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Increasing Stability in the Inverse Source Problem with an Interval (K_1, K_2) of Frequencies

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Abstract: In this paper, we study the increasing stability in the inverse source problem with an interval (K_1, K_2) of frequencies. Our results show that increasing stability can be obtained with larger wavenumber intervals. The stability estimate consists of the Lipschitz type data discrepancy and the frequency tail of the source function, where the latter decreases as the frequency K_2 increases or K_1 decreases. The method is based on the Fourier transform and explicit bounds for analytic continuation.

Keywords: uniqueness; inverse scattering problem; far-field pattern; penetrable obstacle; transmission problem; interior transmission problem; embedded obstacles

MSC: 78A46; 65C30

1. Introduction

1.1. Statement of the Problem

Let us consider the following n-dimensional Helmholtz equation:

$$\Delta u(x) + k^2 u(x) = f(x)g(k), \ \ x \in \mathbb{R}^n, \tag{1}$$

where $n \geq 2$ is an integer, $k > 0$ is the wavenumber, u is the radiated wave field, and the source f stands for the electric current density. We assume that the supp (f) is contained in the ball B_R defined by

$$B_R := \{x \in \mathbb{R}^n : \ |x| < R\}$$

for some $R > 0$. There exists a constant $\delta > 0$ satisfying

$$|g(k)| \geq \delta > 0 \quad \text{for all} \quad k \in (K_1, K_2) \tag{2}$$

without loss of generality; we assume $K_1 > 0$ and $K_2 > 0$.

The following Sommerfeld radiation condition is required to ensure the uniqueness of the wave field u:

$$r^{\frac{n-1}{2}}(\partial_r u - iku) = 0, \ \ r := |x| \to +\infty, \tag{3}$$

uniformly in all directions $\hat{x} = x/|x|$.

Now, we are in the position to discuss our inverse source problem:

IP. Assume ∂B_R is the boundary of B_R. Let f be a complex function with a compact support contained in B_R and g satisfy (2). The inverse problem is to determine f by using the boundary observation data $u(x,k)|_{\partial B_R}$ with an interval of frequencies $k \in (K_1, K_2)$ where $0 < K_1 < 1$ and $K_2 > 1$ are positive constants.

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1.2. Motivations

Let us consider the radiation of acoustic waves from a time-varying source term $F(x)G(t)$, embedded in an infinite and homogeneous medium. The real-valued radiated field is governed by

$$\partial_t^2 U(x,t) - \Delta U(x,t) = F(x)G(t), \quad x \in \mathbb{R}^3, \ t > 0, \tag{4}$$

together with the initial conditions

$$U(x,0) = \partial_t U(x,0) = 0, \quad x \in \mathbb{R}^3. \tag{5}$$

We suppose that F is compactly supported in the space region B_{R_0} and the source radiates only over a finite time period $[0, t_0]$ for some $t_0 > 0$. This implies that $G(t) = 0$ for $t \geq t_0$.

The well-known representation for U to the wave equation (Equation (4)) gives

$$U(x,t) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{f(y)g(t-|x-y|)}{|y-x|} dy. \tag{6}$$

Huygens' principle means that $U(x,t) = 0$ for $x \in B_{R_0}$, $t \geq t_0$.

We denote by $\widehat{G}$ the Fourier transform of G with respect to t; that is,

$$\widehat{G}(k) = \int_{\mathbb{R}} G(t) e^{-ikt} dt. \tag{7}$$

Then, we find that $\hat{U}(x,k) = \int_{\mathbb{R}^3} \frac{e^{i|x-y|}}{4\pi|x-y|} F(y)\hat{G}(k)dy$, satisfying the Helmholtz equation

$$\Delta \widehat{U}(x,k) + k^2 \widehat{U}(x,k) = F(x)\hat{G}(k) \tag{8}$$

and the Sommerfeld radiation condition

$$r(\partial_r \widehat{U} - ik\widehat{U}) = 0, \ r := |x| \to +\infty. \tag{9}$$

Therefore, (1)–(3) is an important mathematical model well worth studying.

1.3. Known Results

The inverse source problems in waves arise in many scientific and industrial areas such as antenna design and synthesis [1], biomedical imaging [2], and photo-acoustic tomography [3,4].

