

Article

Multiplicities and Volumes of Filtrations

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Abstract: In this article, we survey some aspects of the theory of multiplicities of m_R -primary ideals in a local ring (R, m_R) and the extension of this theory to multiplicities of graded families of m_R -primary ideals. We first discuss the existence of multiplicities as a limit. Then, we focus on a theorem of Rees, characterizing when two m_R -primary ideals $I \subset J$ have the same multiplicity, and discuss extensions of this theorem to filtrations of m_R -primary ideals. In the final sections, we give outlines of the proof of existence of the multiplicity of a graded family of m_R -primary ideals as a limit, with mild conditions on R , and the proof of the extension of Rees' theorem to divisorial filtrations.

Keywords: multiplicity; graded family of ideals; divisorial filtration

MSC: 13H15; 13A18; 14C17

1. Introduction

These are the notes for a series of three lectures given at the workshop “Recent trends in Commutative Algebra”, at the Indian Institute of Technology Bombay, on 17–22 June 2024. The notes address the generalization of classical theorems for ideals and their Rees algebras to non-Noetherian filtrations. Other recent papers on non-Noetherian filtrations, with a variety of perspectives, include [1–5]. In these notes, we first review the notion of multiplicity and Rees' theorem on multiplicity of m_R -primary ideals. We then extend multiplicity to graded families of m_R -primary ideals and obtain extensions of Rees' theorem to graded families of m_R -primary ideals.

The multiplicity $e(m_R)$ of a local ring (R, m_R) is the most fundamental invariant of a local ring. This theory has classical origins, as a “local intersection product”. Multiplicity was given its modern formulation by Samuel in [6]. The multiplicity $e(R)$ of a local ring is a positive integer.

The importance of this invariant is seen by the fact that it characterizes non-singularity. If R is a regular local ring, then $e(R) = 1$, as shown by Serre in Appendix II [7], and if R is formally equidimensional, then $e(R) = 1$ if and only if R is a regular local ring, as shown by Nagata, Theorem 40.6 [8]. As such, the multiplicity is the most fundamental invariant in the resolution of singularities.

The multiplicity can be interpreted geometrically, as an intersection number on the blowup of the ideal I , as shown by Ramanujam in [9].

Let I be an m_R -primary ideal in a d -dimensional Noetherian local ring. The multiplicity $e(I)$ can be expressed as a limit:

$$e(I) = \lim_{n \rightarrow \infty} \frac{\lambda_R(R/I^n)}{n^d/d!}.$$

Here, λ_R is the length of an R -module.



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A fundamental result on multiplicity is the theorem of Rees [10] (stated in Theorem 3) showing that, if R is a formally equidimensional Noetherian local ring and $I \subset J$ are two m_R -primary ideals, then $e(I) = e(J)$ if and only if I and J have the same integral closure.

A graded family of ideals is a family of ideals $\mathcal{I} = \{I_n\}_{n \geq 0}$ where $I_m I_n \subset I_{m+n}$ for all m, n . We can define a multiplicity associated to a graded family of m_R -primary ideals $\mathcal{I} = \{I_n\}$ by

$$e(\mathcal{I}) = \limsup_{n \rightarrow \infty} \frac{\lambda_R(R/I_n)}{n^d/d!}. \quad (1)$$

This limsup is always finite.

This definition is a generalization of the multiplicity of an ideal, as an ideal I in a ring R induces a graded family of ideals $\mathcal{I} = \{I^n\}$ of powers of I , and the multiplicity $e(I)$ of the ideal I is equal to the multiplicity $e(\mathcal{I})$ of the filtration $\mathcal{I}$.

The limsup in (1) is always a limit under mild conditions, as stated in Theorem 5 of this survey. In fact, (1) is a limit for all graded filtrations $\mathcal{I}$ of m_R -primary ideals on a local ring R if and only if $\dim N(\hat{R}) < \dim R$, where $N(\hat{R})$ is the nilradical of the m_R -adic completion of R . If R is equidimensional, this condition is just that $\hat{R}$ is generically reduced. If $\hat{R}$ is not generically reduced, then the fact that the filtration is graded is not strong enough to ensure that the limit exists, as is illustrated in Example 5.3 [11]. In the positive direction, the existence of the limit is proven for analytically unramified local rings (the completion is a domain). This assumption is necessary for the proof to work. Then, a little commutative algebra gives the full statement that limits exist when $\dim N(\hat{R}) < d$.

An outline of the proof is given in Section 5. The proof uses the theory of Okounkov bodies as developed in [12–14]. In this section, we explain this method.

In Section 4, we discuss generalizations of Rees' theorem (Theorem 3) to graded families of m_R -primary ideals and give an outline of the proof of the generalization of this theorem to divisorial filtrations of m_R -primary ideals in Section 6.

Okounkov body techniques were first used in algebraic geometry in the computation of volumes of line bundles on projective varieties over an algebraically closed field. This was realized in [12,13]. In these works, especially in [13], many applications to algebraic geometry are given. In [13], the authors apply their theory to the case of local rings, and prove the existence of the multiplicity as a limit for a graded filtration of m_R -primary ideals in a local ring R , but with the restriction that R be the local ring of a closed point on an algebraic variety over an algebraically closed field.

