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Elementary Operators with *m*-Null Symbols

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Abstract: Motivated by Botelho and Jamison's seminal 2010 study on elementary operators that are m-isometries, in this paper, we introduce the concept of m-null pairs of operators and establish some structural properties and characterizations of the class of elementary operators whose symbols are m-null (so-called m-null elementary operators). It is shown that if the symbols of an elementary operator L are, in turn, a p-null elementary operator and a q-null elementary operator, then L is a $(p+q-1)$-null elementary operator. Some extant results on elementary m-isometries can be recovered from this renewed perspective, often providing added value.

Keywords: elementary operator; m-isometry; m-null operators; (m,T)-null operators; Stirling numbers

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1. Introduction

Let $\mathcal{B}(H)$ denote the set of all bounded linear operators on a Hilbert space H, and let m be a fixed positive integer. An operator $T \in \mathcal{B}(H)$ is said to be an m-isometry if

$$\sum_{j=0}^{m} (-1)^j \binom{m}{j} T^{*j} T^j = 0,$$

where, as usual, $T^* \in \mathcal{B}(H)$ is the adjoint of T. Note that a 1-isometry is an isometry. The notion of m-isometry was developed by Agler and Stankus in the trilogy [1–3]. Since then, a number of authors have explored the properties of this class of operators, and multiple generalizations have been proposed in different contexts. The arguments employed in those investigations encompass a wide variety of methods, including combinatorics, arithmetic progressions, Lagrange interpolation, properties of operator roots of polynomials, tensor products, factorization into left and right multiplication operators, and the hereditary functional calculus. For a rather complete account, the interested reader is referred to [4] and references therein.

Fix $d \in \mathbb{N}$. An operator $L : \mathcal{B}(H) \to \mathcal{B}(H)$ is said to be an elementary operator if there exist finite sequences $\{A_i\}_{i=1}^d$ and $\{B_i\}_{i=1}^d$ of operators in $\mathcal{B}(H)$ such that $L(T) = \sum_{i=1}^d A_i T B_i$, where the number d is called the length of the operator. The special cases $L(T) = ATB$ (of length 1) and $L(T) = AT + TB$ (of length 2) have been amply studied in several contexts. Botelho et al. [5,6] characterized the elementary operators of length 1 acting on the Hilbert–Schmidt class of a separable Hilbert space that are 2-isometries and 3-isometries. They also gave sufficient conditions for such elementary operators to be m-isometries for higher m, and formulated the following conjecture: given $p, q \in \mathbb{N}$,

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if there is a nonzero scalar λ such that λA is a p-isometry and $\lambda^{-1}B^*$ is a q-isometry, then the elementary operator $L(T) = ATB$ is a $(p+q-1)$-isometry. This conjecture was subsequently confirmed by Duggal [7] and other authors from different viewpoints and generalizations, cf. [8] and references therein. To our best knowledge, the initial approach of Botelho and Jamison [5], based on a result by Fong and Sourour [9], has not been pursued in later investigations on the subject. In this paper, we provide a proof that combines the ideas in [5] with basic operator algebra and some combinatorics. With this purpose, we introduce the concept of m-null pairs of operators and establish some structural properties and characterizations of the class of elementary operators whose symbols are m-null, which we call m-null elementary operators. This is performed in Section 2. Then, in Section 3, it is shown that if the symbols of an elementary operator L are, in turn, a p-null elementary operator and a q-null elementary operator, then L is a $(p+q-1)$-null elementary operator (Theorem 1). As a consequence, the aforementioned result of Botelho–Jamison–Duggal is retrieved (Corollary 1), as are other improved versions of extant results on elementary m-isometries. In view of the techniques employed in achieving this result, in the last Section 4, we raise the natural question whether its reciprocal, as established by Gu ([10], Theorem 7), could be obtained without resorting to analytic tools.

2. (m, T)-Null Pairs

Let us introduce some notation. To any pair of operators $(A, B) \in \mathcal{B}(H) \times \mathcal{B}(H)$, we associate the elementary operator of length 1 $Q_{A,B} : \mathcal{B}(H) \to \mathcal{B}(H)$, which maps each $T \in \mathcal{B}(H)$ to

$$Q_{A,B}(T) := ATB.$$

The operators A and B are called the symbols of $Q_{A,B}$. It can be easily checked that the adjoint $Q^*_{A,B}$ of $Q_{A,B}$ is given by Q_{A^*,B^*}. Further, for $m \in \mathbb{N}$ and $T \in \mathcal{B}(H)$, we write

$$\begin{aligned} &P_{A,B,0}(T) := T, \\ &P_{A,B,m}(T) := (I - Q_{A,B})^m(T) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} Q^j_{A,B}(T), \\ &\Delta_{A,B,m}(T) := (-1)^{m-1} P_{A,B,m-1}(T). \end{aligned}$$

Here, as usual, $I \in \mathcal{B}(H)$ is the identity operator. Clearly,

$$P_{A,B,m}(T) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} A^j T B^j.$$

Definition 1. *Let $m \in \mathbb{N}$ and $T \in \mathcal{B}(H)$. A pair of symbols $(A, B) \in \mathcal{B}(H) \times \mathcal{B}(H)$ is said to be (m, T)-null provided that $P_{A,B,m}(T) = 0$ (or, equivalently, $\Delta_{A,B,m+1}(T) = 0$). If this is true of all $T \in \mathcal{B}(H)$, then we shall just say that (A, B) is m-null. The elementary operator $Q_{A,B}$ is called (m, T)-null, respectively m-null, whenever so is (A, B). Either one of these conditions will be termed strict if it holds for m, but not for $m-1$.*

Remark 1. *In the notation above, $S \in \mathcal{B}(H)$ is an m-isometry if*

$$P_{S^*,S,m}(I) = 0,$$

or, equivalently,

$$\Delta_{S^*,S,m+1}(I) = 0.$$

Additionally, the operator S is an m-isometry if the pair (S^*, S) is (m, I)-null, and a strict m-isometry if (S^*, S) is (m, I)-null but not $(m-1, I)$-null.