Motivated by these significant applications, the inverse source problems, as an important research subject in inverse scattering theory, have continuously attracted the attention of many researchers [5–17]. Consequently, a great deal of mathematical and numerical results are available, especially for the acoustic waves or the Helmholtz equations. We state in brief some of the existing results that are relevant to the problem under investigation in this paper. It is well known that there is an obstruction to uniqueness for inverse source problems for Helmholtz equations with single-frequency data. This is clear since a single near-field or far-field measurement gives a function of $n-1$ independent variables in an n-dimensional space, while the source function has n independent variables. For the convenience of the reader, we refer to [18] (Chapter 4) and [8]. However, by considering multi-frequency measurements, the uniqueness can be proved. For this, one can see, for example, the recent works [7,19] in which uniqueness and stability results have been proved for the recovery of the source term from the knowledge of multi-frequency boundary measurements. In [9], the authors treated an interior inverse source problem for the Helmholtz equation from boundary Cauchy data for multiple wavenumbers, and they showed an

increasing stability result with growing K for the problem under consideration. Interested readers can also see [19], which claims a uniqueness result and a numerical algorithm for recovering the location and the shape of a supported acoustic source function from boundary measurements at many frequencies. See also [5,10–13,20] and the references therein. As for increasing stability results proved for coefficients inverse problems, we can refer the reader, for example, to [21,22], in which inverse problems of recovering an electric potential appearing in a Schrödinger equation are studied (see also the references therein). In [6], the increasing stability for the one-dimensional inverse medium problem of recovering the refractive index is investigated. The inverse source problem is to determine f from the boundary measurements $u(x,k)|_{\partial B_R}$ corresponding to the wavenumber k given in a finite interval $(0,K)$. The stability estimate consists of the Lipschitz type of data discrepancy and the high-frequency tail of the source function. The latter decreases as the frequency K of the data increases. In this work, we determine f in a finite interval (K_1,K_2) of frequencies. We show that the stability estimate consists of the Lipschitz type of data discrepancy and the high-frequency tail of the source function. The latter decreases as the frequency K_2 of the data increases or K_1 decreases, which implies that the inverse problem is more stable with larger wavenumber intervals. We use the Fourier transform in time to reduce our inverse source problem for Helmholtz equations to the identification of the initial data in the hyperbolic initial value problem by lateral Cauchy data (boundary observability in control theory) and, as a consequence, to obtain the most complete increasing stability results.

The paper is organized as follows. In Section 2, we briefly show the main result. Section 3 is developed to stability analysis of the inverse source problem by using multi-frequency data.

2. Main Result

Let $d > 0$ be an integer and denote a complex-valued functional space:

$$\mathcal{C}_{M,d} = \{f \in H^{2d}(\mathbb{R}^n) |\ \|f\|_{H^{n+1}(B_R)} \le M,\ supp(f) \subset B_R,\ f : B_R \to \mathbb{C}\},$$

where $M > 0$ is a constant. For any $v \in H^{1/2}(\partial B_R)$, we set

$$\|v(x,k)\|_{\partial B_R} = \int_{\partial B_R} (|\mathcal{B}v(x,k)|^2 + k^2|v(x,k)|^2) ds(x).$$

Now, we show the main increasing stability result.

Theorem 1. *Let $f \in \mathcal{C}_{M,d}$ and g satisfy (2). Then, there exists a constant $C > 0$ depending on n, d, δ, K_1, K_2 and R such that*

$$\|f\|^2_{L^2(\mathbb{R}^n)} \le C\Big(\epsilon^2 + \frac{M^2}{(K_1^{-1}\ln|\ln\epsilon|)^{\frac{n}{2}}} + \frac{M^2}{(K_2\ln|\ln\epsilon|)^{\frac{4d-n}{2}}}\Big) \tag{10}$$

where $0 < K_1 < 1$, $K_2 > 1$, $4d > n$ and

$$\epsilon = \Big(\int_{K_1}^{K_2} k^{n-1}\|u(x,k)\|_{\partial B_R} dk\Big)^{\frac{1}{2}}.$$

Remark 1. *There are three parts in the stability estimates (10): the first part is the data discrepancy, and the second and the third part come from the frequency tail of the function. It is clear to see that the stability increases as K_2 increases or K_1 decreases, i.e., the problem is more stable as more frequency data are used.*

