0,z_0)$, *and* $y_0 \succcurlyeq F_2(x_0,y_0,z_0)$, *and* $z_0 \preccurlyeq F_3(x_0,y_0,z_0)$.

Proof. Let the partial ordering $w=(u,v,z) \preccurlyeq (p,q,s)=u$ be assigned to $(X\times X\times X,\preccurlyeq)$, given that $p \preccurlyeq x$, $q \succcurlyeq v$, and $s \preccurlyeq z$. Let the function T be defined by

$$T(x,y,z)=(F_1(x,y,z),F_2(x,y,z),F_3(x,y,z)).$$

From Proposition 4, if $X\times X\times X$ is endowed with the partial ordering $(x,y,z) \preccurlyeq (p,q,s)$, then T is an increasing map.

We will endow the partially ordered Cartesian product $(X\times X\times X,\preccurlyeq)$ with the metric $\rho((x_1,x_2,x_3),(y_1,y_2,y_3))=\sum_{k=1}^{3} d(x_k,y_k)$, which satisfies the property

$$\lim_{n\to\infty}\rho\left(\left(x_1^{(n)},x_2^{(n)},x_3^{(n)}\right),(x_1,x_2,x_3)\right)=0$$

if and only if $\lim_{n\to\infty} d\left(x_k^{(n)},x_k\right)=0$ for all $k=1,2,3$.

We will rewrite (5) for the map T and the metric $\rho(\cdot,\cdot)$. Let us denote

$$x=(x_1,x_2,x_3),\ u=(u_1,u_2,u_3)\in X\times X\times X$$

and $x \succcurlyeq u$. Then

$$\begin{aligned}\rho(Tx,Tu) &= \rho((F_1(x),F_2(x),F_3(x)),(F_1(u),F_2(u),F_3(u)))\\ &= \sum_{k=1}^{3} d(F_k(x_1,x_2,x_3),F_k(u_1,u_2,u_3)) \leq \alpha\sum_{k=1}^{3} d(x_k,u_k) \qquad (6)\\ &= k\rho((x_1,x_2x_3),(u_1,u_2,u_3))=k\rho(x,u).\end{aligned}$$

Consequently, $T: X\times X\times X \to X\times X\times X$ satisfies (4), i.e., T is a monotone contractive map.

If F_k, $k=1,2,3$ are continuous in (X,d), then T is continuous in $(X\times X\times X,\rho)$.

Assumption (th5.ii) in Theorem 5 is the assumption for the map f in Theorem 2 to be a contractive one.

If

$$x_1^{(0)} \preccurlyeq F_1\left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right), x_2^{(0)} \succcurlyeq F_2\left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right), \text{and } x_3^{(0)} \preccurlyeq F_3\left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right)$$

then $\left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right) \preccurlyeq T\left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right)$.

If

$$x_1^{(0)} \succcurlyeq F_1\left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right), x_2^{(0)} \preccurlyeq F_2\left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right), \text{and } x_3^{(0)} \succcurlyeq F_3\left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right)$$

then $\left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right) \succcurlyeq T\left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right)$.

As a result, it can be concluded that there is a tripled fixed point for the ordered triple of maps (F_1, F_2, F_3), or that there is a fixed point $(\xi_1, \xi_2, \xi_3) \in X \times X \times X$ of T, because all the conditions in Theorem 2 are fulfilled.

The error estimates follow directly by the definition of the metric function $\rho(\cdot, \cdot)$ in $X \times X \times X$. □

Theorem 6. *Adding to the hypotheses of Theorem 5 the following conditions:*

(th6.i) $F_2(\xi, \eta, \zeta) = F_1(\eta, \zeta, \xi)$ *and* $F_3(\xi, \eta, \zeta) = F_1(\zeta, \xi, \eta)$

(th6.ii) *if each pair of components* $(x, y, z), (u, v, w) \in (X \times X \times X, \preccurlyeq)$, *where* $(x_1, x_2, x_3) \preccurlyeq (y_1, y_2, y_3)$ *if* $x_1 \preccurlyeq y_1$, $x_2 \succcurlyeq y_2$, *and* $x_3 \preccurlyeq y_3$, *has either a lower bound or an upper bound.*

Then the components of the tripled fixed point (ξ_1, ξ_2, ξ_3) *are equal, i.e.,* $\xi_1 = \xi_2 = \xi_3$.

Proof. The proof follows directly from [33]. Indeed, let define the F invariant set M by $(x, y, z, u, v, w) \in M$, if and only if either $(x, y, z) \preccurlyeq (u, v, w)$ or $(x, y, z) \succcurlyeq (u, v, w)$, matching all the conditions of Theorem 3. □

3.2. Tripled Fixed Points for Maps with the Total Monotone Property in Partially Ordered Metric Spaces

Definition 8. *Let* $(X, \preccurlyeq)$ *be a partially ordered set and* (F_1, F_2, F_3) *be an ordered triple of maps. We say that* (F_1, F_2, F_3) *has the total monotone property if for any* $x, x_1, x_2, y, y_1, y_2, z, z_1, z_2 \in X$ *the following holds:*

$$F_i(x_1, y, z) \preccurlyeq F_i(x_2, y, z), \text{ holds for every } i = 1, 2, 3, \text{ provided that } x_1 \preccurlyeq x_2$$

$$F_i(x, y_1, z) \preccurlyeq F_i(x, y_2, z), \text{ holds for every } i = 1, 2, 3, \text{ provided that } y_1 \preccurlyeq y_2$$

and

$$F_i(x, y, z_1) \preccurlyeq F_i(x, y, z_2), \text{ holds for every } i = 1, 2, 3, \text{ provided that } z_1 \preccurlyeq z_2.$$

Till the end of this subsection, we will endow the Cartesian product set $X \times X \times X$ with the partial order $w = (x, y, z) \preccurlyeq (p, q, s) = u$, whenever $x \preccurlyeq p$, $y \preccurlyeq q$, provided that $(X, \preccurlyeq)$ be a partially ordered set.

Theorem 7. *Let* $(X, \rho, \preccurlyeq)$ *be a partially ordered complete metric space, and* (F_1, F_2, F_3) *be an ordered triple of continuous maps with the total monotone property. Let there exist* $\alpha \in [0, 1)$, *so that the inequality*

$$\sum_{k=1}^{3} \rho(F_k(x, y, z), F_k(u, v, w)) \leq \alpha(\rho(x, u) + \rho(y, v) + \rho(z, w)) \tag{7}$$

holds for all $x \succcurlyeq u$, $y \succcurlyeq v$ *and* $z \succcurlyeq w$.

Let one of the following hold:

(th7.i) F_k, $k = 1, 2, 3$ *are continuous maps*

(th7.ii) *for any convergent sequence* $\lim_{n\to\infty}(x_n, y_n, z_n) = (x, y, z)$, $(x_n, y_n, z_n) \in X \times X \times X$

- *if* $(x_n, y_n, z_n) \preccurlyeq (x_{n+1}, y_{n+1}, z_{n+1})$ *then* $(x_n, y_n, z_n) \preccurlyeq (x, y, z)$
- *if* $(x_n, y_n, z_n) \succcurlyeq (x_{n+1}, y_{n+1}, z_{n+1})$ *then* $(x_n, y_n, z_n) \succcurlyeq (x, y, z)$.

If there are $x_1^{(0)}, x_2^{(0)}, x_3^{(0)} \in X$ *so that one of the following holds*

- $x_1^{(0)} \preccurlyeq F_1\left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right)$, $x_2^{(0)} \preccurlyeq F_2\left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right)$, *and* $x_3^{(0)} \preccurlyeq F_3\left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right)$
- $x_1^{(0)} \succcurlyeq F_1\left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right)$, $x_2^{(0)} \succcurlyeq F_2\left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right)$, *and* $x_3^{(0)} \succcurlyeq F_3\left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right)$

then a tripled fixed point $(\xi_1, \xi_2, \xi_2) \in X \times X \times X$ *exists.*

The following error estimates hold:

(th7.I) *a priori error estimate*

$$\max_{k=1,2,3}\left\{d\left(x_k^{(n)}, \xi_k\right)\right\} \leq \frac{k^n}{1-k}\sum_{k=1}^{3} d\left(x_k^{(0)}, x_k^{(1)}\right)$$

(th7.II) *a posteriori error estimate*

$$\max_{k=1,2,3}\left\{d\left(x_k^{(n+1)}, \xi_k\right)\right\} \leq \frac{k}{1-k}\sum_{k=1}^{3} d\left(x_k^{(n)}, x_k^{(n+1)}\right)$$

(th7.III) *the rate of convergence*

$$\max_{k=1,2,3}\left\{d\left(x_k^{(n+1)}, \xi_k\right)\right\} \leq \frac{k}{1-k}\sum_{k=1}^{3} d\left(x_k^{(n)}, \xi_k\right).$$

If each pair of components $(x, y, z), (u, v, w) \in (X \times X \times X, \preccurlyeq)$ *also has a lower bound or an upper bound, then* (ξ_1, ξ_2, ξ_3) *is a unique coupled fixed point.*

Proof. Let the partial ordering $w = (u, v, z) \preccurlyeq (p, q, s) = u$ be assigned to $(X \times X, \preccurlyeq)$, given that $p \preccurlyeq x$, $q \preccurlyeq v$, and $s \preccurlyeq z$. Let the function T be defined by

$$T(x, y, z) = (F_1(x, y, z), F_2(x, y, z), F_3(x, y, z)).$$

From Proposition 4, if $X \times X \times X$ is endowed with the partial ordering $(x, y, z) \preccurlyeq (p, q, s)$, then T is an increasing map.

We will endow the partially ordered Cartesian product $(X \times X \times X, \preceq)$ with the metric $\rho((x_1, x_2, x_3), (y_1, y_2, y_3)) = \sum_{k=1}^{3} d(x_k, y_k)$, that satisfies the property

$$\lim_{n\to\infty} \rho((x_1^{(n)}, x_2^{(n)}, x_3^{(n)}), (x_1, x_2, x_3)) = 0$$

if and only if $\lim_{n\to\infty} d(x_k^{(n)}, x_k) = 0$ for all $k = 1, 2, 3$.

We will rewrite (7) for the map T and the metric $\rho(\cdot, \cdot)$. Let us denote $x = (x_1, x_2, x_3), u = (u_1, u_2, u_3) \in X \times X \times X$ and $x \succcurlyeq u$. Then

$$\begin{aligned} \rho(Tx, Tu) &= \rho((F_1(x), F_2(x), F_3(x)), (F_1(u), F_2(u), F_3(u))) \\ &= \sum_{k=1}^{3} d(F_k(x_k, x_2, x_3), F_k(u_1, u_2, u_3)) \leq \alpha \sum_{k=1}^{3} d(x_k, u_k) \\ &= \alpha\rho((x_1, x_2x_3), (u_1, u_2, u_3)) = \alpha\rho(x, u). \end{aligned} \tag{8}$$

Consequently, $T : X \times X \times X \to X \times X \times X$ satisfies (4), i.e., T is a monotone contractive map.

If F_k, $k = 1, 2, 3$ are continuous in (X, d) then T is continuous in $(X \times X \times X, \rho)$.