The following simple recurrence relation, valid for all $A, B, T \in \mathcal{B}(H)$, will be useful:

$$\begin{aligned} P_{A,B,m+1}(T) &= (I - Q_{A,B})^{m+1}(T) \\ &= (I - Q_{A,B})(I - Q_{A,B})^{m}(T) \\ &= (I - Q_{A,B})P_{A,B,m}(T) \\ &= P_{A,B,m}(T) - Q_{A,B}P_{A,B,m}(T). \end{aligned} \tag{1}$$

Remark 2. *In particular, (1) shows that if (A, B) is (m, T)-null (respectively, m-null), then it is $(m + j, T)$-null (respectively, $(m + j)$-null) for all $j \in \mathbb{N}_0$. Replacing A and B in (1) with S^* and S, respectively, retrieves the well-known fact that every m-isometry S is also an $m + j$-isometry for all $j \in \mathbb{N}_0$.*

Proposition 1. *Let $A, B, T \in \mathcal{B}(H)$. There holds*

$$Q^k_{A,B}(T) = \sum_{j=0}^{k} k^{\underline{j}} \left[\frac{(-1)^j}{j!} P_{A,B,j}(T) \right] \quad (k \in \mathbb{N}_0), \tag{2}$$

where

$$k^{\underline{j}} = \begin{cases} 1, & j = 0 \\ k(k-1)\cdots(k-j+1), & j \geq 1 \end{cases}$$

is the falling factorial. Furthermore, if (A, B) is (m, T)-null, (2) becomes

$$Q^k_{A,B}(T) = \sum_{j=0}^{m-1} k^{\underline{j}} \left[\frac{(-1)^j}{j!} P_{A,B,j}(T) \right] \quad (k \in \mathbb{N}_0). \tag{3}$$

Proof. Given $k \in \mathbb{N}_0$, write

$$\alpha_{A,B,T}(k) = \sum_{j=0}^{k} k^{\underline{j}} \left[\frac{(-1)^j}{j!} P_{A,B,j}(T) \right].$$

Then,

$$\begin{aligned} &\alpha_{A,B,T}(k+1) - \alpha_{A,B,T}(k) \\ &= \sum_{j=0}^{k+1} (k+1)^{\underline{j}} \left[\frac{(-1)^j}{j!} P_{A,B,j}(T) \right] - \sum_{j=0}^{k} k^{\underline{j}} \left[\frac{(-1)^j}{j!} P_{A,B,j}(T) \right] \\ &= (k+1)^{\underline{k+1}} \left[\frac{(-1)^{k+1}}{(k+1)!} P_{A,B,k+1}(T) \right] + \sum_{j=1}^{k} \left[(k+1)^{\underline{j}} - k^{\underline{j}} \right] \left[\frac{(-1)^j}{j!} P_{A,B,j}(T) \right] \\ &= (k+1)! \left[\frac{(-1)^{k+1}}{(k+1)!} P_{A,B,k+1}(T) \right] \\ &\quad + \sum_{j=0}^{k-1} \left[(k+1)^{\underline{j+1}} - k^{\underline{j+1}} \right] \left[\frac{(-1)^{j+1}}{(j+1)!} P_{A,B,j+1}(T) \right] \\ &= (k+1)! \left[\frac{(-1)^{k+1}}{(k+1)!} P_{A,B,k+1}(T) \right] \\ &\quad + \sum_{j=0}^{k-1} \left[\binom{k+1}{j+1} - \binom{k}{j+1} \right] \left[(-1)^{j+1} P_{A,B,j+1}(T) \right] \\ &= k! \left[\frac{(-1)^{k+1}}{k!} P_{A,B,k+1}(T) \right] + \sum_{j=0}^{k-1} \binom{k}{j} \left[(-1)^{j+1} P_{A,B,j+1}(T) \right] \\ &= k^{\underline{k}} \left[\frac{(-1)^{k+1}}{k!} P_{A,B,k+1}(T) \right] + \sum_{j=0}^{k-1} k^{\underline{j}} \left[\frac{(-1)^{j+1}}{j!} P_{A,B,j+1}(T) \right] \\ &= -\sum_{j=0}^{k} k^{\underline{j}} \left[\frac{(-1)^j}{j!} P_{A,B,j+1}(T) \right]. \end{aligned}$$

Using (1), we infer that

$$\begin{aligned}\alpha_{A,B,T}(k+1) - \alpha_{A,B,T}(k) &= -\sum_{j=0}^{k} \frac{(-1)^j k^{\underline{j}}}{j!}\left[P_{A,B,j}(T) - Q_{A,B}P_{A,B,j}(T)\right] \\ &= -\alpha_{A,B,T}(k) + Q_{A,B}[\alpha_{A,B,T}(k)].\end{aligned}$$

Thus, we find

$$\alpha_{A,B,T}(k+1) = Q_{A,B}[\alpha_{A,B,T}(k)].$$

Since $\alpha_{A,B,T}(0) = T$, we obtain

$$\alpha_{A,B,T}(k) = Q_{A,B}^k(T),$$

which is (2).