Remark 2. *The idea was firstly proposed in [9] by separating the stability into the data discrepancy and high-frequency tail where the latter was estimated by the unique continuation for the three-dimensional inverse source scattering problem. Our stability result in this work is consistent with the one in [13,23] for both the two- and three-dimensional inverse scattering problems.*

3. Proof of Increasing Stability for Sources

Let $\xi \in \mathbb{R}^n$ with $|\xi| = k$. Multiplying $e^{-i\xi\cdot x}$ on both sides of (1) and integrating over B_R, we obtain

$$\int_{B_R} f(x)g(k)e^{-i\xi\cdot x}dx = \int_{\partial B_R} e^{-i\xi\cdot x}(\partial_\nu u + i\xi\cdot\nu u)ds(x),\ |\xi| = k \in (0,+\infty).$$

Since supp (f) is contained in the ball B_R, we have

$$\int_{\mathbb{R}^n} f(x)e^{-i\xi\cdot x}dx\cdot g(k) = \int_{\partial B_R} e^{-i\xi\cdot x}(\partial_\nu u + i\xi\cdot\nu u)ds(x),\ |\xi| = k \in (0,+\infty)$$

which gives

$$\left|\widehat{f}(\xi)\right|^2 \le C\int_{\partial B_R}(|\partial_\nu u|^2 + k^2|u|^2)ds(x),\ |\xi| = k \in (0,+\infty), \tag{11}$$

where $C > 0$ is a constant and depends on n, δ and R. Using the spherical polar coordinates

$$\xi = k\hat{\xi} = k(\cos\varphi_1, \sin\varphi_1\cos\varphi_2, \cdots, \sin\varphi_1\cdots\sin\varphi_{n-2}\sin\varphi_{n-1}),$$

we obtain that

$$\int_{\mathbb{R}^n}\left|\widehat{f}(\xi)\right|^2 d\xi = \int_0^{2\pi} d\varphi_{n-1}\int_0^{\pi}\sin\varphi_{n-2}d\varphi_{n-2}\cdots\int_0^{\pi}\sin\varphi_1^{n-2}d\varphi_1\int_0^{+\infty}k^{n-1}\left|\widehat{f}(\xi)\right|^2 dk.$$

It follows from the Plancherel theorem that

$$\begin{aligned}(2\pi)^n\|f\|^2_{L^2(\mathbb{R}^n)} &= \|\widehat{f}\|^2_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n}|\widehat{f}(\xi)|^2 d\xi\\ &= \int_{|\xi|\le\omega}|\widehat{f}(\xi)|^2 d\xi + \int_{\omega\le|\xi|\le s}|\widehat{f}(\xi)|^2 d\xi + \int_{|\xi|>s}|\widehat{f}(\xi)|^2 d\xi.\end{aligned}$$

Denote

$$\begin{aligned}I(\omega;s) &= \int_{\omega\le|\xi|\le s}|\widehat{f}(\xi)|^2 d\xi\\ &= \int_0^{2\pi} d\varphi_{n-1}\int_0^{\pi}\sin\varphi_{n-2}d\varphi_{n-2}\cdots\int_0^{\pi}\sin\varphi_1^{n-2}d\varphi_1\int_\omega^s k^{n-1}\left|\widehat{f}(k\hat{\xi})\right|^2 dk\end{aligned} \tag{12}$$

Combining (11) and (12), we obtain

$$|I(\omega;s)| \le C\epsilon^2 \ \text{ for all } s,\omega \in [K_1,K_2], \tag{13}$$

where $C > 0$ depends on R, δ and n. Observe that, as known [24], $I(\omega;s)$ is an entire analytic function of $s \in \mathbb{C}$ or $\omega \in \mathbb{C}$.

Lemma 1. *Let $\|f\|_{L^2(\mathbb{R}^n)} \le M$. Then, we have for all $s = s_1 + is_2 \in \mathbb{C}$ that*

$$|I(\omega;s)| \le Cs^n e^{2R|s_2|}M^2, \tag{14}$$

where $C > 0$ depends on n, δ and R.

Proof. This proof can be seen in [24] (Lemma 3.1), and so we omit it. □

Denote

$$I_2(s) = \int_{|\xi|>s} |\widehat{f}(\xi)|^2 d\xi = \int_0^{2\pi} d\varphi_{n-1} \int_0^{\pi} \sin\varphi_{n-2} d\varphi_{n-2} \cdots \int_0^{\pi} \sin\varphi_1^{n-2} d\varphi_1 \int_s^{+\infty} k^{n-1} |\widehat{f}(k\hat{\xi})|^2 dk.$$

The following estimate can also be found in [24] (Lemma 3.2).