In this article, we extend the use of Okounkov body methods to apply to local rings with the least possible assumptions, as well as extending some classical results of Rees to arbitrary filtrations or divisorial filtrations.

2. The Classical Theory of Multiplicity for Ideals

A comprehensive reference on this topic is the book [15] by Irena Swanson and Craig Huneke.

2.1. Integral Closure of Ideals

Definition 1. Let I be an ideal in a ring R . An element $x \in R$ is said to be integral over I if there exists an integer n and elements $a_i \in I^i$ for $i = 1, \dots, n$ such that

$$x^n + a_1 x^{n-1} + \dots + a_{n-1} x + a_n = 0.$$

The set of all elements of R which are integral over I is called the integral closure of I and is denoted by $\bar{I}$. If $I = \bar{I}$, then I is called integrally closed. If $I \subset J$ are ideals, we say that J is integral over I if $J \subset \bar{I}$.

Theorem 1 (Corollary 1.3.1 [15]). *The integral closure of an ideal in a ring R is an integrally closed ideal.*

The Rees algebra of an ideal I in R is the graded R -algebra $\oplus_{n \geq 0} I^n$. The integral closure of the Rees algebra $\oplus_{n \geq 0} I^n \cong \sum_{n \geq 0} I^n t^n$ in $R[t]$ is

$$\sum_{n \geq 0} \overline{I^n} t^n \cong \oplus_{n \geq 0} \overline{I^n}.$$

2.2. Multiplicity of an m_R -Primary Ideal

Let R be a Noetherian local ring of dimension d with maximal ideal m_R , and I be an m_R -primary ideal.

Theorem 2 (Hilbert, Samuel, Theorem 11.1.3 [15]). *There exists a polynomial $P(n)$ of degree d with rational coefficients such that for all $m \gg 0$,*

$$P(n) = P_I(n) = \lambda_R(R/I^n).$$

Here, λ_R is length as an R -module.

Multiplicity was first defined and investigated by Samuel.

Definition 2. *The multiplicity of I , denoted by $e(I)$, is $d!$ times the leading coefficient of $P(n)$; that is,*

$$e(I) = \lim_{n \rightarrow \infty} \frac{\lambda_R(R/I^n)}{n^d/d!}.$$

$e(I)$ is a non-negative integer (by Lemma 11.1.1 [15]).

Definition 3. *If A is a ring of finite Krull dimension, we say that A is equidimensional if $\dim A/P = \dim A$ for every minimal prime P of A . A Noetherian local ring A is formally equidimensional if its completion $\hat{A}$ with respect to the maximal ideal m_A is equidimensional.*

The following is a celebrated theorem of Rees.

Theorem 3 (Rees [10], Theorem 11.3.1 [15]). *Let R be a formally equidimensional Noetherian local ring and let $I \subset J$ be two m_R -primary ideals. Then, $\bar{I} = \bar{J}$ if and only if $e(I) = e(J)$.*

This condition is equivalent to the statement that the Rees algebra $\oplus_{n \geq 0} J^n$ is contained in $\oplus_{n \geq 0} \overline{I^n}$.

2.3. Valuations

We review material that can be found in Chapter 6 [15]. Let K be a field. A valuation v of K is a (surjective) map $v : K \setminus \{0\} \rightarrow vK$ where vK is a totally ordered abelian group such that $v(ab) = v(a) + v(b)$ and $v(a + b) \geq \min\{v(a), v(b)\}$ for $a, b \in K \setminus \{0\}$. By convention, $v(0) = \infty$, which is larger than anything in vK . The valuation ring of v is $\mathcal{O}_v = \{x \in K \mid v(x) \geq 0\}$, a not necessarily Noetherian local ring with maximal ideal $m_v = \{x \in K \mid v(x) > 0\}$. The residue field of $\mathcal{O}_v$ is $Kv = \mathcal{O}_v/m_v$. If R is a subring of $\mathcal{O}_v$, then we say that the center of v on R is the subring $m_v \cap R$. If R is a local ring contained in $\mathcal{O}_v$ such that $m_v \cap R = m_R$, we say that v dominates R . For $\tau \in vK$, we have valuation ideals in R defined by

$$I(v)_\tau = \{x \in R \mid v(x) \geq \tau\}.$$

Definition 4 (Definition 9.3.1 [15]). Let R be a Noetherian integral domain with field of fractions K and let v be a valuation of K such that $R \subset \mathcal{O}_v$. Let $P = m_v \cap R$ be the center of v on R . If $\text{trdeg}_{\text{QF}(R/P)} Kv = \text{ht}(P) - 1$, then v is said to be a divisorial valuation with respect to R .

Here, $\text{QF}(A)$ denotes the quotient field of a domain A . In excellent local rings, there is a more natural description of divisorial valuations.

Lemma 1 (Lemma 6.1 [16]). Suppose that R is an excellent local domain. Then, a valuation v of the quotient field K of R which is non-negative on R is a divisorial valuation of R if and only if $\mathcal{O}_v$ is essentially of finite type over R (a localization of a finitely generated R -algebra).

