

Article

Structural Properties of Optimal Maintenance Policies for k -out-of- n Systems with Interdependence Between Internal Deterioration and External Shocks [†]

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Abstract: Many modern engineering systems, such as offshore wind turbines, rely on k -out-of- n configurations to ensure reliability. These systems are exposed to both internal deterioration and external shocks, which can significantly impact operational efficiency and maintenance costs, necessitating optimal maintenance policies. This study investigates an optimal condition-based maintenance policy for a k -out-of- n system, where each unit deteriorates independently following a gamma process and is subject to random external shocks that cause sudden jumps in deterioration. This study considers (1) stochastic dependencies among units, where shock-induced cumulative deterioration in one unit affects others, and (2) interdependencies between external shocks and internal deterioration, where internal deterioration influences external factors and vice versa. Using a Markov decision process framework, we derive an optimal maintenance policy that minimizes expected maintenance costs while incorporating these interdependencies. Under reasonable assumptions, we establish key structural properties of the optimal policy, enabling its efficient identification. A case study on offshore wind turbines demonstrates the effectiveness of the proposed approach, achieving up to a 9.9% reduction in maintenance costs compared to alternative policies. This cost reduction is achieved by optimizing the timing of preventive maintenance while incorporating the two aforementioned types of dependence into the decision-making process. Sensitivity analyses further explore the effects of cost parameters, deterioration rates, and shock characteristics, offering valuable insights into designing maintenance strategies for systems influenced by shocks and interdependent deterioration.

Keywords: condition-based maintenance; Markov decision process; stochastic order; stochastic process; value iteration algorithm

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1. Introduction

Systems inevitably deteriorate over time, especially when operated under stressful and high-load environments, regardless of their initial performance levels. Such deterioration leads to a decline in availability, safety, and reliability, posing significant challenges for modern infrastructure systems. For example, offshore wind turbines rely heavily on effective maintenance strategies to ensure operational reliability and cost efficiency. These systems

often face dual challenges: gradual internal deterioration and sudden external shocks, such as lightning strikes or strong wind. While condition-based maintenance (CBM) has proven effective in single-unit systems by leveraging real-time data to optimize maintenance actions, its application to multi-unit systems with complex interdependencies remains limited [1,2]. This limitation is particularly concerning for industries where downtime or failure can lead to significant economic losses and safety risks. Addressing this gap requires innovative approaches that explicitly consider the interdependence between deterioration and external shocks. To address these challenges, developing effective maintenance policies that explicitly account for the interdependence between internal deterioration and external shocks is essential.

Research on optimizing CBM decision-making has extensively explored the structural properties of optimal policies. Derman [3] established the foundational structural properties of optimal policies for single-unit systems with two actions: operation or replacement. Building on this work, Ohnishi et al. [4] analyzed systems with incomplete information and identified structural properties for policies involving three actions: operation, inspection, and replacement. Their study was among the first to address uncertainties in observed data related to system deterioration, thus advancing CBM decision-making. Lovejoy [5] extended Ohnishi et al. [4]’s results to systems with multiple actions. Elwany et al. [6] utilized a Wiener process to model deterioration and confirmed the presence of structural properties for continuous states. Chen et al. [7] later addressed heterogeneity in deterioration rates across spare units, offering valuable insights for managing systems with varied deterioration factors. While these studies have focused on single-unit systems, real-world systems such as offshore wind turbines and fuel supply systems are typically multi-unit systems with interacting components. Developing efficient maintenance policies for these systems remains critical and challenging.

Despite the need for efficient maintenance policies in multi-unit systems, their inherent complexity poses significant challenges [8,9]. Two primary obstacles hinder progress: the curse of dimensionality and interdependencies among units. The curse of dimensionality [10] arises as the state and action spaces expand exponentially with the number of units, making computationally feasible solutions challenging. Interdependencies among units, such as economic, structural, and stochastic, further complicate the problem. Economic dependencies arise from shared fixed setup costs when maintaining multiple units simultaneously [11]. Structural dependency is divided into technical and performance dependence [8]. Technical dependence implies that to maintain one unit, other units may also need maintenance or may not be maintainable if a specific unit is operational. Performance dependence relates to the system’s design, such as a redundant k -out-of- n configuration to enhance availability. A k -out-of- n system is a reliability model that functions successfully if at least k out of n components are operational. These systems are widely used in engineering to represent redundancy and fault-tolerant designs. As special cases, when $k = n$, the system corresponds to a series system, where all components must function for the system to operate. Conversely, when $k = 1$, the system corresponds to a parallel system, where the operation of any single component is sufficient for the system to function. Stochastic dependencies, such as load-sharing effects, influence deterioration and failure rates across units [12–14]. Krishnamoorthy [15] analyzed the reliability of multi-unit systems where the life of the unit depends on other units’ lives. Ignoring these interdependencies risks compromising system reliability. For example, in wind turbines, the failure of one blade can increase stress on others, accelerating their deterioration. Effective maintenance policies must, therefore, account for these dependencies to ensure reliable system performance.

Research on maintenance policies for multi-unit systems can be categorized as numerical or analytical. Numerical approaches involve computational optimization to derive

policies. For example, Xu et al. [16] used a copula function to model dependent deterioration in k -out-of- n systems and investigated the policy properties. Oakley et al. [17] demonstrated that their policy, which accounts for stochastic dependence among units, is more cost-effective than conventional policies. Analytical approaches primarily investigate the structural properties of optimal policies. Sun et al. [18] proved that k -out-of- n systems without stochastic dependence have structural properties, while Zhang et al. [19] extended these findings to include stochastic dependencies. When an optimal policy has known structural properties, the search space for candidates is reduced, enabling more efficient solutions. Additionally, analytical properties can guide the assessment of numerically derived policies, revealing potential improvements when they do not meet these criteria.

External shocks further complicate systems, causing disruption alongside internal deterioration [20,21]. Examples include battery overcharging in electric vehicles [22] and lightning strikes damaging wind turbine blades [23]. Maintenance policies addressing shocks have been proposed, such as those by Rafiee et al. [24] for single-unit systems and Dui et al. [25] for multi-unit systems. Yousefi et al. [26] applied deep reinforcement learning to manage maintenance under shocks. Most studies assume that system deterioration and shock intensity are independent. However, in reality, each shock alters the deterioration rate, while the system's deterioration state can, in turn, influence shock impact [27]. Tajiani et al. [28] considered that increasing deterioration amplifies shock effects, while Kurt and Maillart [29] demonstrated that the deterioration rate depends on shock damage, revealing structural properties in single-unit systems. Wang et al. [30] found interdependence between deterioration state and shock damage, establishing structural properties for single-unit policies. Qi and Huang [31] proposed an inspection policy considering the interdependence. In multi-unit systems, not all units experience shocks equally; various shock types impact each unit differently [32,33]. For instance, micro-electromechanical system (MEMS) oscillators, composed of multiple resonators, may experience thermal and vibration shocks. These shocks affect the deterioration rates of both directly and indirectly impacted resonators [34]. Pipelines, which are k -out-of- n systems, are subject to multiple types of shocks, such as excess load, vibration, geological deformation, and seismic waves, with each unit affected differently [35]. Thus, maintenance policies must account for variability in shock impacts across units.

While CBM has been extensively studied for single-unit systems or systems where units are assumed to deteriorate independently, its application to multi-unit systems that consider interdependencies remains an underexplored area. Ignoring these dependencies, such as those among units or between units and external factors like shocks, can compromise the optimality of the proposed maintenance policies. To address this gap, we focus on the following two challenges, which are also research questions of this study: (1) proposing an optimal maintenance policy that explicitly models and incorporates these dependencies, and (2) analyzing the structural properties of the proposed optimal policy to uncover insights into its characteristics.

To clearly illustrate the positioning of this study, Table 1 summarizes the relationship between this study and related research on optimal maintenance policies. For clarity and easier comparison, dependencies are classified into two types: (1) Stochastic dependence among units: this category refers to the effect of factors from other units, such as shock-induced cumulative deterioration, on a unit's deterioration. (2) Interdependencies between internal deterioration and external shocks: this category includes the influence of internal deterioration on external shocks, represented as “internal deterioration → external shocks”, and the impact of external shocks on a unit's deterioration, represented as “internal deterioration ← external shocks”. Type (1) is specific to multi-unit systems, while Type (2) applies to single-unit systems.

Although recent studies such as Wang et al. [36] and Huang et al. [37] have assessed the reliability of k -out-of- n systems while considering the dependency between deterioration and shocks, research on maintenance policies for k -out-of- n systems that explicitly account for shocks, as summarized in Table 1, remains insufficiently explored. Our previous study [38] examined k -out-of- n systems with identical units subjected to external shocks, incorporating stochastic dependencies among units as well as interdependencies between deterioration and shock intensity. In this extended study, we mathematically prove the structural properties of the optimal maintenance policy and conduct sensitivity analysis to examine the impact of various parameters on the policy. This series of studies contributes to the theoretical understanding of optimal maintenance policies as follows:

- We propose an optimal maintenance policy for k -out-of- n systems that accounts for variability in shock impacts across units and addresses the interdependence between unit deterioration and external shocks, making it applicable to real-world systems for efficient operation.
- We derive a structural property of the optimal policy, offering key insights for developing algorithms that enable efficient searching for the optimal solution.

Table 1. Research on maintenance policy.

	System Design	Interdependence Among Units	Interdependence Between External Shocks and Internal Deterioration	Optimization Target	Optimization Framework	Structural Property
Derman [3]	single-unit	-	-	maintenance action	MDP	✓
Ohnishi et al. [4]	single-unit	-	-	maintenance action	partially observable MDP	✓
Lovejoy et al. [5]	single-unit	-	-	maintenance action	partially observable MDP	✓
Elwany et al. [6]	single-unit	-	-	maintenance action	MDP	✓
Chen et al. [7]	single-unit	-	-	maintenance action	MDP	✓
Zhu and Xiang [11]	n -unit	×	-	maintenance action	stochastic programming	✓
Ashizawa and Jin [13]	2-unit	✓	-	maintenance action	MDP	✓
Liu et al. [14]	series	✓	-	maintenance action	MDP	✓
Xu et al. [16]	k -out-of- n	✓	-	maintenance action	MDP	-
Oakley et al. [17]	series-parallel	✓	-	maintenance action	Bayesian sequential decision	✓
Sun et al. [18]	k -out-of- n	×	-	maintenance action	MDP	✓
Zhang et al. [19]	k -out-of- n	✓	-	maintenance action	MDP	✓
Rafiee et al. [24]	single-unit	-	×	inspection interval	cost rate	-
Dui et al. [25]	n -unit	×	age→internal deterioration age→shocks	maintenance action	integer programming	-
Yousefi et al. [26]	series parallel	×	×	maintenance action	MDP	-
Yousefi et al. [27]	series	×	internal deterioration ↔ external shocks	inspection interval	cost rate	-
Tajiani [28]	single-unit	-	internal deterioration ← external shocks	maintenance threshold	cost rate	-

Table 1. Cont.

