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Convergence of Infinite Products of Uniformly Locally Nonexpansive Mappings

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Abstract: The generic convergence of infinite products of nonexpansive mappings was established in a 1999 paper of ours. In the present paper, such results are extended to infinite products of uniformly locally nonexpansive mappings. In particular, the convergence of infinite products of uniformly locally contractive mappings, as well as its stability, are proved. Moreover, the Baire category approach and the porosity notion are used to show that most sequences of uniformly locally nonexpansive mappings are, in fact, uniformly locally contractive.

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1. Introduction

For more than sixty years now, there has been a lot of research activity regarding the fixed-point theory of nonexpansive (that is, 1-Lipschitz) mappings. See, for example, Refs. [1–12] and the references cited therein. This activity stems from Banach’s classical theorem [13] concerning the existence of a unique fixed point for a strict contraction. It also concerns the convergence of (inexact) iterates of a nonexpansive mapping to one of its fixed points. In particular, the existence of a fixed point was studied in [1,3–6], while convergence results were obtained in [2,11,12]. Since that seminal result, many developments have taken place in this area, including, in particular, studies of feasibility [14,15] and common fixed points [12,16,17], which find important applications in engineering, the medical sciences, and the natural sciences.

In 1961, E. Rakotch [18] showed that a uniformly locally contractive mapping has a fixed point. In [19], it was proved that for a uniformly locally contractive mapping the fixed point problem is well posed and that its inexact iterates converge to its fixed point, uniformly on bounded sets. Using the porosity notion, it was shown that most uniformly locally nonexpansive mappings are, in fact, uniformly locally contractive.

In [20], the generic convergence of infinite products of nonexpansive mappings was established. In the present paper, the results of [20] are extended to infinite products of uniformly locally nonexpansive mappings. In particular, the convergence of infinite products of uniformly locally contractive mappings, as well as its stability, are established. Moreover, the Baire category approach and the porosity notion are used to show that most sequences of uniformly locally nonexpansive mappings are, in fact, uniformly locally contractive. In this connection, note that infinite products find various applications in

engineering and medical sciences and, in particular, in computer tomography and radiation planning. For recent studies of infinite products of nonexpansive mappings, see [12,14–17].

2. Preliminaries

Assume that (X, ρ) is a complete metric space, $\theta \in X$, and that for each $\gamma \in (0, 1)$, there is a point

$$(1 - \gamma)x \oplus \gamma\theta \in X$$

such that for each $x, y \in X$ and each $\gamma \in (0, 1)$,

$$\rho((1 - \gamma)x \oplus \gamma\theta, (1 - \gamma)y \oplus \gamma\theta) \leq (1 - \gamma)\rho(x, y), \quad (1)$$

$$(1 - \gamma)\theta \oplus \gamma\theta = \theta$$

and

$$\rho((1 - \gamma)x \oplus \gamma\theta, x) \leq \gamma\rho(x, \theta). \quad (2)$$

Note that this assumption indeed holds if X is a convex subset of a hyperbolic space [6].

Assume that

$$\text{diam}(X) = \sup\{\rho(x, y) : x, y \in X\} < \infty \quad (3)$$

and that

$$\Delta \in (0, \min\{1, \text{diam}(X)\}). \quad (4)$$

Denote by $\mathcal{M}$ the set of all mappings $T : X \rightarrow X$ such that for each $x, y \in X$ satisfying $\rho(x, y) \leq \Delta$, the inequality

$$\rho(T(x), T(y)) \leq \rho(x, y) \quad (5)$$

holds. Elements of $\mathcal{M}$ are said to be uniformly locally nonexpansive mappings. We remark in passing that the smaller class of uniformly local contractions was introduced in [4], while the larger class of locally nonexpansive mappings was studied in [2]. The work [4] also contains an example of a uniformly local contraction that is not nonexpansive.

For each $S, T \in \mathcal{M}$, set

$$d_{\mathcal{M}}(S, T) = \sup\{\rho(S(x), T(x)) : x \in X\}. \quad (6)$$

It is not difficult to see that $(\mathcal{M}, d_{\mathcal{M}})$ is a complete metric space.

Denote by $\mathfrak{M}$ the set of all sequences of operators $\{A_t\}_{t=1}^{\infty} \subset \mathcal{M}$. For each $\{A_t\}_{t=1}^{\infty}, \{B_t\}_{t=1}^{\infty} \in \mathfrak{M}$, set

$$d_{\mathfrak{M}}(\{A_t\}_{t=1}^{\infty}, \{B_t\}_{t=1}^{\infty}) = \sup\{d_{\mathcal{M}}(A_t, B_t) : t = 1, 2, \dots\}. \quad (7)$$

It is not difficult to see that $(\mathfrak{M}, d_{\mathfrak{M}})$ is a complete metric space. Choose a positive number

$$\gamma_* < \min\{2^{-1}, \Delta(\text{diam}(X) + 1)^{-1}\} \quad (8)$$

such that

$$n_* := \gamma_*^{-1} \quad (9)$$

is an integer. Clearly, $n_* \geq 2$.