Finally, when (A, B) is (m, T)-null, we must have (3). Indeed, if $k \geq m$, this derives from the fact that (A, B) is (j, T)-null for all $j \geq m$, while if $k < m$, then $j \geq m$ implies $k \leq j - 1$, whence $k^{\underline{j}} = 0$. □

Proposition 2. *Let $m \in \mathbb{N}$ and $A, B, T \in \mathcal{B}(H)$. Assume that (A, B) is (m, T)-null and $Q_{A,B}^k(T) \geq 0$ for sufficiently large $k \in \mathbb{N}$. Then, $\Delta_{A,B,m}(T) \geq 0$.*

Proof. From Proposition 1,

$$\lim_{k\to\infty} \frac{1}{k^{\underline{m-1}}} Q_{A,B}^k(T) = \lim_{k\to\infty} \sum_{j=0}^{m-1} \frac{k^{\underline{j}}}{k^{\underline{m-1}}}\left[\frac{(-1)^j}{j!} P_{A,B,j}(T)\right].$$

Now

$$\lim_{k\to\infty} \frac{k^{\underline{j}}}{k^{\underline{m-1}}} = \begin{cases} 0, & j < m-1 \\ 1, & j = m-1 \end{cases}$$

implies

$$\lim_{k\to\infty} \frac{1}{k^{\underline{m-1}}} Q_{A,B}^k(T) = \frac{(-1)^{m-1}}{(m-1)!} P_{A,B,m-1}(T) = \frac{1}{(m-1)!}\Delta_{A,B,m}(T).$$

Upon assuming $Q_{A,B}^k(T) \geq 0$ for large k, the result follows. □

Given $T \in \mathcal{B}(H)$, the next proposition characterizes those pairs $(A, B) \in \mathcal{B}(H) \times \mathcal{B}(H)$ that are (m, T)-null.

Proposition 3. *Let $m \in \mathbb{N}$ and $A, B, T \in \mathcal{B}(H)$. The pair (A, B) is (m, T)-null if, and only if,*

$$Q_{A,B}^k(T) = \sum_{i=0}^{m-1} (-1)^{m-i-1} \binom{k}{i} \binom{k-i-1}{m-i-1} Q_{A,B}^i(T) \tag{4}$$

$$= \sum_{i=0}^{m-1} \binom{k}{i} \binom{m-k-1}{m-i-1} Q_{A,B}^i(T) \quad (k \in \mathbb{N}_0). \tag{5}$$

Proof. Assume that (A, B) is (m, T)-null, and fix $k \in \mathbb{N}_0$. Then, from Proposition 1,

$$Q_{A,B}^k(T) = \sum_{j=0}^{m-1} k^{\underline{j}}\left[\frac{(-1)^j}{j!} P_{A,B,j}(T)\right].$$

Since

$$\binom{k}{j}\binom{j}{i} = \binom{k}{i}\binom{k-i}{j-i} \quad (j \in \mathbb{N}_0,\ 0 \leq j \leq m-1),$$

we have

$$\begin{aligned} Q_{A,B}^k(T) &= \sum_{j=0}^{m-1} \frac{k^{\underline{j}}}{j!}\left[\sum_{i=0}^{j}(-1)^{j-i}\binom{j}{i}Q_{A,B}^i(T)\right] \\ &= \sum_{i=0}^{m-1}\left[\sum_{j=i}^{m-1}(-1)^{j-i}\binom{k}{j}\binom{j}{i}\right]Q_{A,B}^i(T) \\ &= \sum_{i=0}^{m-1}(-1)^i\binom{k}{i}\left[\sum_{j=i}^{m-1}(-1)^j\binom{k-i}{j-i}\right]Q_{A,B}^i(T) \\ &= \sum_{i=0}^{m-1}(-1)^{m-i-1}\binom{k}{i}\binom{k-i-1}{m-i-1}Q_{A,B}^i(T). \end{aligned}$$

The last identity is obtained by appropriately redefining and shifting the indices in the horizontal recurrence relation for the binomial coefficients ([11], Corollary 2.3). This yields (4). Equations (4) and (5) are seen to be equivalent on account of ([11], Equation (3.12)).

Conversely, if (4) holds then, particularizing $k = m$ there, gives

$$(I - Q_{A,B})^m(T) = (-1)^m Q_{A,B}^m(T) + \sum_{i=0}^{m-1}(-1)^i\binom{m}{i}Q_{A,B}^i(T) = 0,$$

so that (A, B) is (m, T)-null. □

We close this section with a second characterization of m-null pairs of operators, this time involving Stirling numbers.

Let $n, k \in \mathbb{N}_0$, $0 \leq k \leq n$. The Stirling number of the first kind $s(n,k)$ is defined such that the number of permutations of n elements that contain exactly k permutation cycles is the non-negative number $|s(n,k)| = (-1)^{n-k}s(n,k)$. The Stirling number of the second kind $\left\{ {n \atop k} \right\}$ is the number of ways of partitioning a set of n elements into k disjoint non-empty subsets. The Stirling numbers of the first and second kind turn out to be the coefficients of the expansions of the falling factorials into powers, and of the powers into falling factorials, respectively. They arise frequently in combinatorial, probabilistic, and statistical applications, and have been generalized in several directions, one of them being the so-called non-central Stirling numbers. For further insight on Stirling and non-central Stirling numbers of the first and second kind, the interested reader is referred to ([11], Chapter 8).