	System Design	Interdependence Among Units	Interdependence Between External Shocks and Internal Deterioration	Optimization Target	Optimization Framework	Structural Property
Kurt and Maillart [29]	single-unit	-	internal deterioration $\leftarrow$ external shocks	maintenance action	MDP	✓
Wang et al. [30]	single-unit	-	internal deterioration $\leftrightarrow$ external shocks	maintenance action	MDP	✓
Qi and Huang et al. [31]	single-unit	-	internal deterioration $\leftrightarrow$ external shocks	inspection interval	cost rate	-
Lorvand and Kelkinnama. [33]	k -out-of- n	$\times$	age $\rightarrow$ external shocks	maintenance interval	cost rate	-
Cao et al. [39]	single-unit	-	internal deterioration $\leftarrow$ external shocks age $\rightarrow$ external shocks	inspection interval	multi objective programming	-
Shafiee et al. [40]	series	$\times$	-	maintenance threshold maintenance interval	cost rate	-
This research	k -out-of- n	✓	internal deterioration $\leftrightarrow$ external shocks	maintenance action	MDP	✓

✓ indicates that the factor is considered, $\times$ indicates that the factor is not considered, and - indicates that the factor is not applicable.

To the best of our knowledge, these two contributions represent the first exploration of these topics for k -out-of- n systems, filling a critical gap in the existing literature.

The remainder of this paper is organized as follows. Section 2 outlines the system model and formulates the CBM optimization problem using the Markov decision process (MDP). Section 3 presents the structural properties of the optimal policy. Section 4 applies the policy to a case study on offshore wind turbines and demonstrates the policy's effectiveness. Section 5 conducts a sensitivity analysis to investigate the influence of cost parameters, deterioration rates, and shock characteristics on the proposed policy. Finally, Section 6 concludes with a summary and discusses future research directions.

2. Model

2.1. System Deterioration

We consider a k -out-of- n : G system with n identical units, which functions if and only if at least k out of the n units are operational. When more than $n - k + 1$ units fail, the system becomes non-functional and cannot meet operational requirements. For example, in 3-out-of-5 systems, consider a wind farm consisting of $n = 5$ wind turbines. The farm is designed to maintain power generation requirements if at least $k = 3$ turbines are operational. In this model, each unit is subject to both internal deterioration and external shocks. Let $S_i(t)$ denote the deterioration state of unit i at time t , with the initial state $S_i(0) = 0$. We assume that the system experiences M types of random external shocks, and each shock type affects the specific units. Let $\mathbf{U}_m$ represent the subset of units directly impacted by shock type $m \in \{1, \dots, M\}$. Each shock type occurs independently and at a constant rate, with the interval between successive type m shocks following an exponential distribution $g_m(t)$. When a type m shock occurs at time t , it affects each unit

$i \in \mathbf{U}_m \subseteq \{1, \dots, n\}$, causing an incremental deterioration Δ_i , which represents a sudden jump in $S_i(t)$. Δ_i is a random variable that depends on $S_i(t)$, as the same type of shock may lead to a larger increment in deterioration for a system in a more deteriorated state. Given $S_i(t) = s_i$, Δ_i is assumed to follow the probability density function $f_i^m(\Delta_i|s_i)$. Let $Z_i(t)$ denote the cumulative incremental deterioration of unit i due to all shocks up to time t , which aggregates Δ_i up to time t . Each unit fails when its deterioration state $S_i(t)$ reaches or exceeds a predetermined threshold L .

Let vectors $\mathbf{S}(t) = (S_1(t), \dots, S_n(t))$ and $\mathbf{Z}(t) = (Z_1(t), \dots, Z_n(t))$ represent the system's deterioration state and shock-induced cumulative deterioration, respectively. We assume each unit's deterioration follows a gamma process, which is most widely used in engineering applications for its properties such as monotonically increasing deterioration and independent increment [41]. In this study, we assume each unit's deterioration is independent of the states of other units but dependent on their shock-induced cumulative deterioration. Given $S_i(t) = s_i$ and $\mathbf{Z}(t) = \mathbf{z} = (z_1, \dots, z_n)$, the transition probability density function from s to s' over a small interval Δt is given by:

$$p_i(s'_i|s_i, \mathbf{z}, \Delta t) = \frac{(s'_i - s_i)^{\alpha\Delta t - 1}}{\Gamma(\alpha\Delta t)\beta_i(\mathbf{z})^{\alpha\Delta t}} e^{-\frac{s'_i - s_i}{\beta_i(\mathbf{z})}} \mathbb{1}_{\{s'_i - s_i > 0\}}, \quad (1)$$

where α is the shape parameter, and $\beta_i(\mathbf{z})$ is the scale parameter, which depends on $\mathbf{z}$. We assume α is uniform across all units, whereas $\beta_i(\mathbf{z})$ varies. In Equation (1), $\mathbb{1}_{\{\cdot\}}$ is an indicator function that returns 1 if the condition inside the brackets holds true, and 0 otherwise. Figure 1 illustrates the two types of dependencies described above and in Section 1. For example, Shock 1 affects Unit 1, and following its occurrence, the deterioration rate of Unit 1 increases due to the interdependence between deterioration and shocks. Simultaneously, the deterioration rates of Units 2 and 3 also increase because of the stochastic dependence among units. Shock 2 affects all three units, causing a jump in their deterioration states, and the deterioration rates of all three units increase further due to the interdependence between deterioration and shocks. Finally, when Shock 3 occurs, the intensity of this shock increases because the system is in a more deteriorated state, and

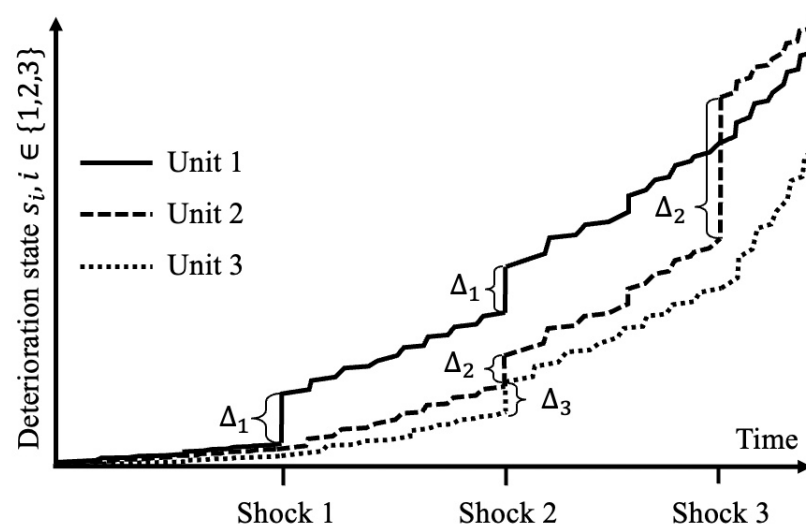


Figure 1. Sample path of deterioration process for a three unit system.

System deterioration state $\mathbf{s}$ and cumulative incremental deterioration $\mathbf{z}$ are inspected perfectly at discrete intervals $t = k\tau$ ($k = 1, 2, \dots$), where τ is the fixed inspection interval. Let $q(s', z' | s, z, \tau)$ represent the conditional transition probability density function that the system state transitions from $\mathbf{s} = (s_1, s_2, \dots, s_n)$ and $\mathbf{z} = (z_1, z_2, \dots, z_n)$ to

$\mathbf{s}' = (s'_1, s'_2, \dots, s'_n)$ and $\mathbf{z}' = (z'_1, z'_2, \dots, z'_n)$ after a time interval τ . This transition probability density function is expressed as follows:

$$\begin{aligned}
 & q(\mathbf{s}', \mathbf{z}' | \mathbf{s}, \mathbf{z}, \tau) \\
 &= \sum_{m=1}^M \int_{z_n}^{z'_n} \cdots \int_{z_1}^{z'_1} \int_{s_n+y_n-z_n}^{s'_n} \cdots \int_{s_1+y_1-z_1}^{s'_1} \int_0^\tau q(\mathbf{s}', \mathbf{z}' | \mathbf{x}, \mathbf{y}, \tau-u) g_m(u) \prod_{l=1, l \neq m}^M [1 - G_l(\tau)] \\
 &\quad \times \prod_{i=1}^n \left\{ \mathbb{1}_{\{i \in \mathbf{U}_m\}} \times f_i^m(y_i - z_i | x_i) p_i(x_i | s_i, \mathbf{z}, u) + \mathbb{1}_{\{i \notin \mathbf{U}_m, y_i = z_i\}} \times p_i(x_i | s_i, \mathbf{z}, u) \right\} du dx_1 \cdots dx_n dy_1 \cdots dy_n \\
 &\quad + \mathbb{1}_{\{\mathbf{z}' = \mathbf{z}\}} \prod_{m=1}^M [1 - G_m(\tau)] \prod_{i=1}^n p_i(s'_i | s_i, \mathbf{z}, \tau),
 \end{aligned} \tag{2}$$

where $1 - G_m(\tau)$ is the probability that no shock occurs within the interval τ . Note that $Z_i(t) \leq S_i(t)$ always holds, as $Z_i(t)$ represents only the deterioration due to external shocks, while $S_i(t)$ encompasses both internal and external deterioration. If the deterioration states of at least $n - k + 1$ units exceed or reach L , the system is considered to have failed. Let T_f be the elapsed time from the last inspection to system failure. The cumulative distribution function of T_f is expressed as follows:

$$\begin{aligned}
 & F_{T_f}(t | \mathbf{s}, \mathbf{z}) \\
 &= \Pr(T_f \leq t | \mathbf{s}, \mathbf{z}) \\
 &= \Pr\left(\sum_{i=1}^n \mathbb{1}_{\{S_i(t) \geq L\}} \geq n - k + 1 | \mathbf{s}, \mathbf{z}\right) \\
 &= \sum_{j=1}^{\kappa} \int_{l_{s_n}(C_j)}^{u_{s_n}(C_j)} \cdots \int_{l_{s_1}(C_j)}^{u_{s_1}(C_j)} \int_{z_n}^{s'_n - s_n + z_n} \cdots \int_{z_1}^{s'_1 - s_1 + z_1} q(\mathbf{s}', \mathbf{z}' | \mathbf{s}, \mathbf{z}, t) dz'_1 \cdots dz'_n ds'_1 \cdots ds'_n,
 \end{aligned} \tag{3}$$

where κ denotes the total number of possible combinations of unit failures that result in system failure, and C_j is the cut set, which is a specific combination of unit failures that causes the failure of the k -out-of- n system. $u_{s_i}(C_j)$, $l_{s_i}(C_j)$, $u_{z_i}(C_j)$ and $l_{z_i}(C_j)$ are defined as follows

$$l_{s_i}(C_j) = \begin{cases} L & i \in C_j \\ 0 & i \notin C_j \end{cases}, \quad u_{s_i}(C_j) = \begin{cases} \infty & i \in C_j \\ L & i \notin C_j \end{cases}, \quad i \in \{1, 2, \dots, n\}. \tag{4}$$