The valuation ring $\mathcal{O}_v$ of a divisorial valuation is Noetherian (Theorem 9.3.2 [15]). In fact, a valuation ring $\mathcal{O}_v$ is Noetherian if and only if the value group vK of v is isomorphic to $\mathbb{Z}$ (Proposition 6.3.4 [15]).

3. Graded Families of Ideals

Let (R, m_R) be a local Noetherian ring. A family of ideals $\mathcal{I} = \{I_n\}_{n \in \mathbb{N}}$ of R is a filtration of R if $R = I_0 \supset I_1 \subset I_2 \supset \cdots$. A graded family of ideals $\mathcal{I}$ of R is family $\mathcal{I} = \{I_n\}_{n \in \mathbb{N}}$ of ideals of R such that $I_m I_n \subset I_{m+n}$ for all m, n .

Example 1. Let $I \subset R$ be an ideal. Then, the I -adic filtration of R is $\mathcal{I} = \{I^n\}$. The filtration $\mathcal{I}$ is a graded filtration of R .

Example 2. Let R be a domain and v a divisorial valuation of R . Then, $\mathcal{I} = \{I(v)_n\}$ is graded filtration of R .

Example 3. Given divisorial valuations $v_1, \dots, v_r$ of R and $\lambda_1, \dots, \lambda_r \in \mathbb{R}_{\geq 0}$, $\mathcal{I} = \{I_n\}$ where

$$I_n = I(v_1)_{n\lambda_1} \cap \cdots \cap I(v_r)_{n\lambda_r}$$

is a graded filtration of R , which is called a divisorial filtration. If $\lambda_i \in \mathbb{Q}_{\geq 0}$ for all i , then $\mathcal{I}$ is a $\mathbb{Q}$ -divisorial filtration. If $\lambda_i \in \mathbb{Z}_{\geq 0}$ for all i , then $\mathcal{I}$ is called a $\mathbb{Z}$ -divisorial (or just divisorial) filtration.

Theorem 4 (Rees [17], Chapter 10 [15]). If R is a Noetherian local domain and $I \subset R$ is an ideal, then $\mathcal{I} = \{\overline{I^n}\}$, the filtration of integral closures of ordinary powers, is a divisorial filtration. There exist irredundant expressions

$$\overline{I^n} = I(v_1)_{na_1} \cap \cdots \cap I(v_r)_{na_r}$$

for some $a_1, \dots, a_r \in \mathbb{Z}_{>0}$ and all $n \geq 0$. The divisorial valuations $v_1, \dots, v_r$ are called the Rees valuations of I . Rees (Chapter 10 [15]) generalizes this to arbitrary (Noetherian) local rings.

Example 4. Let P be a prime ideal in a domain R . Then, the n -th symbolic power of P is $P^{(n)} = P^n R_P \cap R$. $\{P^{(n)}\}$ is a graded filtration of R ; it is a symbolic filtration. If R_P is regular, then $\{P^{(n)}\}$ is a divisorial filtration.

Geometric Interpretation of Divisorial Filtrations

Let J be an ideal in a ring R , and $X = \text{Proj}(\oplus_{n \geq 0} I^n)$. Then, there is a natural projective morphism $\pi : X \rightarrow \text{Spec}(R)$. The normalization $\overline{X}$ of X is the scheme $\text{Proj}(\oplus_{n \geq 0} \overline{I^n})$. Recall that $\oplus_{n \geq 0} \overline{I^n}$ is the integral closure of the Rees algebra $\oplus_{n \geq 0} I^n$. If $\oplus_{n \geq 0} \overline{I^n}$ is a finitely

generated R -algebra (which will be the case if R is universally Nagata), then the natural projection $\overline{X} \rightarrow \operatorname{Spec}(R)$ is projective.

Suppose that R is a normal and excellent local ring, $\pi : X \rightarrow \operatorname{Spec}(R)$ is the blowup of an ideal and X is normal with prime exceptional divisors $E_1, \dots, E_r$. Let $K = \operatorname{QF}(R)$ and let μ_i be the canonical valuation of K which has the valuation ring $\mathcal{O}_{X, E_i}$ for $1 \leq i \leq r$. That is, if t_i is a generator of the maximal ideal of the one-dimensional regular local ring $\mathcal{O}_{X, E_i}$, then $\mu_i(x) = n$ if $x \in K$ and $\frac{x}{t_i^n}$ is a unit in $\mathcal{O}_{X, E_i}$. Let $D = \sum_{i=1}^r a_i E_i$ with $a_i \in \mathbb{N}$ be an effective Weil divisor on X . Then, $\mathcal{I}(D) = \{I(nD)\}$ where

$$I(nD) = \Gamma(X, \mathcal{O}_X(-nD)) = I(\mu_1)_{a_1 n} \cap \dots \cap I(\mu_r)_{a_r n}$$

is a divisorial filtration of R .