Lemma 1. *Let $x \in X \setminus \{\theta\}$ and $\gamma \in (0, 1)$. Then,*

$$\rho(x, (1 - \gamma)x \oplus \gamma\theta) = \gamma\rho(x, \theta)$$

and

$$\rho((1-\gamma)x \oplus \gamma\theta, \theta) = (1-\gamma)\rho(x, \theta).$$

Proof. By (1) and (2),

$$\rho(x, (1-\gamma)x \oplus \gamma\theta) \leq \gamma\rho(x, \theta)$$

and

$$\begin{aligned} & \rho((1-\gamma)x \oplus \gamma\theta, \theta) \\ &= \rho((1-\gamma)x \oplus \gamma\theta, (1-\gamma)\theta \oplus \gamma\theta) \leq (1-\gamma)\rho(x, \theta). \end{aligned}$$

In view of the above relations, we have

$$\begin{aligned} \rho(x, \theta) &= \gamma\rho(x, \theta) + (1-\gamma)\rho(x, \theta) \\ &\geq \rho(x, (1-\gamma)x \oplus \gamma\theta) + \rho((1-\gamma)x \oplus \gamma\theta, \theta) \geq \rho(x, \theta). \end{aligned}$$

This completes the proof of Lemma 1. $\square$

Lemma 2. Assume that $x \in X \setminus \{\emptyset\}$,

$$\rho(x, \theta) > \Delta,$$

$$x_0 = x, \tag{10}$$

for each nonnegative integer $i < n_* - 1$,

$$x_{i+1} = ((1 - (i + 1)\gamma_*)(1 - i\gamma_*)^{-1}x_i \oplus \gamma_i(1 - i\gamma_*)^{-1}\theta), \tag{11}$$

and

$$x_{n_*} = \theta.$$

Then,

$$\rho(x_i, x_{i+1}) \leq \Delta, \quad i = 0, \dots, n_* - 1, \quad \sum_{i=1}^{n_*-1} \rho(x_i, x_{i+1}) = \rho(x, \theta).$$

Proof. Lemma 1 and (10), (11) imply that

$$x_1 = (1 - \gamma_*)x \oplus \gamma_*\theta,$$

$$\rho(x_0, x_1) = \gamma_*\rho(x, \theta), \quad \rho(x_1, \theta) = (1 - \gamma_*)\rho(x, \theta). \tag{12}$$

Assume now that $k < n_* - 1$ is a natural number and that for each $i \in \{1, \dots, k\}$, we have

$$\rho(x_{i-1}, x_i) = \gamma_*\rho(x, \theta), \quad \rho(x_i, \theta) = (1 - i\gamma_*)\rho(x, \theta). \tag{13}$$

(In view of (12), (13) holds for $i = 1$). By (13),

$$\begin{aligned} \rho(x, \theta) &\leq \rho(x, x_k) + \rho(x_k, \theta) \\ &\leq \sum_{j=0}^{k-1} \rho(x_j, x_{j+1}) + \rho(x_k, \theta) \\ &= k\gamma_*\rho(x, \theta) + (1 - k\gamma_*)\rho(x, \theta). \end{aligned}$$

Thus,

$$\rho(x, x_k) = k\gamma_*\rho(x, \theta). \tag{14}$$

Now, Lemma 1 and (11), (13) imply that

$$\begin{aligned}\rho(x_k, x_{k+1}) &= \rho(x_k, (1 - (k+1)\gamma_*)(1 - k\gamma_*)^{-1}x_k \oplus \gamma_*(1 - k\gamma_*)^{-1}\theta) \\ &= \gamma_*(1 - k\gamma_*)^{-1}\rho(x_k, \theta) = \gamma_*\rho(x, \theta), \\ \rho(x_{k+1}, \theta) &= (1 - (k+1)\gamma_*)(1 - k\gamma_*)^{-1}\rho(x_k, \theta) \\ &= (1 - (k+1)\gamma_*)\rho(x, \theta)\end{aligned}$$

and so our assumption holds for $k+1$ too. Thus, we have shown by using mathematical induction that for each $i \in \{1, \dots, n_* - 1\}$, relations (13) and (14) hold. By (13),

$$\rho(x_{n_*-1}, \theta) = (1 - (n_* - 1)\gamma_*)\rho(x, \theta) = \gamma_*\rho(x, \theta) < \Delta$$

and

$$\sum_{i=1}^{n_*} \rho(x_i, x_{i-1}) = n_*\gamma_*\rho(x, \theta) = \rho(x, \theta).$$