2.2. Actions and Costs in Condition-Based Maintenance

One of two maintenance actions, namely do nothing (DN) or replacement (RP), is selected for each unit based on system information $(\mathbf{s}, \mathbf{z})$, where $\mathbf{s}$ represents the system deterioration state and $\mathbf{z}$ represents the system shock-induced cumulative deterioration. Let $\mathbf{r} = (r_1, r_2, \dots, r_n)$ denote the maintenance action for the entire system, indicating whether each unit is replaced or not. Here, $r_i = 0$ indicates that unit i is replaced, and $r_i = 1$ indicates that unit i undergoes no action. The action space of $\mathbf{r}$ is defined as follows,

$$\mathcal{A} = \{(r_1, r_2, \dots, r_n) | r_i \in \{0, 1\}, i \in \{1, 2, \dots, n\}\}.$$

When DN is selected for unit i , unit i is operated for one more time interval. Conversely, when RP is selected for unit i , the operating unit is replaced with an as-good-as-new unit. In this situation, replacement cost is incurred depending on s_i . If $s_i < L$, a preventive replacement cost c_p is incurred. On the other hand, a corrective replacement cost c_f is incurred if $s_i \geq L$. Please note that c_f is always higher than c_p in all units to capture the harm caused by failure. A setup cost c_s is incurred when one or more units are replaced. Maintenance time is assumed to be negligible; thus, the unit is immediately operated after

replacement completion. Furthermore, we assume that the system cannot be self-detected but can be identified through inspection. If the system fails during operation until the next inspection reveals the failure, a downtime cost c_d is incurred per unit time.

2.3. Optimal Maintenance Policy

The optimal decision-making problem for CBM is formulated as a discrete-time MDP. An MDP is applicable to sequential decision-making problems with fully observable states and probabilistic transitions. The optimal maintenance policy is defined as a sequence of actions that minimizes the total expected discounted cost based on the system's deterioration state and shock-induced cumulative deterioration. Specifically, the objective is to minimize the total expected discounted cost given the system information (s, z) .

Let $V^N(s, z)$ denote the total expected discounted cost over the remaining N decision periods, defined as follows,

$$V^N(s, z) = \min_r \{V_r^N(s, z)\}, \quad (5)$$

where

$$\begin{aligned} V_r^N(s, z) &= \sum_{i=1}^n c_p \mathbb{1}_{\{r_i=0, s_i < L\}} + \sum_{i=1}^n c_f \mathbb{1}_{\{r_i=0, s_i \geq L\}} + W(s \circ r, z \circ r) + c_s \mathbb{1}_{\{\sum_{i=1}^n r_i \neq n\}} \\ &\quad + e^{-\gamma\tau} \int_{z_n r_n}^{\infty} \cdots \int_{z_1 r_1}^{\infty} \int_{s_n r_n}^{\infty} \cdots \int_{s_1 r_1}^{\infty} q(s', z' | s \circ r, z \circ r, \tau) V^{N-1}(s', z') ds'_1 \cdots ds'_n dz'_1 \cdots dz'_n, \end{aligned} \quad (6)$$

and $e^{-\gamma\tau}$ ($\gamma > 0$) is the discount factor. Here, $(s \circ r, z \circ r) = ((s_1 r_1, \dots, s_n r_n), (z_1 r_1, \dots, z_n r_n))$ represents the updated system information following maintenance action r . The function $W(s, z)$ denotes the expected downtime cost, which depends on the time T_f elapsed from the last inspection to system failure. $W(s, z)$ is given by

$$\begin{aligned} W(s, z) &= \int_0^{\tau} \frac{c_d(e^{-\gamma t} - e^{-\gamma\tau})}{\gamma} dF_{T_f}(t | s, z) \\ &= \int_0^{\infty} \frac{c_d \mathbb{1}_{\{t \leq \tau\}}(e^{-\gamma t} - e^{-\gamma\tau})}{\gamma} dF_{T_f}(t | s, z). \end{aligned} \quad (7)$$

When remaining periods $N = 0$, $V^0(s, z) = 0$. As N approaches infinity, the contraction mapping theorem guarantees that $\lim_{N \rightarrow \infty} V^N(s, z) = V(s, z)$ [42]. Thus, we define

$$V(s, z) = \min_r \{V_r(s, z)\}, \quad (8)$$

where

$$\begin{aligned} V_r(s, z) &= \sum_{i=1}^n c_p \mathbb{1}_{\{r_i=0, s_i < L\}} + \sum_{i=1}^n c_f \mathbb{1}_{\{r_i=0, s_i \geq L\}} + W(s \circ r, z \circ r) + c_s \mathbb{1}_{\{\sum_{i=1}^n r_i \neq n\}} \\ &\quad + e^{-\gamma\tau} \int_{z_n r_n}^{\infty} \cdots \int_{z_1 r_1}^{\infty} \int_{s_n r_n}^{\infty} \cdots \int_{s_1 r_1}^{\infty} q(s', z' | s \circ r, z \circ r, \tau) V(s', z') ds'_1 \cdots ds'_n dz'_1 \cdots dz'_n. \end{aligned}$$

To solve the optimization problem, we employ a value iteration algorithm based on the Markov decision process framework, iteratively computing the optimal maintenance policy. The details of the algorithm are presented in Section 4.

3. Structural Properties of the Optimal Maintenance Policy

In an MDP, the number of candidate maintenance policies increases exponentially as the number of units and state dimensions grow. This high dimensionality can make it impractical to search exhaustively for the optimal policy within a reasonable timeframe. To address this, we examine properties of the optimal policy that can help reduce the computational complexity. By identifying the structural properties of the optimal policy, we can limit the search space to policies satisfying these properties, thereby alleviating the curse of dimensionality. Below, we introduce definitions and lemmas that form the foundation for these structural properties.

Definition 1. For n -dimensional vectors $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n)$, we say that $\mathbf{x}$ is smaller than $\mathbf{y}$ in the product order if $x_i \leq y_i$ for all $i \in \{1, \dots, n\}$. This is written as $\mathbf{x} \leq_c \mathbf{y}$.

Definition 2. Stochastic Increasing (SI) (Shaked and Shanthikumar [43])

For random variables X and Y , if $\Pr\{X > t\} \leq \Pr\{Y > t\}$ for all t , then X is said to be stochastically smaller than Y , written $X \preceq_{\text{SI}} Y$. For random vectors $\mathbf{X} = (X_1, \dots, X_n)$ and $\mathbf{Y} = (Y_1, \dots, Y_n)$ in $\mathbb{R}_+^n = [0, \infty)^n$, if $\Pr\{\mathbf{X} \in U\} \leq \Pr\{\mathbf{Y} \in U\}$ for all upper sets $U \subseteq \mathbb{R}_+^n$, then $\mathbf{X}$ is stochastically smaller than $\mathbf{Y}$, written $\mathbf{X} \preceq_{\text{SI}} \mathbf{Y}$.

Lemma 1. Shaked and Shanthikumar [43]

If $\mathbf{X} \preceq_{\text{SI}} \mathbf{Y}$, then $E[f(\mathbf{X})] \leq E[f(\mathbf{Y})]$ for any non-decreasing function $f(\cdot)$, and $E[f(\mathbf{X})] \geq E[f(\mathbf{Y})]$ for any non-increasing function $f(\cdot)$, provided that the expectations exist.

Lemma 2. Shaked and Shanthikumar [43]

For random vectors $\mathbf{X} = (X_1, \dots, X_n)$ and $\mathbf{Y} = (Y_1, \dots, Y_n)$ in $\mathbb{R}_+^n = [0, \infty)^n$, if $X_1 \preceq_{\text{SI}} Y_1$, and $\langle X_i | X_1 = x_1, \dots, X_{i-1} = x_{i-1} \rangle \preceq_{\text{SI}} \langle Y_i | Y_1 = y_1, \dots, Y_{i-1} = y_{i-1} \rangle$ whenever $x_j \leq y_j$ for all $j = 1, \dots, i-1$, then $\mathbf{X} \preceq_{\text{SI}} \mathbf{Y}$. Here, $\langle X_i | X_1 = x_1, \dots, X_{i-1} = x_{i-1} \rangle$ represents the conditional distribution of X_i given $X_1 = x_1, \dots$, and $X_{i-1} = x_{i-1}$.

We make the following assumptions to consider the interdependence between deterioration and external shocks:

Assumption 1. If $\mathbf{z}^- \leq_c \mathbf{z}^+$, then $\beta_i(\mathbf{z}^-) \leq \beta_i(\mathbf{z}^+)$ for all $i \in \{1, \dots, n\}$.

Assumption 2. If $s_i^- \leq s_i^+$, then $f_i^m(\Delta_i | s_i^-) \preceq_{\text{SI}} f_i^m(\Delta_i | s_i^+)$ for all $i \in \{1, \dots, n\}$.

Assumption 1 indicates that all units are more susceptible to deterioration as the system experiences cumulative external shocks. Assumption 2 implies that as units deteriorate, they become more vulnerable to the impacts of external shocks. These assumptions are reasonable as they align with observed behaviors and are supported by previous research [27,29,30,32,34,39,44].

We now present Lemma 3, which concerns the transition of system information.

