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Characterization of Monotone Sequences of Positive Numbers Prescribed by Means †

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Abstract: The aim of this article is to investigate the relations between the exponent of the convergence of sequences and other characteristics defined for monotone sequences of positive numbers. Another main goal is to characterize such monotone sequences (a_n) of positive numbers that, for each $n \geq 2$, satisfy the equality $a_n = K(a_{n-1}, a_{n+1})$, where the function $K : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is the mean, i.e., each value of $K(x,y)$ lies between $\min\{x,y\}$ and $\max\{x,y\}$. Well-known examples of such sequences are, for example, arithmetic (geometric) progression, because starting from the second term, each of its terms is equal to the arithmetic (geometric) mean of its neighboring terms. Furthermore, this accomplishment generalized and extended previous results, where the properties of the logarithmic sequence (a_n) are referred to, i.e., in such a sequence that every $n \geq 2$ satisfies $a_n = L(a_{n-1}, a_{n+1})$, where $L(x,y)$ is the logarithmic mean of positive numbers x, y defined as follows: $L(x,y) := \begin{cases} \frac{y-x}{\ln y - \ln x} & \text{if} \quad x \neq y\,, \\ x & \text{if} \quad x = y\,. \end{cases}$

Keywords: means of two variables; logarithmic mean; power mean; f-mean; exponent of convergence of sequences; sequences prescribed by means

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1. Introduction

The core focus of this paper is to investigate the relations between the convergence exponent of sequences and other characteristics specified for monotone sequences of positive numbers. We also focus on the characterization of such monotone sequences (a_n) of positive numbers that for each $n \geq 2$ satisfy the equality $a_n = K(a_{n-1}, a_{n+1})$, for some mean $K : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$. Such sequences that satisfy this equality for each $n \geq 2$ are called K-sequences. In the present paper, some basic estimations and limits of the terms of K-sequences are investigated.

Denote by $\mathbb{N}$ and $\mathbb{R}^+$ the set of all positive integers and positive real numbers, respectively. Let us denote $\mathcal{S}$ (respectively, $\mathcal{T}$) the system of all nondecreasing (nonincreasing) sequences of positive real numbers and for $h \in (0, \infty]$, $l \in [0, \infty)$, we denote

$$\mathcal{S}_h = \{(a_n) \in \mathcal{S} : \lim_{n\to\infty} a_n = h\} \text{ and } \mathcal{T}_l = \{(a_n) \in \mathcal{T} : \lim_{n\to\infty} a_n = l\}\,.$$

So,

$$\mathcal{S} = \bigcup_{h\in(0,\infty]} \mathcal{S}_h \text{ and } \mathcal{T} = \bigcup_{l\in[0,\infty)} \mathcal{T}_l\,.$$

This area has been studied by many mathematicians. For this paper, we were inspired by [1–9].

2. Definitions and Notations

In this part, we recall some basic definitions. The following definitions are from papers [1,2,7,9–11].

- A function $M : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is called a *mean* on $\mathbb{R}^+$ if for all $x, y \in \mathbb{R}^+$ we have

$$\min\{x, y\} \le M(x, y) \le \max\{x, y\}.$$

It is obvious that $M(x, x) = x$ for all $x \in \mathbb{R}^+$.
The mean $M : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is called *symmetric* if

$$M(x, y) = M(y, x)$$

for all $x, y \in \mathbb{R}^+$.
The mean $M : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is called a *strict* mean on $\mathbb{R}^+$ if for all $x, y \in \mathbb{R}^+$ with $x \neq y$ we have

$$\min\{x, y\} < M(x, y) < \max\{x, y\}.$$

The mean $M : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is called *homogeneous* if

$$M(zx, zy) = zM(x, y),$$

for all $x, y, z \in \mathbb{R}^+$.
Classical examples of two-variable strict means on $\mathbb{R}^+$ are the arithmetic, the geometric, and the harmonic mean:

$$A(x, y) := \frac{x + y}{2}, \quad G(x, y) := \sqrt{xy}, \quad H(x, y) := \frac{2xy}{x + y}\,.$$

- The power mean of degree $\alpha \in \mathbb{R}$ of two positive numbers x, y is as follows:

$$M_\alpha(x, y) = \left(\frac{x^\alpha + y^\alpha}{2}\right)^{\frac{1}{\alpha}} \text{ if } \alpha \neq 0 \text{ and } M_0(x, y) = \lim_{\alpha\to 0} M_\alpha(x, y).$$

The case $\alpha = 1$ corresponds to the arithmetic mean, $\alpha = 0$ to the geometric mean, and $\alpha = -1$ to the harmonic mean. It is well known that

$$\lim_{\alpha\to-\infty} M_\alpha(x, y) = \lim_{\alpha\to-\infty} \left(\frac{x^\alpha + y^\alpha}{2}\right)^{\frac{1}{\alpha}} = \min\{x, y\}$$

and

$$\lim_{\alpha\to\infty} M_\alpha(x, y) = \lim_{\alpha\to\infty} \left(\frac{x^\alpha + y^\alpha}{2}\right)^{\frac{1}{\alpha}} = \max\{x, y\}\,.$$

The means $\min\{x, y\}, \max\{x, y\}$ are called the minimum and maximum mean, respectively. Obviously, the mean M_α is a homogeneous strict mean such that

$$\lim_{y\to\infty} M_\alpha(x, y) = \infty \text{ for } \alpha > 0\,,$$

$$\lim_{x\to 0} M_\alpha(x,y) = 0 \text{ for } \alpha < 0\,,$$

moreover, $M_\alpha(x,y)$ is continuous with respect to x and y.
It is well known that $M_0(x,y)$ and $M_\alpha(x,y)$ is increasing with respect to α for a given $x,y \in \mathbb{R}^+$ (see [12]).

- The logarithmic mean (see [9]) of positive numbers x,y is

$$L(x,y) := \begin{cases} \frac{y-x}{\ln y - \ln x} & \text{if} \quad x \neq y\,, \\ x & \text{if} \quad x = y\,. \end{cases}$$

Obviously, the mean L is a homogeneous strict mean such that

$$\lim_{y\to\infty} L(x,y) = \infty \text{ and } \lim_{x\to 0} L(x,y) = 0\,.$$

Moreover, $L(x,y)$ is continuous with respect to x and y.
In [4,5], there is a generalization of the logarithmic mean and in [6,7], the following relation between $L(a,b)$ and $M_\alpha(a,b)$ was proven in various ways for arbitrary positive numbers a,b:

$$M_0(a,b) \le L(a,b) \le M_{\frac{1}{3}}(a,b),$$

where the equality occurs if and only if $a = b$.

- For $p \in \mathbb{R}$ the generalized logarithmic mean L_p of two positive numbers and are defined as

$$L_p(a,b) = \begin{cases} a & \text{if} \quad a = b, \\ \left(\frac{b^p - a^p}{p(b-a)}\right)^{\frac{1}{p-1}} & \text{if} \quad a \neq b, p \neq 0, p \neq 1, \\ \frac{b-a}{\ln b - \ln a} & \text{if} \quad a \neq b, p = 0, \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} & \text{if} \quad a \neq b, p = 1. \end{cases}$$

It is well known that $L_p(a,b)$ is continuous and increasing with respect to p for fixed a and b (see [4]). Notice that $L_{-1}(a,b) = M_0(a,b)$.