Given $A, B, T \in \mathcal{B}(H)$, consider the falling factorial operator

$$Q_{A,B}^{\underline{k}}(T) := \begin{cases} I, & k = 0 \\ Q_{A,B}(Q_{A,B} - I)(Q_{A,B} - 2I)\cdots[Q_{A,B} - (k-1)I](T), & k \in \mathbb{N}. \end{cases}$$

Falling factorial operators can be expanded into a sum of operator powers whose coefficients are the Stirling numbers of the first kind. In fact, as $Q_{A,B}$ commutes with itself, in analogy with [11] (Equation (8.2)), we have

$$\begin{aligned} Q^{k}_{A,B}(T) &= Q_{A,B}(Q_{A,B}-I)(Q_{A,B}-2I)\cdots[Q_{A,B}-(k-1)I](T) \\ &= [Q_{A,B}-(k-1)I]\cdots(Q_{A,B}-2I)(Q_{A,B}-I)Q_{A,B}(T) \\ &= \sum_{j=0}^{k} s(k,j)Q^{j}_{A,B}(T) \end{aligned}$$

for each $k \in \mathbb{N}$. Since $s(0,0) = 1$, the identity

$$Q^{k}_{A,B}(T) = \sum_{j=0}^{k} s(k,j)Q^{j}_{A,B}(T)$$

actually holds true for all $k \in \mathbb{N}_0$.

The following well-known lemma will be useful in the sequel. We include a proof for the sake of completeness.

Lemma 1. *Let $r, s \in \mathbb{N}_0$, and let $f(x) = a_0x^r + a_1x^{r-1} + ... + a_r$ $(a_0 \neq 0)$ be a polynomial of degree r. Then*

$$\sum_{k=0}^{s}(-1)^{s-k}\binom{s}{k}f(x+k) = \begin{cases} r!a_0, & s = r \\ 0, & s \geq r+1 \end{cases}$$

for all $x \in \mathbb{R}$.

Proof. This can be easily proved using the calculus of finite differences. In fact, let $\mathbf{\Delta} = \mathbf{E} - \mathbf{I}$ be the forward difference operator, where $\mathbf{E}$ denotes the translation operator, given by $(\mathbf{E}g)(x) = g(x+1)$ $(x \in \mathbb{R})$, and $\mathbf{I}$ is the identity operator, acting on the space of all polynomials g. On the one hand,

$$\mathbf{\Delta}^s = \sum_{k=0}^{s}(-1)^{s-k}\binom{s}{k}\mathbf{E}^k,$$

so that

$$(\mathbf{\Delta}^s f)(x) = \sum_{k=0}^{s}(-1)^{s-k}\binom{s}{k}f(x+k) \quad (x \in \mathbb{R}).$$

On the other hand, an induction process gives

$$(\mathbf{\Delta}^r f)(x) = r!a_0,$$

whence

$$(\mathbf{\Delta}^s f)(x) = 0 \quad (s \geq r+1).$$

This establishes the result. □

Proposition 4. *Assume $m \in \mathbb{N}$ and $A, B, T \in \mathcal{B}(H)$. Then,*

(i) There holds

$$\Delta_{A,B,m+1}(T) = \sum_{i=0}^{m}\left\{ \begin{matrix} m \\ i \end{matrix} \middle| -1 \right\} Q^{i}_{A,B}(T), \tag{6}$$

where

$$\left\{ \begin{matrix} m \\ i \end{matrix} \middle| -1 \right\} = \frac{1}{i!}\sum_{j=0}^{i}(-1)^{i-j}\binom{i}{j}(j-1)^m \quad (i \in \mathbb{N}_0,\ 0 \leq i \leq m)$$

are non-central Stirling numbers of the second kind with parameter -1 *([11], Equation (8.62)).*

(ii) (A, B) *is* (m, T)*-null if, and only if,*

$$\sum_{i=0}^{m} \left\{ \begin{matrix} m \\ i \end{matrix} \middle| -1 \right\} Q_{A,B}^{i}(T) = 0.$$

(iii) (A, B) *is strictly* $(m+1, T)$*-null if, and only if,*

$$\sum_{i=0}^{m} \left\{ \begin{matrix} m \\ i \end{matrix} \right\} Q_{A,B}^{i+1}(T) = \sum_{i=0}^{m} \left\{ \begin{matrix} m \\ i \end{matrix} \middle| -1 \right\} Q_{A,B}^{i}(T) \neq 0,$$

where

$$\left\{ \begin{matrix} m \\ i \end{matrix} \right\} = \frac{1}{i!} \sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j} j^m \quad (i \in \mathbb{N}_0,\ 0 \leq i \leq m)$$

are Stirling numbers of the second kind ([11], Theorem 8.4).

Proof. Let $T \in \mathcal{B}(H)$. By induction on $r \in \mathbb{N}_0$, we will show that

$$Q_{A,B}^{r}(T) = \sum_{k=0}^{r} \left\{ \begin{matrix} r \\ k \end{matrix} \right\} Q_{A,B}^{k}(T). \tag{7}$$

Since

$$\left\{ \begin{matrix} 0 \\ 0 \end{matrix} \right\} = \left\{ \begin{matrix} 1 \\ 1 \end{matrix} \right\} = 1 \quad \text{and} \quad \left\{ \begin{matrix} r \\ 0 \end{matrix} \right\} = 0 \quad (r \in \mathbb{N}),$$

the identity (7) holds trivially for $r = 0$ and $r = 1$. Assuming (7) is true for r, we shall prove it for $r+1$. Indeed, using the induction hypothesis and the triangular recurrence formula for the Stirling numbers of the second kind ([11], Theorem 8.7), we obtain