Lemma 3. If $\mathbf{s}^- = (s_1^-, \dots, s_n^-) \leq_c \mathbf{s}^+ = (s_1^+, \dots, s_n^+)$ and $\mathbf{z}^- = (z_1^-, \dots, z_n^-) \leq_c \mathbf{z}^+ = (z_1^+, \dots, z_n^+)$, then $\langle \mathbf{s}', \mathbf{z}' | \mathbf{s}^-, \mathbf{z}^-, \tau \rangle \preceq_{\text{SI}} \langle \mathbf{s}', \mathbf{z}' | \mathbf{s}^+, \mathbf{z}^+, \tau \rangle$ for any τ .

Proof. We prove Lemma 3 using mathematical induction.

Let $q^k(\mathbf{s}', \mathbf{z}' | \mathbf{s}, \mathbf{z}, \tau)$ represent the conditional probability density function that $\mathbf{s}$ transitions to $\mathbf{s}'$ and $\mathbf{z}$ transitions to $\mathbf{z}'$ when the system experiences k external shocks within interval τ .

1. Base Case ($k = 0$): By Equation (1) and Assumption 1, $\langle s'_i | s^-, z^-, \tau \rangle \preceq_{\text{SI}} \langle s'_i | s^+, z^+, \tau \rangle$ holds. Since each unit's deterioration is independent of other unit deterioration states, it follows from Lemma 2 that $\langle s', z' | s^-, z^-, \tau, k \rangle \preceq_{\text{SI}} \langle s', z' | s^+, z^+, \tau, k \rangle$.
2. Inductive Step: Assume that $\langle s', z' | s^-, z^-, \tau, k \rangle \preceq_{\text{SI}} \langle s', z' | s^+, z^+, \tau, k \rangle$ holds for all τ when $k = l$.

For $k = l + 1$, consider that the first external shock occurs after time u , with shock type m . Let s and z transition to x and y , respectively, due to this shock. The probability density function of this event, denoted $a^{l+1}(s', z', x, y | s, z, \tau, m, u)$, can be expressed as

$$\begin{aligned} & a^{l+1}(s', z', x, y | s, z, \tau, m, u) \\ &= q^l(s', z' | x, y, \tau - u) g_m(u) \prod_{l=1, l \neq m}^M [1 - G_l(u)] \\ & \times \prod_{i=1}^n \left\{ \mathbb{1}_{\{i \in \mathbf{u}_m\}} f_i^m(y_i - z_i | x_i) p_i(x_i | s_i, z, u) + \mathbb{1}_{\{i \notin \mathbf{u}_m, y_i = z_i\}} p_i(x_i | s_i, z, u) \right\}. \end{aligned} \quad (9)$$

By Lemma 2 and $\langle x, y | s^-, z^-, u \rangle \preceq_{\text{SI}} \langle x, y | s^+, z^+, u \rangle$, we have $\langle s', z' | s^-, z^-, \tau, k = l + 1 \rangle \preceq_{\text{SI}} \langle s', z' | s^+, z^+, \tau, k = l + 1 \rangle$. By the independence and stationarity of external shock arrivals, $\langle s', z' | s^-, z^-, \tau, k \rangle \preceq_{\text{SI}} \langle s', z' | s^+, z^+, \tau, k \rangle$ holds for $k = l + 1$. Thus, Lemma 3 is proved. $\square$

We now present Theorem 1 regarding the expected downtime cost and Theorem 2 regarding the total expected discounted cost.

Theorem 1. *The expected downtime cost $W(s, z)$ is non-decreasing in s and z , ordered by the product order.*

Proof. By Lemma 3, we have $\langle s', z' | s^-, z^-, t \rangle \preceq_{\text{SI}} \langle s', z' | s^+, z^+, t \rangle$, implying

$$\Pr(T_f \leq t | s^-, z^-) \leq \Pr(T_f \leq t | s^+, z^+) \Leftrightarrow \langle t | s^+, z^+ \rangle \preceq_{\text{SI}} \langle t | s^-, z^- \rangle.$$

for $f_{T_f}(t | s, z)$ in Equation (7).

Since $\frac{c_d \mathbb{1}_{\{t \leq \tau\}} (e^{-\gamma t} - e^{-\gamma \tau})}{\gamma}$ in Equation (7) is non-increasing in t , it follows from Lemma 1 that $W(s, z)$ is non-decreasing in s and z . This completes the proof of Theorem 1. $\square$

Theorem 2. *The total expected discounted cost $V(s, z)$ is non-decreasing in s and z , ordered by the product order.*

Proof. Theorem 2 is proved by mathematical induction.

Define $V^N(s, z)$ as the total expected discounted cost for the remaining N periods given (s, z) . For $N = 0$, $V^0(s, z) = 0$ for any s and z , which implies that $V^0(s, z)$ is non-decreasing in s and z . Assuming that $V^N(s, z)$ is non-decreasing in s and z for $N = l$, we prove that $V^{l+1}(s, z)$ is also non-decreasing.

By Lemmas 1 and 3, and Theorem 1, it follows for $s^- \leq_c s^+$ and $z^- \leq_c z^+$:

$$\begin{aligned} V_r^{l+1}(s^+, z^+) - V_r^{l+1}(s^-, z^-) &\geq e^{-\gamma \tau} \int_0^\infty \cdots \int_0^\infty q(s', z' | s^+ \circ r, z^+ \circ r, \tau) V^l(s', z') ds'_1 \cdots dz'_n \\ &\quad - e^{-\gamma \tau} \int_0^\infty \cdots \int_0^\infty q(s', z' | s^- \circ r, z^- \circ r, \tau) V^l(s', z') ds'_1 \cdots dz'_n \\ &\geq 0. \end{aligned}$$

Since $V_r^{l+1}(s, z)$ for any r is non-decreasing in s and z , so is $V^{l+1}(s, z)$. Thus, by the contraction mapping theorem, $\lim_{N \rightarrow \infty} V^N(s, z) = V(s, z)$, completing the proof of Theorem 2. $\square$

Next, we provide Theorem 3 on optimal actions concerning information s and z . This is our main result, which leads to alleviating the curse of dimensionality since only policies with Theorem 3 are candidates for the optimal policy.

Theorem 3. We consider two actions $r = (r_1, \dots, r_n)$ and $\tilde{r} = (\tilde{r}_1, \dots, \tilde{r}_n)$, where $r_i = 0$ and $\tilde{r}_i = 1$. For any s and z , the maximum number of changes in the order of $V_r(s, z)$ and $V_{\tilde{r}}(s, z)$ is three with respect to s_i , and one with respect to z_i .

Proof. We first prove that at most three switching points exist between r and $\tilde{r}$ with respect to s_i . When $r_i = 0$, $V_r(s, z)$ is constant in $s_i (< L)$ and $s_i (\geq L)$ from Equation (8). When $\tilde{r}_i = 1$, $V_{\tilde{r}}(s, z)$ is non-decreasing in s_i from Theorem 2, so there may exist $s_i = a_i^* (< L)$ and $s_i = b_i^* (\geq L)$ such that:

$$\begin{aligned} V_r(s, z) &> V_{\tilde{r}}(s, z), & 0 \leq s_i < a_i^*, \\ V_r(s, z) &\leq V_{\tilde{r}}(s, z), & a_i^* \leq s_i < L, \\ V_r(s, z) &> V_{\tilde{r}}(s, z), & L \leq s_i \leq b_i^*, \\ V_r(s, z) &\leq V_{\tilde{r}}(s, z), & b_i^* \leq s_i. \end{aligned}$$

For z_i , there is at most one switching point since $V_r(s, z)$ is constant in z_i from Equation (8) when $r_i = 0$. When $\tilde{r}_i = 1$, $V_{\tilde{r}}(s, z)$ is non-decreasing in z_i from Theorem 2. Thus, if z_i^* satisfies $V_r(s, z) = V_{\tilde{r}}(s, z)$, $\tilde{r}$ is preferable for $z_i \leq z_i^*$, while r is preferable for $z_i > z_i^*$. $\square$

4. Case Study: Three-Bladed Rotor System of Offshore Wind Turbines

4.1. System and Parameter Specifications

Offshore wind energy has gained importance as a clean and stable energy source compared to onshore wind [12]. Actually, operation and maintenance costs constitute approximately one-fourth of the investment in offshore wind turbines [45], making efficient maintenance strategies essential for cost reduction [46].

This study applies the proposed CBM policy to a three-bladed rotor system in offshore wind turbines, modeled as a 3-out-of-3 system. Offshore wind turbine blades are subject to sustained tensile stress in a corrosive environment, which can lead to stress corrosion cracks [47]. These cracks pose significant risks, including environmental damage and financial losses, making them critical targets for condition monitoring [48]. Referring to [40], blade deterioration is modeled as a gamma process with parameters $\alpha = 0.542$ and $\beta_i(z) = 0.872$ for $z = (0, \dots, 0)$ for each blade $i \in \{1, 2, 3\}$. Also, a blade fails if the crack length exceeds the threshold $L = 20$ cm [40]. If L is set to a small value, the optimal policy will be conservative. On the other hand, the big value will make the optimal policy aggressive. External shocks, such as lightning and strong wind with a speed of 25m/s or more, can further accelerate system deterioration. Thus, we express $\beta_i(z) = 0.872(1 + (2z_i + \sum_{j=1, j \neq i}^3 z_j)/16)$, $\forall i \in \{1, 2, 3\}$ in the presence of cumulative deterioration.

External shocks, such as lightning and strong wind, accelerate deterioration. Lightning affects a single blade, while strong wind impacts all blades [23,49]. The deterioration from these shocks is modeled using exponential distributions with parameters $\kappa(s_i) = \max(0.01, 0.2 - 0.0075s_i)$ for lightning and $h_i(s_i) = \max(0.1, 0.4 - 0.015s_i)$ for strong wind. Referring to [50], we assume that each blade experiences an average monthly

lightning strike with probability $1/600$; then, the probability density function for the arrival time of lightning strike is $g_m(x) = \frac{1}{600}e^{-\frac{1}{600}x}$ for $m \in \{1, 2, 3\}$. Similarly, referring to [49], strong winds with speeds of 25m/s or more are assumed to have an average monthly arrival probability of $1/12$, and the probability density function for arrival time for strong wind is $g_4(x) = \frac{1}{12}e^{-\frac{1}{12}x}$. These probabilities indicate that shocks are relatively infrequent over the turbine's lifecycle of approximately 20 years.