- A sequence (a_n) of positive real numbers is called a logarithmic sequence if

$$a_n = L(a_{n-1}, a_{n+1}) \text{ for every } n \ge 2.$$

- Let $K : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ be a mean. Then, the sequence (a_n) of positive real numbers is called a K-sequence if

$$a_n = K(a_{n-1}, a_{n+1}) \text{ for every } n \ge 2.$$

If the mean K is chosen to be the logarithmic mean, i.e., $K = L$, then the logarithmic sequence is obtained. Thus, the logarithmic sequence is an L-sequence. Of course, we can choose any of the means M_α and L_p; in this case, we obtain the M_α- and L_p-sequences, respectively.

- Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a monotone function. Then the f-mean of two positive numbers a,b is as follows:

$$M_f(a,b) = f^{-1}\left(\frac{f(a) + f(b)}{2}\right).$$

In the case of $f(x) = x^\alpha$, we have $M_f = M_\alpha$.

- In [12], the *exponent of convergence* of a sequence $(a_n) \in \mathcal{S}_\infty$ is defined as the infimum of all positive real numbers α such that the series $\sum_{n=1}^{\infty} a_n^{-\alpha}$ is convergent and, in case no such α exist, we define the exponent of convergence as ∞. Let us denote the exponent

of convergence of $(a_n) \in \mathcal{S}_\infty$ by $\lambda((a_n))$. It can be easily seen that $\lambda((a_n))$ can be interpreted as a measure of rate of convergence of (a_n) to infinity. If $q > \lambda((a_n))$, then $\sum_{n=1}^{\infty} a_n^{-q} < \infty$ and if $q < \lambda((a_n))$, then $\sum_{n=1}^{\infty} a_n^{-q} = \infty$. In case $q = \lambda((a_n))$, the series $\sum_{n=1}^{\infty} a_n^{-q}$ can be either convergent or divergent.
From ([12], p. 26, Exercises 113, 114), it follows that the set of all possible values of λ forms the whole interval $[0, \infty]$, i.e., $\{\lambda((a_n)) : (a_n) \in \mathcal{S}_\infty\} = [0, \infty]$ and if $(a_n) \in \mathcal{S}_\infty$, then $\lambda((a_n))$ can be calculated by

$$\lambda((a_n)) = \limsup_{n\to\infty} \frac{\log n}{\log a_n}.$$

- For $0 < q$, we define the set

$$\mathcal{I}_{1/x^q} = \Big\{(a_n) \in \mathcal{S}_\infty : \sum_{n=1}^{\infty} \frac{1}{a_n^q} < \infty\Big\}.$$

By values of λ, we define the following sets (similarly to [10,13]):
$\mathcal{I}_{<q} = \{(a_n) \in \mathcal{S}_\infty : \lambda((a_n)) < q\}$ for $0 < q \leq \infty$,
$\mathcal{I}_{\leq q} = \{(a_n) \in \mathcal{S}_\infty : \lambda((a_n)) \leq q\}$ for $0 \leq q \leq \infty$ and
$\mathcal{I}_0 = \{(a_n) \in \mathcal{S}_\infty : \lambda((a_n)) = 0\}$.
Obviously, $\mathcal{I}_{\leq 0} = \mathcal{I}_0$ and $\mathcal{I}_{\leq \infty} = \mathcal{S}_\infty$.
Families $\mathcal{I}_{<q}, \mathcal{I}_{\leq q}$, and $\mathcal{I}_{1/x^q}$ are related to $0 < q < q'$ by following inclusions (similarly to [13], Th.2.1.)

$$\mathcal{I}_0 \subsetneq \mathcal{I}_{<q} \subsetneq \mathcal{I}_{1/x^q} \subsetneq \mathcal{I}_{\leq q} \subsetneq \mathcal{I}_{<q'},$$

and the difference of successive sets is infinite, so equality does not hold in any of the inclusions.

3. Overview of Known Results

In this section, we mention well-known results related to the topic of this paper and some of them are generalized in the proofs of our theorems.

$(S1)$ Let a_1, a_2 be two positive numbers.

(i) If $a_1 = a_2$, then a_1, a_2 are the first two terms of the constant sequence $a_1, a_1, \cdots, a_1, \cdots$ which is logarithmic.

(ii) Let $a_1 \neq a_2$. Then, there exists a logarithmic sequence (a_k) such that if $a_1 < a_2$, then $a_1 < a_2 < \cdots < a_k < a_{k+1} < \cdots$, and if $a_1 > a_2$, then $a_1 > a_2 > \cdots > a_k > a_{k+1} > \cdots$. ([2], Theorem 2.1.)

$(S2)$ Let (a_k) be a logarithmic sequence of positive numbers. Then, we have

(i) If $a_1 > a_2$, then the series $\sum_{k=1}^{\infty} a_k$ converges and

$$\sum_{k=1}^{\infty} a_k \leq \frac{a_1^2}{a_1 - a_2}.$$

([2], Theorem 2.2.)

(ii) If $a_1 < a_2$, then $\lim_{k\to\infty} a_k = \infty$. ([2], Theorem 2.3.)

$(S3)$ Let (a_n) be an increasing logarithmic sequence of positive numbers. Then, the following statements hold.

(i) The series $\sum_{n=1}^{\infty} \frac{1}{a_n}$ converges. Moreover,

$$a_n > \left(\frac{a_2^\alpha - a_1^\alpha}{2}\right)^{\frac{1}{\alpha}} n^{\frac{1}{\alpha}} \text{ for every } \alpha \geq \frac{1}{3} \text{ and } n \geq 2.$$

([1], Corollary 2.3., and Theorem 2.1.(i))

(ii) The inequality

$$a_{n+1} - a_n > (\sqrt{a_2} - \sqrt{a_1})^2 (n+1)$$

holds for every $n \geq 2$.

([1], Theorem 2.6.)

(iii) We have

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1 \text{ and } \lim_{n\to\infty} \frac{a_n}{q^n} = 0 \text{ for every real } q > 1.$$

([1], Theorem 2.7., and Corollary 2.8.)

$(S4)$ Let (a_n) be a decreasing logarithmic sequence of positive numbers. Then, the following statements hold.

(i) We have

$$a_n < \left(\frac{a_2^\alpha - a_1^\alpha}{2}\right)^{\frac{1}{\alpha}} n^{\frac{1}{\alpha}} \text{ for every } \alpha < 0 \text{ and } n \geq 2.$$

([1], Theorem 2.1.(ii))

(ii) We have

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0 \text{ and } \lim_{n\to\infty} \frac{a_n}{q^n} = 0 \text{ for every real } q > 0.$$

([1], Theorem 2.7., and Corollary 2.8.)

4. Results

In this section, we generalize the results of $(S1)$–$(S4)$ for a suitable class of means. The following Theorem 1 generalizes the result $(S1)$, and it holds for all known means from part 2.

The next theorem describes for given positive real numbers $b \neq c$ such means $K(x,y)$ for which there is a sequence (a_n), which will be a K-sequence and $a_1 = b, a_2 = c$. If $b = c$, then b, c are the first two terms of the constant sequence $b, b, \cdots, b, \cdots$, which is a K-sequence for every mean $K(x,y)$.