Assumption (th7.ii) in Theorem 5 is the assumption for the map f in Theorem 2 to be a contractive one.

If

$$x_1^{(0)} \preccurlyeq F_1\left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right), x_2^{(0)} \preccurlyeq F_2\left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right), \text{and } x_3^{(0)} \preccurlyeq F_3\left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right)$$

then $\left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right) \preccurlyeq T\left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right)$.

If

$$x_1^{(0)} \succcurlyeq F_1\left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right), x_2^{(0)} \succcurlyeq F_2\left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right), \text{and } x_3^{(0)} \succcurlyeq F_3\left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right)$$

then $\left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right) \succcurlyeq T\left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right)$.

As a result, it can be concluded that there is a tripled fixed point for the ordered triple of maps (F_1, F_2, F_3), or that there is a fixed point $(\xi_1, \xi_2, \xi_3) \in X \times X \times X$ of T, because all the conditions in Theorem 2 are fulfilled.

The error estimates follow directly by the definition of the metric function $\rho(\cdot, \cdot)$ in $X \times X \times X$. □

Theorem 8. *Adding to the hypotheses of Theorem 7 the following conditions:*

(th8.i) $F_2(\xi, \eta, \zeta) = F_1(\eta, \zeta, \xi)$ *and* $F_3(\xi, \eta, \zeta) = F_1(\zeta, \xi, \eta)$

(th8.ii) *any two triples* $(x, y, z), (u, v, w) \in X \times X \times X$ *there is a triple* $(p, q, r) \in X \times X \times X$*), comparable with both* $(x, y, z), (u, v, w)$.

The components of the tripled fixed point (ξ_1, ξ_2, ξ_3) *are equal, i.e.,* $\xi_1 = \xi_2 = \xi_3$.

Proof. The proof follows directly from [33]. Indeed, let us define the F invariant set M by $(x, y, z, u, v, w) \in M$, if and only if either $(x, y, z) \preccurlyeq (u, v, w)$ or $(x, y, z) \succcurlyeq (u, v, w)$, matching all the conditions of Theorem 3. □

4. Applications of Theorems 5 and 7 in Solving of Systems of Matrix Equations

4.1. Application of Theorem 5

We will use the notation $\Omega_{aI} = \{X \in \mathcal{H}(N) : X \succcurlyeq aI\}$. Let the system of matrix Equation (9)

$$X_i = \mathcal{Q}_i + \sum_{k=1}^{3} (-1)^k \mathcal{A}_{ik}^* X_k^{-\theta_k} \mathcal{A}_{ik}, \quad i, k = 1, 2, 3 \tag{9}$$

be considered, where $\theta_k \in (0, 1]$, $\mathcal{A}_{ik} \in \mathcal{X}(N)$, $\mathcal{Q}_i \in \mathcal{H}(N)$ and $\mathcal{Q}_i, \mathcal{A}_{ik}^* \mathcal{A}_{ik}, \in \mathcal{P}(N)$ for $i, k = 1, 2, 3$.

Theorem 9. *Let there be* $0 < a_1, a_3 < a_2$, *so that*

(th9.i) $\mathcal{Q}_i - a_1^{-\theta_1} \mathcal{A}_{i1}^* \mathcal{A}_{i1} - a_1^{-\theta_3} \mathcal{A}_{i3}^* \mathcal{A}_{i3} \succcurlyeq a_1 I$, *for* $i = 1, 2, 3$

(th9.ii)

$$\begin{aligned}
&a_2^{\theta_2} a_3^{\theta_3} \mathcal{A}_{11}^* \mathcal{A}_{11} - a_1^{\theta_1} a_3^{\theta_3} \mathcal{A}_{12}^* \mathcal{A}_{12} + a_1^{\theta_1} a_2^{\theta_2} \mathcal{A}_{13}^* \mathcal{A}_{13} \preccurlyeq a_1^{\theta_1} a_2^{\theta_2} a_3^{\theta_3} (\mathcal{Q}_1 - a_1 I), \\
&a_2^{\theta_2} a_3^{\theta_3} \mathcal{A}_{21}^* \mathcal{A}_{21} - a_1^{\theta_1} a_3^{\theta_3} \mathcal{A}_{22}^* \mathcal{A}_{22} + a_1^{\theta_1} a_2^{\theta_2} \mathcal{A}_{23}^* \mathcal{A}_{23} \succcurlyeq a_1^{\theta_1} a_2^{\theta_2} a_3^{\theta_3} (\mathcal{Q}_2 - a_2 I), \\
&a_2^{\theta_2} a_3^{\theta_3} \mathcal{A}_{31}^* \mathcal{A}_{31} - a_1^{\theta_1} a_3^{\theta_3} \mathcal{A}_{32}^* \mathcal{A}_{32} + a_1^{\theta_1} a_2^{\theta_2} \mathcal{A}_{33}^* \mathcal{A}_{33} \preccurlyeq a_1^{\theta_1} a_2^{\theta_2} a_3^{\theta_3} (\mathcal{Q}_3 - a_3 I)
\end{aligned} \tag{10}$$

(th9.iii) $\delta = \max\left\{ \frac{\theta_1 \| \sum_{i=1}^{3} \mathcal{A}_{i1}^* \mathcal{A}_{i1} \|}{a_1^{\theta_1+1}}, \frac{\theta_2 \| \sum_{i=1}^{3} \mathcal{A}_{i2}^* \mathcal{A}_{i2} \|}{a_1^{\theta_2+1}}, \frac{\theta_3 \| \sum_{i=1}^{3} \mathcal{A}_{i3}^* \mathcal{A}_{i3} \|}{a_1^{\theta_3+1}} \right\} < 1$

are satisfied. Then the system (9) has a unique solution $\widetilde{X}_i, \in \Omega_{aI}$ *for* $i = 1, 2, 3$.

Moreover the sequences $\left\{ X_i^{(n)} \right\}_{n=0}^{\infty}$, $i = 1, 2, 3$, *defined as follows*

$$\begin{cases} X_i^{(0)} = a_i I, \\ X_i^{(n+1)} = \mathcal{Q}_i - \mathcal{A}_{i1}^* \left(X_1^{(n)} \right)^{-\theta_1} \mathcal{A}_{i1} + \mathcal{A}_{i2}^* \left(X_2^{(n)} \right)^{-\theta_2} \mathcal{A}_{i2} - A_{i3}^* \left(X_3^{(n)} \right)^{-\theta_3} \mathcal{A}_{i3}, \\ \text{for } i = 1, 2, 3 \end{cases} \tag{11}$$

converge to the solution $\widetilde{X}_i$, $i = 1, 2, 3$ *of considered system of matrix equations, in the space* $(\mathcal{H}(N), \| \cdot \|_{\mathrm{tr}}, \preccurlyeq)$, *respectively.*

The following error estimations are listed below:

(th9.I) *a priori error estimate*

$$\max_{i=1,2,3} \left\{ \left\| X_i^{(n)} - \widetilde{X}_i \right\|_{\mathrm{tr}} \right\} \leq \frac{\delta^n}{1-\delta} M\left(X_1^{(1)}, (X_1^{(0)}, X_2^{(1)}, X_2^{(0)}, X_3^{(1)}, X_3^{(0)} \right)$$

(th9.II) *a posteriori error estimate*

$$\max_{i=1,2,3} \left\{ \left\| X_i^{(n+1)} - \widetilde{X}_i \right\|_{\mathrm{tr}} \right\} \leq \frac{\delta}{1-\delta} M\left(X_1^{(n+1)}, X_1^{(n)}, X_2^{(n+1)}, X_2^{(n)}, X_3^{(n+1)}, X_3^{(n)} \right)$$

(th9.III) *the convergence rate*

$$\max_{i=1,2,3} \left\{ \left\| X_i^{(n+1)} - \widetilde{X}_i \right\|_{\mathrm{tr}} \right\} \leq \frac{\delta}{1-\delta} M\left(X_1^{(n)}, \widetilde{X}_1, X_2^{(n)}, \widetilde{X}_2, X_3^{(n)}, \widetilde{X}_3 \right),$$

where we use the notation $M(U_1, V_1, U_2, V_2, U_3, V_3) = \sum_{i=1}^{3} \| U_i - V_i \|_{\mathrm{tr}}$.

If, in addition, for $i = 1, 2, 3$ $\mathcal{Q}_i$, $\mathcal{Q}_i - a_i^{-\theta_1} \mathcal{A}_{i1}^* \mathcal{A}_{i1} - a_i^{-\theta_3} \mathcal{A}_{i3}^* \mathcal{A}_{i3} \succcurlyeq a_i I \in \mathcal{P}(N)$ *and* F_i *are continuous, then there is* $n_0 \in \mathbb{N}$ *so that* $(\widetilde{X}_1, \widetilde{X}_2, \widetilde{X}_i) \in [a_1 I, n_0 \mathcal{Q}_1] \times [a_1 I, n_0 \mathcal{Q}_2] \times [a_1 I, n_0 \mathcal{Q}_3]$.

Proof. Let us define the maps $F_i : \mathcal{H}(N) \times \mathcal{H}(N) \times \mathcal{H}(N) \to \mathcal{H}(N)$ by

$$F_i(X_1, X_2, X_3) = \mathcal{Q}_i + \sum_{k=1}^{3} (-1)^k \mathcal{A}_{ik}^* X_k^{-\theta_k} \mathcal{A}_{ik}.$$

Let $X_i \in \Omega_{a_1 I}$, i.e., $X_i \succcurlyeq a_1 I$, $i = 1, 2, 3$. Then, from $\mathcal{Q}_i, \mathcal{A}_{ik}^* \mathcal{A}_{ik}, \in \mathcal{P}(N)$ for $i, k = 1, 2, 3$ and (th9.i) we obtain

$$\begin{aligned} F_i(X_1, X_2, X_3) &= \mathcal{Q}_i + \textstyle\sum_{k=1}^{3} (-1)^k \mathcal{A}_{ik}^* X_k^{-\theta_k} \mathcal{A}_{ik} \\ &\succcurlyeq \mathcal{Q}_i - \mathcal{A}_{i1}^* X_1^{-\theta_1} \mathcal{A}_{i1} - \mathcal{A}_{i3}^* X_3^{-\theta_3} \mathcal{A}_{i3} \\ &\succcurlyeq \mathcal{Q}_i - a_1^{-\theta_1} \mathcal{A}_{i1}^* \mathcal{A}_{i1} - a_1^{-\theta_3} \mathcal{A}_{i3}^* \mathcal{A}_{i3} \succcurlyeq a_1 I. \end{aligned}$$

Thus $F_i : \Omega_{a_1 I} \times \Omega_{a_1 I} \times \Omega_{a_1 I} \to \Omega_{a_1 I}$.