Referring to Besnard and Bertling [51], the cost parameters are as follows: down-time cost $c_d = 216,000$ EUR, setup cost $c_s = 40,000$ EUR, preventive replacement cost $c_p = 130,000$ EUR, and corrective replacement cost $c_f = 400,000$ EUR. The inspection interval is $\tau = 2$ months, with a discount factor of $e^{-0.01\tau}$.

4.2. Optimal Maintenance Policy for a Three-Bladed Rotor System of Offshore Wind Turbines

An optimal maintenance policy for a three-bladed rotor system of offshore wind turbines is derived using a value iteration algorithm (see Algorithm 1). In Algorithm 1, ϵ is the predefined small constant used as a criterion for determining the convergence of the total expected discounted cost. In the following experiments, ϵ is set to 0.01. To simplify the computation, we discretize both the continuous deterioration states and the shock-induced cumulative deterioration into discrete intervals. Specifically, if the size of the stress corrosion crack (in cm) on unit i falls within the interval $[0.0, 5.0)$, the deterioration level is set to $s_i = 0$. For intervals $[5.0, 10.0)$, $[10.0, 15.0)$, $[15.0, 20.0)$, and $[20.0, \infty)$, the deterioration levels are defined as $s_i = 1, 2, 3$, and 4, respectively. Similarly, if the shock-induced cumulative deterioration (in cm) of unit i falls within the interval $[0.0, 5.0)$, the shock-induced cumulative deterioration level is set to $z_i = 0$, and for the intervals $[5.0, 10.0)$, $[10.0, 15.0)$, $[15.0, 20.0)$, and $[20.0, \infty)$, the levels are $z_i = 1, 2, 3$, and 4, respectively. Since $L = 20$, a unit is considered to be failed when its deterioration level reaches $s_i = 4$.

Algorithm 1 The value iteration algorithm utilized to compute $V(s, z)$.

```

input Deterioration and shock parameters, and cost parameters
output Optimal value function  $V(s, z)$ 
calculate  $q(s', z'|s, z)$ 
calculate  $W(s, z)$ 
initialization  $V_r^0(s, z), \forall s, z$ 
while  $\max |V^N(s, z) - V^{N-1}(s, z)| < \epsilon$  do
  for  $r, s, z$  do
    Compute  $V_r^{N+1}(s, z)$  thorough Equation (6)
  end for
   $N = N + 1$ 
end while

```

Figure 2 illustrates the optimal actions for each s in the three-bladed rotor system for cases where $z = (0, 0, 0)$, $(1, 1, 1)$, $(2, 2, 2)$, $(3, 3, 3)$, and $(4, 4, 4)$, respectively. Each action in plots Figure 2a–e is the one that minimizes the total expected discounted cost when each (s, z) is given. For example, the action $r = (1, 0, 0)$, which replaces units 2 and 3, is the optimal action when $s = (1, 2, 4)$ and $z = (0, 0, 0)$. The color in each plot represents the optimal number of units to be replaced, with darker colors indicating a greater number of units replaced. Since $z_i \leq s_i$ always holds, actions for regions without any color do not need to be considered.

From plots Figure 2a,c, the number of units to be replaced increases as s_1 increases. Furthermore, in $s = (1, 2, 2)$, $(2, 1, 2)$, $(2, 2, 1)$, and $(2, 2, 2)$ of plots Figure 2a,c, the number of units to be replaced increases as z increases. This indicates that the optimal policy becomes more conservative to avoid system failure as s and z increase. Furthermore, Figure 2 demonstrates that the optimal action switches between $r_i = 1$ and $r_i = 0$ is at most

one in s_i and z_i , respectively. This observation confirms that the optimal policy exhibits the structural property discussed in Theorem 3.

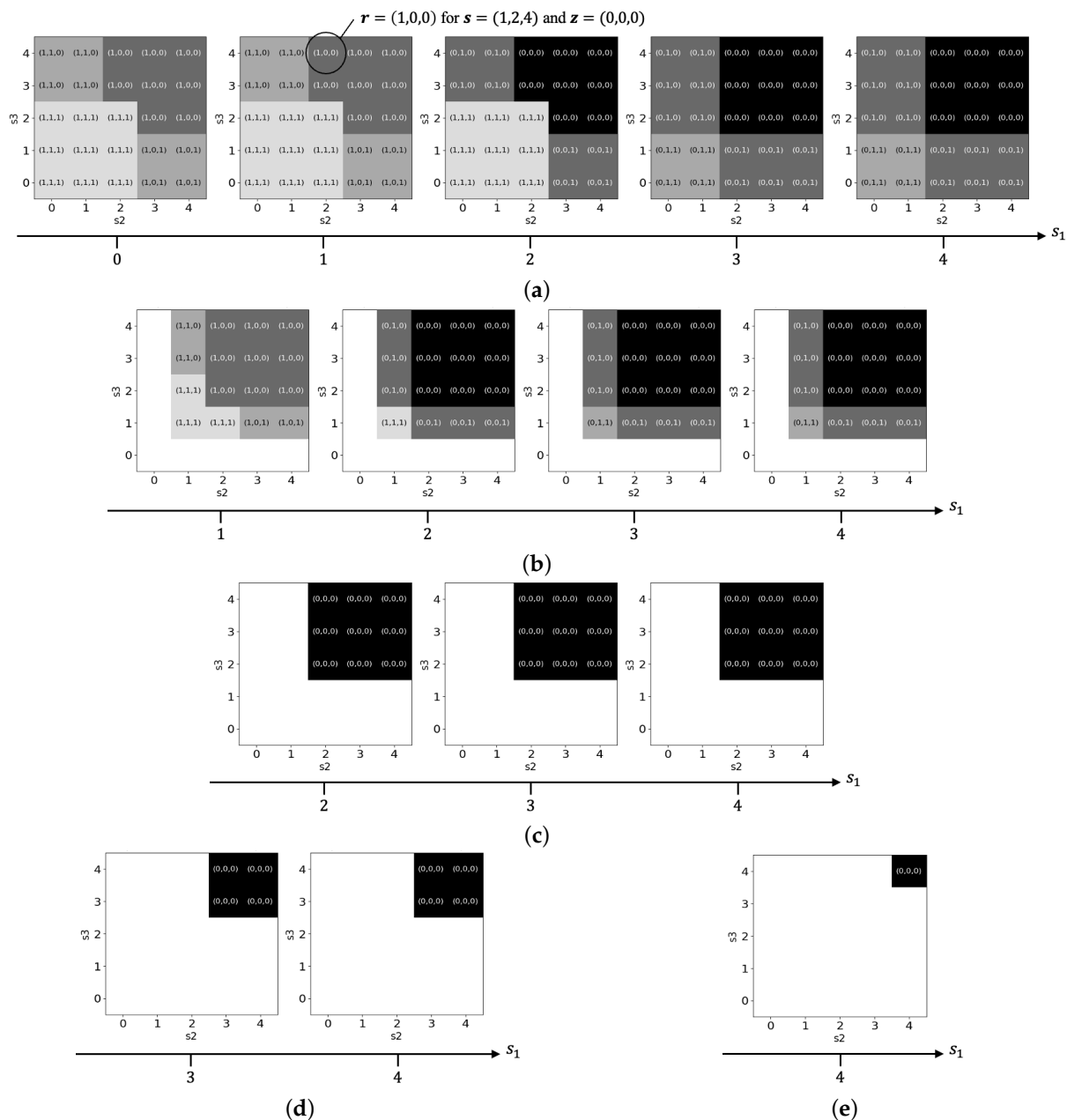


Figure 2. Optimal maintenance policy for a three-bladed rotor system. (a) Shock-induced cumulative deterioration $z = (0,0,0)$. (b) Shock-induced cumulative deterioration for $z = (1,1,1)$. (c) Shock-induced cumulative deterioration $z = (2,2,2)$. (d) Shock-induced cumulative deterioration $z = (3,3,3)$. (e) Shock-induced cumulative deterioration $z = (4,4,4)$.

4.3. Performance of the Proposed Policy

To evaluate the effectiveness of the proposed policy, we compare it with four alternative policies: three of (i) to (iii) that account for interdependence among units and one of (iv) that does not. These alternatives are as follows: (i) a conventional policy that ignores interdependence between deterioration and shocks, (ii) a policy that considers only the dependency of unit deterioration on shock-induced cumulative deterioration (Shock→Deterioration Policy), (iii) a policy that considers only the dependency of shock-induced increments on the deterioration state (Deterioration→Shock Policy), and (iv) an individual optimal policy proposed by Wang et al. [30], which incorporates the interde-

pendence between deterioration and shocks but neglects interdependence among units. Specifically, for (i) the conventional policy, we assume $\beta_i(z) = \beta_i((0, \dots, 0))$ for all z , disregarding the dependency of unit deterioration on shock-induced cumulative deterioration. Similarly, for the Deterioration→Shock Policy, we set $f_i^m(\Delta_i|s_i) = f_i^m(\Delta_i|0)$ for all s_i , neglecting the dependency of shock-induced increments on the deterioration state. Furthermore, when deriving the individual optimal policy, the three-unit system is decomposed into three single-unit systems, and the optimal maintenance policies of each single-unit system are derived separately. For a three-bladed rotor system, this approach does not account for economic and stochastic dependencies. In other words, a setup cost is always incurred when RP is performed, and $\beta_i(z) = 0.872(1 + z_i/8), \forall i \in \{1, 2, 3\}$.

The effectiveness of each policy is assessed by comparing the average total discounted costs, the availability and the frequency of preventive and corrective RP under the proposed and conventional policies. The following simulation procedure in Algorithm 2 is used to obtain this average for each policy: starting with an initial state ($s = (0, \dots, 0), z = (0, \dots, 0)$), we simulate each policy over a sufficiently long, finite episode to cover the lifecycle of the offshore wind turbines. The total discounted cost and the total downtime within one episode is the sum of all incurred costs and downtime. This process is repeated 2000 times, and the average total discounted cost, the availability from the total downtime, and the frequency of preventive and corrective RP across all simulations are calculated.