Lemma 2 (Remark 6.6 to Lemma 6.5 [16]). *Let $\mathcal{I}$ be divisorial filtration on R . Then, there exists $\pi : X \rightarrow \operatorname{Spec}(R)$ and D on X as above such that $\mathcal{I} = \mathcal{I}(D)$.*

We will call $X, D, \mathcal{I}(D)$ a representation of $\mathcal{I}$. Even on a fixed X , there may be infinitely many different representations of a particular $\mathcal{I}$.

4. Multiplicities of Graded Families of m_R -Primary Ideals

A graded family of ideals $\mathcal{I} = \{I_n\}$ is m_R -primary if I_n is m_R -primary for $n > 0$. Let (R, m_R) be a Noetherian local ring of dimension d and $\mathcal{I} = \{I_n\}$ be a graded family of m_R -primary ideals. The multiplicity $e(\mathcal{I})$ of the family is

$$e(\mathcal{I}) = \limsup_{n \rightarrow \infty} \frac{\lambda_R(R/I_n)}{n^d/d!}.$$

This limsup is always finite. To see this, observe that I_1 m_R -primary implies there exists $c > 0$ such that $m_R^c \subset I_1$. Then, $I_1^n \subset I_n$ implies $m_R^{cn} \subset I_n$ for all n . Thus, $\lambda_R(R/I_n) \leq \lambda_R(R/m_R^{cn})$ for all n , and so the existence of the limsup follows from Theorem 2, as the multiplicity $e(m_R^c) < \infty$.

We remark that, if $\mathcal{I} = \{I^n\}$ is an I -adic filtration, then $e(\mathcal{I}) = e(I)$ is the ordinary multiplicity of the ideal I .

The following theorem characterizes the local rings for which this limsup is always a limit.

Theorem 5 (Theorem 1.1 [11]). *Suppose that R is a Noetherian local ring of dimension d and $N(\hat{R})$ is the nilradical of the m_R -adic completion $\hat{R}$ of R . Then, the limit*

$$\lim_{n \rightarrow \infty} \frac{\ell_R(R/I_n)}{n^d}$$

exists for any graded family $\{I_n\}$ of m_R -primary ideals of R if and only if $\dim N(\hat{R}) < d$.

The nilradical $N(R)$ of a d -dimensional ring R is

$$N(R) = \{x \in R \mid x^n = 0 \text{ for some positive integer } n\}.$$

$\dim N(R) = -1$ if $N(R) = 0$. If $N(R) \neq 0$, then $\dim N(R) = \dim R/\operatorname{ann}(N(R))$, so that $\dim N(R) = d$ if and only if there exists a minimal prime P of R such that $\dim R/P = d$ and R_P is not reduced. Thus, if R is equidimensional, $\dim N(R) < \dim R$ if and only if R is generically reduced. If R is excellent, then $N(\hat{R}) = N(R)\hat{R}$ so that $\dim N(R) = \dim N(\hat{R})$, and Theorem 5 is true with the condition $\dim N(\hat{R}) < d$ replaced with $\dim N(R) < d$.

There exist Noetherian local domains R (so that $N(R) = 0$) such that $\dim N(\hat{R}) = \dim R$ (Appendix to [8]).

Example 5. In general, $e(\mathcal{I})$ can be an irrational number. Let k be a field. For $\lambda \in \mathbb{R}_{>0}$, let I_n be the ideal in $R = k[[x, y]]$ generated by monomials $x^i y^j$ such that $\frac{1}{\lambda}i + j \geq n$ and $\mathcal{I} = \{I_n\}$. Then,

$$e(\mathcal{I}) = \lim_{n \rightarrow \infty} \frac{\lambda_R(R/I_n)}{n^2/2!} = \lim_{n \rightarrow \infty} \frac{\lceil n\lambda \rceil n}{n^2} = \lambda,$$

where $\lceil x \rceil$ is the round up of a real number x .

There exist examples of divisorial filtrations $\mathcal{I}$ of m_R -primary ideals such that $e(\mathcal{I})$ is irrational (Example 6 [18] and Equation (12) after Theorem 1.4 [19]).

Earlier, Lazarsfeld and Mustață [13] proved that the limit exists if R is a domain which is essentially of finite type over an algebraically closed field k , with $R/m_R = k$. All these assumptions are necessary in their proof. There are earlier results by Ein, Lazarsfeld, and Smith [20]. We will outline a proof of the sufficiency of the condition $\dim N(\hat{R}) < d$ in Theorem 5 in Section 5.

One direction of Theorem 3 extends to arbitrary graded families of ideals.

Theorem 6 (Theorem 6.9 [21], Theorem 1.4 [22]). Suppose that R is a Noetherian d -dimensional local rings such that $\dim N(\hat{R}) < d$. Suppose that $\mathcal{I} = \{I_n\}$ and $\mathcal{J} = \{J_n\}$ are graded filtrations of R with $\mathcal{I} \subset \mathcal{J}$ ($I_n \subset J_n$ for all n) and the Rees algebra $\bigoplus_{n \geq 0} J_n$ is integral over $\bigoplus_{n \geq 0} I_n$. Then, $e(\mathcal{I}) = e(\mathcal{J})$.