This completes the proof of Lemma 2. $\square$

3. Contractive Sequences of Mappings

A sequence $\{A_t\}_{t=1}^\infty \in \mathfrak{M}$ is called contractive if there is a decreasing function $\phi : [0, \Delta] \rightarrow [0, 1]$ such that

$$\phi(t) < 1 \text{ for all } t \in (0, \Delta] \quad (15)$$

and for each $x, y \in X$ satisfying $\rho(x, y) \leq \Delta$ and each integer $t \geq 1$,

$$\rho(A_t(x), A_t(y)) \leq \phi(\rho(x, y))\rho(x, y). \quad (16)$$

Theorem 1. Assume that $\{A_t\}_{t=1}^\infty \in \mathfrak{M}$ is contractive and that $\epsilon > 0$. Then, there is a natural number n_ϵ such that for each $r : \{1, 2, \dots\} \rightarrow \{1, 2, \dots\}$, each $x, y \in X$, and each integer $n \geq n_\epsilon$,

$$\rho(A_{r(n)} \cdots A_{r(1)}(x), A_{r(n)} \cdots A_{r(1)}(y)) \leq \epsilon. \quad (17)$$

Proof. We may assume without any loss of generality that $\epsilon < \Delta$. Let $\phi : [0, \Delta] \rightarrow [0, 1]$ be a decreasing function satisfying (15) and (16) for each $x, y \in X$ satisfying $\rho(x, y) \leq \Delta$ and each integer $t \geq 1$. Choose a natural number

$$n_\epsilon > 2n_*\text{diam}(X)\epsilon^{-1}(1 - \phi((2n_*)^{-1}\epsilon))^{-1}. \quad (18)$$

Assume that $r : \{1, 2, \dots\} \rightarrow \{1, 2, \dots\}$. In order to prove the theorem, it is sufficient to show that for each $x, y \in X$ and each integer $n \geq n_\epsilon$, inequality (17) holds. In order to meet this goal, it is sufficient to show that for each $x \in X$ and each integer $n \geq n_\epsilon$, we have

$$\rho(A_{r(n)} \cdots A_{r(1)}(x), A_{r(n)} \cdots A_{r(1)}(\theta)) \leq \epsilon/2.$$

Let $x \in X$. In view of (5) and the inequality $\epsilon \leq \Delta$, it is sufficient to show that there is an integer $k \in \{1, \dots, n_\epsilon\}$ such that

$$\rho(A_{r(k)} \cdots A_{r(1)}(x), A_{r(k)} \cdots A_{r(1)}(\theta)) \leq \epsilon/2.$$

Suppose to the contrary that this does not hold. Then,

$$\rho(x, \theta) > \epsilon/2 \quad (19)$$

and for each integer $k \in \{1, \dots, n_\epsilon\}$,

$$\rho(A_{r(k)} \cdots A_{r(1)}(x), A_{r(k)} \cdots A_{r(1)}(\theta)) > \epsilon/2. \quad (20)$$

Lemma 2 implies that there are $x_i \in X, i = 0, \dots, n_*$, such that

$$x_0 = x, x_{n_*} = \theta, \quad \rho(x_i, x_{i+1}) \leq \Delta, i = 0, \dots, n_* - 1, \sum_{i=1}^{n_*} \rho(x_i, x_{i-1}) = \rho(x, \theta). \quad (21)$$

It follows from (19)–(21) that

$$\max\{\rho(x_i, x_{i+1}) : i = 0, \dots, n_* - 1\} > (2n_*)^{-1}\epsilon \quad (22)$$

and for each integer $k \in \{1, \dots, n_\epsilon\}$,

$$\max\{\rho(A_{r(k)} \cdots A_{r(1)}(x_i), A_{r(k)} \cdots A_{r(1)}(x_{i+1})) : i = 0, \dots, n_* - 1\} > (2n_*)^{-1}\epsilon. \quad (23)$$

By (5) and (21), for each $i \in \{0, \dots, n_\epsilon - 1\}$ and for each $k \in \{1, \dots, n_\epsilon - 1\}$, we have

$$\rho(A_{r(1)}(x_i), A_{r(1)}(x_{i+1})) \leq \rho(x_i, x_{i+1}) \quad (24)$$

and

$$\begin{aligned} & \rho(A_{r(k+1)} \cdots A_{r(1)}(x_i), A_{r(k+1)} \cdots A_{r(1)}(x_{i+1})) \\ & \leq \rho(A_{r(k)} \cdots A_{r(1)}(x_i), A_{r(k)} \cdots A_{r(1)}(x_{i+1})). \end{aligned} \quad (25)$$