Algorithm 2 Calculate average total discounted cost and availability for each policy.

```

input deterioration and shock parameters, cost parameters, and  $V_r(s, z)$ 
output average total discounted cost
for  $e$  in  $\{1, 2, \dots, 2000\}$  do
  initialize  $s = (0, \dots, 0), z = (0, \dots, 0)$ 
  initialize total discounted cost, total downtime
  initialize preventive and corrective RP times
  for  $t$  in  $\{1, 2, \dots\}$  do
    select action  $r = \arg \min_r \{V_r(s, z)\}$ 
    interact with action  $r$  to obtain the next  $s, z$ , cost, and downtime for one interval
    calculate total discounted cost and total downtime
    calculate total preventive and corrective RP times
  end for
  memorize total discounted cost and total downtime
  calculate frequency of preventive and corrective RP
end for
calculate average total discounted cost and average availability base on total downtime
calculate average frequency of preventive and corrective RP

```

Table 2 presents the average total maintenance costs and cost reduction rates (CRRs) of the proposed policy relative to each alternative policy. The CRR is computed as $(v_2 - v_1)/v_2$, where v_1 and v_2 are the average total costs of the proposed and alternative policies, respectively.

As shown in Table 2, the effectiveness of the proposed policy is evaluated by comparing the average total discounted costs over simulated lifecycles. The proposed policy achieved the lowest average cost of EUR 3,226,835 among the four policies considered. A comparison between the proposed policy and the deterioration→shock policy revealed that the proposed policy resulted in a slightly lower availability (0.1% decrease). However, when analyzing the frequency of preventive and corrective RP, the proposed policy required slightly fewer RPs, suggesting that the system operated closer to its failure threshold under the proposed policy. This indicates that the proposed policy minimizes costs by effectively reducing unnecessary RPs while maintaining system reliability. Furthermore,

a comparison between the proposed policy and the individual optimal policy based on previous research [30] demonstrated the effectiveness of considering economic and stochastic dependencies among units. These results highlight the advantage of incorporating interdependencies into the maintenance decision framework. The findings of this study can be generalized to other industries that operate systems subject to shocks. For example, systems in industries such as electric vehicles [22], micro-electromechanical system [34], and electromechanical systems of train and aircraft [39] are susceptible to failures caused by shocks. The proposed policy in this study is expected to enhance the economic operation of such systems by mitigating the impact of shocks and optimizing maintenance policies.

Table 2. Comparison of average total discounted costs and availability for optimal policies in a three-bladed rotor system.

Policy	Cost (EUR)	CRR (%)	Availability (%)	Frequency of Preventive RP (times/ τ)	Frequency of Corrective RP (times/ τ)
Proposed Policy	3,226,835	-	96.7%	0.282	0.048
(i) Deterioration $\nrightarrow$ Shock Policy (Conventional)	3,579,910	9.9	95.4%	0.236	0.066
(ii) Shock $\rightarrow$ Deterioration Policy	3,566,466	9.5	95.4%	0.236	0.065
(iii) Deterioration $\rightarrow$ Shock Policy	3,269,933	1.3	96.8%	0.290	0.046
(iv) Individual Optimal Policy [30]	3,556,059	9.3	95.5%	0.237	0.065

4.4. The 2-out-of-3 System

To provide further insights for readers, we extended our analysis to include a more general 2-out-of-3 system, using the same parameters specified in Section 4.1. The optimal maintenance policy is derived and illustrated in Figure 3.

From Figure 3a–c, it is observed that the number of units to be replaced increases as s_1 increases. Furthermore, in states $s = (2, 2, 3)$, $(2, 3, 2)$, and $(4, 2, 2)$ shown in Figure 3b,c, the number of units to be replaced also increases as z increases. Figure 3 confirms that the optimal policy exhibits the structural property discussed in Theorem 3. Comparing Figures 2 and 3, the optimal policy for 2-out-of-3 systems has narrower regions where RP is selected. This indicates that redundant systems reduce the number of units to be replaced in any given situation. To explore the impact of variable factors, a sensitivity analysis is conducted in Section 5.

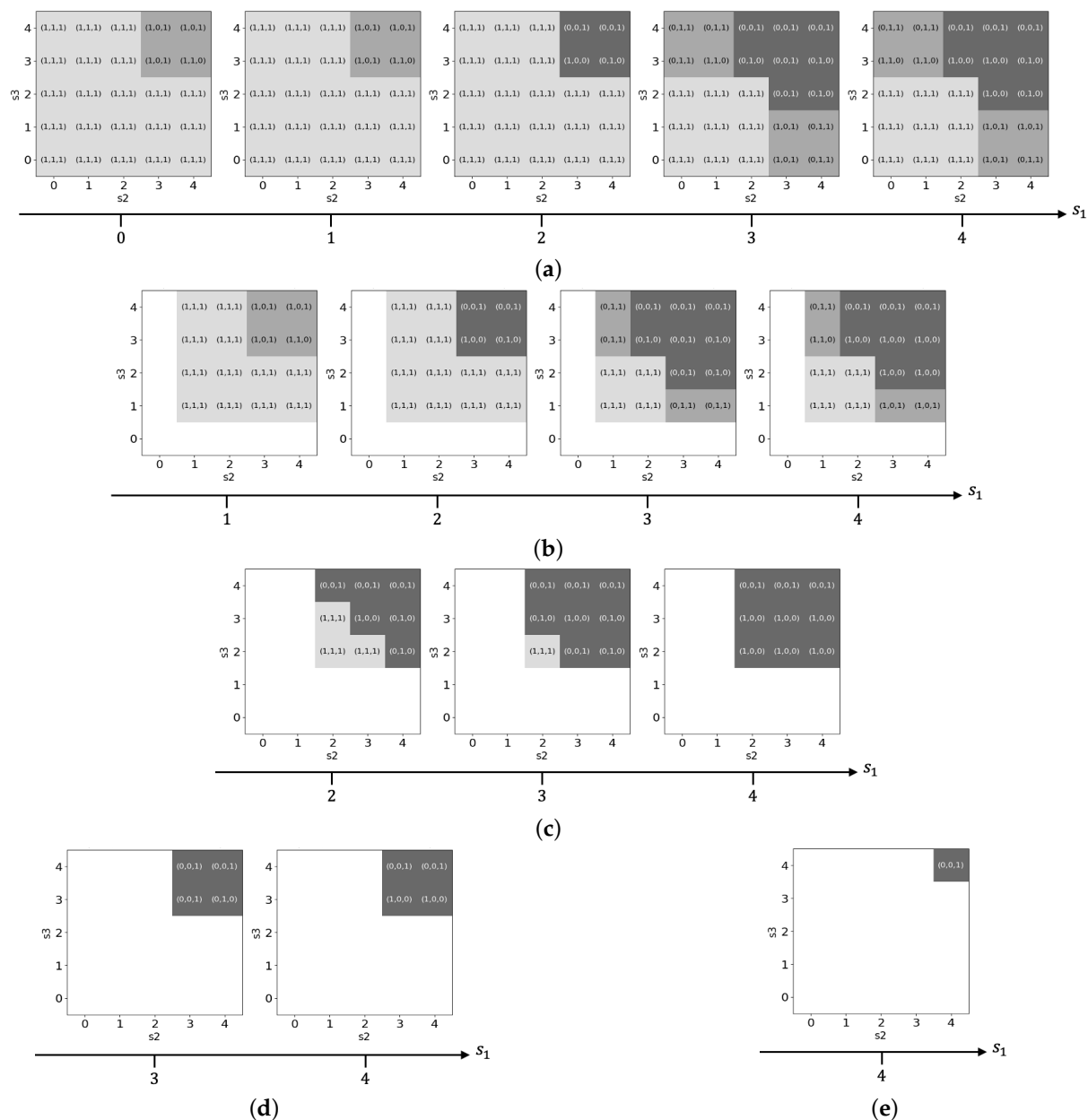


Figure 3. Optimal maintenance policy for a 2-out-of-3 system. (a) Shock-induced cumulative deterioration $z = (0, 0, 0)$. (b) Shock-induced cumulative deterioration $z = (1, 1, 1)$. (c) Shock-induced cumulative deterioration $z = (2, 2, 2)$. (d) Shock-induced cumulative deterioration $z = (3, 3, 3)$. (e) Shock-induced cumulative deterioration $z = (4, 4, 4)$.

5. Sensitivity Analysis

This section focuses on a 1-out-of-2 system and presents a sensitivity analysis to investigate the impact of parameters related to costs, deterioration, and shocks on the proposed maintenance policy. To visualize the behavior of the proposed policy in the (s, z) space, we limit the number of units to two and consider a 1-out-of-2 system.

In this section, parameters are set under Assumptions 1 and 2, ensuring their validity within the range that appropriately represents system deterioration, shocks, and costs. The parameters subject to sensitivity analysis include the preventive replacement cost (c_p), corrective replacement cost (c_f), downtime cost (c_d), setup cost (c_s), scale parameter model coefficient (b), and the frequency of external shocks (λ). The shape parameters for both units are set as $\alpha = 1.2$. The scale parameters are defined by $\beta_1(z) = b + b(2z_1 + z_2)/3$ and $\beta_2(z) = b + b(z_1 + 2z_2)/3$, where b representing the coefficient in the scale parameter

model. In this section, b is specified as shown in Table 3. We consider three types of external shocks, with shock probabilities defined by a Poisson distribution with $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$, where λ is listed in Table 3. Each shock induces a jump following an exponential distribution: $f_1^1(\Delta_1|s_1) = 0.5s_1e^{-0.5s_1\Delta_1}$, $f_2^2(\Delta_2|s_2) = 0.5s_2e^{-0.5s_2\Delta_2}$, $f_1^3(\Delta_1|s_1) = 0.8s_1e^{-0.8s_1\Delta_1}$, and $f_2^3(\Delta_2|s_2) = 0.8s_2e^{-0.8s_2\Delta_2}$. Cost parameters are also specified in Table 3. The inspection interval is set to $\tau = 1$ with a discount factor $e^{-0.05\tau}$.

Table 3. Parameter settings for sensitivity analysis.

	b	λ	c_p	c_f	c_d	c_s
Low	0.4	0.125	4	12	4	1
Base	0.5	0.250	7	16	8	2
High	0.6	0.500	10	20	12	3

The sensitivity analysis examines how the optimal maintenance policy changes with respect to each parameter in Table 3. Figures 4–9 illustrate these results, with (a), (b), and (c) in each figure corresponding to increasing parameter values. In each figure, the horizontal axis represents the shock-induced cumulative deterioration level z_1 , while the vertical axis represents the shock-induced cumulative deterioration level z_2 . Each figure contains nine plots arranged to visualize parameter impacts across different states, where the horizontal axis in each plot corresponds to s_1 , and the vertical axis to s_2 . The color in each plot denotes the optimal maintenance action for each specific state $(s, z) = ((s_1, s_2), (z_1, z_2))$. This layout provides a clear view of how the optimal actions shift with changes in shock-induced cumulative deteriorations and deterioration states across a range of parameter settings.