Example 6. The converse of Theorem 6 is false for general graded filtrations. Let k be a field and $R = k[[x_1, \dots, x_d]]$ be a power series ring. Let $J_n = m_R^n$ and $I_n = m_R^{n+1}$. Then, $e(\mathcal{I}) = e(\mathcal{J})$ but $\bigoplus_{n \geq 0} J_n$ is not integral over $\bigoplus_{n \geq 0} I_n$.

The converse of Theorem 6 for divisorial filtrations, and thus Rees' theorem extends to general divisorial filtrations. If $\mathcal{I} = \{I_n\}$ is a divisorial filtration, then $\bigoplus_{n \geq 0} I_n$ is integrally closed.

Theorem 7 (Theorem 3.5 [22]). Suppose that R is a d -dimensional excellent local domain. Suppose that $\mathcal{I} = \{I_n\} \subset \mathcal{J} = \{J_n\}$ are m_R -primary divisorial filtrations on R such that $e(\mathcal{I}) = e(\mathcal{J})$. Then, $I_n = J_n$ for all n .

Saturation of a Graded Family of m_R -Primary Ideals

This subsection recounts work of Harold Blum, Yuchen Liu, and Lu Qi in [23]. Let R be a d -dimensional analytically irreducible Noetherian local domain (meaning $\hat{R}$ is a domain). Let $I \subset R$ be an m_R -primary ideal. The integral closure $\bar{I}$ is equal to

$$\bar{I} = \{x \in m_R \mid v(x) \geq v(I) \text{ for all } v \in \text{DivVal}_{R, m_R}\} \quad (2)$$

where DivVal_{R, m_R} is the set of divisorial valuations of R with center m_R and

$$v(I) = \min\{v(x) \mid x \in I\}.$$

The minimum exists since R is Noetherian. A proof of Equation (2) is given in Section 6.8 [15].

Definition 5. The saturation $\tilde{\mathcal{I}} = \{\tilde{I}_n\}$ of a graded family of m_R -primary ideals is defined by

$$\tilde{I}_n = \{x \in m_R \mid v(x) \geq nv(\mathcal{I}) \text{ for all } v \in \text{DivVal}_{R, m_R}\},$$

where $v(\mathcal{I}) = \lim_{m \rightarrow \infty} \frac{v(I_m)}{m}$.

$\mathcal{I}$ is said to be saturated if $\mathcal{I} = \tilde{\mathcal{I}}$.

Theorem 8 (Theorem 1.4 [23]). Let R be an analytically irreducible local domain and $\mathcal{I} \subset \mathcal{J}$ be graded m_R -filtrations. Then, $e(\mathcal{I}) = e(\mathcal{J})$ if and only if $\tilde{\mathcal{I}} = \tilde{\mathcal{J}}$.

Question 1. How should we understand the saturation $\tilde{\mathcal{I}}$ of $\mathcal{I}$?

There is a nice answer to this question when $\mathcal{I} = \{I^n\}$ is an I -adic filtration. In this case, $\tilde{\mathcal{I}} = \{\overline{I^n}\}$. However, the saturation can be larger than the integral closure. Recall Example 6, where $\mathcal{I} = \{I_n = m_R^{n+1}\} \subset \mathcal{J} = \{J_n = m_R^n\}$. Then, $e(\mathcal{I}) = e(\mathcal{J})$ but $\bigoplus_{n \geq 0} J_n$ is not integral over $\bigoplus_{n \geq 0} I_n$. In this example, we have that $\tilde{\mathcal{I}} = \tilde{\mathcal{J}} = \mathcal{J}$.

5. Overview of the Proof of Sufficiency in Theorem 5

Suppose that R is a Noetherian local ring of dimension d such that $\dim N(\hat{R}) < d$ and $\mathcal{I} = \{I_n\}$ is a graded family of m_R -primary ideals. We will outline a hybrid proof that $e(\mathcal{I})$ exists as a limit, incorporating material from [11,24,25]. We will prove that

$$e(\mathcal{I}) = \lim_{n \rightarrow \infty} \frac{\lambda_R(R/I_n)}{n^d/d!}. \quad (3)$$

We can reduce to the case where R is a complete local domain, so we can assume that R is excellent and analytically irreducible.

5.1. Semigroups and Cones

Suppose that $S \subset \mathbb{N}^{d+1}$ is a semigroup. Let $\Sigma = \Sigma(S)$ be the convex cone in $\mathbb{R}^{d+1}$ which is the closure of the set of all linear combinations $\sum \lambda_i s_i$ with $\lambda_i \in \mathbb{R}_{\geq 0}$ and $s_i \in S$. Set

$$\Delta = \Delta(S) = \Sigma \cap (\mathbb{R}^d \times \{1\}).$$

For $m \in \mathbb{N}$, let $S_m = S \cap (\mathbb{N}^d \times \{m\})$, which can be viewed as a subset of $\mathbb{N}^d$ (by projecting onto the first d components). Let $G(S)$ be the subgroup of $\mathbb{Z}^{d+1}$ generated by S .