Lemma 2. *Let $f \in \mathcal{C}_{M,d}$. For any $s > 0$, we have*

$$|I_2(s)| \leq C \frac{M^2}{s^{4d-n}}, \tag{15}$$

where $C > 0$ depends on n and R.

The following lemma gives a link between the values of an analytical function for small and large arguments, which is proved in [14].

Lemma 3. *Let $p(z)$ be analytic in the infinite rectangular slab*

$$\mathcal{R} = \{z \in \mathbb{C} : (A, +\infty) \times (-d, d)\},$$

where A is s positive constant, and continuous in $\overline{\mathcal{R}}$, satisfying

$$\begin{cases} |p(z)| \leq \epsilon, & z \in (A, A_1], \\ |p(z)| \leq M, & z \in \mathcal{R}, \end{cases} \tag{16}$$

where A, A_1, ϵ and M are positive constants. Then, there exists a function $\mu(z)$ with $z \in (A_1, +\infty)$ satisfying

$$\mu(z) \geq \frac{64ad}{3\pi^2(a^2 + 4d^2)} e^{\frac{\pi}{2d}(\frac{a}{2} - z)},$$

where $a = A_1 - A$, such that

$$|p(z)| \leq M\epsilon^{\mu(z)} \quad \forall z \in (A_1, +\infty).$$

Using Lemma 3, we show the relation between $I(\omega; s)$ for $s \in (K_2, \infty)$ with $I(\omega; K_2)$.

Lemma 4. *Let $\|f\|_{L^2(\mathbb{R}^n)} \leq M$. Then, there exists a function $\mu(s)$ satisfying*

$$\mu(s) \geq \frac{64ad}{3\pi^2(a^2 + 4d^2)} e^{\frac{\pi}{2d}(\frac{a}{2} - s)}, \quad s \in (K_2, +\infty)$$

such that

$$|I(\omega; s)| \leq CM^2 e^{(2R+1)s} \epsilon^{\mu(s)} \quad \text{for all } K_2 < s < +\infty, \qquad \omega \in [K_1, K_2] \tag{17}$$

where $C > 0$ depends on n, δ and R.

Proof. Let the sector $S = \{z = x + iy \in \mathbb{C} : -\frac{\pi}{4} < \arg z < \frac{\pi}{4}\}$. Choose the appropriate d satisfying $\mathcal{R} \subset S$. Observe that $|s_2| \leq s_1$ when $s \in S$. It follows from Lemma 1 that

$$|I(\omega; s)e^{-(2R+1)s}| \leq CM^2, \quad s \in \mathcal{R}$$

where $C > 0$ depends on n, δ and R. Recalling from (13) a prior estimate $|I(\omega; s)| \leq C\epsilon^2$, $s, \omega \in [K_1, K_2]$. Then applying Lemma 3 with $A = K_1$ and $A_1 = K_2$ to be function $p(s) := I(\omega; s)e^{-(2R+1)s}$, we conclude that there exists a function $\mu(s)$ satisfying

$$\mu(s) \geq \frac{64ad}{3\pi^2(a^2 + 4d^2)} e^{\frac{\pi}{2d}(\frac{a}{2} - s)}, \quad s \in (K_2, +\infty) \tag{18}$$

such that

$$|I(\omega; s)e^{-(2R+1)s}| \leq CM^2\epsilon^{\mu(s)},$$

where $K_2 < s < +\infty$ and C depending on n, δ and R. Thus we complete the proof. □

Using Lemma 3, we deduce the following Lemma.