We will show that the ordered triple of maps (F_1, F_2, F_3) satisfies the mixed monotone property. Let $X, U \in \Omega_{a_1 I}$ be such that $X \succcurlyeq U$ then $X^{-\theta_i} \preccurlyeq U^{-\theta_i}$. Therefore, by $\mathcal{A}_{ik}^* \mathcal{A}_{ik} \in \mathcal{P}(N)$ we get

$$F_i(X, Y, Z) - F_i(U, Y, Z) = -\mathcal{A}_{i1}^* (X^{-\theta_1} - U^{-\theta_1}) \mathcal{A}_{i1} = \mathcal{A}_{i1}^* (U^{-\theta_1} - X^{-\theta_1}) \mathcal{A}_{i1} \succcurlyeq 0,$$

$$F_i(Y, X, Z) - F_i(Y, U, Z) = \mathcal{A}_{i2}^* (X^{-\theta_2} - U^{-\theta_2}) \mathcal{A}_{i2} = -\mathcal{A}_{i2}^* (U^{-\theta_2} - X^{-\theta_2}) \mathcal{A}_{i2} \preccurlyeq 0$$

and

$$F_i(Y,Z,X) - F_i(Y,Z,U) = -\mathcal{A}_{i3}^*(X^{-\theta_3} - U^{-\theta_3})\mathcal{A}_{i3} = \mathcal{A}_{i3}^*(U^{-\theta_3} - X^{-\theta_3})\mathcal{A}_{i3} \succcurlyeq 0.$$

Thus, the ordered pair of maps satisfies the mixed monotone property.

If we set $X_i^{(0)} = a_i I$, for $i = 1,2,3$ using (th9.ii), we can easily show that $X_1^{(0)} \preccurlyeq F_1\left(X_1^{(0)}, X_2^{(0)}, X_3^{(0)}\right)$, $X_2^{(0)} \succcurlyeq F_2\left(X_1^{(0)}, X_2^{(0)}, X_3^{(0)}\right)$ and $X_3^{(0)} \preccurlyeq F_3\left(X_1^{(0)}, X_2^{(0)}, X_3^{(0)}\right)$.

The inequalities

$$F_1(a_1 I, a_2 I, a_3 I) = \mathcal{Q}_1 - a_1^{-\theta_1}\mathcal{A}_{11}^*\mathcal{A}_{11} + a_2^{-\theta_2}\mathcal{A}_{12}^*\mathcal{A}_{12} - a_3^{-\theta_3}\mathcal{A}_{13}^*A_{13} \succcurlyeq a_1 I,$$

$$F_2(a_1 I, a_2 I, a_3 I) = \mathcal{Q}_2 - a_1^{-\theta_1}\mathcal{A}_{21}^*\mathcal{A}_{21} + a_2^{-\theta_2}\mathcal{A}_{22}^*\mathcal{A}_{22} - a_3^{-\theta_3}\mathcal{A}_{23}^*\mathcal{A}_{23} \preccurlyeq a_2 I,$$

$$F_3(a_1 I, a_2 I, a_3 I) = \mathcal{Q}_3 - a_1^{-\theta_1}\mathcal{A}_{31}^*\mathcal{A}_{31} + a_2^{-\theta_2}\mathcal{A}_{32}^*\mathcal{A}_{32} - a_3^{-\theta_3}\mathcal{A}_{33}^*\mathcal{A}_{33} \succcurlyeq a_3 I$$

are equivalent (th9.ii). Thus there holds $x \preccurlyeq F_i(x,y,z)$, $y \succcurlyeq F_2(x,y,z)$ and $z \preccurlyeq F_3(x,y,z)$ for some (x,y,z).

There exists a greatest lower bound and a least upper bound for each $X, Y \in \mathcal{H}(N)$.

The maps F_i, $i = 1,2,3$ are continuous.

Let us put

$$T(x,y,z) = (F_1(x,y,z), F_2(x,y,z), F_3(x,y,z)) : \Omega_{a_1 I} \times \Omega_{a_1 I} \times \Omega_{a_1 I} \to \Omega_{a_1 I} \times \Omega_{a_1 I} \times \Omega_{a_1 I}$$

and $(\Omega_{a_1 I} \times \Omega_{a_1 I} \times \Omega_{a_1 I}, |||\cdot|||, \preccurlyeq)$, where $(x,y,z) \preccurlyeq (u,v,w)$ if $x \preccurlyeq u$, $y \succcurlyeq v$, and $z \preccurlyeq w$ and $|||(x,y,z)||| = \|x\|_{\mathrm{tr}} + \|y\|_{\mathrm{tr}} + \|z\|_{\mathrm{tr}}$. We can write down the chain of inequalities

$$\begin{aligned}
S &= |||T(X_1,X_2,X_3) - T(U_1,U_2,U_3)||| \\
&= |||(F_1(X) - F_1(U), F_2(X) - F(U), F_3(X) - F(U))||| \\
&= \sum_{i=1}^{3} \|F_i(X_1,X_2,X_3) - F_i(U_1,U_2,U_3)\|_{\mathrm{tr}} \\
&= \sum_{i=1}^{3} \left\| \sum_{j=1}^{3} \mathcal{A}_{ij}^* \left(U_j^{-\theta_j} - X_j^{-\theta_j} \right) \mathcal{A}_{ij} \right\| \\
&= \sum_{i=1}^{3} \mathrm{tr} \left(\sum_{j=1}^{3} A_{ij}^* A_{ij} \left(U_j^{-\theta_j} - X_j^{-\theta_j} \right) \right) \\
&\leq \sum_{j=1}^{3} \left\| \sum_{i=1}^{3} A_{ij}^* A_{ij} \right\| . \mathrm{tr}(U_j^{-\theta_j} - X_j^{-\theta_j}) \\
&\leq \sum_{j=1}^{3} \left\| \sum_{i=1}^{3} A_{ij}^* A_{ij} \right\| . \frac{\theta_j}{a_1^{\theta_j+1}} \mathrm{tr}(X_j - U_j) \\
&\leq \max_{j=1,2,3} \left\{ \left\| \sum_{i=1}^{3} A_{ij}^* A_{ij} \right\| \right\} \sum_{j=1}^{3} \frac{\theta_j}{a_1^{\theta_j+1}} \mathrm{tr}(X_j - U_j) \\
&= \max_{j=1,2,3} \left\{ \frac{\theta_j \| \sum_{i=1}^{3} A_{ij}^* A_{ij} \|}{a_1^{\theta_j+1}} \right\} \sum_{j=1}^{3} \mathrm{tr}(X_j - U_j) \\
&= \delta \sum_{j=1}^{3} \|X_j - U_j\|_{\mathrm{tr}} \\
&= \delta |||(X_1,X_2,X_3) - (U_1,U_2,U_3)|||,
\end{aligned}$$

where $\delta = \max\limits_{j=1,2,3} \left\{ \dfrac{\theta_j \| \sum_{i=1}^{3} A_{ij}^* A_{ij} \|}{a_1^{\theta_j+1}} \right\} < 1.$

Consequently, the ordered pair of maps (F_1, F_2, F_3) satisfies the assumption in Theorem 5, and thus a unique ordered pair $(\widetilde{X}_1, \widetilde{X}_2, \widetilde{X}_3) \in \Omega_{a_1 I} \times \Omega_{a_1 I} \times \Omega_{a_1 I}$ exists, which is a solution to system (9).

We have proven the existence of unique solutions $\widetilde{X}_1, \widetilde{X}_2, \widetilde{X}_3 \succcurlyeq a_1 I$. We will give a shorter interval where the solutions are situated.

Let there hold the inequalities

$$Q_i,\ Q_i - a_1^{-\theta_1} A_{i1}^* A_{i1} - a_1^{-\theta_3} A_{i3}^* A_{i3} \succcurlyeq a_1 I \in \mathcal{P}(N) \text{ for } i = 1, 2, 3 \tag{12}$$

For any $Q_i \in \mathcal{P}(N)$ there is $n_0 \in \mathbb{N}$ so that

$$-n_0^{-\theta_1 N} \mathcal{A}_{11}^* \mathcal{Q}_1^{-\theta_1} \mathcal{A}_{11} + a_1^{-\theta_2} \mathcal{A}_{12}^* \mathcal{A}_{12} - n_0^{-\theta_3 N} \mathcal{A}_{13}^* \mathcal{Q}_1^{-\theta_3} \mathcal{A}_{13} \preccurlyeq (n_0 - 1)\mathcal{Q}_1,$$

$$-n_0^{-\theta_1 N} \mathcal{A}_{21}^* \mathcal{Q}_1^{-\theta_1} \mathcal{A}_{21} + a_1^{-\theta_2} \mathcal{A}_{22}^* \mathcal{A}_{22} - n_0^{\theta_3 N} \mathcal{A}_{23}^* \mathcal{Q}_1^{-\theta_3} \mathcal{A}_{23} \preccurlyeq (n_0 - 1)\mathcal{Q}_2,$$

and

$$-n_0^{-\theta_1 N} \mathcal{A}_{31}^* \mathcal{Q}_1^{-\theta_1} \mathcal{A}_{31} + a_1^{-\theta_2} \mathcal{A}_{32}^* \mathcal{A}_{32} - n_0^{\theta_3 N} \mathcal{A}_{33}^* \mathcal{Q}_3^{-\theta_3} \mathcal{A}_{33} \preccurlyeq (n_0 - 1)\mathcal{Q}_3,$$

The last three inequalities are equivalent to

$$\begin{aligned} S_1 &= F_1(n_0\mathcal{Q}_1, a_1 I, n_0\mathcal{Q}_3) \\ &= \mathcal{Q}_1 - \mathcal{A}_{11}^* (n_0\mathcal{Q}_1)^{-\theta_1} \mathcal{A}_{11} + \mathcal{A}_{12}^* (a_1 I)^{-\theta_2} \mathcal{A}_{12} - \mathcal{A}_{13}^* (n_0\mathcal{Q}_3)^{-\theta_3} \mathcal{A}_{13} \\ &= \mathcal{Q}_1 - n_0^{-\theta_1 N} \mathcal{A}_{11}^* \mathcal{Q}_1^{-\theta_1} \mathcal{A}_{11} + a_1^{-\theta_2} \mathcal{A}_{12}^* \mathcal{A}_{12} - n_0^{-\theta_3 N} \mathcal{A}_{13}^* \mathcal{Q}_3^{-\theta_3} \mathcal{A}_{13} \\ &\preccurlyeq n_0\mathcal{Q}_1, \end{aligned} \tag{13}$$

$$\begin{aligned} S_2 &= F_2(n_0\mathcal{Q}_1, a_1 I, n_0\mathcal{Q}_3) \\ &= \mathcal{Q}_2 - \mathcal{A}_{21}^* (n_0\mathcal{Q}_1)^{-\theta_1} \mathcal{A}_{21} + \mathcal{A}_{22}^* (a_1 I)^{-\theta_2} \mathcal{A}_{22} - \mathcal{A}_{23}^* (n_0\mathcal{Q}_3)^{-\theta_3} \mathcal{A}_{23} \\ &= \mathcal{Q}_2 - n_0^{-\theta_1 N} \mathcal{A}_{21}^* \mathcal{Q}_1^{-\theta_1} \mathcal{A}_{21} + a_1^{-\theta_2} \mathcal{A}_{22}^* \mathcal{A}_{22} - n_0^{-\theta_3 N} \mathcal{A}_{23}^* \mathcal{Q}_3^{-\theta_3} \mathcal{A}_{23} \\ &\preccurlyeq n_0\mathcal{Q}_2, \end{aligned} \tag{14}$$

and

$$\begin{aligned} S_3 &= F_3(n_0\mathcal{Q}_1, a_1 I, n_0\mathcal{Q}_3) \\ &= \mathcal{Q}_3 - A_{31}^* (n_0\mathcal{Q}_1)^{-\theta_1} A_{31} + A_{32}^* (a_1 I)^{-\theta_2} A_{32} - A_{33}^* (n_0\mathcal{Q}_3)^{-\theta_3} A_{33} \\ &= Q_3 - n_0^{-\theta_1 N} A_{31}^* \mathcal{Q}_1^{-\theta_1} A_{31} + a_1^{-\theta_2} A_{32}^* A_{32} - n_0^{-\theta_3 N} A_{33}^* \mathcal{Q}_3^{-\theta_3} A_{33} \\ &\preccurlyeq n_0\mathcal{Q}_3. \end{aligned} \tag{15}$$