Theorem 9 (Cone Theorem). Lazarsfeld and Mustață [13], Kaveh and Khovanskii [12]. Suppose that a subsemigroup S of $\mathbb{Z}^d \times \mathbb{N}$ satisfies the following conditions:

- (C1) There exist finitely many vectors $(v_i, 1)$ spanning a semigroup $B \subset \mathbb{N}^{d+1}$ such that $S \subset B$
- (C2) $G(S) = \mathbb{Z}^{d+1}$

Then,

$$\lim_{n \rightarrow \infty} \frac{\#S_n}{n^d} = \text{vol}(\Delta(S)).$$

Here, $\#T$ is the cardinality of a set T .

5.2. Proof of the Existence of the Limit (3)

A proof of the following theorem using only commutative algebra is given in Lemma 4.2 [21]. A short geometric proof is given in [11]. It may be that S/m_S is not separable for a particular R and any regular local ring S satisfying the conclusions of the theorem.

Theorem 10. *There exists a regular local ring S such that S is essentially of finite type over R and $QF(S) = QF(R)$.*

Let $y_1, \dots, y_d$ be a regular system of parameters in S . Let $\lambda_1, \dots, \lambda_d$ be rationally independent real numbers with $\lambda_i \geq 1$ for all i . Define a valuation v on $K := QF(R)$ which dominates S by prescribing

$$v(y_1^{a_1} \cdots y_d^{a_d}) = a_1 \lambda_1 + \cdots + a_d \lambda_d$$

for $a_1, \dots, a_d \in \mathbb{Z}_{\geq 0}$. The value group vK is the ordered subgroup $\mathbb{Z}\lambda_1 + \cdots + \mathbb{Z}\lambda_d$ of $\mathbb{R}$ and the residue field of $\mathcal{O}_v$ is S/m_S . Let $k = R/m_R$ and $k' = S/m_S = \mathcal{O}_v/m_v = Kv$. Since S is essentially of finite type over R , we have that $[k' : k] < \infty$.

For $\lambda \in \mathbb{R}$, define

$$K_\lambda = \{x \in K \mid v(x) \geq \lambda\} \text{ and } K_\lambda^+ = \{x \in K \mid v(x) > \lambda\}.$$

Let $\mathcal{A} = \{A_n\}$ be a graded family of ideals of R (not necessarily m_R -primary). For $\beta \in \mathbb{Z}_{>0}$, define

$$\Gamma_\beta(\mathcal{A})^{(t)} = \{(n_1, \dots, n_d, i) \in \mathbb{N}^{d+1} \mid \dim_k A_i \cap K_{n_1\lambda_1 + \dots + n_d\lambda_d} / A_i \cap K_{n_1\lambda_1 + \dots + n_d\lambda_d}^+ \geq t \text{ and } n_1 + \dots + n_d \leq \beta i\}.$$

Here, $\dim_k$ denotes k -vector space dimension.

Define $\Gamma_\beta(\mathcal{A}) = \Gamma_\beta(\mathcal{A})^{(1)}$. Let $\lambda = n_1\lambda_1 + \cdots + n_d\lambda_d$ be such that $n_1 + \cdots + n_d \leq \beta i$. Then,

$$\dim_k K_\lambda \cap A_i / K_\lambda^+ \cap A_i = \#\{t \mid (n_1, \dots, n_d, i) \in \Gamma_\beta(\mathcal{A})^{(t)}\}. \quad (4)$$

We necessarily have that $\Gamma_\beta(\mathcal{A})^{(t)} = \emptyset$ if $t > [k' : k]$.

Lemma 3. *For $1 \leq i \leq [k' : k]$ and β sufficiently large, $\Gamma_\beta(\mathcal{A})^{(i)}$ satisfies (C1) and (C2) of the Cone Theorem and $\Delta(\Gamma_\beta(\mathcal{A})^{(i)}) = \Delta(\Gamma_\beta(\mathcal{A}))$.*

Lemma 3 is proved in the course of the proof of Theorem 3.11 [25].

Theorem 11. *For sufficiently large β ,*

$$\lim_{n \rightarrow \infty} \frac{\lambda_R(A_n / A_n \cap K_{\beta n})}{n^d} = [k' : k] \text{vol}(\Delta(\Gamma_\beta(\mathcal{A}))).$$

Proof. This follows from (4), Lemma 3 and the Cone Theorem. $\square$

Theorem 12. *There exists $c \in \mathbb{Z}_{>0}$ such that*

$$m_R^c \subset I_1 \quad (5)$$

and there exists $\alpha \in \mathbb{Z}_{>0}$ such that

$$K_{\alpha n} \cap R \subset m_R^n \quad (6)$$

for all n .

The proof of (6) is by Lemma 4.3 [26].

For $\beta \geq c\alpha$, $K_{\beta i} \cap R \subset K_{\alpha ci} \cap R \subset m_R^{ic} \subset I_i$ by (5) and (6). Thus,

$$\lambda_R(R/I_n) = \lambda_R(R/K_{\beta n}) - \lambda_R(I_n/I_n \cap K_{\beta n})$$

for all n . Let $\mathcal{F}$ be the filtration $\mathcal{F} = \{F_i\}$ where $F_i = R$ for all i . Let β be sufficiently large that $\beta \geq c\alpha$ and the conclusions of Theorem 11 hold for $\mathcal{I}$ and $\mathcal{F}$. Then,

$$\lim_{n \rightarrow \infty} \frac{\lambda_R(R/I_n)}{n^d} = [k' : k] (\text{vol}(\Delta(\Gamma_\beta(\mathcal{F})) - \text{vol}(\Delta(\Gamma_\beta(\mathcal{I}))),$$

establishing sufficiency in Theorem 5.