Theorem 1. *Suppose that $K : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a strict mean such that $K(x,y)$ is continuous with respect to x and y. Let $a_1 \neq a_2$ be two positive numbers. Then, we have*

(i) if $a_1 < a_2$ and $\lim_{y\to\infty} K(x,y) = \infty$, then there exists a sequence (a_n) which is a K-sequence and, in such a case, the sequence (a_n) is increasing.

(ii) if $a_1 > a_2$ and $\lim_{y\to 0} K(x,y) = 0$, then there exists a sequence (a_n) which is a K-sequence and, in such a case, the sequence (a_n) is decreasing.

Proof. (i) Let $a_1 < a_2$. We shall prove that there exists $y > 0$, such that $a_2 = K(a_1, y)$. Let us consider the function $f(y) = K(a_1, y)$ on the interval (a_1, ∞). From the statements of this theorem, we demonstrate that the function f is continuous on (a_1, ∞) and there holds

$$\lim_{y\to a_1} f(y) = f(a_1) = a_1 \text{ and } \lim_{y\to\infty} f(y) = \infty. \tag{1}$$

Since function f is continuous, f has the Darboux property on (a_1, ∞). Thus, (1) implies the existence of $a_3 \in (a_1, \infty)$ such that $f(a_3) = a_2$, i.e., $K(a_1, a_3) = a_2$. Since K is a strict mean, then $a_1 < a_2 < a_3$.

(ii) It can be shown similarly that if $a_1 > a_2$, then there exists $a_3 < a_2$ such that $K(a_1, a_3) = a_2$. In this case, let us consider the function $f(x) = K(a_1, y)$ defined on the interval $(0, a_1)$. So, we have constructed the term a_3 of the sequence (a_n).

Now, it is easy to construct the sequence (a_n) corresponding to (i) and (ii) by mathematical induction. □

Example 1. *It is easy to verify that* $\lim_{y\to\infty} L_p(x,y) = \infty$ *for any* $p \in \mathbb{R}$ *and if* $p \leq 0$, *then* $\lim_{y\to 0} L_p(x,y) = 0$. *Therefore, by the previous theorem:*
For any $p \in \mathbb{R}$ *and* $a_1 < a_2$ *positive numbers, there exists an* (a_n) *increasing* L_p*-sequence.*
For any $p \leq 0$ *and* $a_1 > a_2$ *positive numbers, there exists an* (a_n) *decreasing* L_p*-sequence.*

The following Theorems 2 and 3 and Corollary 1 generalize and refine, among other things, the results $(S2)$–$(S4)$, with the exception of $(S3)$-(ii).

Theorem 2. *Suppose that* $K : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ *is a mean and a sequence* (a_n) *of positive numbers is a K-sequence. Then, the following statements hold.*

(i) *If* $a_1 > a_2$ *and*
$$M_0(a,b) \leq K(a,b) \text{ for all } a,b \in \mathbb{R}^+,$$
then $\left(\frac{1}{a_n}\right) \in \mathcal{S}_\infty$ *(i.e.,* $\lim_{n\to\infty} a_n = 0$*) and* $\lambda\left(\left(\frac{1}{a_n}\right)\right) = 0$. *Moreover, for every* $\alpha > 0$, *we have*
$$\sum_{n=1}^{\infty} a_n^\alpha \leq \frac{a_1^{2\alpha}}{a_1^\alpha - a_2^\alpha}.$$
This is a refined and generalized (S2)-(i).

(ii) *If* $a_1 < a_2$ *and*
$$K(a,b) \leq M_0(a,b) \text{ for all } a,b \in \mathbb{R}^+,$$
then $(a_n) \in \mathcal{S}_\infty$ *(i.e.,* $\lim_{n\to\infty} a_n = \infty$*) and* $\lambda((a_n)) = 0$. *Moreover, for every* $\alpha > 0$, *we have*
$$\sum_{n=1}^{\infty} \frac{1}{a_n^\alpha} \leq \frac{\left(\frac{a_2}{a_1}\right)^\alpha}{a_2^\alpha - a_1^\alpha}.$$

(iii) *If* $a_1 < a_2$ *and for some increasing function* $f : \mathbb{R}^+ \to \mathbb{R}^+$ *there holds*
$$K(a,b) \leq M_f(a,b) \text{ for all } a,b \in \mathbb{R}^+,$$
then $(f(a_n)) \in \mathcal{S}_\infty$ *(i.e.,* $\lim_{n\to\infty} f(a_n) = \infty$*). Moreover,* $f(a_n) > n\left(\frac{f(a_2)-f(a_1)}{2}\right)$ *for every* $n \geq 2$ *and* $(f(a_n)) \in \mathcal{I}_{\leq 1}$.
This is a generalization of (S2)-(ii) and (S3)-(i) (in the case of $K = L$ *and* $f(x) = x^\alpha$, *where* $\alpha \geq \frac{1}{3}$*). In this case, function* f *cannot be decreasing.*

(iv) *If* $a_1 > a_2$ *and for some decreasing function* $f : \mathbb{R}^+ \to \mathbb{R}^+$ *there holds*
$$M_f(a,b) \leq K(a,b) \text{ for all } a,b \in \mathbb{R}^+,$$
then $(f(a_n)) \in \mathcal{S}_\infty$ *(i.e.,* $\lim_{n\to\infty} f(a_n) = \infty$*). Moreover,* $f(a_n) > n\left(\frac{f(a_2)-f(a_1)}{2}\right)$ *for every* $n \geq 2$ *and* $(f(a_n)) \in \mathcal{I}_{\leq 1}$.
This is a generalization of (S4)-(i) (in the case of $K = L$ *and* $f(x) = x^\alpha$, *where* $\alpha < 0$*). In this case, function* f *cannot be increasing.*

Proof. (i) According to the assumption, for every $n \geq 2$, we have
$$\sqrt{a_{n-1}a_{n+1}} \leq K(a_{n-1}, a_{n+1}) = a_n \text{ thus } \frac{a_{n+1}}{a_n} \leq \frac{a_n}{a_{n-1}}.$$

From this, for every $n \in \mathbb{N}$, we obtain

$$\frac{a_{n+1}}{a_n} \leq \frac{a_2}{a_1} = q < 1,$$

i.e., the sequence (a_n) is decreasing. Multiplying these inequalities for $n = 1, 2, \ldots, k$, we obtain

$$a_{k+1} = \frac{a_{k+1}}{a_k} \cdot \frac{a_k}{a_{k-1}} \cdots \frac{a_3}{a_2} \cdot \frac{a_2}{a_1} \cdot a_1 \leq a_1 \cdot q^k \text{ for all } k = 0, 1, 2, \ldots. \tag{2}$$

From (2), we have

$$\frac{1}{a_n} \geq \frac{1}{a_1} \left(\frac{a_1}{a_2}\right)^{n-1} \text{ for all } n = 1, 2, \ldots.$$