Next, we consider

$$\sum_{i=0}^{n_*-1} (\rho(x_i, x_{i+1}) - \rho(A_{r(1)}(x_i), A_{r(1)}(x_{i+1})))$$

and

$$\begin{aligned} & \sum_{i=0}^{n_*-1} (\rho(A_{r(k)} \cdots A_{r(1)}(x_i), A_{r(k)} \cdots A_{r(1)}(x_{i+1})) \\ & - \rho(A_{r(k+1)} \cdots A_{r(1)}(x_i), A_{r(k+1)} \cdots A_{r(1)}(x_{i+1}))) \end{aligned}$$

for $k = 1, \dots, n_\epsilon$. By (22), there is $j \in \{0, \dots, n_* - 1\}$ such that

$$\rho(x_j, x_{j+1}) > (2n_*)^{-1}\epsilon. \quad (26)$$

It follows from (5), (16), (21), and (26) that

$$\begin{aligned} & \rho(x_j, x_{j+1}) - \rho(A_{r(1)}(x_j), A_{r(1)}(x_{j+1})) \\ & \geq \rho(x_j, x_{j+1}) - \phi(\rho(x_j, x_{j+1}))\rho(x_j, x_{j+1}) \\ & \geq \rho(x_j, x_{j+1})(1 - \phi(\rho(x_j, x_{j+1}))) \\ & \geq \epsilon(2n_*)^{-1}(1 - \phi((2n_*)^{-1}\epsilon)) \end{aligned}$$

and

$$\begin{aligned} & \sum_{i=0}^{n_*-1} (\rho(x_i, x_{i+1}) - \rho(A_{r(1)}(x_i), A_{r(1)}(x_{i+1}))) \\ & \geq \epsilon(2n_*)^{-1}(1 - \phi((2n_*)^{-1}\epsilon)). \end{aligned} \quad (27)$$

Let $k \in \{1, \dots, n_\epsilon - 1\}$. By (20) and (21), there is

$$j_k \in \{0, \dots, n_* - 1\}$$

such that

$$\rho(A_{r(k)} \cdots A_{r(1)}(x_{j_k}), A_{r(k)} \cdots A_{r(1)}(x_{j_{k+1}})) > (2n_*)^{-1}\epsilon.$$

It follows from (22), (24), (25), and the above inequality that

$$\begin{aligned} & \rho(A_{r(k)} \cdots A_{r(1)}(x_{j_k}), A_{r(k)} \cdots A_{r(1)}(x_{j_{k+1}})) \\ & - \rho(A_{r(k+1)} \cdots A_{r(1)}(x_{j_k}), A_{r(k+1)} \cdots A_{r(1)}(x_{j_{k+1}})) \\ & \geq \rho(A_{r(k)} \cdots A_{r(1)}(x_{j_k}), A_{r(k)} \cdots A_{r(1)}(x_{j_{k+1}})) \\ & - \phi(\rho(A_{r(k)} \cdots A_{r(1)}(x_{j_k}), A_{r(k)} \cdots A_{r(1)}(x_{j_{k+1}}))) \\ & \quad \times \rho(A_{r(k)} \cdots A_{r(1)}(x_{j_k}), A_{r(k)} \cdots A_{r(1)}(x_{j_{k+1}})) \\ & \geq \rho(A_{r(k)} \cdots A_{r(1)}(x_{j_k}), A_{r(k)} \cdots A_{r(1)}(x_{j_{k+1}}))(1 - \\ & \quad - \phi(\rho(A_{r(k)} \cdots A_{r(1)}(x_{j_k}), A_{r(k)} \cdots A_{r(1)}(x_{j_{k+1}})))) \\ & \geq \epsilon(2n_*)^{-1}(1 - \phi((2n_*)^{-1}\epsilon)) \end{aligned}$$

and

$$\begin{aligned} & \sum_{i=0}^{n_*-1} (\rho(A_{r(k)} \cdots A_{r(1)}(x_i), A_{r(k)} \cdots A_{r(1)}(x_{i+1}))) \\ & - \sum_{i=0}^{n_*-1} (\rho(A_{r(k+1)} \cdots A_{r(1)}(x_i), A_{r(k+1)} \cdots A_{r(1)}(x_{i+1}))) \\ & \geq \epsilon(2n_*)^{-1}(1 - \phi((2n_*)^{-1}\epsilon)). \end{aligned}$$

In view of (21), (27), and the above relation, we have

$$\begin{aligned} \text{diam}(X) & \geq \rho(x, \theta) \\ & = \sum_{i=0}^{n_*-1} \rho(x_i, x_{i+1}) \\ & - \sum_{i=0}^{n_*-1} (\rho(A_{r(n_\epsilon)} \cdots A_{r(1)}(x_i), A_{r(n_\epsilon)} \cdots A_{r(1)}(x_{i+1}))) \\ & = \sum_{i=0}^{n_*-1} (\rho(x_i, x_{i+1}) - \rho(A_{r(1)}(x_i), A_{r(1)}(x_{i+1}))) \\ & + \sum_{k=1}^{n_\epsilon-1} \sum_{i=0}^{n_*-1} (\rho(A_{r(k)} \cdots A_{r(1)}(x_i), A_{r(k)} \cdots A_{r(1)}(x_{i+1}))) \\ & \quad - \rho(A_{r(k+1)} \cdots A_{r(1)}(x_i), A_{r(k+1)} \cdots A_{r(1)}(x_{i+1}))) \\ & \geq n_\epsilon \epsilon (2n_*)^{-1} (1 - \phi((2n_*)^{-1}\epsilon)) \end{aligned}$$

and

$$n_\epsilon \leq 2n_* \text{diam}(X) \epsilon^{-1} (1 - \phi((2n_*)^{-1}\epsilon))^{-1}.$$

This, however, contradicts (18). The contradiction we have reached proves Theorem 1. $\square$

4. A Porosity Result

We begin this section by recalling the notion of porosity.