Remark 1. The above proof can be modified to work for any valuation v of $QF(R)$ of maximal rank equal to the dimension of the domain R which dominates m_R ($R \subset \mathcal{O}_v$ and $m_v \cap R = m_R$ where $\mathcal{O}_v$ is the valuation ring of v and the maximal ideal m_v of $\mathcal{O}_v$ satisfies $m_v \cap R = m_R$) as long as local uniformization is true for R along v . That is, there must exist a regular local ring S such that $QF(R) = QF(S)$ which is essentially of finite type over R and dominates R and such that S is dominated by v . Local uniformization is known to hold in many cases for such Abhyankar valuations [27], and holds in particular whenever R is essentially of finite type over an arbitrary field [28].

Although the volume of the associated Okounkov body will be the same for all choices of such a valuation v , the actual relationship between the Okounkov bodies themselves is very subtle.

Example 7. In this example, we apply the algorithm of the proof of Theorem 5 to Example 5, illustrating the use of the Okounkov body method in the simplest possible case. The more subtle elements of the proof of Theorem 5 do not appear in this example.

With the notation of the proof of Theorem 5, let $R = k[[x, y]]$ and $\mathcal{I} = \{I_n\}$ where I_n is the ideal in R generated by the monomials $x^i y^j$ such that $\frac{1}{\lambda}i + j \geq n$. In the notation of the proof, we can take $S = R$ since R is already regular, and take our regular system of parameters to be $y_1 = x$ and $y_2 = y$. Let us take the valuation v defined by $\lambda_1 = 1$ and $\lambda_2 = \pi$, so that $v(\sum a_{ij}x^i y^j) = \min\{i + j\pi \mid a_{ij} \neq 0\}$. Now, let β be chosen so that the line $n_1 + n_2 = \beta$ is above the line $\frac{1}{\lambda}n_1 + n_2 = 1$ in $\mathbb{N}^2$. Then, with the notation of the above proof,

$$\Gamma_\beta(\mathcal{I}) = \{(n_1, n_2, i) \in \mathbb{N}^3 \mid \frac{1}{\lambda}n_1 + n_2 \geq i \text{ and } n_1 + n_2 \leq \beta i\}$$

and

$$\Gamma_\beta(\mathcal{F}) = \{(n_1, n_2, i) \in \mathbb{N}^3 \mid n_1 + n_2 \leq \beta i\}$$

so that the Okounkov body $\Delta(\Gamma_\beta(\mathcal{I}))$ is the region between the line $\frac{1}{\lambda}n_1 + n_2 = 1$ and the line $n_1 + n_2 = \beta$ in the first quadrant of $\mathbb{R}^2$ and $\Delta(\Gamma_\beta(\mathcal{F}))$ is the region below the line $n_1 + n_2 = \beta$ in $\mathbb{R}^2$. Thus,

$$\lim_{n \rightarrow \infty} \frac{\lambda_R(R/I_n)}{n^2} = \text{vol}(\Delta(\Gamma_\beta(\mathcal{F})) - \text{vol}(\Delta(\Gamma_\beta(\mathcal{I})))$$

is the volume of the region below the line $\frac{1}{\lambda}n_1 + n_2 = 1$ in the first quadrant of $\mathbb{R}^2$, which is $\frac{1}{2}\lambda$, from which we see that

$$e(\mathcal{I}) = 2 \lim_{n \rightarrow \infty} \frac{\lambda_R(R/I_n)}{n^2} = \lambda,$$

in agreement with Example 5.

Here, we give a geometric interpretation of the volume of Theorem 5 when $\mathcal{I}$ is a divisorial filtration on a d -dimensional normal algebraic local ring (R is essentially of finite type over a field). Let $X, D, \mathcal{I}(D)$ be a representation of $\mathcal{I}$ (as explained in

Section 3). In Section 8 [22], an anti-positive intersection product is defined. It follows from Theorem 8.3 [22] or Equation (9) of [19], that

$$\lim_{n \rightarrow \infty} \frac{\lambda_R(R/I_n)}{n^d} = -\frac{\langle (-D)^d \rangle}{d!} \quad (7)$$

where $\langle (-D)^d \rangle$ is the self intersection product of $-D$. This product is computed on the Zariski Riemann manifold of $\text{Spec}(R)$. In the case that $-D$ is nef, then this is just the ordinary self intersection product $((-D)^d)$ of $-D$ on X . In particular, if there exists a representation of $\mathcal{I}$ where $-D$ is nef, then the limit of (7) can be calculated using ordinary intersection theory.

6. Outline of the Proof of Rees' Theorem for Divisorial Filtrations

In this section, we give an outline of the proof of Theorem 7 (Theorem 3.5 [22]).