Since $\frac{a_1}{a_2} > 1$, then $\left(\frac{1}{a_n}\right) \in \mathcal{S}_\infty$ and

$$0 \leq \lambda\left(\left(\frac{1}{a_n}\right)\right) = \limsup_{n\to\infty} \frac{\log n}{\log \frac{1}{a_n}} \leq \limsup_{n\to\infty} \frac{\log n}{\log \frac{1}{a_1} + (n-1)\log \frac{a_1}{a_2}} = 0.$$

Then, from (2), for every $\alpha > 0$, we have

$$\sum_{n=1}^{\infty} a_n^\alpha = \sum_{k=0}^{\infty} a_{k+1}^\alpha \leq \sum_{k=0}^{\infty} a_1^\alpha \cdot (q^\alpha)^k = a_1^\alpha \cdot \frac{1}{1-q^\alpha} = \frac{a_1^{2\alpha}}{a_1^\alpha - a_2^\alpha}.$$

(ii) According to the assumption, for every $n \geq 2$, we have

$$\sqrt{a_{n-1}a_{n+1}} \geq K(a_{n-1}, a_{n+1}) = a_n \text{ thus } \frac{a_{n+1}}{a_n} \geq \frac{a_n}{a_{n-1}}.$$

From this, for every $n \in \mathbb{N}$, we obtain

$$\frac{a_{n+1}}{a_n} \geq \frac{a_2}{a_1} = q > 1,$$

i.e., the sequence (a_n) is increasing. Multiplying these inequalities for $n = 1, 2, \ldots, k$, we obtain

$$a_{k+1} = \frac{a_{k+1}}{a_k} \cdot \frac{a_k}{a_{k-1}} \cdots \frac{a_3}{a_2} \cdot \frac{a_2}{a_1} \cdot a_1 \geq a_1 \cdot q^k \text{ for all } k = 0, 1, 2, \ldots. \tag{3}$$

Then, from (3), we have $(a_n) \in \mathcal{S}_\infty$ ($\lim_{n\to\infty} a_n = \infty$) and

$$0 \leq \lambda((a_n)) = \limsup_{n\to\infty} \frac{\log n}{\log a_n} \leq \limsup_{n\to\infty} \frac{\log n}{\log a_1 + (n-1)\log q} = 0.$$

Thus, $\lambda((a_n)) = 0$ and for every $\alpha > 0$ we have

$$\sum_{n=1}^{\infty} \frac{1}{a_n^\alpha} = \sum_{k=0}^{\infty} \frac{1}{a_{k+1}^\alpha} \leq \sum_{k=0}^{\infty} \frac{1}{a_1^\alpha} \cdot \left(\frac{a_1}{a_2}\right)^{\alpha k} = \frac{1}{a_1^\alpha} \cdot \frac{1}{1 - \left(\frac{a_1}{a_2}\right)^\alpha} = \frac{\left(\frac{a_2}{a_1}\right)^\alpha}{a_2^\alpha - a_1^\alpha}.$$

(iii) Since (a_n) is a K-sequence, if $a_{n-1} < a_n$ holds for some $n \geq 2$, then $a_n \leq a_{n+1}$ necessarily holds. Thus, $a_{n-1} < a_{n+1}$ and

$$a_n = K(a_{n-1}, a_{n+1}) \leq M_f(a_{n-1}, a_{n+1}) < a_{n+1} \text{ so } a_{n-1} < a_n < a_{n+1},$$

since the mean M_f is a strict one. Then, from the condition $a_1 < a_2$, we obtain $a_2 < a_3$, etc., so the sequence (a_n) is increasing. According to the assumptions of this theorem, for every $n \geq 2$, we have

$$a_n = K(a_{n-1}, a_{n+1}) \leq M_f(a_{n-1}, a_{n+1}) = f^{-1}\left(\frac{f(a_{n-1}) + f(a_{n+1})}{2}\right).$$

Therefore, for a $f : \mathbb{R}^+ \to \mathbb{R}^+$ increasing function the sequence $(f(a_n))$ is increasing and for every $n \geq 2$ we have

$$2f(a_n) \leq f(a_{n-1}) + f(a_{n+1}) \text{ i.e., } f(a_n) - f(a_{n-1}) \leq f(a_{n+1}) - f(a_n).$$

Then, for every $n \in \mathbb{N}$, we have

$$f(a_{n+1}) = \sum_{j=1}^{n}(f(a_{j+1}) - f(a_j)) + f(a_1) > n.(f(a_2) - f(a_1)).$$

From this, for $n \geq 2$ we obtain $f(a_n) > (n-1) \cdot c \geq n \cdot \frac{c}{2}$, where $c = f(a_2) - f(a_1)$ is a positive constant. Thus, $(f(a_n)) \in \mathcal{S}_\infty$ and

$$\lambda((f(a_n))) = \limsup_{n\to\infty} \frac{\log n}{\log f(a_n)} \leq \limsup_{n\to\infty} \frac{\log n}{\log \frac{c}{2} + \log n} = 1.$$

On the other hand, if $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a decreasing function, then the sequence $(f(a_n))$ is decreasing and for every $n \geq 2$, we have

$$2f(a_n) \geq f(a_{n-1}) + f(a_{n+1}) \text{ i.e., } f(a_{n-1}) - f(a_n) \leq f(a_n) - f(a_{n+1}).$$

Then, for every $n \in \mathbb{N}$, we have

$$f(a_1) - f(a_{n+1}) = \sum_{j=1}^{n}(f(a_j) - f(a_{j+1})) \geq n.(f(a_1) - f(a_2)), \text{ i.e.,}$$

$$f(a_{n+1}) \leq f(a_1) - n \cdot (f(a_1) - f(a_2)).$$

Since, if $f(a_1) > f(a_2)$, then we obtain from the above inequality $\limsup_{n\to\infty} f(a_n) = -\infty$, which is a contradiction to the fact that f is a positive function.

(iv) Since (a_n) is a K-sequence, then if $a_{n-1} > a_n$ holds for some $n \geq 2$, then $a_n \geq a_{n+1}$ necessarily holds. Thus, $a_{n-1} > a_{n+1}$ and

$$a_n = K(a_{n-1}, a_{n+1}) \geq M_f(a_{n-1}, a_{n+1}) > a_{n+1} \text{ so } a_{n-1} > a_n > a_{n+1},$$

since the mean M_f is a strict one. From the condition $a_1 > a_2$, we obtain $a_2 > a_3$, etc., so the sequence (a_n) is decreasing. According to the assumptions of the theorem, for every $n \geq 2$, the mean Er,s(x, y) is symmetric in its parameters r and s and its variables x and y as well.

$$a_n = K(a_{n-1}, a_{n+1}) \geq M_f(a_{n-1}, a_{n+1}) = f^{-1}\left(\frac{f(a_{n-1}) + f(a_{n+1})}{2}\right).$$

Therefore, for a $f : \mathbb{R}^+ \to \mathbb{R}^+$ decreasing function, the sequence $(f(a_n))$ is increasing and for every $n \geq 2$, we have

$$2f(a_n) \leq f(a_{n-1}) + f(a_{n+1}) \text{ i.e., } f(a_n) - f(a_{n-1}) \leq f(a_{n+1}) - f(a_n).$$

From this similarly to (iii), we demonstrate that for $n \geq 2$ we have $f(a_n) > n.\frac{c}{2}$, where $c = f(a_2) - f(a_1)$ is a positive constant. Thus, $(f(a_n)) \in \mathcal{S}_\infty$ and $\lambda((f(a_n))) \leq 1$.