Lemma 5. *Let $q(z)$ be analytic in the infinite rectangular slab*

$$\mathcal{R} = \{z = x + iy \in \mathbb{C} : \frac{x}{x^2 + y^2} > \frac{1}{A}, -d \leq \frac{y}{x^2 + y^2} \leq d\},$$

where A is s positive constant, and continuous in $\overline{\mathcal{R}}$ satisfying

$$\begin{cases} |p(z)| \leq \epsilon, & z \in [A, A_1), \\ |p(z)| \leq M, & z \in \mathcal{R}, \end{cases} \tag{19}$$

where A, A_1, ϵ and M are positive constants. Then, there exists a function $\mu(z)$ with $z \in (0, A)$ satisfying

$$\mu(\frac{1}{z}) \geq \frac{64ad}{3\pi^2(a^2 + 4d^2)} e^{\frac{\pi}{2d}(\frac{a}{2} - \frac{1}{z})},$$

where $a = \frac{1}{A} - \frac{1}{A_1}$, such that

$$|q(z)| \leq M\epsilon^{\mu(\frac{1}{z})} \quad \forall z \in (0, A).$$

Using Lemma 5, we also show the relation between $I(\omega; K_2)$ for $\omega \in (0, K_1)$ with $I(K_1; K_2)$.

Lemma 6. *Let $\|f\|_{L^2(\mathbb{R}^n)} \leq M$. Then, there exists a function $\mu(s)$ satisfying*

$$\mu(\frac{1}{s}) \geq \frac{64ad}{3\pi^2(a^2 + 4d^2)} e^{\frac{\pi}{2d}(\frac{a}{2} - \frac{1}{s})}, \quad s \in (0, K_1) \tag{20}$$

such that

$$|I(\omega; K_2)| \leq CM^2 e^{(2R+1)\omega} \epsilon^{2\mu(\frac{1}{\omega})} \quad \text{for all } 0 < \omega < K_1, \tag{21}$$

where $C > 0$ depends on n, δ and R.

Proof. Observe that $|s_2| \leq s_1$ when $s \in S$. It follows from Lemma 1 that

$$|I(\omega; K_2)e^{-(2R+1)\omega}| \leq CM^2, \quad s \in \mathcal{R} \cap S$$

where $C > 0$ depends on n, δ and R. Recalling from (13) a prior estimate $|I(\omega; K_2)| \leq C\epsilon^2$, $\omega \in [K_1, K_2]$. Then, applying Lemma 5 with $A = K_1$ and $A_1 = K_2$ to be function $q(\omega) := I(\omega; K_2)e^{-(2R+1)s}$, we conclude that there exists a function $\mu(s)$ satisfying

$$\mu(\frac{1}{s}) \geq \frac{64ad}{3\pi^2(a^2+4d^2)} e^{\frac{\pi}{2d}(\frac{a}{2}-\frac{1}{s})}, \quad s \in (0, K_1) \tag{22}$$

such that

$$|I(\omega; K_2)e^{-(2R+1)\omega}| \leq CM^2\epsilon^{\mu(\frac{1}{\omega})},$$

where $0 < \omega < K_1$ and C depending on n, δ and R. Thus, we complete the proof. □

Now, we show the proof of Theorem 1. If $\epsilon \geq e^{-1}$, the estimate is obvious. If $\epsilon < e^{-1}$, we discuss (10) in two cases.

Case (i): $K_2 \leq \frac{1}{2c_1} \ln|\ln\epsilon|$. Choose $s_0 = \frac{1}{2c_1} \ln|\ln\epsilon|$. A direct application of estimate (17) shows that

$$\begin{aligned} I(\omega; s_0) &\leq M^2 e^{(2R+1)s_0} \epsilon^{\mu(s_0)} \\ &\leq M^2 e^{(2R+1)s_0 - \frac{c_2 a}{a^2+c_3} e^{c_1(\frac{a}{2}-s_0)} |\ln\epsilon|} \\ &\leq M^2 e^{-\frac{c_2 a}{a^2+c_3} e^{c_1(\frac{a}{2}-s_0)} |\ln\epsilon| (1 - \frac{c_4 s(a^2+c_3)}{a} e^{c_1(\frac{a}{2}-s_0)} |\ln\epsilon|^{-1})} \\ &\leq M^2 e^{-b_0 e^{-c_1 s_0} |\ln\epsilon| (1 - b_1 s_0 e^{c_1 s_0} |\ln\epsilon|^{-1})}, \end{aligned}$$

where c_i, $i = 1, 2, 3, 4$ are constants and $b_0 = \frac{c_2 a}{a^2+c_3} e^{c_1 \frac{a}{2}}$. For a sufficiently small ϵ, we obtain that