We can reduce to the case where R is normal and excellent. There exists $\varphi : X \rightarrow \text{Spec}(R)$ where X is the blowup of an m_R -primary ideal and is normal with prime exceptional divisors $E_1, \dots, E_r$ such that $\mathcal{I}$ and $\mathcal{J}$ are represented on X . Letting μ_i be the canonical valuation of $\mathcal{O}_{X,E_i}$, there exist effective Cartier divisors $D_1 = \sum a_i E_i$ and $D_2 = \sum b_i E_i$ such that $a_i, b_i \in \mathbb{N}$ with $a_i \geq b_i$ for all i and $\mathcal{I} = \mathcal{I}(D_1)$ and $\mathcal{J} = \mathcal{I}(D_2)$.

Now, let $D = \sum c_i E_i$ with $c_i \in \mathbb{N}$ be an effective Cartier divisor on X . For $1 \leq i \leq r$ and $m \in \mathbb{N}$, let

$$\tau_{m,i} = \tau_{E_i,m}(D) = \min\{\mu_i(x) \mid x \in \Gamma(X, \mathcal{O}_X(-mD))\}.$$

We have $\tau_{mn,i} \leq n\tau_{m,i}$ for all i, m, n . Let

$$\gamma_{E_i}(D) = \inf_m \frac{\tau_{m,i}}{m} \in \mathbb{R}_{\geq 0}.$$

We have $\gamma_{E_i}(D) \geq c_i$ for all i and

$$\Gamma(X, \mathcal{O}_X(-mD)) = \Gamma(X, \mathcal{O}_X(-[\sum_{i=1}^r \gamma_{E_i}(D)E_i])) \quad (8)$$

for all $m \geq 0$. For $1 \leq i \leq r$, there exists a valuation v_i of $K = \text{QF}(R)$ with value group $v_i K = \mathbb{Z}^d$ with the lexicographic order such that $v_i(x) = (\mu_i(f), \dots)$ for $x \in K$. For $\delta \in \mathbb{Z}_{>0}$, let

$$\Gamma_\delta^i(D) = \{(v_i(x), n) \mid x \in I_n \text{ and } \mu_i(x) \leq n\delta\} \subset \mathbb{N}^{d+1}.$$

For $\delta \gg 0$, $\Gamma_\delta(D)$ satisfies (C1) and (C2) of the Cone Theorem so that

$$\lim_{n \rightarrow \infty} \frac{\#\Gamma_\delta^i(D)_n}{n^d} = \text{vol}(\Delta_\delta^i(D)) \quad (9)$$

where $\Delta_\delta^i(D) = \Delta(\Gamma_\delta^i(D))$. In fact, for $1 \leq i \leq r$ and $\delta \gg 0$, we have by Equation (17) [22] that

$$\lim_{n \rightarrow \infty} \frac{\lambda_R(R/I(D)_n)}{n^d} = [Kv_i : R/m_R] \left(\text{vol}(\Delta_\delta^i(0)) - \text{vol}(\Delta_\delta^i(D)) \right) \quad (10)$$

where 0 denotes the trivial divisor $0 = 0E_1 + \dots + 0E_r$.

Lemma 4 (Lemma 3.2 [22]). Suppose that Δ_1 and Δ_2 are compact convex subsets of $\mathbb{R}^d$, $\Delta_1 \subset \Delta_2$ and $\text{vol}(\Delta_1) = \text{vol}(\Delta_2) > 0$. Then $\Delta_1 = \Delta_2$.

We have reduced the proof of Theorem 7 to the following.

Theorem 13. (Rees' theorem for divisorial filtrations on an excellent normal local domain) with the above notation, suppose that $e(\mathcal{I}(D_1)) = e(\mathcal{I}(D_2))$. Then, $I(nD_1) = I(nD_2)$ for all $n \in \mathbb{N}$.

Proof. Let $\pi_1 : \mathbb{R}^{d+1} \rightarrow \mathbb{R}$ be projection onto the first factor. Since $\Delta_\delta^i(D_j)$ is compact for all i and $j = 1, 2$, we have that

$$\pi_1^{-1}(\gamma_{E_i}(D_j)) \cap \Delta_\delta^i(D_j) \neq \emptyset$$

and

$$\pi_1^{-1}(a) \cap \Delta_\delta^i(D_j) = \emptyset$$

if $a < \gamma_{E_i}(D_j)$. Now $D_2 \leq D_1$ implies $\Delta_\delta^i(D_1) \subset \Delta_\delta^i(D_2)$ for $1 \leq i \leq r$. Since $\text{vol}(\Delta_\delta^i(D_2)) = \text{vol}(\Delta_\delta^i(D_1)) > 0$ by (9), we have that $\Delta_\delta^i(D_1) = \Delta_\delta^i(D_2)$ by Lemma 4. Thus, $\gamma_{E_i}(D_1) = \gamma_{E_i}(D_2)$ for $1 \leq i \leq r$ and so

$$\sum_{i=1}^r \gamma_{E_i}(D_2) E_i = \sum_{i=1}^r \gamma_{E_i}(D_1) E_i$$

which implies that $I(nD_1) = I(nD_2)$ for all $n \in \mathbb{N}$ by (8). $\square$

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