From (12) it follows

$$\begin{aligned} S_4 &= F_1(a_1 I, n_0\mathcal{Q}_2, a_1 I) \\ &= \mathcal{Q}_1 - \mathcal{A}_{11}^* (a_1 I)^{-\theta_1} \mathcal{A}_{11} + \mathcal{A}_{12}^* (n_0\mathcal{Q}_2)^{-\theta_2} \mathcal{A}_{12} - \mathcal{A}_{13}^* (a_1 I)^{-\theta_3} \mathcal{A}_{13} \\ &\succcurlyeq \mathcal{Q}_1 - a_1^{-\theta_1} \mathcal{A}_{11}^* \mathcal{A}_{11} - a_1^{-\theta_3} \mathcal{A}_{13}^* \mathcal{A}_{13} \succcurlyeq a_1 I, \end{aligned} \tag{16}$$

and

$$\begin{aligned} S_5 &= F_2(a_1 I, n_0\mathcal{Q}_2, a_1 I) \\ &= \mathcal{Q}_2 - \mathcal{A}_{21}^* (a_1 I)^{-\theta_1} \mathcal{A}_{21} + \mathcal{A}_{22}^* (n_0\mathcal{Q}_2)^{-\theta_2} \mathcal{A}_{22} - \mathcal{A}_{23}^* (a_1 I)^{-\theta_3} \mathcal{A}_{23} \\ &\succcurlyeq \mathcal{Q}_2 - a_1^{-\theta_1} \mathcal{A}_{21}^* \mathcal{A}_{21} - a_1^{-\theta_3} \mathcal{A}_{23}^* \mathcal{A}_{23} \succcurlyeq a_1 I, \end{aligned} \tag{17}$$

and

$$\begin{aligned} S_6 &= F_3(a_1 I, n_0\mathcal{Q}_2, a_1 I) \\ &= \mathcal{Q}_3 - \mathcal{A}_{31}^* (a_1 I)^{-\theta_1} \mathcal{A}_{31} + \mathcal{A}_{32}^* (n_0\mathcal{Q}_2)^{-\theta_2} \mathcal{A}_{32} - \mathcal{A}_{33}^* (a_1 I)^{-\theta_3} \mathcal{A}_{33} \\ &\succcurlyeq \mathcal{Q}_3 - a_1^{-\theta_1} \mathcal{A}_{31}^* \mathcal{A}_{31} - a_1^{-\theta_3} \mathcal{A}_{33}^* \mathcal{A}_{33} \succcurlyeq a_1 I. \end{aligned} \tag{18}$$

Let $X \in [a_1 I, n_0 Q_1]$, $Y \in [a_1 I, n_0 Q_2]$, and $Z \in [a_1 I, n_0 Q_3]$, i.e.,

$$a_1 I \preccurlyeq X \preccurlyeq n_0 Q_1 \text{ and } a_1 I \preccurlyeq Y \preccurlyeq n_0 Q_2 \text{ and } a_1 I \preccurlyeq Z \preccurlyeq n_0 Q_3.$$

Using the mixed monotone property of (F_1, F_2, F_3) and (13)–(18), we obtain

$$a_1 I \preccurlyeq F_1(a_1 I, n_0 \mathcal{Q}_2, a_1 I) \preccurlyeq F_1(X, Y, Z) \preccurlyeq F_1(n_0 \mathcal{Q}_1, a_1 I, n_0 \mathcal{Q}_3) \preccurlyeq n_0 \mathcal{Q}_1, \tag{19}$$

$$a_1 I \preccurlyeq F_2(a_1 I, n_0 \mathcal{Q}_2, a_1 I) \preccurlyeq F_2(X, Y, Z) \preccurlyeq F_2(n_0 \mathcal{Q}_1, a_1 I, n_0 \mathcal{Q}_3) \preccurlyeq n_0 \mathcal{Q}_2, \tag{20}$$

and

$$a_1 I \preccurlyeq F_3(a_1 I, n_0 \mathcal{Q}_2, a_1 I) \preccurlyeq F_3(X, Y, Z) \preccurlyeq F_3(n_0 \mathcal{Q}_1, a_1 I, n_0 \mathcal{Q}_3) \preccurlyeq n_0 \mathcal{Q}_3. \tag{21}$$

Consequently, from (19)–(21), it follows that

$$T : [a_1 I, n_0 Q_1] \times [a_1 I, n_0 Q_2] \times [a_1 I, n_0 Q_3] \to [a_1 I, n_0 Q_1] \times [a_1 I, n_0 Q_2] \times [a_1 I, n_0 Q_3],$$

where $T(X, Y, Z) = (F_1(X, Y, Z), F_2(X, Y, Z), F_3(X, Y, Z))$. As far as T is a continuous map and $[a_1 I, n_0 Q_1] \times [a_1 I, n_0 Q_2] \times [a_1 I, n_0 Q_3]$ is compact and convex according to Theorem 4, it follows that T has at least one fixed point in $[a_1 I, n_0 Q_1] \times [a_1 I, n_0 Q_2] \times [a_1 I, n_0 Q_3]$. □

We will use the notation $\Omega_{a_1 I} = \{X \in \mathcal{H}(N) : X \succcurlyeq a_1 I\}$. Let the matrix Equation (22)

$$X = \mathcal{Q} - \mathcal{A}^* X^{-\theta} \mathcal{A} + \mathcal{B}^* X^{-\theta} \mathcal{B} - \mathcal{C}^* X^{-\theta} \mathcal{C} \tag{22}$$

be considered, where, $\theta \in (0, 1]$, $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathcal{X}(N)$, $\mathcal{Q} \in \mathcal{H}(N)$ and $\mathcal{Q}, \mathcal{A}^*\mathcal{A}, \mathcal{B}^*\mathcal{B}, \mathcal{C}^*\mathcal{C} \in \mathcal{P}(N)$.

Theorem 10. *Let there be* $0 < a < b$, *so that*

(th10.i) $\mathcal{Q} - a^{-\theta}\mathcal{A}^*\mathcal{A} - a^{-\theta}\mathcal{C}^*\mathcal{C} \succcurlyeq aI$

(th10.ii)

$$b^\theta \mathcal{A}^*\mathcal{A} - a^\theta \mathcal{B}^*\mathcal{B} + b^\theta \mathcal{C}^*\mathcal{C} \preccurlyeq a^\theta b^\theta (\mathcal{Q} - aI)$$

and

$$b^\theta \mathcal{A}^*\mathcal{A} - a^\theta \mathcal{B}^*\mathcal{B} + b^\theta \mathcal{C}^*\mathcal{C} \succcurlyeq a^\theta b^\theta (\mathcal{Q} - bI),$$

(th10.iii) $\delta = \max\left\{ \frac{\theta \|\mathcal{A}^*\mathcal{A}\|}{a^{\theta+1}}, \frac{\theta \|\mathcal{B}^*\mathcal{B}\|}{a^{\theta+1}}, \frac{\theta \|\mathcal{C}^*\mathcal{C}\|}{a^{\theta+1}} \right\} < 1/3.$

Then the system

$$\begin{cases} X_0 = aI, Y_0 = bI, Z_0 = aI, \\ X_{n+1} = \mathcal{Q} - \mathcal{A}^* X_n^{-\theta} \mathcal{A} + \mathcal{B}^* Y_n^{-\theta} \mathcal{B} - \mathcal{C}^* Z_n^{-\theta} \mathcal{C} \\ Y_{n+1} = \mathcal{Q} - \mathcal{A}^* Y_n^{-\theta} \mathcal{A} + \mathcal{B}^* Z_n^{-\theta} \mathcal{B} - \mathcal{C}^* X_n^{-\theta} \mathcal{C} \\ Z_{n+1} = \mathcal{Q} - \mathcal{A}^* Z_n^{-\theta} \mathcal{A} + \mathcal{B}^* X_n^{-\theta} \mathcal{B} - \mathcal{C}^* Y_n^{-\theta} \mathcal{C} \end{cases}$$

has a unique solution (X, Y, Z), *such that* $X = Y = Z$, *which is the unique solution of Equation (22).*

4.2. Application of Theorem 7

We will use the notation $\Omega_{a_1 I} = \{X \in \mathcal{H}(N) : X \succcurlyeq a_1 I\}$. Let the system of matrix Equation (23)

$$X_i = \mathcal{Q}_i - \sum_{k=1}^{3} \mathcal{A}_{ik}^* X_k^{-\theta_k} \mathcal{A}_{ik}, \ \ i = 1, 2, 3 \tag{23}$$

be considered, where $\theta_k \in (0, 1]$, $\mathcal{A}_{ik} \in \mathcal{X}(N)$, $\mathcal{Q}_i \in \mathcal{H}(N)$ and $\mathcal{Q}_i, \mathcal{A}_{ik}^*\mathcal{A}_{ik}, \in \mathcal{P}(N)$ for $i, k = 1, 2, 3$.

Theorem 11. *Let there be* $0 < a_1, a_3 < a_2$, *so that*

(th11.i) $Q_i - \sum_{k=1}^{3} a_1^{-\theta_k} \mathcal{A}_{ik}^* \mathcal{A}_{ik} \succcurlyeq a_1 I$, *for* $i = 1, 2, 3$

(th11.ii) $a_2^{\theta_2}a_3^{\theta_3}\mathcal{A}_{i1}^*\mathcal{A}_{i1} + a_1^{\theta_1}a_3^{\theta_3}\mathcal{A}_{i2}^*\mathcal{A}_{i2} + a_1^{\theta_1}a_2^{\theta_2}\mathcal{A}_{i3}^*\mathcal{A}_{i3} \preccurlyeq a_1^{\theta_1}a_2^{\theta_2}a_3^{\theta_3}(\mathcal{Q}_i - a_1 I)$, *for* $i = 1,2,3$

(th11.iii) $\delta = \max\left\{\frac{\theta_1\|\sum_{i=1}^3 \mathcal{A}_{i1}^*\mathcal{A}_{i1}\|}{a_1^{\theta_1+1}}, \frac{\theta_2\|\sum_{i=1}^3 \mathcal{A}_{i2}^*\mathcal{A}_{i2}\|}{a_1^{\theta_2+1}}, \frac{\theta_3\|\sum_{i=1}^3 \mathcal{A}_{i3}^*\mathcal{A}_{i3}\|}{a_1^{\theta_3+1}}\right\} < 1$

are satisfied.