On the other hand, if $f : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing function, then the sequence $(f(a_n))$ is decreasing and for every $n \geq 2$, we have

$$2f(a_n) \geq f(a_{n-1}) + f(a_{n+1}) \text{ i.e., } f(a_{n-1}) - f(a_n) \leq f(a_n) - f(a_{n+1}).$$

Then, for every $n \in \mathbb{N}$, we have

$$f(a_1) - f(a_{n+1}) = \sum_{j=1}^{n} (f(a_j) - f(a_{j+1})) \geq n.(f(a_1) - f(a_2)) \text{ i.e.,}$$

$$f(a_{n+1}) \leq f(a_1) - n \cdot (f(a_1) - f(a_2)).$$

Since $f(a_1) > f(a_2)$, we obtain the inequality $\limsup_{n\to\infty} f(a_n) = -\infty$, which is a contradiction to the fact that f is a positive function. □

Example 2. *The conditions of (i) of Theorem 2 are satisfied for L_p means, where $p \in [-1, 0]$.*
The conditions of (ii) of Theorem 2 are satisfied for L_p means, where $p \leq -1$.

Corollary 1. *Suppose that $K : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a mean and a sequence (a_n) of positive numbers is a K-sequence. Then, the following statements hold.*

(i) If $a_1 < a_2$ and for some $\alpha > 0$ there holds

$$K(a, b) \leq M_\alpha(a, b) \text{ for all } a, b \in \mathbb{R}^+,$$

then $(a_n) \in \mathcal{S}_\infty$ ($\lim_{n\to\infty} a_n = \infty$) and

$$a_n > \left(\frac{a_2^\alpha - a_1^\alpha}{2}\right)^{\frac{1}{\alpha}} n^{\frac{1}{\alpha}} \text{ for every } \alpha > 0 \text{ and } n \geq 2.$$

Moreover, $(a_n) \in \mathcal{I}_{\leq\alpha}$.
This is a generalization and refinement of (S2)-(ii) and (S3)-(i) (in case $K = L$, $\alpha \geq \frac{1}{3}$).

(ii) If $a_1 > a_2$ and for some $\alpha < 0$ there holds

$$M_\alpha(a, b) \leq K(a, b) \text{ for all } a, b \in \mathbb{R}^+,$$

then $(a_n^\alpha) \in \mathcal{S}_\infty$ ($\lim_{n\to\infty} a_n = 0$) and

$$a_n < \left(\frac{a_2^\alpha - a_1^\alpha}{2}\right)^{\frac{1}{\alpha}} n^{\frac{1}{\alpha}} \text{ for every } \alpha < 0 \text{ and } n \geq 2.$$

Moreover, $(a_n^\alpha) \in \mathcal{I}_{\leq 1}$.
This is a generalization and refinement of (S4)-(i) (since in case $K = L$ and $\alpha < 0$ there holds $M_\alpha(a, b) \leq M_0(a, b) \leq K(a, b)$ for all $a, b \in \mathbb{R}^+$).

Proof. (i) Follows from Theorem 2 (iii) by choosing $f(x) = x^\alpha$, for $\alpha > 0$.
(ii) Follows from Theorem 2 (iv) by choosing $f(x) = x^\alpha$, for $\alpha < 0$. □

Theorem 3. *Suppose that $K : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a homogeneous mean such that $K(x, y)$ is continuous with respect to y and a sequence (a_n) of positive numbers is a K-sequence. Then, the following statements hold.*

(i) *If a sequence* (a_n) *is increasing and* $M_0(x,y) < K(x,y)$ *for all* $x, y \in \mathbb{R}^+$, $x \neq y$, *then*

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1 \text{ and } \lim_{n\to\infty} \frac{a_n}{q^n} = 0 \text{ for every real } q > 1.$$

This is a generalization of (S3)-(iii).

(ii) *If a sequence* (a_n) *is decreasing and* $M_0(x,y) < K(x,y)$ *for all* $x, y \in \mathbb{R}^+$, $x \neq y$, *then*

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0 \text{ and } \lim_{n\to\infty} \frac{a_n}{q^n} = 0 \text{ for every real } q > 0.$$

This is a generalization of (S4)-(ii).

(iii) *If a sequence* (a_n) *is increasing and* $M_0(x,y) > K(x,y)$ *for all* $x, y \in \mathbb{R}^+$, $x \neq y$, *then*

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \infty \text{ and } \sum_{n=1}^{\infty} a_n x^n \text{ is divergent for every real } x \neq 0.$$

(iv) *If sequence* (a_n) *is decreasing and* $M_0(x,y) > K(x,y)$ *for all* $x, y \in \mathbb{R}^+$, $x \neq y$, *then*

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1 \text{ and } \lim_{n\to\infty} \frac{a_n}{q^n} = 0 \text{ for every real } q > 1.$$

Proof. According to the assumptions, for $n \geq 2$, we obtain

$$a_n = K(a_{n-1}, a_{n+1}) = a_{n-1} K\left(1, \frac{a_{n+1}}{a_{n-1}}\right).$$

Therefore,

$$\frac{a_n}{a_{n-1}} = K\left(1, \frac{a_{n+1}}{a_n} \frac{a_n}{a_{n-1}}\right). \tag{4}$$

On the other hand, if $M_0(x,y) < K(x,y)$ for all $x, y \in \mathbb{R}^+$, $x \neq y$, from the assumption of the theorem, for every $n \geq 2$, we have

$$\sqrt{a_{n-1}a_{n+1}} < K(a_{n-1}, a_{n+1}) = a_n \text{ , thus } \frac{a_{n+1}}{a_n} < \frac{a_n}{a_{n-1}}.$$

Since the sequence $\left(\frac{a_{n+1}}{a_n}\right)$ is decreasing and $1 < \frac{a_{n+1}}{a_n}$ for all $n \geq 2$, if (a_n) is increasing, then there exists $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = x$, where $1 \leq x < \frac{a_2}{a_1}$. If the sequence (a_n) is decreasing, then obviously $\frac{a_{n+1}}{a_n} < 1$ for all $n \geq 2$, and there exists finite $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = x$, where $0 \leq x < \frac{a_2}{a_1} < 1$.

Since $K(x,y)$ is continuous with respect to y, then taking $n \to \infty$ in (4) gives us $x = K(1, x^2)$. In the case of $x \in (0,1) \cup (1,\infty)$, it cannot be true, because from the assumption of (i) and (ii) of this theorem, we have $K(1,x^2) > \sqrt{1 \cdot x^2} = x$. Thus, $x = 1$ if (a_n) is increasing and $x = 0$ if (a_n) is decreasing. Further, consider the power series

$$\sum_{n=1}^{\infty} a_n x^n.$$

(i) If (a_n) is an increasing sequence, then from above we have $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$, which implies that the radius of its convergence is $R = 1$. Thus, for every $0 < x < 1$, the series $\sum_{n=1}^{\infty} a_n x^n$ converges. Consequently, $\lim_{n\to\infty} a_n x^n = 0$. Denoting $q = \frac{1}{x}$, we have $\lim_{n\to\infty} \frac{a_n}{q^n} = 0$.