$$\begin{aligned}
Q_{A,B}^{r+1}(T) &= Q_{A,B}^{r} Q_{A,B}(T) \\
&= \sum_{k=0}^{r} \left\{ \begin{matrix} r \\ k \end{matrix} \right\} Q_{A,B}^{k}[(Q_{A,B} - kI) + kI](T) \\
&= \sum_{k=0}^{r} \left\{ \begin{matrix} r \\ k \end{matrix} \right\} Q_{A,B}^{k}(Q_{A,B} - kI)(T) + \sum_{k=0}^{r} k \left\{ \begin{matrix} r \\ k \end{matrix} \right\} Q_{A,B}^{k}(T) \\
&= \sum_{k=0}^{r} \left\{ \begin{matrix} r \\ k \end{matrix} \right\} Q_{A,B}^{k+1} + \sum_{k=0}^{r} k \left\{ \begin{matrix} r \\ k \end{matrix} \right\} Q_{A,B}^{k}(T) \\
&= \sum_{k=1}^{r+1} \left\{ \begin{matrix} r \\ k-1 \end{matrix} \right\} Q_{A,B}^{k}(T) + \sum_{k=1}^{r+1} k \left\{ \begin{matrix} r \\ k \end{matrix} \right\} Q_{A,B}^{k}(T) \\
&= \sum_{k=1}^{r+1} \left[\left\{ \begin{matrix} r \\ k-1 \end{matrix} \right\} + k \left\{ \begin{matrix} r \\ k \end{matrix} \right\} \right] Q_{A,B}^{k}(T) \\
&= \sum_{k=0}^{r+1} \left\{ \begin{matrix} r+1 \\ k \end{matrix} \right\} Q_{A,B}^{k}(T).
\end{aligned}$$

Now we may write

$$\begin{aligned} P_{A,B,m}(T) &= \sum_{j=0}^{m}(-1)^j\binom{m}{j}Q_{A,B}^j(T) \\ &= \sum_{j=0}^{m}(-1)^j\binom{m}{j}\left[\sum_{i=0}^{j}\left\{\begin{matrix} j \\ i \end{matrix}\right\}Q_{A,B}^i(T)\right] \\ &= \sum_{i=0}^{m}\left[\sum_{j=i}^{m}(-1)^j\binom{m}{j}\left\{\begin{matrix} j \\ i \end{matrix}\right\}\right]Q_{A,B}^i(T). \end{aligned}$$

To get (6), it suffices to observe that for each $i \in \mathbb{N}_0$, $0 \leq i \leq m$, Lemma 1, with $x = 0$, implies

$$\frac{1}{i!}\sum_{k=0}^{i}(-1)^{i-k}\binom{i}{k}\left[\sum_{j=0}^{i-1}(-1)^j\binom{m}{j}k^j\right] = 0,$$

so that

$$\begin{aligned} \sum_{j=i}^{m}(-1)^j\binom{m}{j}\left\{\begin{matrix} j \\ i \end{matrix}\right\} &= \sum_{j=i}^{m}(-1)^j\binom{m}{j}\left[\frac{1}{i!}\sum_{k=0}^{i}(-1)^{i-k}\binom{i}{k}k^j\right] \\ &= \frac{1}{i!}\sum_{k=0}^{i}(-1)^{i-k}\binom{i}{k}\left[\sum_{j=i}^{m}(-1)^j\binom{m}{j}k^j\right] \\ &= \frac{1}{i!}\sum_{k=0}^{i}(-1)^{i-k}\binom{i}{k}\left[(1-k)^m - \sum_{j=0}^{i-1}(-1)^j\binom{m}{j}k^j\right] \\ &= \frac{(-1)^m}{i!}\sum_{k=0}^{i}(-1)^{i-k}\binom{i}{k}(k-1)^m \\ &= (-1)^m\left\{\begin{matrix} m \\ i \end{matrix}\middle|\, -1\right\} \end{aligned}$$

(cf. [11], p. 316); this gives Part (i). Part (ii) is a straightforward consequence. Lastly, the same argument as above, applied to $I - Q_{A,B}$ instead of $Q_{A,B}$, shows that

$$P_{A,B,m}(T) = (-1)^m(Q_{A,B} - I)^m(T) = (-1)^m\sum_{i=0}^{m}\left\{\begin{matrix} m \\ i \end{matrix}\right\}(Q_{A,B} - I)^{\underline{i}}(T).$$

Therefore,

$$\begin{aligned} Q_{A,B}P_{A,B,m}(T) &= (-1)^m\sum_{i=0}^{m}\left\{\begin{matrix} m \\ i \end{matrix}\right\}Q_{A,B}(Q_{A,B} - I)^{\underline{i}}(T) \\ &= (-1)^m\sum_{i=0}^{m}\left\{\begin{matrix} m \\ i \end{matrix}\right\}Q_{A,B}^{\underline{i+1}}(T). \end{aligned}$$

Bearing in mind Definition 1, Equations (1) and (6) complete the proof. □

3. Elementary Operators with Elementary Symbols

Theorem 1. *Let $p, q \in \mathbb{N}$, let $S \in \mathcal{B}(H)$, and assume that the pairs (A_1, A_2) and (B_2, B_1) of operators in $\mathcal{B}(H)$ are (p, S)-null and (q, S)-null, respectively. Define the elementary operators*

$$A(S) := Q_{A_1,A_2}(S) = A_1SA_2, \qquad B(S) := Q_{B_2,B_1}(S) = B_2SB_1.$$

Then, the pair $(A(S), B(S))$ *is* $(p+q-1)$*-null, that is,*

$$\sum_{k=0}^{p+q-1}(-1)^k\binom{p+q-1}{k}A_1^kSA_2^kTB_2^kSB_1^k=0\quad(T\in\mathcal{B}(H)).$$