Then the system (23) has a unique solution $\widetilde{X}_i, \in \Omega_{aI}$ *for* $i = 1,2,3$.

Moreover, the sequences $\left\{X_i^{(n)}\right\}_{n=0}^{\infty}$, $i = 1,2,3$, *defined as follows*

$$\begin{cases} X_i^{(0)} = a_i I, \\ X_i^{(n+1)} = \mathcal{Q}_i - \mathcal{A}_{i1}^*\left(X_1^{(n)}\right)^{-\theta_1}\mathcal{A}_{i1} - \mathcal{A}_{i2}^*\left(X_2^{(n)}\right)^{-\theta_2}\mathcal{A}_{i2} - A_{i3}^*\left(X_3^{(n)}\right)^{-\theta_3}\mathcal{A}_{i3}, \\ \text{for } i = 1,2,3 \end{cases} \tag{24}$$

converge to the solution $\widetilde{X}_i$, $i = 1,2,3$ *of considered system of matrix equations, in the space* $(\mathcal{H}(N), \|\cdot\|_{\mathrm{tr}}, \preccurlyeq)$, *respectively.*

The error estimations are listed below:

(th11.I) *a priori error estimate*

$$\max_{i=1,2,3}\left\{\left\|X_i^{(n)} - \widetilde{X}_i\right\|_{\mathrm{tr}}\right\} \le \frac{\delta^n}{1-\delta}M\left(X_1^{(1)}, (X_1^{(0)}, X_2^{(1)}, X_2^{(0)}, X_3^{(1)}, X_3^{(0)}\right)$$

(th11.II) *a posteriori error estimate*

$$\max_{i=1,2,3}\left\{\left\|X_i^{(n+1)} - \widetilde{X}_i\right\|_{\mathrm{tr}}\right\} \le \frac{\delta}{1-\delta}M\left(X_1^{(n+1)}, X_1^{(n)}, X_2^{(n+1)}, X_2^{(n)}, X_3^{(n+1)}, X_3^{(n)}\right)$$

(th11.III) *the convergence rate*

$$\max_{i=1,2,3}\left\{\left\|X_i^{(n+1)} - \widetilde{X}_i\right\|_{\mathrm{tr}}\right\} \le \frac{\delta}{1-\delta}M\left(X_1^{(n)}, \widetilde{X}_1, X_2^{(n)}, \widetilde{X}_2, X_3^{(n)}, \widetilde{X}_3\right),$$

where we use the notation $M(U_1, V_1, U_2, V_2, U_3, V_3) = \sum_{i=1}^3 \|U_i - V_i\|_{\mathrm{tr}}$.

Proof. Let us define the maps $F_i : \mathcal{H}(N) \times \mathcal{H}(N) \times \mathcal{H}(N) \to \mathcal{H}(N)$ by

$$F_i(X_1, X_2, X_3) = \mathcal{Q}_i - \sum_{k=1}^{3} \mathcal{A}_{ik}^* X_k^{-\theta_k}\mathcal{A}_{ik}.$$

Let $X_i \in \Omega_{a_1 I}$, i.e., $X_i \succcurlyeq a_1 I$, $i = 1,2,3$. Then, from $\mathcal{Q}_i, \mathcal{A}_{ik}^*\mathcal{A}_{ik}, \in \mathcal{P}(N)$ for $i,k = 1,2,3$ and (th11.i) we obtain

$$F_i(X_1, X_2, X_3) = Q_i - \sum_{k=1}^{k} \mathcal{A}_{ik}^* X_k^{-\theta_k}\mathcal{A}_{ik} \succcurlyeq a_1 I.$$

Thus, $F_i : \Omega_{a_1 I} \times \Omega_{a_1 I} \times \Omega_{a_1 I} \to \Omega_{a_1 I}$.

We will show that the ordered triple of maps (F_1, F_2, F_3) satisfies the total monotone property. Let $X, U \in \Omega_{a_1 I}$ be such that $X \succcurlyeq U$ then $X^{-\theta_i} \preccurlyeq U^{-\theta_i}$. Therefore, by $\mathcal{A}_{ik}^*\mathcal{A}_{ik} \in \mathcal{P}(N)$ we obtain

$$F_i(X,Y,Z) - F_i(U,Y,Z) = -\mathcal{A}_{i1}^*(X^{-\theta_1} - U^{-\theta_1})\mathcal{A}_{i1} = -\mathcal{A}_{i1}^*(U^{-\theta_1} - X^{-\theta_1})\mathcal{A}_{i1} \succcurlyeq 0,$$

$$F_i(Y,X,Z) - F_i(Y,U,Z) = -\mathcal{A}_{i2}^*(X^{-\theta_2} - U^{-\theta_2})\mathcal{A}_{i2} = \mathcal{A}_{i2}^*(U^{-\theta_2} - X^{-\theta_2})\mathcal{A}_{i2} \succcurlyeq 0,$$

and

$$F_i(Y,Z,X) - F_i(Y,Z,U) = -\mathcal{A}_{i3}^*(X^{-\theta_3} - U^{-\theta_3})\mathcal{A}_{i3} = \mathcal{A}_{i3}^*(U^{-\theta_2} - X^{-\theta_2})\mathcal{A}_{i3} \succcurlyeq 0.$$

Thus, the ordered pair of maps satisfies the total monotone property.

If we set $X_i^{(0)} = a_i I$ for $i = 1,2,3$ using (th11.ii), we can easily show that

$$X_1^{(0)} \preccurlyeq F_1\left(X_1^{(0)}, X_2^{(0)}, X_3^{(0)}\right),$$

$$X_2^{(0)} \preccurlyeq F_2\left(X_1^{(0)}, X_2^{(0)}, X_3^{(0)}\right),$$

and

$$X_3^{(0)} \preccurlyeq F_3\left(X_1^{(0)}, X_2^{(0)}, X_3^{(0)}\right).$$

The inequalities

$$F_i(a_1 I, a_2 I, a_3 I) = \mathcal{Q}_i - a_1^{-\theta_1}\mathcal{A}_{i1}^*\mathcal{A}_{i1} - a_2^{-\theta_2}\mathcal{A}_{i2}^*\mathcal{A}_{i2} - a_3^{-\theta_3}\mathcal{A}_{i3}^* A_{i3} \succcurlyeq a_i I$$

for every $i = 1,2,3$ are equivalent (th11.ii). Thus, there holds $x \preccurlyeq F_i(x,y,z)$, $y \preccurlyeq F_2(x,y,z)$ and $z \preccurlyeq F_3(x,y,z)$ for some (x,y,z).

There exists a greatest lower bound and a least upper bound for each $X, Y \in \mathcal{H}(N)$.

The maps F_i, $i = 1,2,3$ are continuous.

Let us put

$$T(x,y,z) = (F_1(x,y,z), F_2(x,y,z), F_3(x,y,z)) : \Omega_{a_1 I} \times \Omega_{a_1 I} \times \Omega_{a_1 I} \to \Omega_{a_1 I} \times \Omega_{a_1 I} \times \Omega_{a_1 I}$$

and $(\Omega_{a_1 I} \times \Omega_{a_1 I} \times \Omega_{a_1 I}, |||\cdot|||, \preccurlyeq)$, where $(x,y,z) \preccurlyeq (u,v,w)$ if $x \preccurlyeq u$, $y \preccurlyeq v$, and $z \preccurlyeq w$ and $|||(x,y,z)||| = \|x\|_{\mathrm{tr}} + \|y\|_{\mathrm{tr}} + \|z\|_{\mathrm{tr}}$. We can write down the chain of inequalities

$$\begin{aligned}
S &= |||T(X_1,X_2,X_3) - T(U_1,U_2,U_3)||| \\
&= |||(F_1(X) - F_1(U), F_2(X) - F(U), F_3(X) - F(U))||| \\
&= \sum_{i=1}^{3} \|F_i(X_1,X_2,X_3) - F_i(U_1,U_2,U_3)\|_{\mathrm{tr}} \\
&= \sum_{i=1}^{3} \left\| \sum_{j=1}^{3} \mathcal{A}_{ij}^* \left(U_j^{-\theta_j} - X_j^{-\theta_j}\right) \mathcal{A}_{ij} \right\| \\
&= \sum_{i=1}^{3} \mathrm{tr}\left(\sum_{j=1}^{3} A_{ij}^* A_{ij} \left(U_j^{-\theta_j} - X_j^{-\theta_j}\right) \right) \\
&\leq \sum_{j=1}^{3} \left\| \sum_{i=1}^{3} A_{ij}^* A_{ij} \right\| . \mathrm{tr}(U_j^{-\theta_j} - X_j^{-\theta_j}) \\
&\leq \sum_{j=1}^{3} \left\| \sum_{i=1}^{3} A_{ij}^* A_{ij} \right\| . \frac{\theta_j}{a_1^{\theta_j+1}} \mathrm{tr}(X_j - U_j) \\
&\leq \max_{j=1,2,3} \left\{ \left\| \sum_{i=1}^{3} A_{ij}^* A_{ij} \right\| \right\} \sum_{j=1}^{3} \frac{\theta_j}{a_1^{\theta_j+1}} \mathrm{tr}(X_j - U_j) \\
&= \max_{j=1,2,3} \left\{ \frac{\theta_j \| \sum_{i=1}^{3} A_{ij}^* A_{ij} \|}{a_1^{\theta_j+1}} \right\} \sum_{j=1}^{3} \mathrm{tr}(X_j - U_j) \\
&= \delta \sum_{j=1}^{3} \|X_j - U_j\|_{\mathrm{tr}} \\
&= \delta |||(X_1,X_2,X_3) - (U_1,U_2,U_3)|||,
\end{aligned}$$

where $\delta = \max\limits_{j=1,2,3} \left\{ \dfrac{\theta_j \| \sum_{i=1}^{3} A_{ij}^* A_{ij} \|}{a_1^{\theta_j+1}} \right\} < 1.$

Consequently, the ordered pair of maps (F_1, F_2, F_3) satisfies the assumption in Theorem 7, and thus a unique ordered pair $(\widetilde{X}_1, \widetilde{X}_1, \widetilde{X}_3) \in \Omega_{aI} \times \Omega_{aI} \times \Omega_{aI}$ exists, which is a solution to system (23). □

Following the ideas from [24] we will try to solve the matrix equation

$$X = \mathcal{Q} - \mathcal{A}^* X^{-\theta} \mathcal{A} - \mathcal{B}^* X^{-\theta} \mathcal{B} - \mathcal{C}^* X^{-\theta} \mathcal{C}. \tag{25}$$

Let us put $\mathcal{A}_{11} = \mathcal{A}_{21} = \mathcal{A}_{31} = \mathcal{A}$, $\mathcal{A}_{12} = \mathcal{A}_{22} = \mathcal{A}_{32} = \mathcal{B}$, $\mathcal{A}_{13} = \mathcal{A}_{23} = \mathcal{A}_{33} = \mathcal{C}$, $\theta_1 = \theta_2 = \theta_3 = \theta$, and $\mathcal{Q}_1 = \mathcal{Q}_2 = \mathcal{Q}_3 = \mathcal{Q}$ in (23). Let us denote $F(X, Y, Z) = \mathcal{Q} - \mathcal{A}^* X^{-\theta} \mathcal{A} + \mathcal{B}^* Y^{-\theta} \mathcal{B} - \mathcal{C}^* Z^{-\theta} \mathcal{C}$. Then, it holds that $F_1(X, Y, Z) = F_1(X, Y, Z)$, $F_2(X, Y, Z) = F(Y, Z, X)$, and $F_3(X, Y, Z) = F(Z, X, Y)$, and consequently, we can apply Theorems 7 and 11.