(ii) If (a_n) is a decreasing sequence, then from above we have $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0$, which implies that the radius of convergence R of the considered power series is infinity. Thus, for every real $x > 0$, we have $\lim_{n\to\infty} a_n x^n = 0$.

In the case of $M_0(x,y) > K(x,y)$ for all $x, y \in \mathbb{R}^+$, $x \neq y$, from the assumption of the theorem, for every $n \geq 2$ we have

$$\sqrt{a_{n-1}a_{n+1}} > K(a_{n-1}, a_{n+1}) = a_n \text{ , thus } \frac{a_{n+1}}{a_n} > \frac{a_n}{a_{n-1}}.$$

Since the sequence $\left(\frac{a_{n+1}}{a_n}\right)$ is increasing and $1 < \frac{a_{n+1}}{a_n}$ for all $n \in \mathbb{N}$, in the case when (a_n) is increasing there exists $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = x$, where $1 < \frac{a_2}{a_1} \leq x \leq \infty$. If the sequence (a_n) is decreasing, then obviously $\frac{a_{n+1}}{a_n} < 1$ for all $n \in \mathbb{N}$ and there exists $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = x$, where $0 < \frac{a_2}{a_1} \leq x \leq 1$. Then, taking $n \to \infty$ in (4), we obtain $x = K(1, x^2)$. This, in the case of $x \in (0,1) \cup (1,\infty)$ cannot be true, because from assumption of (iii), (iv) of the theorem we have $K(1,x^2) < \sqrt{1 \cdot x^2} = x$. Thus, $x = \infty$ if (a_n) is increasing and $x = 1$ if (a_n) is decreasing.

(iii) If (a_n) is an increasing sequence, then $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \infty$, which implies that the radius of convergence of the considered power series is $R = 0$. Thus, for every real $x \neq 0$, the series $\sum_{n=1}^{\infty} a_n x^n$ diverges.

(iv) If (a_n) is a decreasing sequence, then $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$. Further, we perform similarly to (i). □

Example 3. *The conditions of (i) of Theorem 3 are satisfied for L_p means, where $p > -1$.*
The conditions of (ii) of Theorem 3 are satisfied for L_p means, where $p \in (-1, 0]$.
The conditions of (iii) of Theorem 3 are satisfied for L_p means, where $p < -1$.
The conditions of (iv) of Theorem 3 are satisfied for L_p means, where $p < -1$.

In the context of (S3)-(ii), the following question arises.

Problem 1. *For which means K, there holds that if (a_n) is an increasing K-sequence, then there exist constants $c > 0$ and $\alpha > 0$ such that $a_{n+1} - a_n > cn^\alpha$ for every $n \in \mathbb{N}$?*

In the next part of this section, we will find necessary and sufficient conditions for sequences $(a_n) \in \mathcal{S}$ for which there is such a mean K that (a_n) will be a K-sequence.

Theorem 4. *For every increasing sequence (a_n) there exists a mean $K(x,y)$ defined on $\mathbb{R}^+ \times \mathbb{R}^+$, such that $\lim_{y\to\infty} K(x,y) = \infty$, $K(x,y)$ is continuous with respect to x and y, and the sequence (a_n) is a K-sequence.*

Proof. Let (a_n) be a given increasing sequence. So, there exists an unique $h \in (0, \infty]$ such that $(a_n) \in \mathcal{S}_h$, i.e., $\lim_{n\to\infty} a_n = h$. Therefore, $0 < a_1 < a_2 < a_3 < \cdots < h$.

Let $f : (0,h) \to \mathbb{R}^+$ be an increasing continuous function such that $f(a_n) = n$ for all $n \in \mathbb{N}$. Such a function can be, for example:

$$f(x) = \begin{cases} \frac{x-a_n}{a_{n+1}-a_n} + n & \text{if } x \in [a_n, a_{n+1}) \text{ and } n \in \mathbb{N}, \\ \frac{x}{a_1} & \text{if } x \in (0, a_1). \end{cases} \tag{5}$$

Now, we choose the mean K defined on $\mathbb{R}^+ \times \mathbb{R}^+$ by function f in the following way:

$$K(x,y) = \begin{cases} M_f(x,y) & \text{if } \max\{x,y\} < h, \\ \max\{x,y\} & \text{if } \max\{x,y\} \geq h. \end{cases}$$

Obviously, the mean $K(x,y)$ is a mean such that $\lim_{y\to\infty} K(x,y) = \infty$.

We show that $K(x,y)$ is continuous with respect to x and y. It is sufficient to prove that, both for x and for y, the proof is the same.

Let $y \in \mathbb{R}^+$ be arbitrary but fixed. Then, two cases are possible.

If $y \geq h$, then $K(x,y) = \max\{x,y\}$, which is continuous with respect to x.

If $y < h$, then

$$K(x,y) = \begin{cases} f^{-1}\left(\frac{f(x)+f(y)}{2}\right) & \text{if} \quad x < h\,, \\ x & \text{if} \quad x \geq h\,. \end{cases}$$

Since f is continuous on $(0,h)$, therefore f^{-1} is also continuous on the interval $(0,\infty)$. So $K(x,y)$ is continuous on the intervals $(0,h)$ and (h,∞).

On the other hand, $\lim\limits_{x\to h^-} \frac{f(x)+f(y)}{2} = \infty$ and $\lim\limits_{z\to\infty} f^{-1}(z) = h$, so that

$$\lim_{x\to h^-} K(x,y) = \lim_{x\to h^-} f^{-1}\left(\frac{f(x)+f(y)}{2}\right) = h = K(h,y)\,.$$

So, $K(x,y)$ is continuous at $x = h$.

Since $f(a_n) = n$ for all $n \in \mathbb{N}$, then for all $n \geq 2$, we have

$$K(a_{n-1},a_{n+1}) = M_f(a_{n-1},a_{n+1}) = f^{-1}\left(\frac{f(a_{n-1})+f(a_{n+1})}{2}\right) = \\ = f^{-1}\left(\frac{n-1+n+1}{2}\right) = f^{-1}(n) = a_n.$$

□

Further, we denote $\mathcal{S}'$ the system of all nondecreasing sequences (a_n) of positive real numbers, such that for every $n \geq 2$, the terms of sequence (a_n) satisfy the following condition.

$$\text{If } a_{n-1} = a_n < a_{n+1}, \text{ then } a_{n+1} < a_{n+2}. \tag{6}$$

Theorem 5. *The sequence $(a_n) \in \mathcal{S}$ is a K-sequence for some mean $K(x,y)$ defined on $\mathbb{R}^+ \times \mathbb{R}^+$ if and only if $(a_n) \in \mathcal{S}'$.*

Proof. "⇒" We assume that $(a_n) \in \mathcal{S}$ is a K-sequence for some mean $K(x,y)$ defined on $\mathbb{R}^+ \times \mathbb{R}^+$ and $(a_n) \notin \mathcal{S}'$. Thus, for every $n \geq 2$, we have $K(a_{n-1},a_{n+1}) = a_n$ and there exists an integer $k \geq 2$ such that $a_{k-1} = a_k < a_{k+1} = a_{k+2}$. Then $a_k = K(a_{k-1},a_{k+1}) = K(a_k,a_{k+1})$ and $a_{k+1} = K(a_k,a_{k+2}) = K(a_k,a_{k+1})$. Therefore, $a_k = a_{k+1}$, which is a contradiction.