Proof. Fix an arbitrary $T\in\mathcal{B}(H)$, and set

$$Q_S(T):=Q_{A(S),B(S)}(T)=A(S)TB(S)=Q_{A_1,A_2}(S)TQ_{B_2,B_1}(S). \tag{8}$$

We want to show that

$$\begin{aligned}&\sum_{k=0}^{p+q-1}(-1)^k\binom{p+q-1}{k}Q_S^k(T)\\&=\sum_{k=0}^{p+q-1}(-1)^k\binom{p+q-1}{k}Q_{A_1,A_2}^k(S)TQ_{B_2,B_1}^k(S)\\&=0.\end{aligned} \tag{9}$$

Without loss of generality, we may assume $q\geq p$. Since (A_1,A_2) is (p,S)-null, Proposition 3 yields

$$Q_{A_1,A_2}^k(S)=\sum_{i=0}^{p-1}(-1)^{p-i-1}\binom{k}{i}\binom{k-i-1}{p-i-1}Q_{A_1,A_2}^i(S)\quad(k\in\mathbb{N}_0). \tag{10}$$

Similarly, (B_2,B_1) being (q,S)-null, also

$$Q_{B_2,B_1}^k(S)=\sum_{j=0}^{q-1}(-1)^{q-j-1}\binom{k}{j}\binom{k-j-1}{q-j-1}Q_{B_2,B_1}^j(S)\quad(k\in\mathbb{N}_0). \tag{11}$$

Inserting (10) and (11) into (9), we obtain

$$\begin{aligned}&\sum_{k=0}^{p+q-1}(-1)^k\binom{p+q-1}{k}Q_{A_1,A_2}^k(S)TQ_{B_2,B_1}^k(S)\\&=\sum_{k=0}^{p+q-1}(-1)^k\binom{p+q-1}{k}\left[\sum_{i=0}^{p-1}(-1)^{p-i-1}\binom{k}{i}\binom{k-i-1}{p-i-1}Q_{A_1,A_2}^i(S)\right]\\&\quad T\left[\sum_{j=0}^{q-1}(-1)^{q-j-1}\binom{k}{j}\binom{k-j-1}{q-j-1}Q_{B_2,B_1}^j(S)\right]\\&=-\sum_{i=0}^{p-1}(-1)^iQ_{A_1,A_2}^i(S)T\sum_{j=0}^{q-1}(-1)^j\left[\sum_{k=0}^{p+q-1}(-1)^{p+q-k-1}\binom{p+q-1}{k}\binom{k}{i}\right.\\&\quad\left.\times\binom{k-i-1}{p-i-1}\binom{k}{j}\binom{k-j-1}{q-j-1}\right]Q_{B_2,B_1}^j(S).\end{aligned}$$

To complete the proof, it suffices to establish that

$$\sum_{k=0}^{p+q-1}(-1)^{p+q-k-1}\binom{p+q-1}{k}\binom{k}{i}\binom{k-i-1}{p-i-1}\binom{k}{j}\binom{k-j-1}{q-j-1}=0$$

for every $i, j \in \mathbb{N}_0$ with $0 \leq i \leq p-1$ and $0 \leq j \leq q-1$. This amounts to showing that

$$\sum_{k=0}^{p+q-1} (-1)^{p+q-k-1} \binom{p+q-1}{k} \times \frac{k^2(k-1)^2 \cdots (k-p+1)^2 (k-p) \cdots (k-q+1)}{(k-i)(k-j)} = 0.$$

Since

$$f(x) = \frac{x^2(x-1)^2 \cdots (x-p+1)^2 (x-p) \cdots (x-q+1)}{(x-i)(x-j)} \quad (x \in \mathbb{R})$$

is a polynomial in x of degree less than $p+q-1$, Lemma 1, with $x = 0$, yields the desired conclusion. □

Corollary 1. *Let $p, q \in \mathbb{N}$ and $M, N \in \mathcal{B}(H)$. Assume M is a p-isometry and N^* is a q-isometry. Then, the operator $L(T) = MTN$ $(T \in \mathcal{B}(H))$ is a $(p+q-1)$-isometry.*

Proof. In fact, (M^*, M) is (p, I)-null, and (N, N^*) is (q, I)-null (Remark 1). Apply Theorem 1, with $S = I$, $(A_1, A_2) = (M^*, M)$, and $(B_2, B_1) = (N, N^*)$, to obtain

$$\sum_{k=0}^{p+q-1} (-1)^k \binom{p+q-1}{k} M^{*k} M^k T N^k N^{*k} = 0 \quad (T \in \mathcal{B}(H)),$$

as required. □

The following result by Fong and Sourour ([9], Theorem 1) will allow us to give a final property of m-null elementary operators.

Lemma 2. *Let $\{M_k\}_{k=1}^m$ and $\{N_k\}_{k=1}^m$ be bounded operators on the Hilbert space H, and let Φ be the operator given by*

$$\Phi(T) = \sum_{k=1}^{m} M_k T N_k \quad (T \in \mathcal{B}(H)),$$

with not all the M_k's equal to 0. If $\Phi(T) = 0$ for all $T \in \mathcal{B}(H)$, then $\{N_1, N_2, \ldots, N_m\}$ is linearly dependent. Furthermore, if $\{N_1, N_2, \ldots, N_q\}$ $(q \leq m)$ is a maximal linearly independent subset of $\{N_1, N_2, \ldots, N_m\}$, and c_{ij} $(j, i \in \mathbb{N},\ 1 \leq j \leq q,\ q+1 \leq i \leq m)$ denote constants for which

$$N_i = \sum_{j=1}^{q} c_{ij} N_j \quad (q+1 \leq i \leq m), \tag{12}$$

then $\Phi(T) = 0$ for all $T \in \mathcal{B}(H)$ if, and only if,

$$M_j = -\sum_{i=q+1}^{m} c_{ij} M_i \quad (1 \leq j \leq q). \tag{13}$$

(In case $m = q$, identity (12) becomes vacuous and condition (13) should be interpreted as $M_1 = M_2 = \ldots = M_m = 0$).