Theorem 12. *Let there be* $0 < a$, *so that*

(th12.i) $\mathcal{Q} - a^{-\theta} \mathcal{A}^* \mathcal{A} - a^{-\theta} \mathcal{B}^* \mathcal{B} - a^{-\theta} \mathcal{C}^* \mathcal{C} \succcurlyeq aI$,

(th12.ii) $\delta = \max\left\{ \frac{\theta \|\mathcal{A}^* \mathcal{A}\|}{a^{\theta+1}}, \frac{\theta \|\mathcal{B}^* \mathcal{B}\|}{a^{\theta+1}}, \frac{\theta \|\mathcal{C}^* \mathcal{C}\|}{a^{\theta+1}} \right\} < 1/3$

are satisfied.

Then, Equation (25) has a unique solution $\widetilde{X}$, *which is the solutions of the system of matrix equations*

$$\begin{cases} X_0 = Y_0 = Z_0 = aI, \\ X_{n+1} = \mathcal{Q} - \mathcal{A}^* X_n^{-\theta} \mathcal{A} - \mathcal{B}^* Y_n^{-\theta} \mathcal{B} - \mathcal{C}^* Z_n^{-\theta} \mathcal{C} \\ Y_{n+1} = \mathcal{Q} - \mathcal{A}^* Y_n^{-\theta} \mathcal{A} - \mathcal{B}^* Z_n^{-\theta} \mathcal{B} - \mathcal{C}^* X_n^{-\theta} \mathcal{C} \\ Z_{n+1} = \mathcal{Q} - \mathcal{A}^* Z_n^{-\theta} \mathcal{A} - \mathcal{B}^* X_n^{-\theta} \mathcal{B} - \mathcal{C}^* Y_n^{-\theta} \mathcal{C}, \end{cases} \tag{26}$$

where $\widetilde{X} = \widetilde{Y} = \widetilde{Z} \in \Omega_{aI}$. *The error estimations from Theorem 11 hold.*

5. Illustrative Examples

Let us put

$$\mathcal{A}_{11} = \begin{pmatrix} 0.021 & 0.01 & 0.1 \\ 0.1 & 0.03 & 0.1 \\ 1 & 0.01 & 0.2 \end{pmatrix}, \mathcal{A}_{21} = \begin{pmatrix} 0.018\,75 & 0.004\,38 & 0.125 \\ 0.125 & 0.05 & 0.006\,25 \\ 0.0025 & 0.0125 & 0.0625 \end{pmatrix},$$

$$\mathcal{A}_{31} = \begin{pmatrix} 0.036\,67 & 0.003\,33 & 0.003\,33 \\ -0.016\,67 & 0.66 & -0.06 \\ -0.033\,33 & 0.063\,33 & 0.793\,33 \end{pmatrix}, \mathcal{A}_{12} = \begin{pmatrix} 0.083\,33 & -0.016\,67 & -0.016\,67 \\ 0.025 & 0.083\,33 & -0.025 \\ 0.033\,33 & 0.033\,33 & 0.083\,33 \end{pmatrix},$$

$$\mathcal{A}_{22} = \begin{pmatrix} 0.18 & 0.54 & -0.54 \\ 0.72 & 1.8 & 0.72 \\ 0.9 & 0.9 & 1.8 \end{pmatrix}, \mathcal{A}_{32} = \begin{pmatrix} 0.015 & 0.0005 & 0.0005 \\ 0 & 0.014 & -0.001 \\ -0.001 & 0.0015 & 0.017 \end{pmatrix},$$

$$\mathcal{A}_{13} = \begin{pmatrix} 0.2 & -0.07 & 0.11 \\ 0.07 & 0.06 & -0.13 \\ 0.11 & 0.13 & 0.14 \end{pmatrix}, \mathcal{A}_{23} = \begin{pmatrix} 0.0066 & 0.0066 & 0.016 \\ 0.0024 & 0.0072 & 0.01 \\ 0.064 & 0.1426 & 0.002\,68 \end{pmatrix},$$

$$\mathcal{A}_{33} = \begin{pmatrix} 0.02 & 0.025 & 0.006 \\ 0 & 0.07 & 0.06 \\ 0.005 & 0.005 & -0.16 \end{pmatrix}, \mathcal{Q}_1 = \begin{pmatrix} 3.5 & 0.5 & 0.5 \\ 0.5 & 3.5 & 1 \\ 0.5 & 1 & 4 \end{pmatrix},$$

$$\mathcal{Q}_2 = \begin{pmatrix} 7.5 & 0.5 & 2.5 \\ 0.5 & 7.5 & 4.5 \\ 2.5 & 4.5 & 34 \end{pmatrix}, \mathcal{Q}_3 = \begin{pmatrix} 10.5 & 7.5 & 1.5 \\ 7.5 & 12.45 & 1.5 \\ 1.5 & 1.5 & 12 \end{pmatrix},$$

and $a_1 = 2$, $a_2 = 45$, $a_3 = 2.5$, $\theta_1 = \frac{1}{5}$, $\theta_2 = \frac{1}{13}$, and $\theta_3 = \frac{1}{7}$.

We give four numerical examples to illustrate that the matrix sequences X_k, Y_k and Z_k, generated by the relevant systems, converge to the positive definite solutions $(\widetilde{X}, \widetilde{Y}, \widetilde{Z})$. We use the stopping criterion

$$\max\left\{\left\|X_{n+1} - \widetilde{X}\right\|_{\mathrm{tr}}, \left\|Y_{n+1} - \widetilde{Y}\right\|_{\mathrm{tr}}, \left\|Z_{n+1} - \widetilde{Z}\right\|_{\mathrm{tr}}\right\} \leq 10^{-10}.$$

5.1. Application of Theorem 9

Example 1. *Let us consider the system of matrix equations*

$$\begin{cases} X = \mathcal{Q}_1 - \mathcal{A}_{11}^* X^{-\theta_1} \mathcal{A}_{11} + \mathcal{A}_{12}^* Y^{-\theta_2} \mathcal{A}_{12} - \mathcal{A}_{13}^* Z^{-\theta_3} \mathcal{A}_{13} \\ Y = \mathcal{Q}_2 - \mathcal{A}_{21}^* X^{-\theta_1} \mathcal{A}_{21} + \mathcal{A}_{22}^* Y^{-\theta_2} \mathcal{A}_{22} - \mathcal{A}_{23}^* Z^{-\theta_3} \mathcal{A}_{23} \\ Z = \mathcal{Q}_3 - \mathcal{A}_{31}^* X^{-\theta_1} \mathcal{A}_{31} + \mathcal{A}_{32}^* Y^{-\theta_2} \mathcal{A}_{32} - \mathcal{A}_{33}^* Z^{-\theta_3} \mathcal{A}_{33}. \end{cases} \tag{27}$$

To solve (27) we will apply Theorem 9. It is easy to calculate $A_{ik}^* A_{ik} \succ 0$, $Q_i, A_{ik}^* A_{ik}, \in \mathcal{P}(N)$.

$$\mathcal{Q}_1 - a_1^{-\theta_1} \mathcal{A}_{11}^* \mathcal{A}_{11} - a_1^{-\theta_3} \mathcal{A}_{13}^* \mathcal{A}_{13} - a_1 I \approx \begin{pmatrix} 0.568\,73 & 0.484\,42 & 0.289\,72 \\ 0.484\,42 & 1.476\,04 & 0.992\,33 \\ 0.289\,72 & 0.992\,33 & 1.903\,75 \end{pmatrix} \succ 0,$$

$$\mathcal{Q}_2 - a_1^{-\theta_1} \mathcal{A}_{21}^* \mathcal{A}_{21} - a_1^{-\theta_3} \mathcal{A}_{23}^* \mathcal{A}_{23} - a_1 I \approx \begin{pmatrix} 5.482\,33 & 0.491\,04 & 2.496\,87 \\ 0.491\,04 & 5.481\,33 & 4.498\,32 \\ 2.496\,87 & 4.498\,32 & 31.982\,63 \end{pmatrix} \succ 0,$$

$$\mathcal{Q}_3 - a_1^{-\theta_1} \mathcal{A}_{31}^* \mathcal{A}_{31} - a_1^{-\theta_3} \mathcal{A}_{33}^* \mathcal{A}_{33} - a_1 I \approx \begin{pmatrix} 8.497\,24 & 7.510\,83 & 1.522\,66 \\ 7.510\,83 & 10.062\,26 & 1.487\,51 \\ 1.522\,66 & 1.487\,51 & 9.422\,47 \end{pmatrix} \succ 0,$$

$$\begin{aligned} S_7 &= \mathcal{Q}_1 - a_1^{-\theta_1} \mathcal{A}_{11}^* \mathcal{A}_{11} + a_1^{-\theta_2} \mathcal{A}_{12}^* \mathcal{A}_{12} - a_1^{-\theta_3} \mathcal{A}_{13}^* \mathcal{A}_{13} - a_1 I \\ &\approx \begin{pmatrix} 4.576\,83 & 0.485\,90 & 0.291\,10 \\ 0.485\,90 & 5.482\,98 & 0.993\,13 \\ 0.291\,10 & 0.993\,13 & 5.910\,99 \end{pmatrix} \succ 0, \end{aligned}$$

$$\begin{aligned} S_8 &= \mathcal{Q}_2 - a_1^{-\theta_1} \mathcal{A}_{21}^* \mathcal{A}_{21} + a_1^{-\theta_2} \mathcal{A}_{22}^* \mathcal{A}_{22} - a_1^{-\theta_3} \mathcal{A}_{23}^* \mathcal{A}_{23} - a_2 I \\ &\approx \begin{pmatrix} -36.502\,18 & 2.135\,23 & 4.019\,93 \\ 2.135\,23 & -34.278\,58 & 6.456\,55 \\ 4.019\,93 & 6.456\,55 & -7.995\,42 \end{pmatrix} \prec 0, \end{aligned}$$

and

$$\begin{aligned} S_9 &= \mathcal{Q}_3 - a_1^{-\theta_1} \mathcal{A}_{31}^* \mathcal{A}_{31} + a_1^{-\theta_2} \mathcal{A}_{32}^* \mathcal{A}_{32} - a_1^{-\theta_3} \mathcal{A}_{33}^* \mathcal{A}_{33} - a_3 I \\ &\approx \begin{pmatrix} 7.997\,42 & 7.510\,85 & 1.522\,63 \\ 7.510\,85 & 9.562\,57 & 1.487\,62 \\ 1.522\,63 & 1.487\,62 & 8.923\,52 \end{pmatrix} \succ 0. \end{aligned}$$