"⇐" We assume that $(a_n) \in \mathcal{S}'$ is a given nondecreasing sequence. Hence, there exists an unique $h \in (0,\infty]$ such that $(a_n) \in \mathcal{S}_h$, i.e., $\lim_{n\to\infty} a_n = h$. We may assume that $0 < a_1 \leq a_2 \leq a_3 \leq \cdots \leq h$ and the sequence (a_n) satisfies the condition (6) for all $n \geq 2$. We choose a nondecreasing function $f : (0,h] \to \mathbb{R}^+$ such that we construct it by a method similar to (5) considering the condition (6), in the following way:

Let $f(a_1) = 1$ and for every $n \in \mathbb{N}$, we have

$$f(a_{n+1}) = \begin{cases} f(a_n) & \text{if } a_n = a_{n+1}, \\ f(a_n)+1 & \text{if } a_n < a_{n+1}. \end{cases}$$

Now, let $x \in (0,h)$ be such that $x \neq a_n$, $n = 1,2,3,\ldots$. Thus, $0 < x < a_1$ or $a_n < x < a_{n+1}$ for some $n \in \mathbb{N}$, where $a_n < h$. If $0 < x < a_1$, then let $f(x) = 1$. For

simplicity, let us denote $a_0 = 0$ and for every $n \in \mathbb{N}$ such that $a_n < x < a_{n+1}$ with condition $a_n < h$, we choose a function f on (a_n, a_{n+1}) in the following way:

$$f(x) = \begin{cases} f(a_n) & \text{if } a_{n-1} < a_n < x < a_{n+1}, \\ f(a_n) + \frac{1}{2} & \text{if } a_{n-1} = a_n < x < a_{n+1}. \end{cases}$$

By function f for $h \in (0, \infty]$, we define the following function $K(x, y)$ on $\mathbb{R}^+ \times \mathbb{R}^+$:

$$K(x,y) = \begin{cases} \max\Big\{\min\{x,y\}, \inf f^{-1}\Big(\Big[\frac{f(x)+f(y)}{2}, \infty\Big)\Big)\Big\} & \text{if } \max\{x,y\} < h, \\ \max\{x,y\} & \text{if } \max\{x,y\} \geq h. \end{cases}$$

We shall prove that the function $K(x, y)$ is a mean. If $\max\{x, y\} < h$, then

$$\min\{x,y\} \leq K(x,y) \leq \max\Big\{\max\{x,y\}, \inf f^{-1}([f(\max\{x,y\}), \infty))\Big\} \leq \max\{x,y\}.$$

If $\max\{x, y\} \geq h$, then trivially $\min\{x, y\} \leq K(x, y) \leq \max\{x, y\}$.

Further, we show that the sequence (a_n) is a K-sequence. Let $n \geq 2$. We have the following four subcases.

(i) If $a_{n-1} < a_n < a_{n+1}$, then $f(a_{n-1}) = k - 1$, $f(a_n) = k$, $f(a_{n+1}) = k + 1$, where $k \geq 2$ is an integer; obviously, we have $a_{n-2} < a_{n-1} < a_n < a_{n+1}$ or $a_{n-2} = a_{n-1} < a_n < a_{n+1}$. Therefore,

$$K(a_{n-1}, a_{n+1}) = \max\{a_{n-1}, \inf f^{-1}([k, \infty))\}\,.$$

Since $f(a_n) = k$, then $a_n \in f^{-1}([k, \infty))$. Thus $\inf f^{-1}([k, \infty)) \leq a_n$. On the other hand, if $x \in (a_{n-1}, a_n)$, then

$$f(x) = \begin{cases} f(a_{n-1}) = k - 1 & \text{if } a_{n-2} < a_{n-1} < x < a_n < a_{n+1}, \\ f(a_{n-1}) + \frac{1}{2} = k - \frac{1}{2} & \text{if } a_{n-2} = a_{n-1} < x < a_n < a_{n+1}. \end{cases}$$

Therefore, $x \notin f^{-1}([k, \infty))$, that is why $x < \inf f^{-1}([k, \infty)) \leq a_n$. From this as $x \to a_n$, we obtain $\inf f^{-1}([k, \infty)) = a_n$. Thus,

$$K(a_{n-1}, a_{n+1}) = \max\{a_{n-1}, a_n\} = a_n\,.$$

(ii) If $a_{n-1} = a_n = a_{n+1}$, then $f(a_{n-1}) = f(a_n) = f(a_{n+1}) = k$, where $k \in \mathbb{N}$. Therefore,

$$K(a_{n-1}, a_{n+1}) = \max\{a_{n-1}, \inf f^{-1}([k, \infty))\} = \max\{a_n, \inf f^{-1}([k, \infty))\}.$$

Since $f(a_n) = k$, then $a_n \in f^{-1}([k, \infty))$, i.e., $\inf f^{-1}([k, \infty)) \leq a_n$. Therefore,

$$K(a_{n-1}, a_{n+1}) = a_n.$$

(iii) If $a_{n-1} < a_n = a_{n+1}$, then $f(a_{n-1}) = k - 1$, $f(a_n) = f(a_{n+1}) = k$, where $k \geq 2$ is an integer and by (6) we have $a_{n-2} < a_{n-1} < a_n = a_{n+1}$. Thus

$$K(a_{n-1}, a_{n+1}) = \max\Big\{a_{n-1}, \inf f^{-1}\Big(\Big[k - \frac{1}{2}, \infty\Big)\Big)\Big\}.$$

Since $f(a_n) = k$, then $a_n \in f^{-1}\Big(\Big[k - \frac{1}{2}, \infty\Big)\Big)$. So, $\inf f^{-1}\Big(\Big[k - \frac{1}{2}, \infty\Big)\Big) \leq a_n$. On the other hand, if $x \in (a_{n-1}, a_n)$, then $f(x) = f(a_{n-1}) = k - 1$. There-

fore, $x \notin f^{-1}\left(\left[k-\frac{1}{2},\infty\right)\right)$, that is why $x \leq \inf f^{-1}\left(\left[k-\frac{1}{2},\infty\right)\right) \leq a_n$. From this, we obtain

$$\inf f^{-1}\left(\left[k-\frac{1}{2},\infty\right)\right) = a_n.$$

Therefore, $K(a_{n-1}, a_{n+1}) = \max\{a_{n-1}, a_n\} = a_n$.