Assume that (Y, d) is a complete metric space equipped with a metric d . For each point $\xi \in Y$ and each positive number Δ , set

$$B(\xi, \Delta) = \{\eta \in Y : d(\xi, \eta) \leq \Delta\}.$$

A set $F \subset Y$ is said to be porous (regarding the complete metric d) if there are constants $\beta \in (0, 1)$ and $\Delta_0 > 0$ such that for every number $\Delta \in (0, \Delta_0]$ and each point $\xi \in Y$, there is a point $\eta \in Y$ satisfying

$$B(\eta, \beta\Delta) \subset B(\xi, \Delta) \setminus F.$$

A subset of the metric space Y is said to be σ -porous (regarding the metric d) if it is a countable union of porous sets in the space Y .

It is clear that porous sets are nowhere dense. Therefore, they are of the first Baire category. If Y is a Euclidean space (hence of finite dimension), then any σ -porous set in it is of Lebesgue measure zero. As a matter of fact, the collection of σ -porous sets in a Euclidean space is much smaller than the collection of sets that possess Lebesgue measure 0 and are also of the first Baire category. In addition, we remark in passing that each Banach space contains a set of the first Baire category, which, however, is not σ -porous.

In order to bring out the difference between nowhere dense and porous sets, we note that if a set $E \subset Y$ is nowhere dense, a point y belongs to Y , and r is a positive number, then a point $z \in Y$ and a positive number s satisfying $B(z, s) \subset B(y, r) \setminus E$ exist. However, if the set E is also porous, then for small enough positive numbers r we can choose $s = \alpha r$, where $\alpha \in (0, 1)$ is a certain constant that only depends on E .

The study of mathematical problems by using the porosity notion is now a well-established area of research. Many examples of porosity results can be found in [11]. Using the porosity notion, we show that most sequences of uniformly locally nonexpansive mappings are, in fact, uniformly locally contractive.

Denote by $\mathcal{F}$ the set of all contractive sequences $\{A_t\}_{t=1}^\infty \in \mathfrak{M}$.

Theorem 2. $\mathfrak{M} \setminus \mathcal{F}$ is a σ -porous set in $(\mathfrak{M}, d_{\mathfrak{M}})$.

Proof. For every natural number n , define

$$\begin{aligned} \mathcal{F}_n = \{ \{A_t\}_{t=1}^\infty \in \mathfrak{M} : \sup \{ \rho(A_t(x), A_t(y)) \rho(x, y)^{-1} : \\ t \in \{1, 2, \dots\}, x, y \in X, \rho(x, y) \in [(2n)^{-1}\Delta, \Delta] \} < 1 \}. \end{aligned} \quad (28)$$

Clearly, for each integer $n \geq 1$,

$$\mathcal{F} \subset \mathcal{F}_n.$$

We claim that

$$\mathcal{F} = \bigcap_{n=1}^\infty \mathcal{F}_n. \quad (29)$$

To show this, let

$$\{A_t\}_{t=1}^\infty \in \bigcap_{n=1}^\infty \mathcal{F}_n.$$

In order to prove (29), it is sufficient to show that $\{A_t\}_{t=1}^\infty \in \mathcal{F}$. To this end, let $\phi(0) = 1$, and for each $s \in (0, \Delta]$, set

$$\begin{aligned} \phi(s) = \sup \{ \rho(A_t(x), A_t(y)) \rho(x, y)^{-1} : \\ t \in \{1, 2, \dots\}, x, y \in X, \rho(x, y) \in [s, \Delta] \}. \end{aligned}$$

In view of (28), $\phi(t) < 1$ for each $t > 0$ and the function ϕ is decreasing. It is not difficult to see that for each $x, y \in X$ satisfying $0 < \rho(x, y) \leq \Delta$ and each integer $t \geq 1$, we have

$$\rho(A_t(x), A_t(y))\rho(x, y)^{-1} \leq \phi(\rho(x, y)).$$

Thus, $\{A_t\}_{t=1}^\infty \in \mathcal{F}$ and (29) holds.