It holds that

$$\lambda^+(\mathcal{A}_{12}^* \mathcal{A}_{12} + \mathcal{A}_{22}^* \mathcal{A}_{22} + \mathcal{A}_{32}^* \mathcal{A}_{32}) < 8.16296,$$

and

$$\lambda^+(\mathcal{A}_{13}^* \mathcal{A}_{13} + \mathcal{A}_{23}^* \mathcal{A}_{23} + \mathcal{A}_{33}^* \mathcal{A}_{33}) < 0.10286.$$

Consequently, we obtain

$$\frac{\theta_1 \|\mathcal{A}_{11}^* \mathcal{A}_{11} + \mathcal{A}_{21}^* \mathcal{A}_{21} + \mathcal{A}_{31}^* \mathcal{A}_{31}\|}{a^{k+1}} < 0.0921,$$

$$\frac{\theta_2\|\mathcal{A}_{12}^*\mathcal{A}_{12}+\mathcal{A}_{22}^*\mathcal{A}_{22}+\mathcal{A}_{32}^*\mathcal{A}_{32}\|}{a^{l+1}}<0.1042,$$

and

$$\frac{\theta_3\|\mathcal{A}_{13}^*\mathcal{A}_{13}+\mathcal{A}_{23}^*\mathcal{A}_{23}+\mathcal{A}_{33}^*\mathcal{A}_{33}\|}{a^{r+1}}<0.0207.$$

Thus, the assumptions of Theorem 9 are satisfied. It remains to find the interval, where the coupled fixed point is situated. From

$$\begin{aligned}F_1(a_1I,n_0\mathcal{Q}_2,a_1I) &= \mathcal{Q}_1-a_1^{-\theta_1}\mathcal{A}_{11}^*\mathcal{A}_{11}+n_0^{-\theta_2 N}\mathcal{A}_{12}^*\mathcal{Q}_2^{-\theta_2}\mathcal{A}_{12}-a_1^{-\theta_3}\mathcal{A}_{13}^*\mathcal{A}_{13}\\ &\succcurlyeq \begin{pmatrix}2.571\,12 & 0.484\,87 & 0.289\,83\\ 0.484\,87 & 3.478\,33 & 0.992\,49\\ 0.289\,83 & 0.992\,49 & 3.905\,77\end{pmatrix}\succcurlyeq a_1I,\end{aligned}$$

$$\begin{aligned}F_1(n_0\mathcal{Q}_1,a_1I,n_0\mathcal{Q}_3) &= \mathcal{Q}_1-n_0^{-\theta_1 N}\mathcal{A}_{11}^*\mathcal{Q}_1^{-\theta_1}\mathcal{A}_{11}+a_1^{-\theta_2}\mathcal{A}_{12}^*\mathcal{A}_{12}-n_0^{-\theta_3 N}\mathcal{A}_{13}^*\mathcal{Q}_3^{-\theta_3}\mathcal{A}_{13}\\ &\preccurlyeq \begin{pmatrix}3.461\,08 & 0.501\,02 & 0.489\,38\\ 0.501\,02 & 3.505\,47 & 1.000\,68\\ 0.489\,38 & 1.000\,68 & 4.000\,30\end{pmatrix}\preccurlyeq n_0\mathcal{Q}_1,\end{aligned}$$

$$\begin{aligned}F_2(a_1I,n_0\mathcal{Q}_2,a_1I) &= \mathcal{Q}_2-a_1^{-\theta_1}\mathcal{A}_{21}^*\mathcal{A}_{21}+n_0^{-\theta_2 N}\mathcal{A}_{22}^*\mathcal{Q}_2^{-\theta_2}\mathcal{A}_{22}-a_1^{-\theta_3}\mathcal{A}_{23}^*\mathcal{A}_{23}\\ &\succcurlyeq \begin{pmatrix}7.832\,96 & 1.073\,06 & 3.012\,49\\ 1.073\,06 & 8.660\,08 & 5.166\,82\\ 3.012\,49 & 5.166\,82 & 35.016\,03\end{pmatrix}\succcurlyeq a_1I,\end{aligned}$$

$$\begin{aligned}F_2(n_0\mathcal{Q}_1,a_1,n_0\mathcal{Q}_3) &= \mathcal{Q}_2-n_0^{-\theta_1 N}\mathcal{A}_{21}^*\mathcal{Q}_1^{-\theta_1}\mathcal{A}_{21}+a_1^{-\theta_2}\mathcal{A}_{22}^*\mathcal{A}_{22}-n_0^{-\theta_3 N}\mathcal{A}_{23}^*\mathcal{Q}_3^{-\theta_3}\mathcal{A}_{23}\\ &\preccurlyeq \begin{pmatrix}8.789\,18 & 2.587\,97 & 4.435\,09\\ 2.587\,97 & 11.614\,36 & 6.988\,05\\ 4.435\,09 & 6.988\,05 & 37.839\,01\end{pmatrix}\preccurlyeq n_0\mathcal{Q}_2,\end{aligned}$$

$$\begin{aligned}F_3(a_1I,n_0\mathcal{Q}_2,a_1I) &= \mathcal{Q}_3-a^{-\theta_1}\mathcal{A}_{31}^*\mathcal{A}_{31}+n_0^{-\theta_2 N}\mathcal{A}_{32}^*\mathcal{Q}_2^{-\theta_2}\mathcal{A}_{32}-a^{-\theta_3}\mathcal{A}_{33}^*\mathcal{A}_{33}\\ &\succcurlyeq \begin{pmatrix}10.497\,30 & 7.510\,83 & 1.522\,66\\ 7.510\,83 & 12.062\,32 & 1.487\,51\\ 1.522\,66 & 1.487\,51 & 11.422\,54\end{pmatrix}\succcurlyeq a_1I,\end{aligned}$$

and

$$\begin{aligned}F_3(n_0\mathcal{Q}_1,a_1I,n_0\mathcal{Q}_3) &= \mathcal{Q}_3-n_0^{-\theta_1 N}\mathcal{A}_{31}^*\mathcal{Q}_1^{-\theta_1}\mathcal{A}_{31}+a_1^{-\theta_2}\mathcal{A}_{32}^*\mathcal{A}_{32}-n_0^{-\theta_3 N}\mathcal{A}_{33}^*\mathcal{Q}_3^{-\theta_3}\mathcal{A}_{33}\\ &\preccurlyeq \begin{pmatrix}10.500\,07 & 7.500\,55 & 1.500\,91\\ 7.500\,55 & 12.434\,14 & 1.499\,72\\ 1.500\,91 & 1.499\,72 & 11.976\,28\end{pmatrix}\preccurlyeq n_0\mathcal{Q}_3,\end{aligned}$$

it follows that

$$\widetilde{X}\in\left[\begin{pmatrix}2.571\,12 & 0.484\,87 & 0.289\,83\\ 0.484\,87 & 3.478\,33 & 0.992\,49\\ 0.289\,83 & 0.992\,49 & 3.905\,77\end{pmatrix},\begin{pmatrix}3.461\,08 & 0.501\,02 & 0.489\,38\\ 0.501\,02 & 3.505\,47 & 1.000\,68\\ 0.489\,38 & 1.000\,68 & 4.000\,30\end{pmatrix}\right],$$

$$\widetilde{Y}\in\left[\begin{pmatrix}7.832\,96 & 1.073\,06 & 3.012\,49\\ 1.073\,06 & 8.660\,08 & 5.166\,82\\ 3.012\,49 & 5.166\,82 & 35.016\,03\end{pmatrix},\begin{pmatrix}8.789\,18 & 2.587\,97 & 4.435\,09\\ 2.587\,97 & 11.614\,36 & 6.988\,05\\ 4.435\,09 & 6.988\,05 & 37.839\,01\end{pmatrix}\right],$$

and

$$\widetilde{Z}\in\left[\begin{pmatrix}10.497\,30 & 7.510\,83 & 1.522\,66\\ 7.510\,83 & 12.062\,32 & 1.487\,51\\ 1.522\,66 & 1.487\,51 & 11.422\,54\end{pmatrix},\begin{pmatrix}10.500\,07 & 7.500\,55 & 1.500\,91\\ 7.500\,55 & 12.434\,14 & 1.499\,72\\ 1.500\,91 & 1.499\,72 & 11.976\,28\end{pmatrix}\right].$$

We calculate the first iteration by putting $X_0 = a_1 I$, $Y_0 = a_2 I$ and $Z_0 = a_3 I$.

From $\delta = \max\left\{\frac{k\|\sum_{i=1}^{3} A_{i1}^* A_{i1}\|}{a^{k+1}}, \frac{l\|\sum_{i=1}^{3} A_{i2}^* A_{i2}\|}{a^{l+1}}, \frac{r\|\sum_{i=1}^{3} A_{i3}^* A_{i3}\|}{a^{r+1}}\right\} = 0.10418 < 1$ the approximate solutions with $\varepsilon = 10^{-10}$, using the a posteriori error estimate, will be

$$X_7 \approx \begin{pmatrix} 2.655\,64 & 0.483\,45 & 0.306\,62 \\ 0.483\,45 & 3.4719 & 0.989\,24 \\ 0.306\,62 & 0.989\,24 & 3.898\,97 \end{pmatrix},$$

$$Y_7 \approx \begin{pmatrix} 8.528\,57 & 2.213\,00 & 4.049\,91 \\ 2.213\,00 & 10.955\,37 & 6.497\,05 \\ 4.049\,91 & 6.497\,05 & 37.113\,90 \end{pmatrix},$$

$$Z_7 \approx \begin{pmatrix} 10.497\,42 & 7.508\,90 & 1.520\,09 \\ 7.508\,90 & 12.098\,89 & 1.509\,65 \\ 1.520\,09 & 1.509\,65 & 11.468\,65 \end{pmatrix},$$

and the a posteriori error is $0.177145 * 10^{-11}$.

5.2. Application of Theorem 10

Example 2. *Let us consider the matrix equation*

$$X = Q - A^* X^{-\theta} A + B^* X^{-\theta} B - C^* X^{-\theta} C \tag{28}$$

with $Q = Q_3$, $A = A_{31}$, $B = A_{32}$, $C = A_{33}$, $a = a_1$, $b = a_2$ *and* $\theta = \theta_1$.