5.1. Preventive Replacement Cost (c_p)

Figure 4 illustrates the optimal maintenance policy as the preventive replacement cost, c_p , varies across values of 4, 7, and 10. As c_p increases, the area designated for RP area (dark shaded) narrows, while the DN area (unshaded) expands, indicating a more conservative policy. This shift reflects a tendency to keep the system operational longer as failure-related costs become relatively smaller compared to the preventive replacement cost, c_p .

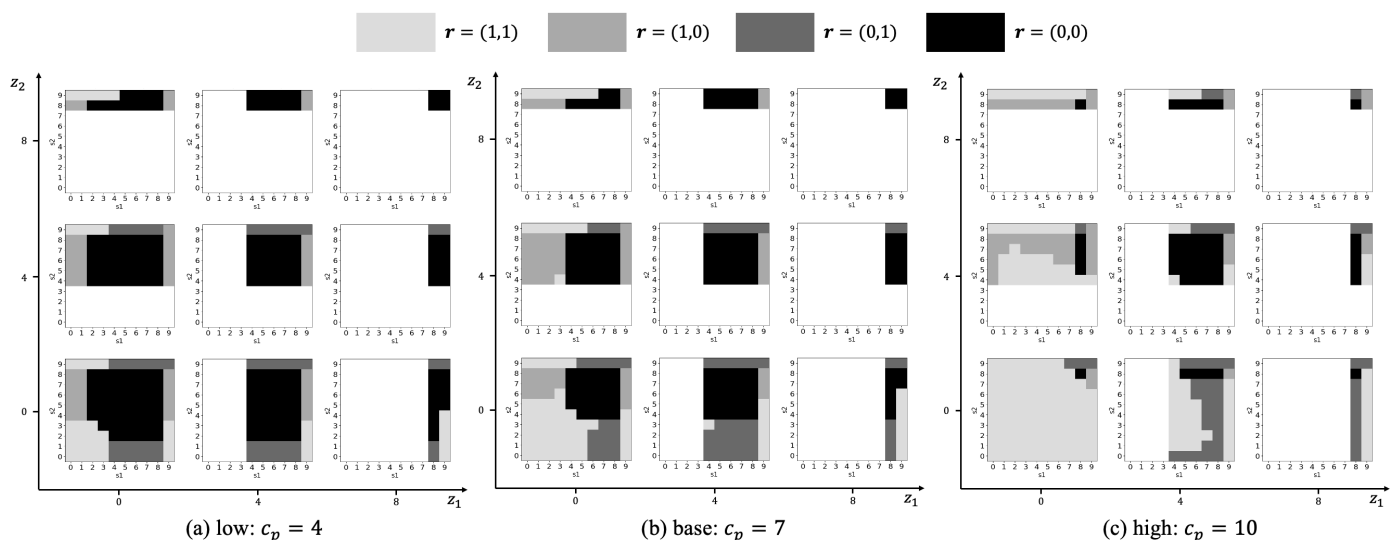


Figure 4. Sensitivity analysis of optimal maintenance policy for preventive replacement costs c_p .

5.2. Corrective Replacement Cost (c_f)

Figure 5 illustrates the optimal policy for corrective replacement costs $c_f = 12, 16$, and 20. Lower c_f values narrow the RP area (dark shaded) and expand the DN area (unshaded), making the policy more aggressive, as the relative cost of failure decreases.

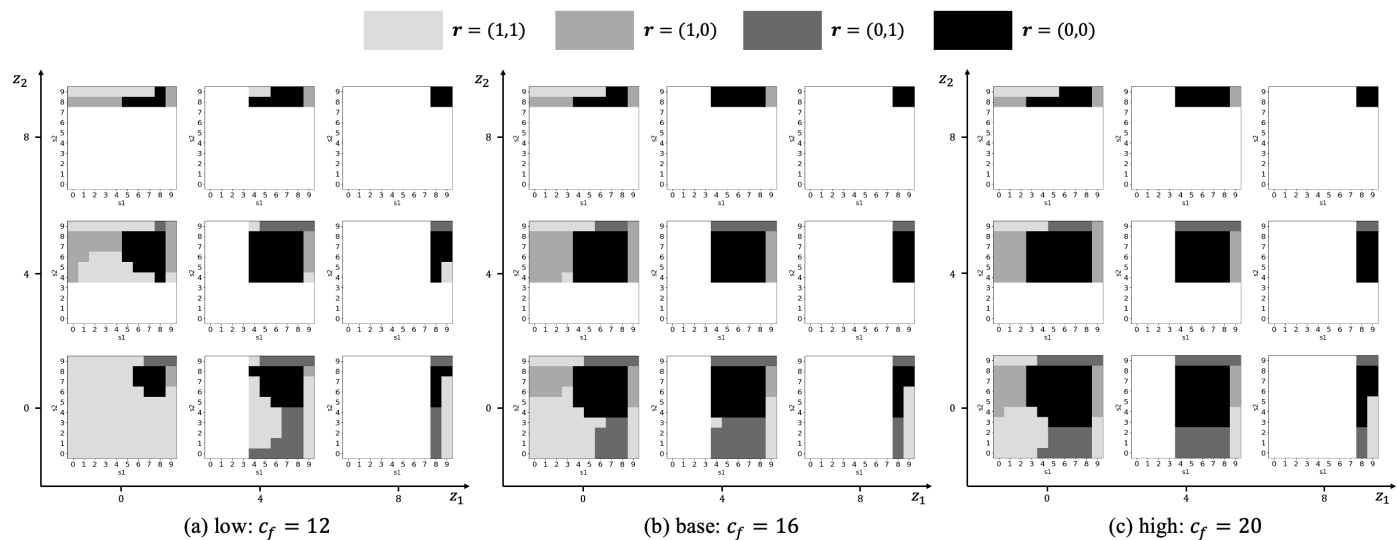


Figure 5. Sensitivity analysis of optimal maintenance policy for corrective replacement costs c_f .

5.3. Downtime Cost (c_d)

In Figure 6, as downtime cost c_d decreases, the policy becomes more aggressive, operating units longer since failure costs are relatively low.

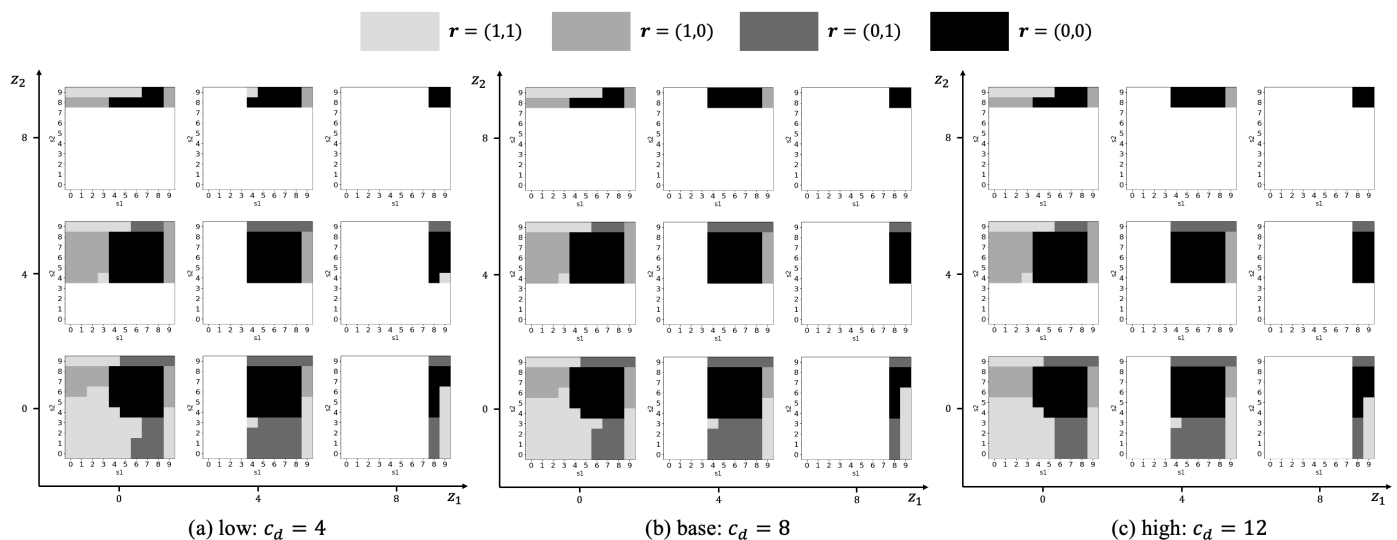


Figure 6. Sensitivity analysis of optimal maintenance policy for downtime cost c_d .

5.4. Setup Cost (c_s)

Figure 7 illustrates the policy behavior for setup costs $c_s = 1, 2$, and 3. As c_s decreases, the area where only one unit is replaced expands, while for high c_s , replacing two units becomes more economical, as seen in configurations like $(z_1, z_2) = (4, 0)$ in Figure 7c.

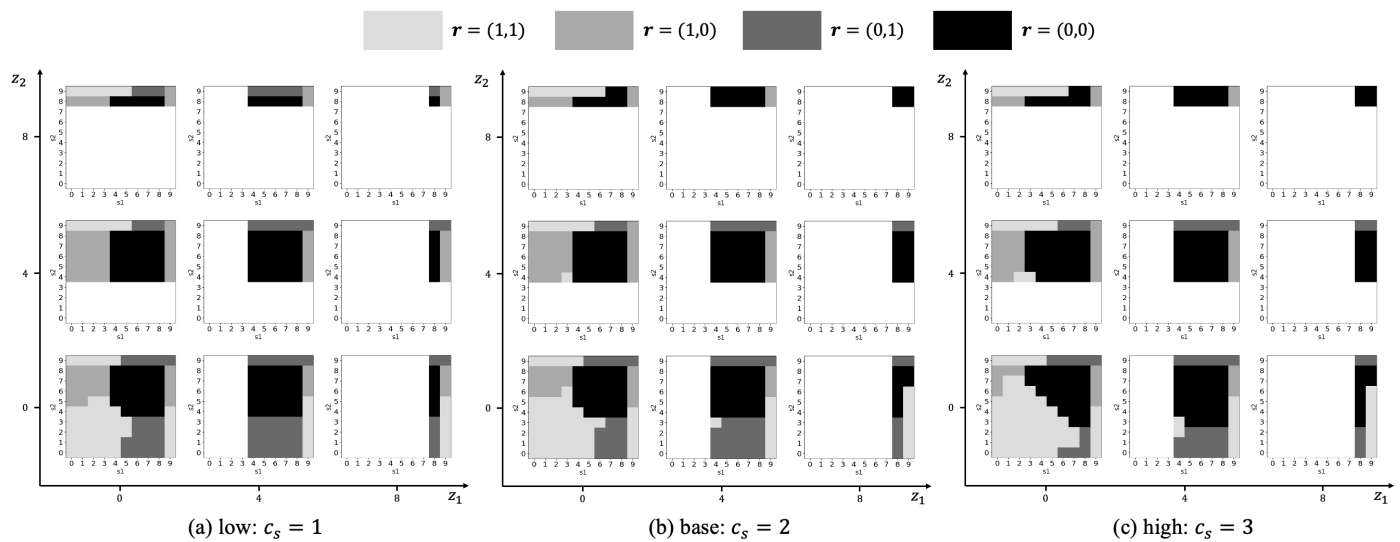


Figure 7. Sensitivity analysis of optimal maintenance policy for setup cost c_s .