We shall also make use of the following basic facts.

Lemma 3. *Let $m \in \mathbb{N}$ and $S \in \mathcal{B}(H)$.*

(i) If the family of powers $\{(I-S)^k\}_{k=0}^m$ is linearly independent, then so is $\{S^k\}_{k=0}^m$.
(ii) If $(I-S)^m = 0$, but $(I-S)^{m-1} \neq 0$, then $\{S^k\}_{k=0}^{m-1}$ is linearly independent.

Proof. First, let $\{\lambda_k\}_{k=0}^m$ be scalars such that $\sum_{k=0}^m \lambda_k S^k = 0$; we want to show that $\lambda_k = 0$ $(k \in \mathbb{N}_0,\ 0 \le k \le m)$. Since

$$S^k = [I - (I - S)]^k = \sum_{r=0}^{k} (-1)^r \binom{k}{r} (I - S)^r,$$

we obtain

$$\sum_{k=0}^{m} \lambda_k \sum_{r=0}^{k} (-1)^r \binom{k}{r} (I - S)^r = \sum_{r=0}^{m} \left[(-1)^r \sum_{k=r}^{m} \binom{k}{r} \lambda_k \right] (I - S)^r = 0.$$

The linear independence of the family $\{(I - S)^r\}_{r=0}^m$ yields the following homogeneous system of linear equations in the unknowns $\{\lambda_k\}_{k=0}^m$:

$$\sum_{k=r}^{m} \binom{k}{r} \lambda_k = 0 \quad (r \in \mathbb{N}_0,\ 0 \le r \le m).$$

The matrix of coefficients of this system is upper triangular, so its determinant can be computed as the product of the entries in the main diagonal, namely,

$$\prod_{r=0}^{m} \binom{r}{r} = 1 \neq 0.$$

This means that the only solution to this system is the trivial one. Consequently, $\{S^k\}_{k=0}^m$ is linearly independent.

Second, to prove that the family $\{(I - S)^k\}_{k=0}^{m-1}$ is linearly independent, let the scalars $\{\mu_k\}_{k=0}^{m-1}$ be such that

$$\sum_{k=0}^{m-1} \mu_k (I - S)^k = 0.$$

Again, we want to show that $\mu_k = 0$ $(k \in \mathbb{N}_0,\ 0 \le k \le m - 1)$. As $(I - S)^m = 0$, but $(I - S)^{m-1} \neq 0$,

$$\begin{aligned} (I - S)^{m-1} \sum_{k=0}^{m-1} \mu_k (I - S)^k &= \mu_0 (I - S)^{m-1} + \sum_{k=1}^{m-1} \mu_k (I - S)^{m+k-1} \\ &= \mu_0 (I - S)^{m-1} \\ &= 0 \end{aligned}$$

necessarily implies that $\mu_0 = 0$. Proceeding by induction, assume $\mu_0 = \mu_1 = \ldots = \mu_{r-1} = 0$ for some $r \in \mathbb{N}_0,\ 0 \le r \le m - 1$. Then,

$$\sum_{k=0}^{m-1} \mu_k (I - S)^k = \sum_{k=r}^{m-1} \mu_k (I - S)^k = 0$$

and

$$\begin{aligned} (I - S)^{m-r-1} \sum_{k=0}^{m-1} \mu_k (I - S)^k &= (I - S)^{m-r-1} \sum_{k=r}^{m-1} \mu_k (I - S)^k \\ &= \mu_r (I - S)^{m-1} + \sum_{k=r+1}^{m-1} \mu_k (I - S)^{m+k-r-1} \\ &= \mu_r (I - S)^{m-1} \\ &= 0, \end{aligned}$$

which entails that $\mu_r = 0$ and completes the induction. □

Theorem 2. *Notation is as in Theorem 1. Fix $m \in \mathbb{N}$ and $S \in \mathcal{B}(H)$. Let the elementary operator Q_S be given by (8).*

(i) Further, let $q \in \mathbb{N}$ be such that $\{B_2^j S B_1^j\}_{j=0}^{q-1}$ is a maximal linearly independent subset of the set $\{B_2^k S B_1^k\}_{k=0}^{m}$, so that

$$B_2^i S B_1^i = \sum_{j=0}^{q-1} c_{ij} B_2^j S B_1^j \quad (q \le i \le m) \tag{14}$$

for some constants c_{ij} $(j, i \in \mathbb{N}_0,\ 0 \le j \le q-1,\ q \le i \le m)$. Then, Q_S is m-null if, and only if,

$$\begin{aligned} &(-1)^j \binom{m}{j} A_1^j S A_2^j \\ &\quad = -\sum_{i=q}^{m} (-1)^i c_{ij} \binom{m}{i} A_1^i S A_2^i \quad (0 \le j \le q-1). \end{aligned} \tag{15}$$

(ii) If q is as in Part (i), then (B_2, B_1) is (q, S)-null if, and only if,

$$\begin{aligned} c_{ij} &= (-1)^{q-j-1} \binom{i}{j} \binom{i-j-1}{q-j-1} \\ &= (-1)^{q-j-1} \frac{q-j}{i-j} \binom{i}{q} \binom{q}{j} \quad (0 \le j \le q-1,\ q \le i \le m). \end{aligned}$$

(iii) Let q be as in Part (i), let $p = m - q + 1$, and assume Q_S is m-null. If (B_2, B_1) is (necessarily strictly) (q, S)-null, then (A_1, A_2) is (p, S)-null. Conversely, if (A_1, A_2) is strictly (p, S)-null, then (B_2, B_1) is (q, S)-null.