In order to complete the proof of the theorem, it is sufficient to show that for each integer $n \geq 1$, the set $\mathfrak{M} \setminus \mathcal{F}_n$ is porous. To this end, let $n \geq 1$ be an integer. Define

$$\alpha = (16n)^{-1}\Delta(\text{diam}(X) + 1)^{-1}. \quad (30)$$

Given $r \in (0, 1]$ and $\{A_t\}_{t=1}^\infty \in \mathfrak{M}$, define

$$\gamma = 2^{-1}r(\text{diam}(X) + 1)^{-1} \quad (31)$$

and for each integer $t \geq 1$ and each $x \in X$, set

$$A_{\gamma,t}(x) = (1 - \gamma)A_t(x) \oplus \gamma\theta. \quad (32)$$

It follows from (1), (5), and (32) that for each integer $t \geq 1$ and each $x, y \in X$ satisfying

$$\rho(x, y) \leq \Delta,$$

we have

$$\rho(A_{\gamma,t}(x), A_{\gamma,t}(y)) \leq (1 - \gamma)\rho(A_t(x), A_t(y)) \leq (1 - \gamma)\rho(x, y). \quad (33)$$

Assume that $\{B_t\}_{t=1}^\infty \in \mathfrak{M}$ satisfies

$$d_{\mathfrak{M}}(\{B_t\}_{t=1}^\infty, \{A_{\gamma,t}\}_{t=1}^\infty) \leq \alpha r. \quad (34)$$

Let $x \in X$ and let t be a natural number. In view of (34),

$$\rho(B_t(x), A_{\gamma,t}(x)) \leq \alpha r. \quad (35)$$

By (1), (30)–(32), and (35),

$$\rho(A_{\gamma,t}(x), A_t(x)) \leq \gamma\rho(\theta, A_t(x)) \leq \gamma\text{diam}(X)$$

and

$$\rho(B_t(x), A_t(x)) \leq \rho(B_t(x), A_{\gamma,t}(x)) + \rho(A_{\gamma,t}(x), A_t(x)) \leq \alpha r + \gamma\text{diam}(X) \leq r. \quad (36)$$

In view of (30), (31), (33), and (35), for each integer $t \geq 1$ and each $x, y \in X$ satisfying

$$\rho(x, y) \in [(2n)^{-1}\Delta, \Delta],$$

we have

$$\begin{aligned} \rho(B_t(x), B_t(y))\rho(x, y)^{-1} &\leq \rho(A_{\gamma,t}(x), A_{\gamma,t}(y))\rho(x, y)^{-1} \\ &\quad + \rho(x, y)^{-1}(\rho(B_t(x), A_{\gamma,t}(x)) + \rho(B_t(y), A_{\gamma,t}(y))) \\ &\leq 1 - \gamma + 2n\Delta^{-1}(2\alpha r) \\ &\leq 1 - 2^{-1}r(\text{diam}(X) + 1)^{-1} + 4^{-1}r(\text{diam}(X) + 1)^{-1} \end{aligned}$$

and

$$\sup\{\rho(B_t(x), B_t(y))\rho(x, y)^{-1} : t \in \{1, \dots, n\}, \\ x, y \in X, \rho(x, y) \in [(2n)^{-1}\Delta, \Delta]\} \leq 1 - 4^{-1}r(\text{diam}(X) + 1)^{-1} < 1.$$

Therefore,

$$\{B_t\}_{t=1}^\infty \in \mathcal{F}_n$$

and

$$\{\{B_t\}_{t=1}^\infty \in \mathfrak{M} : d_{\mathfrak{M}}(\{B_t\}_{t=1}^\infty, \{A_{\gamma,t}\}_{t=1}^\infty) \leq \alpha r\} \\ \subset \{\{B_t\}_{t=1}^\infty \in \mathfrak{M} : d_{\mathfrak{M}}(\{B_t\}_{t=1}^\infty, \{A_t\}_{t=1}^\infty) \leq r\} \cap \mathcal{F}_n$$

and so $\mathfrak{M} \setminus \mathcal{F}_n$ is a porous set. This completes the proof of Theorem 2. $\square$

5. A Stability Result

Theorem 3. Assume that $\{A_t\}_{t=1}^\infty \in \mathfrak{M}$ is contractive and that $\epsilon \in (0, 1)$. Then, there are $\delta \in (0, \epsilon)$ and a natural number n_0 such that for each $r : \{1, 2, \dots\} \rightarrow \{1, 2, \dots\}$, each $\{x_i\}_{i=0}^\infty, \{y_i\}_{i=0}^\infty \subset X$, which satisfy for each integer $i \geq 0$,

$$\rho(x_{i+1}, A_{r(i+1)}(x_i)) \leq \delta, \rho(y_{i+1}, A_{r(i+1)}(y_i)) \leq \delta \quad (37)$$

and for each integer $n \geq n_0$, we have

$$\rho(x_n, y_n) \leq \epsilon.$$

Proof. We may assume without any loss of generality that $\epsilon < \Delta$. By Theorem 1, there is a natural number n_0 such that the following property holds:

(i) for each $r : \{1, 2, \dots\} \rightarrow \{1, 2, \dots\}$, each $x, y \in X$ and each integer $n \geq n_0$,

$$\rho(A_{r(n)} \cdots A_{r(1)}(x), A_{r(n)} \cdots A_{r(1)}(y)) \leq \epsilon/4.$$

Set

$$\delta = (4n_0)^{-1}\epsilon. \quad (38)$$

Assume that $\{x_i\}_{i=0}^\infty, \{y_i\}_{i=0}^\infty \subset X, r : \{1, 2, \dots\} \rightarrow \{1, 2, \dots\}$, (37) holds and that $n \geq n_0$ is an integer. Property (i) implies that

$$\rho(A_{r(n)} \cdots A_{r(n-n_0+1)}(x), A_{r(n)} \cdots A_{r(n-n_0+1)}(y)) \leq \epsilon/4. \quad (39)$$

In view of (37), we have

$$\rho(x_{n-n_0+1}, A_{r(n-n_0+1)}(x_{n-n_0})) \leq \delta, \rho(y_{n-n_0+1}, A_{r(n-n_0+1)}(y_{n-n_0})) \leq \delta. \quad (40)$$

We claim that for each $i \in \{1, \dots, n_0\}$,

$$\rho(x_{n-n_0+i}, A_{r(n-n_0+i)} \cdots A_{r(n-n_0+1)}(x_{n-n_0})) \leq i\delta < \Delta \quad (41)$$

and

$$\rho(y_{n-n_0+i}, A_{r(n-n_0+i)} \cdots A_{r(n-n_0+1)}(y_{n-n_0})) \leq i\delta < \Delta.$$

Clearly, it is sufficient to prove only inequality (41). In view of (40), inequality (41) holds for $i = 1$. Assume now that $i \in \{1, \dots, n_0\} \setminus \{n_0\}$ and that (41) holds. It follows from (5), (37), (38), and (41) that

$$\rho(x_{n-n_0+i+1}, A_{r(n-n_0+i+1)} \cdots A_{r(n-n_0+1)}(x_{n-n_0}))$$

$$\begin{aligned}
&\leq \rho(x_{n-n_0+i+1}, A_{r(n-n_0+i+1)}(x_{n-n_0+i})) \\
&+ \rho(A_{r(n-n_0+i+1)}(x_{n-n_0+i}), A_{r(n-n_0+i+1)}A_{r(n-n_0+i)} \cdots A_{r(n-n_0+1)}(x_{n-n_0})) \\
&\leq \delta + \rho(x_{n-n_0+i}, A_{r(n-n_0+i)} \cdots A_{r(n-n_0+1)}(x_{n-n_0})) \leq (i+1)\delta \leq n_0\delta < \Delta.
\end{aligned}$$

Thus, (41) holds for all $i = 1, \dots, n_0$ and

$$\rho(x_n, A_{r(n)} \cdots A_{r(n-n_0+1)}(x_{n-n_0})) \leq n_0\delta \leq \epsilon/4.$$

Analogously,

$$\rho(y_n, A_{r(n)} \cdots A_{r(n-n_0+1)}(y_{n-n_0})) \leq n_0\delta \leq \epsilon/4.$$

When combined with (39), this implies that

$$\rho(x_n, y_n) \leq \epsilon.$$

This completes the proof of Theorem 3. $\square$

Note that the stability of the convergence of infinite products is important in applications because computational errors are always present in practice.

6. A Generic Result

We equip the space $\mathfrak{M}$ with the topology induced by the metric $d_{\mathfrak{M}}$, which is called the strong topology, and with the uniformity determined by the base

$$\mathcal{U}(N, \epsilon) = \{(\{A_t\}_{t=1}^\infty, \{B_t\}_{t=1}^\infty) \in \mathfrak{M} \times \mathfrak{M} : d_{\mathfrak{M}}(A_t, B_t) \leq \epsilon, t = 1, \dots, N\}, \quad (42)$$

where N is a natural number and $\epsilon > 0$. Note that the space $\mathfrak{M}$ with this uniformity is metrizable and complete. We also equip the space $\mathfrak{M}$ with the topology induced by this uniformity. This topology is called the weak topology.

Theorem 4. *There is a set $\mathcal{F}$, which is a countable intersection of open (in the weak topology) and everywhere dense (in the strong topology) sets, such that for each $\{A_t\}_{t=1}^\infty \in \mathcal{F}$ and each $\epsilon > 0$, an open (in the weak topology) neighborhood $\mathcal{U}$ of $\{A_t\}_{t=1}^\infty$ and a natural number n_0 exist such that for each $\{B_t\}_{t=1}^\infty \in \mathcal{U}$, each $x, y \in X$, and each integer $n \geq n_0$,*

$$\rho(B_n \cdots B_1(x), \rho(B_n \cdots B_1(y))) \leq \epsilon.$$

Proof. Let $\gamma \in (0, 1)$, $\{A_t\}_{t=1}^\infty \in \mathfrak{M}$, and let p be a natural number. Set

$$A_{\gamma,t}(x) := (1 - \gamma)A_t(x) \oplus \gamma\theta, \quad x \in X, \quad t = 1, 2, \dots$$