5.5. Coefficient in Scale Parameter Model (b)

The policy changes for the coefficient $b = 0.4, 0.5$, and 0.6 are illustrated in Figure 8. As b increases, the policy favors continued operation in certain regions, such as $(z_1, z_2) = (0, 0)$ and $(0, 8)$, which indicates that both units have received the same level of cumulative damage or that one unit has experienced severe cumulative damage. This preference arises because replacing both units simultaneously is more cost-effective due to relatively fast deterioration from increasing b and the setup cost savings. Conversely, in some regions, such as $(z_1, z_2) = (0, 4)$ and $(4, 8)$, a larger replacement scale (replacing more units) becomes preferable. This is because as b increases (indicating a relatively high deterioration rate), delaying replacement to align the two units could result in higher penalty costs from system failure, which outweigh the benefits of saving on setup costs.

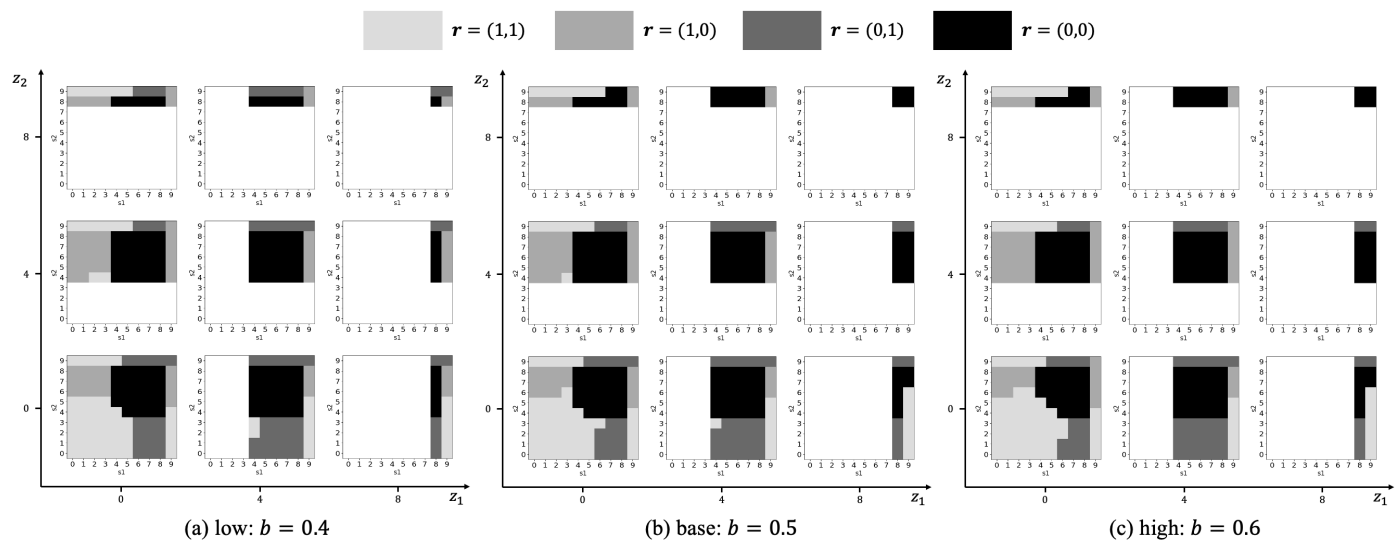


Figure 8. Sensitivity analysis of optimal maintenance policy for coefficient in scale parameter model.

5.6. Shock Likelihood (λ)

Figure 9 shows how the policy adapts as shock likelihood increases from $\lambda = 0.5$ to 2.0 . A higher λ narrows the DN area (unshaded), indicating a more conservative policy under higher shock frequencies, where failure likelihood is amplified.

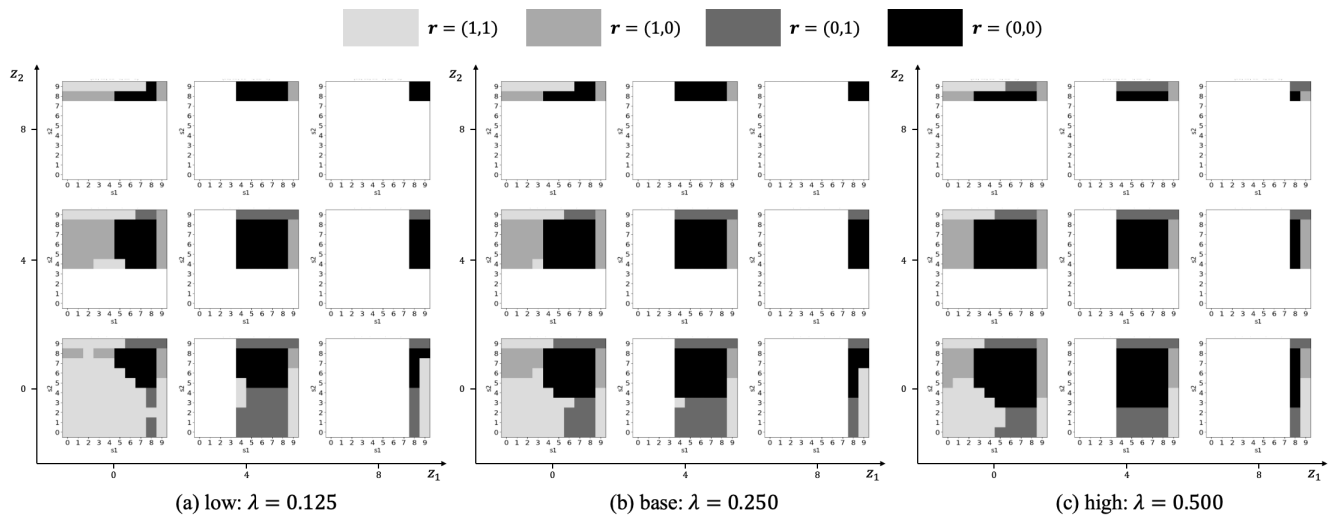


Figure 9. Sensitivity analysis of optimal maintenance policy for likelihoods of external shocks.

In this section, we studied the behavior of the optimal policy for a 1-out-of-2 system under various parameter settings. By considering two types of interdependencies—the interdependence between the shock-induced deterioration of the two units and the interdependence between internal deterioration and random external shocks—the overall deterioration process becomes highly complex, as it incorporates contributions from both internal deterioration and jumps caused by external shocks. Although we analyzed the behavior of the optimal policy for each parameter, isolating the impact of any single parameter on the policy remains challenging. Consequently, the maintenance policy can vary significantly depending on the combination of parameters. It is crucial to determine the maintenance policy using an approach that explicitly accounts for the interdependence between deterioration and random shocks while setting parameters that accurately represent real-world conditions.

6. Conclusions

This study focused on a k -out-of- n system with n identical units, where each unit independently deteriorates following a gamma process and is subjected to various types of random external shocks. We explored CBM policies for this system, explicitly considering the interdependence between unit deterioration and deterioration jumps caused by external shocks. Using an MDP, we developed an optimal CBM policy to minimize the total expected discounted cost, demonstrating analytically that this policy exhibits a structural property.

Additionally, a case study on a three-bladed rotor system in offshore wind turbines was conducted, demonstrating up to a 9.9% reduction in maintenance costs achieved by our proposed policy. Furthermore, the case study validated the structural properties of the proposed policy. Sensitivity analyses further examined the behavior of the optimal policy under different parameter settings, which shows the effects of parameter variations on policy performance.

In this research, we assumed that each unit's deterioration was independent of other units' deterioration states. However, in practical applications, the deterioration or failure of one unit may impact the deterioration rate of others. Extending our model to include such interdependencies offers a valuable direction for future research.

In this optimization problem, the information dimension of the system deterioration state and shock-induced cumulative deterioration increases with the number of units, resulting in an explosive expansion of the solution space. To address this, we provided key insights for developing algorithms that enable efficient searching for the optimal solution.

Future work includes devising and developing algorithms to reduce computational effort for searching the optimal policy by utilizing the derived structural properties.

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Nomenclature

τ	Fixed inspection interval
s_i	Deterioration state of unit i
$\mathbf{s} = (s_1, \dots, s_n)$	Deterioration state of system
z_i	Shock-induced cumulative deterioration of unit i
$\mathbf{z} = (z_1, \dots, z_n)$	Shock-induced cumulative deterioration of system
L	Failure threshold of units
T_f	Elapsed time from last inspection to system failure
α	Shape parameter of units
$\beta_i(\mathbf{z})$	Scale parameter of unit i given $\mathbf{z}$
$W(\mathbf{s}, \mathbf{z})$	Expected down time cost given $\mathbf{s}$ and $\mathbf{z}$
c_p	Constant preventive replacement cost
c_f	Constant corrective replacement cost
c_s	Constant setup cost
c_d	Constant downtime cost
$e^{-\beta\tau}$	Discount factor
$m \in \{1, \dots, M\}$	Shock type
$\mathbf{U}_m$	Set of units whose deterioration state increases due to shocks of type m
$p_i(s'_i s_i, \mathbf{z}, t)$	Transition probability density function from s_i to s'_i given $\mathbf{z}$ after time t
$g_m(t)$	Probability density function that type m shock occurs at time t
$f_i^m(\Delta_i s_i)$	Probability density function that the increment of unit deterioration is Δ_i when a shock of type m occurs in unit i

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