Proof. Part (i) follows immediately from (9) and Lemma 2, while the first identity in Part (ii) is a straightforward consequence of Proposition 3 and the linear independence of $\{B_2^j S B_1^j\}_{j=0}^{q-1}$. Regarding the second, it is easy to check that

$$\begin{aligned} \binom{i}{j} \binom{i-j-1}{q-j-1} &= \frac{q-j}{i-j} \binom{i}{j} \binom{i-j}{q-j} \\ &= \frac{q-j}{i-j} \binom{i}{q} \binom{q}{j} \quad (0 \le j \le q-1,\ q \le i \le m). \end{aligned}$$

To establish the direct implication in Part (iii), first, observe that if (B_2, B_1) is (q, S)-null, then this property must be strict; otherwise, Proposition 3 would render $B_2^{q-1} S B_1^{q-1}$ as a linear combination of $\{B_2^j S B_1^j\}_{j=0}^{q-2}$, contradicting the linear independence of $\{B_2^j S B_1^j\}_{j=0}^{q-1}$. Next, fix $0 \le j \le q-1$. Since Q_S is m-null, Part (i) gives

$$A^j(S) + \sum_{i=q}^{m} (-1)^{i-j} c_{ij} \binom{m}{i} \binom{m}{j}^{-1} A^i(S) = 0.$$

Furthermore, (B_2, B_1) being (q, S)-null, we may apply Part (ii) and substitute c_{ij} to obtain

$$A^j(S) + \sum_{i=q}^{m} (-1)^{i-q-1} \binom{i}{j} \binom{m}{i} \binom{m}{j}^{-1} \binom{i-j-1}{q-j-1} A^i(S) = 0,$$

or

$$A^j(S) + \sum_{i=q}^{m} (-1)^{i-q-1} \binom{m-j}{m-i} \binom{i-j-1}{q-j-1} A^i(S) = 0.$$

Shifting the summation index and recalling that $m = p + q - 1$, we finally arrive at the identity

$$A^j(S) = \sum_{i=0}^{p-1} (-1)^i \binom{p+q-j-1}{q+i-j} \binom{q+i-j-1}{i} A^{q+i}(S), \tag{16}$$

valid for all $0 \leq j \leq q-1$. The particularization of $j = 0$ in (16) gives

$$S = RA^q(S) = A^q R(S),$$

where

$$R := \sum_{i=0}^{p-1} (-1)^i \binom{p+q-1}{q+i} \binom{q+i-1}{i} A^i.$$

On the other hand, particularizing $j = q-1$ in (16) and shifting indices, we obtain

$$\begin{aligned} A^q(S) &= A\, A^{q-1}(S) \\ &= \sum_{i=0}^{p-1} (-1)^i \binom{p}{i+1} A^{q+i+1}(S) \\ &= \sum_{i=1}^{p} (-1)^{i-1} \binom{p}{i} A^{q+i}(S), \end{aligned}$$

whence

$$0 = \left[I + \sum_{i=1}^{p} (-1)^i \binom{p}{i} A^i \right] A^q(S) = \left[\sum_{i=0}^{p} (-1)^i \binom{p}{i} A^i \right] A^q(S).$$

Since R commutes with the operator in brackets,

$$\begin{aligned} 0 = R \left[\sum_{i=0}^{p} (-1)^i \binom{p}{i} A^i \right] A^q(S) &= \left[\sum_{i=0}^{p} (-1)^i \binom{p}{i} A^i \right] RA^q(S) \\ = \sum_{i=0}^{p} (-1)^i \binom{p}{i} A^i(S). \end{aligned}$$

Thus,

$$P_{A_1,A_2,p}(S) = \sum_{i=0}^{p} (-1)^i \binom{p}{i} A_1^i S A_2^i = \sum_{i=0}^{p} (-1)^i \binom{p}{i} A^i(S) = 0,$$

so that (A_1, A_2) is (p, S)-null.

To prove the converse implication in Part (iii), assume that Q_S is m-null and (A_1, A_2) is strictly (p, S)-null. Then, by Part (ii) of Lemma 3 along with Proposition 3, the family $\{A_1^j S A_2^j\}_{j=0}^{p-1}$ is linearly independent maximal. The desired conclusion follows from the direct implication, by symmetry. □

Remark 3. *Again, Remark 1 provides a particular version of Theorem 2 for m-isometries.*

4. An Open Question

The nature of the solutions to the problems discussed in this paper is primarily algebraic and combinatorial rather than analytic, as it is often the case in the literature on

similar problems arising in other contexts, such as m-invertible, m-isometric, m-self-adjoint, and related classes of operators. In ([10], Theorem 7), the following result is established:

Theorem 3. *Assume $A, B \in \mathcal{B}(H)$. Then, the elementary operator with symbols A, B, acting on the Hilbert–Schmidt class of a separable Hilbert space, is a strict m-isometry if, and only if, there is a constant λ such that λA is a strict p-isometry and $\lambda^{-1}B^*$ is a strict q-isometry, where $p + q - 1 = m$.*

Unlike the proof for the *if* part, that for the *only if* part uses analytical arguments relying on properties of the spectral radius and the approximate point spectrum of an operator, which can be traced back to Magajna [12]. The question arises whether a similar converse to Theorem 1, and hence a proof for Theorem 3, can be found that avoids analytic tools.

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