We will apply Theorem 10 using similar equations, and we obtain

$$\mathcal{Q}_3 - a_1^{-\theta_1} \mathcal{A}_{31}^* \mathcal{A}_{31} - a_1^{-\theta_1} \mathcal{A}_{33}^* \mathcal{A}_{33} - a_1 I \succ 0,$$

$$\mathcal{Q}_3 - a_1^{-\theta_1} \mathcal{A}_{31}^* \mathcal{A}_{31} + a_2^{-\theta_1} \mathcal{A}_{32}^* \mathcal{A}_{32} - a_1^{-\theta_1} \mathcal{A}_{33}^* \mathcal{A}_{33} - a_1 I \succ 0,$$

$$\mathcal{Q}_3 - a_1^{-\theta_1} \mathcal{A}_{31}^* \mathcal{A}_{31} + a_2^{-\theta_1} \mathcal{A}_{32}^* \mathcal{A}_{32} - a_1^{-\theta_1} \mathcal{A}_{33}^* \mathcal{A}_{33} - a_2 I \prec 0,$$

and

$$\delta = \max_{i=1,2,3}\left\{\frac{\theta_1 \|\mathcal{A}_{3i}^* \mathcal{A}_{3i}\|}{a_1^{\theta_1+1}}\right\} \approx 0.06935 < \frac{1}{3}.$$

The approximate solutions with $\varepsilon = 10^{-10}$, using the a posteriori error estimate. We calculate the first iteration by using $X_0 = a_1 I$, $Y_0 = a_2 I$, and $Z_0 = a_1 I$. Since X, Y, and Z are solutions in the system of matrix equations

$$\begin{cases} X = Q - A^* X^{-\theta} A + B^* Y^{-\theta} B - C^* Z^{-\theta} C \\ Y = Q - A^* Y^{-\theta} A + B^* Z^{-\theta} B - C^* X^{-\theta} C \\ Z = Q - A^* Z^{-\theta} A + B^* X^{-\theta} B - C^* Y^{-\theta} C, \end{cases} \tag{29}$$

we obtain

$$X_7 \approx \begin{pmatrix} 10.497\,58 & 7.509\,48 & 1.516\,46 \\ 7.509\,48 & 12.156\,27 & 1.495\,21 \\ 1.516\,46 & 1.495\,21 & 11.563\,59 \end{pmatrix},$$

$$Y_7 \approx \begin{pmatrix} 10.498\,14 & 7.510\,53 & 1.516\,03 \\ 7.510\,53 & 12.162\,82 & 1.498\,43 \\ 1.516\,03 & 1.498\,43 & 11.592\,21 \end{pmatrix},$$

$$Z_7 \approx \begin{pmatrix} 10.495\,97 & 7.514\,82 & 1.541\,15 \\ 7.514\,82 & 11.755\,57 & 1.461\,12 \\ 1.541\,15 & 1.461\,12 & 10.965\,44 \end{pmatrix}.$$

The solution of the matrix equation $X = Q - A^* X^{-\theta} A + B^* X^{-\theta} B - C^* X^{-\theta} C$ is

$$\widetilde{X} \approx \begin{pmatrix} 10.498\,52 & 7.510\,85 & 1.515\,04 \\ 7.510\,85 & 12.169\,08 & 1.502\,99 \\ 1.515\,04 & 1.502\,99 & 11.628\,25 \end{pmatrix}$$

and the a posteriori error is $0.73220 * 10^{-12}$.

5.3. Application of Theorem 11

Example 3. *Let us consider the system of matrix equations*

$$\begin{cases} X = \mathcal{Q}_1 - \mathcal{A}_{11}^* X^{-\theta_1} \mathcal{A}_{11} - \mathcal{A}_{12}^* Y^{-\theta_2} \mathcal{A}_{12} - \mathcal{A}_{13}^* Z^{-\theta_3} \mathcal{A}_{13} \\ Y = \mathcal{Q}_2 - \mathcal{A}_{21}^* X^{-\theta_1} \mathcal{A}_{21} - \mathcal{A}_{22}^* Y^{-\theta_2} \mathcal{B}_{22} - \mathcal{A}_{23}^* Z^{-\theta_3} \mathcal{A}_{23} \\ Z = \mathcal{Q}_3 - \mathcal{A}_{31}^* X^{-\theta_1} \mathcal{A}_{31} - \mathcal{A}_{32}^* Y^{-\theta_2} \mathcal{A}_{32} - \mathcal{A}_{33}^* Z^{-\theta_3} \mathcal{A}_{33} \end{cases}. \tag{30}$$

We will apply Theorem 11. It is easy to check the conditions of the theorem. The approximate solutions with $\varepsilon = 10^{-10}$, can be found using the a posteriori error estimate. For $X_0 = a_1 I$, $Y_0 = a_2 I$ and $Z_0 = a_3 I$, after ten iterations, we obtain

$$X_{10} \approx \begin{pmatrix} 2.640\,49 & 0.480\,59 & 0.305\,94 \\ 0.480\,59 & 3.457\,78 & 0.988\,33 \\ 0.305\,94 & 0.988\,33 & 3.886\,57 \end{pmatrix} \succcurlyeq a_1 I,$$

$$Y_{10} \approx \begin{pmatrix} 6.370\,28 & -1.386\,52 & 0.893\,25 \\ -1.386\,52 & 3.641\,97 & 2.3888 \\ 0.893\,25 & 2.3888 & 30.805\,73 \end{pmatrix} \succcurlyeq a_1 I,$$

$$Z_{10} \approx \begin{pmatrix} 10.497\,03 & 7.508\,89 & 1.520\,12 \\ 7.508\,89 & 12.098\,26 & 1.509\,72 \\ 1.520\,12 & 1.509\,72 & 11.467\,85 \end{pmatrix} \succcurlyeq a_1 I,$$

and the a posteriori error is $0.442151 * 10^{-11}$.

5.4. Application of Theorem 12

Example 4. *Let us consider the matrix equation*

$$X = \mathcal{Q} - \mathcal{A}^* X^{-\theta} \mathcal{A} - \mathcal{B}^* X^{-\theta} \mathcal{B} - \mathcal{C}^* X^{-\theta} \mathcal{C} \tag{31}$$

with $\mathcal{Q} = \mathcal{Q}_1$, $\mathcal{A} = \mathcal{A}_{31}$, $\mathcal{B} = \mathcal{A}_{12}$, $\mathcal{C} = \mathcal{A}_{33}$ *and* $\theta = \theta_1$.

We will apply Theorem 12 using similar calculations and obtain

$$\mathcal{Q}_1 - a_1^{-\theta_1} \mathcal{A}_{31}^* \mathcal{A}_{31} - a_1^{-\theta_1} \mathcal{A}_{12}^* \mathcal{A}_{12} - a_1^{-\theta_1} \mathcal{A}_{33}^* \mathcal{A}_{33} - a_1 I \succ 0,$$

and

$$\delta = \max\left\{ \frac{\theta_1 \|\mathcal{A}_{31}^* \mathcal{A}_{31}\|}{a_1^{\theta_1+1}}, \frac{\theta_1 \|\mathcal{A}_{12}^* \mathcal{A}_{12}\|}{a_1^{\theta_1+1}}, \frac{\theta_1 \|\mathcal{A}_{33}^* \mathcal{A}_{33}\|}{a_1^{\theta_1+1}} \right\} \approx 0.06935 < \frac{1}{3}.$$

The approximate solutions with $\varepsilon = 10^{-10}$ are found using the a posteriori error estimate. We calculate the first iteration by using $X_0 = a_1 I$, $Y_0 = a_1 I$, and $Z_0 = a_1 I$. Since X, Y, and Z are solutions in the system of matrix equations

$$\begin{cases} X = Q - \mathcal{A}_{31}^* X^{-\theta} \mathcal{A}_{31} - \mathcal{A}_{12}^* Y^{-\theta} \mathcal{A}_{12} - \mathcal{A}_{33}^* Z^{-\theta} \mathcal{A}_{33} \\ Y = Q - \mathcal{A}_{31}^* Y^{-\theta} \mathcal{A}_{31} - \mathcal{A}_{12}^* Z^{-\theta} \mathcal{A}_{12} - \mathcal{A}_{33}^* X^{-\theta} \mathcal{A}_{33} \\ Z = Q - \mathcal{A}_{31}^* Z^{-\theta} \mathcal{A}_{31} - \mathcal{A}_{12}^* X^{-\theta} \mathcal{A}_{12} - \mathcal{A}_{33}^* Y^{-\theta} \mathcal{A}_{33}, \end{cases} \tag{32}$$

we obtain

$$X_7 \approx \begin{pmatrix} 3.490\,96 & 0.508\,39 & 0.520\,03 \\ 0.508\,39 & 3.138\,66 & 1.013\,89 \\ 0.520\,03 & 1.013\,89 & 3.464\,39 \end{pmatrix},$$

$$Y_7 \approx \begin{pmatrix} 3.490\,96 & 0.508\,39 & 0.520\,03 \\ 0.508\,39 & 3.138\,66 & 1.013\,89 \\ 0.520\,03 & 1.013\,89 & 3.464\,39 \end{pmatrix},$$

$$Z_7 \approx \begin{pmatrix} 3.490\,96 & 0.508\,39 & 0.520\,03 \\ 0.508\,39 & 3.138\,66 & 1.013\,89 \\ 0.520\,03 & 1.013\,89 & 3.464\,39 \end{pmatrix}.$$

The solution of the matrix Equation (31) is

$$\widetilde{X} \approx \begin{pmatrix} 3.491\,62 & 0.509\,11 & 0.519\,00 \\ 0.509\,11 & 3.147\,36 & 1.020\,78 \\ 0.519\,02 & 1.020\,78 & 3.512\,66 \end{pmatrix}$$

and the a posteriori error is $0.425568 * 10^{-11}$.

6. Discussion

We would like to comment on the connection between the presented results and those known so far. Similar results for the existence and uniqueness of fixed point triples were obtained in [35], without requiring the mixed monotone property for the considered triple of maps, but of course the contraction condition (5) needs to be satisfied for all ordered triples. We will mention that the application of the results [35] and the illustrative example [30] are related to linear maps, for which if the inequality is satisfied under the requirements only for comparable triples of elements, then it is satisfied for all ordered triples. In contrast to the above, when applying results from the theory of fixed points in solving matrix equations or systems of matrix equations, the assumption that the contraction holds only for elements that are comparable is of essential importance [17]. This illustrates the deep observation and the proposed technique in [17] for the study of nonlinear matrix equations [17] and systems of nonlinear matrix equations [24].

In [42], comprehensive observation in solving matrix inequalities systems is offered. Is it possible to solve the system of matrix inequalities involved in the current study using the method from [42].

A very natural generalization can involve obtaining sufficient conditions for the existence and uniqueness of n-tupled fixed points introduced in [33], but in the context of the presented investigation, i.e., for n ordered maps $(F_1, \ldots, F_n)$, and applying it in solving matrix equations.

An interesting application in obtaining exact solutions for nonlinear systems of equations [27], where the approximate methods do not find the exact solution as shown in [27], is to change the system of equations to a best proximity points problem. Is it possible to

apply the ideas from [27] in the context of matrix equations, maps with the mixed monotone property, and to obtain the exact solutions?

7. Conclusions

In the present paper, with the help of the deep results in [17,36,43,44], we have constructed a generalization of the coupled fixed points introduced in [25,26]. We have demonstrated that there is a close connection between the type of monotonicity of the maps under study and the definition of partial ordering in the proposed idea in [26] of studying coupled fixed points for maps with the mixed monotone property in partially ordered metric spaces. This allows us to solve various classes of systems of matrix equations. The presented idea of considering an ordered pair of maps instead of one map generalizes the technique proposed by [24] for solving only symmetric systems of matrix equations.

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