(iv) If $a_{n-1} = a_n < a_{n+1}$, then $f(a_{n-1}) = f(a_n) = k$, $f(a_{n+1}) = k+1$, where $k \in \mathbb{N}$ and by (6) we have $a_{n-1} = a_n < a_{n+1} < a_{n+2}$. Hence,

$$K(a_{n-1}, a_{n+1}) = \max\left\{a_{n-1}, \inf f^{-1}\left(\left[k+\frac{1}{2},\infty\right)\right)\right\}.$$

Since $f(a_n) = k < k+\frac{1}{2}$, then $a_n \notin f^{-1}\left(\left[k+\frac{1}{2},\infty\right)\right)$. Thus,

$$a_n \leq \inf f^{-1}\left(\left[k+\frac{1}{2},\infty\right)\right).$$

On the other hand, if $x \in (a_n, a_{n+1})$ then $f(x) = f(a_n) + \frac{1}{2} = k+\frac{1}{2}$. Therefore, $x \in f^{-1}\left(\left[k+\frac{1}{2},\infty\right)\right)$, that is why $a_n \leq \inf f^{-1}\left(\left[k+\frac{1}{2},\infty\right)\right) \leq x$. From this, we obtain

$$\inf f^{-1}\left(\left[k+\frac{1}{2},\infty\right)\right) = a_n.$$

Therefore, $K(a_{n-1}, a_{n+1}) = \max\{a_{n-1}, a_n\} = a_n$.

□

Theorem 6. *The sequence (a_n) of positive numbers is a K-sequence for some mean $K(x,y)$ defined on $\mathbb{R}^+ \times \mathbb{R}^+$ if and only if the sequence $\left(\frac{1}{a_n}\right)$ is an M-sequence for some mean $M(x,y)$ defined on $\mathbb{R}^+ \times \mathbb{R}^+$.*

Proof. "⇒" Let sequence (a_n) be a *K*-sequence for some mean $K(x,y)$ defined on $\mathbb{R}^+ \times \mathbb{R}^+$, i.e., $a_n = K(a_{n-1}, a_{n+1})$ for every $n \geq 2$ integer. We choose a function M defined on $\mathbb{R}^+ \times \mathbb{R}^+$ in the following way:

$$M(x,y) = \frac{1}{K\left(\frac{1}{x},\frac{1}{y}\right)} \text{ for } x, y \in \mathbb{R}^+.$$

We shall prove that function M is a mean. Since K is a mean, then for every $x, y \in \mathbb{R}^+$, we have

$$\min\left\{\frac{1}{x},\frac{1}{y}\right\} \leq K\left(\frac{1}{x},\frac{1}{y}\right) \leq \max\left\{\frac{1}{x},\frac{1}{y}\right\}.$$

Therefore,

$$\min\{x,y\} \leq \frac{1}{K\left(\frac{1}{x},\frac{1}{y}\right)} \leq \max\{x,y\}.$$

Thus, the function M is a mean. Since sequence (a_n) is a *K*-sequence, then for every $n \geq 2$, we have

$$M\left(\frac{1}{a_{n-1}},\frac{1}{a_{n+1}}\right) = \frac{1}{K(a_{n-1}, a_{n+1})} = \frac{1}{a_n}.$$

Thus, sequence $\left(\frac{1}{a_n}\right)$ is an *M*-sequence.

"⇐" The reverse implication can be proved in a similar way. □

Remark 1. *The previous theorem essentially asserts*

$$K(x,y) \text{ is a mean} \iff M(x,y) = \frac{1}{K\left(\frac{1}{x},\frac{1}{y}\right)} \text{ is a mean}$$

hence

$$a_n \text{ is a K-sequence} \iff \frac{1}{a_n} \text{ is a M-sequence.}$$

This gives an alternative way of proving Theorems 1–3. Specifically, it is sufficient to prove only the statements for the increasing cases, and the statements for the decreasing cases can be proved as well, following the previous remark.

For example, the proof of the (iv) part of Theorem 2 would be modified as follows: Let $a_1 > a_2$ *and* $f : \mathbb{R}^+ \to \mathbb{R}^+$ *be a decreasing function such that*

$$M_f(a,b) \le K(a,b) \quad \text{for all} \quad a,b \in \mathbb{R}^+ .$$

Denote by $g : \mathbb{R}^+ \to \mathbb{R}^+$ *the function where*

$$g(x) = f\left(\frac{1}{x}\right).$$

The function g is obviously increasing. Furthermore, since

$$g\left(\frac{1}{f^{-1}(x)}\right) = f\left(f^{-1}(x)\right) = x\,,$$

therefore

$$g^{-1}(x) = \frac{1}{f^{-1}(x)}\,.$$

Denote $b_n = \frac{1}{a_n}$ *and*

$$M(x,y) = \frac{1}{K\left(\frac{1}{x},\frac{1}{y}\right)}\,.$$

Clearly, $b_1 < b_2$ *and*

$$M(a,b) = \frac{1}{K\left(\frac{1}{a},\frac{1}{b}\right)} \le \frac{1}{M_f\left(\frac{1}{a},\frac{1}{b}\right)} = \frac{1}{f^{-1}\left(\frac{f\left(\frac{1}{a}\right)+f\left(\frac{1}{b}\right)}{2}\right)} = g^{-1}\left(\frac{g(a)+g(b)}{2}\right) = M_g(a,b)\,.$$

That is, the series (b_n) *satisfies condition (iii) and* $f(a_n) = g(b_n)$*. So* $f(a_n) = g(b_n) \in \mathcal{S}_\infty$*. Furthermore*

$$f(a_n) = g(b_n) > n\left(\frac{g(b_2)-g(b_1)}{2}\right) = n\left(\frac{f(a_2)-f(a_1)}{2}\right).$$

Corollary 2. *For every decreasing sequence* (a_n) *of positive numbers, there exists a strict mean* $K(x,y)$ *defined on* $\mathbb{R}^+ \times \mathbb{R}^+$ *such that* $\lim_{x\to\infty} K(x,y) = 0$*,* $K(x,y)$ *is continuous with respect to x and y, and the sequence* (a_n) *is a K-sequence.*

Proof. This is a direct corollary of Theorems 4 and 6. □

We denote $\mathcal{T}$ the system of all nonincreasing sequences (a_n) of positive real numbers such that for every $n \ge 2$, the terms of sequence (a_n) satisfy the following condition.

$$\text{If } a_{n-1} = a_n > a_{n+1}, \text{ then } a_{n+1} > a_{n+2}. \tag{7}$$

Corollary 3. *A sequence $(a_n) \in \mathcal{T}$ is a K-sequence for some mean $K(x,y)$ defined on $\mathbb{R}^+ \times \mathbb{R}^+$ if and only if $(a_n) \in \mathcal{T}'$.*

Proof. This is a direct corollary of Theorems 5 and 6. □

Corollary 4. *A monotone sequence $(a_n) \in \mathcal{S} \cup \mathcal{T}$ of positive numbers is a K-sequence for some mean $K(x,y)$ defined on $\mathbb{R}^+ \times \mathbb{R}^+$ if and only if $(a_n) \in \mathcal{S}' \cup \mathcal{T}'$.*

Proof. This is a direct corollary of Theorem 5 and Corollary 3. □

5. Conclusions

In this paper, we investigated sequences of positive real numbers that can be constructed using means.

A condition for the mean K has been determined that is sufficient for the existence of a decreasing K-sequence whose first two terms are given. A similar condition was defined for the increasing case (see Theorem 1).

We have characterized the distribution of the K-sequences in terms of whether the sequences are increasing or decreasing and how the means K relates to the geometric mean (see Theorems 2 and 3).

It has been shown that any strictly monotone sequence can be produced using a certain mean. Specifically, for a given sequence, we defined a mean M_f that produces the sequence (see Theorem 4 and Corollary 2).

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