In view of (33), the sequence $\{A_{\gamma,t}\}_{t=1}^\infty$ is contractive. Theorem 3 implies that there is a natural number

$$n(\{A_t\}_{t=1}^\infty, \gamma, p) > p \text{ and a number } \delta(\{A_t\}_{t=1}^\infty, \gamma, p) \in (0, (2p)^{-1}\Delta)$$

such that the following property holds:

(i) for each $x, y \in X$, each $r : \{1, 2, \dots\} \rightarrow \{1, 2, \dots\}$, and each $\{x_i\}_{i=0}^\infty, \{y_i\}_{i=0}^\infty \subset X$, which satisfy for each integer $i \geq 0$,

$$\rho(x_{i+1}, A_{r(i+1)}(x_i)) \leq \delta(\{A_t\}_{t=1}^\infty, \gamma, p), \quad \rho(y_{i+1}, A_{r(i+1)}(y_i)) \leq \delta(\{A_t\}_{t=1}^\infty, \gamma, p)$$

and for each integer $n \geq n(\{A_t\}_{t=1}^\infty, \gamma, p)$, we have

$$\rho(x_i, y_i) \leq (4p)^{-1}\Delta.$$

there is an open neighborhood $\mathcal{U}(\{A_t\}_{t=1}^\infty, \gamma, p)$ of $\{A_{\gamma,t}\}_{t=1}^\infty$ (in the weak topology) such that for each $\{B_t\}_{t=1}^\infty \in \mathcal{U}(\{A_t\}_{t=1}^\infty, \gamma, p)$, each $x, y \in X$, and each $i \in \{1, \dots, n(\{A_t\}_{t=1}^\infty, \gamma, p)\}$, we have

$$\rho(A_{\gamma,t}(x), B_t(x)) \leq \delta(\{A_t\}_{t=1}^\infty, \gamma, p).$$

When combined with property (i), this implies that the following property holds:

(ii) for each $x, y \in X$ and each $\{B_t\}_{t=1}^\infty \in \mathcal{U}(\{A_t\}_{t=1}^\infty, \gamma, p)$,

$$\rho(B_{n(\{A_t\}_{t=1}^\infty, \gamma, p)} \cdots B_1(x), B_{n(\{A_t\}_{t=1}^\infty, \gamma, p)} \cdots B_1(y)) \leq (4p)^{-1}\Delta.$$

Define

$$\mathcal{F} = \bigcap_{p=1}^\infty \{\mathcal{U}(\{A_t\}_{t=1}^\infty, \gamma, p) : \{A_t\}_{t=1}^\infty \in \mathfrak{M}, \gamma \in (0, 1)\}. \quad (43)$$

In view of (43), $\mathcal{F}$ is an everywhere dense set in the strong topology and a countable intersection of sets that are open in the weak topology.

Let

$$\{B_t\}_{t=1}^\infty \in \mathcal{F} \quad (44)$$

and $\epsilon \in (0, \Delta)$. Fix a natural number

$$p > 4\epsilon^{-1}. \quad (45)$$

By (43) and (44), there are $\{A_t\}_{t=1}^\infty \in \mathfrak{M}$ and $\gamma \in (0, 1)$ such that

$$\{B_t\}_{t=1}^\infty \in \mathcal{U}(\{A_t\}_{t=1}^\infty, \gamma, p). \quad (46)$$

Let

$$\{C_t\}_{t=1}^\infty \in \mathcal{U}(\{A_t\}_{t=1}^\infty, \gamma, p).$$

Property (ii) implies that for each $x, y \in X$, we have

$$\rho(C_{n(\{A_t\}_{t=1}^\infty, \gamma, p)} \cdots C_1(x), C_{n(\{A_t\}_{t=1}^\infty, \gamma, p)} \cdots C_1(y)) \leq (4p)^{-1}\Delta.$$

When combined with (5), this implies that for each $n > n(\{A_t\}_{t=1}^\infty, \gamma, p)$,

$$\rho(C_n \cdots C_1(x), C_n \cdots C_1(y)) \leq \epsilon.$$

This completes the proof of Theorem 4. $\square$

7. Conclusions

In this paper, the generic convergence of infinite products of uniformly locally non-expansive mappings has been analyzed. The results that have been obtained provide an extension of the results of [20], which were established for infinite products of nonexpansive mappings. A complete space of sequences of uniformly locally nonexpansive mappings has been considered, and it has been shown that a typical (generic) sequence of operators is, in fact, uniformly locally contractive and enjoys a convergence property that is stable under small perturbations. Note that the results of this paper have been established for self-mappings of a bounded complete metric space. One of the possible future directions of this research is to obtain analogs of these results for self-mappings of a complete metric space that is not necessarily bounded.

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