$$I(\omega; s_0) \leq CM^2 e^{-\frac{1}{2} b_0 e^{-c_1 s_0} |\ln\epsilon|}.$$

Using the inequality $e^{-t} \leq \frac{(8d-2n)!}{t^{2(4d-n)}}$ for $t > 0$, we obtain

$$|I(\omega; s_0)| \leq M^2 |\ln\epsilon|^{4d-n} |\ln\epsilon|^{-2(4d-n)}.$$

Noting that $|\ln\epsilon| \geq \ln|\ln\epsilon|$ and $K_2 \leq \frac{1}{2c_1} \ln|\ln\epsilon|$, we have

$$|I(\omega; s_0)| \leq C \frac{M^2}{(|\ln\epsilon|)^{4d-n}} \leq C \frac{M^2}{(\ln|\ln\epsilon|)^{4d-n}} \leq C \frac{M^2}{K_2^{\frac{4d-n}{2}} (\ln|\ln\epsilon|)^{\frac{4d-n}{2}}}.$$

Hence,

$$\begin{aligned} (2\pi)^n ||f||^2_{L^2(\mathbb{R}^n)} &= ||\widehat{f}||^2_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 d\xi \\ &= \int_{|\xi|\leq\omega} |\widehat{f}(\xi)|^2 d\xi + \int_{\omega\leq|\xi|\leq s_0} |\widehat{f}(\xi)|^2 d\xi + \int_{|\xi|>s_0} |\widehat{f}(\xi)|^2 d\xi \\ &\leq \int_{|\xi|\leq\omega} |\widehat{f}(\xi)|^2 d\xi + I(\omega; s_0) + \frac{M^2}{(\ln|\ln\epsilon|)^{4d-n}} \\ &\leq \int_{|\xi|\leq\omega} |\widehat{f}(\xi)|^2 d\xi + C \frac{M^2}{K_2^{\frac{4d-n}{2}} (\ln|\ln\epsilon|)^{\frac{4d-n}{2}}}. \end{aligned} \tag{23}$$

Case (ii): $K_2 > \frac{1}{2c_1}\ln|\ln\epsilon|$. In this case, we choose $s_0 = K_2$. Then,

$$\begin{aligned}(2\pi)^n||f||^2_{L^2(\mathbb{R}^n)} &= ||\widehat{f}||^2_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n}|\widehat{f}(\xi)|^2d\xi \\ &= \int_{|\xi|\le\omega}|\widehat{f}(\xi)|^2d\xi + \int_{\omega\le|\xi|\le K_2}|\widehat{f}(\xi)|^2d\xi + \int_{|\xi|>s}|\widehat{f}(\xi)|^2d\xi \\ &\le \int_{|\xi|\le\omega}|\widehat{f}(\xi)|^2d\xi + I(\omega;K_2) + \frac{M^2}{K_2^{4d-n}} \\ &\le \int_{|\xi|\le\omega}|\widehat{f}(\xi)|^2d\xi + I(\omega;K_2) + \frac{M^2}{K_2^{\frac{4d-n}{2}}(\ln|\ln\epsilon|)^{\frac{4d-n}{2}}}.\end{aligned} \tag{24}$$

Combining (23) and (24), we finally obtain

$$||f||^2_{L^2(\mathbb{R}^n)} \le C\Big(\int_{|\xi|\le\omega}|\widehat{f}(\xi)|^2d\xi + I(\omega;K_2) + \frac{M^2}{K_2^{\frac{4d-n}{2}}(\ln|\ln\epsilon|)^{\frac{4d-n}{2}}}\Big).$$

Similarly, we discuss ω in two cases. It is known from [24] that

$$\int_{|\xi|\le\omega}|\widehat{f}(\xi)|^2d\xi \le CM^2\omega^n.$$

If $K_1 \le \frac{2c_1}{\ln|\ln\epsilon|}$. Choose $w_0 = K_1$; then,

$$\begin{aligned}\int_{|\xi|\le\omega_0}|\widehat{f}(\xi)|^2d\xi + I(\omega_0;K_2) &= K_1^n + I(K_1,K_2) \\ &\le \frac{K_1^{\frac{n}{2}}}{\ln|\ln\epsilon|^{\frac{n}{2}}} + I(K_1;K_2).\end{aligned} \tag{25}$$

If $K_1 > \frac{2c_1}{\ln|\ln\epsilon|}$. Let $w_0 = \frac{2c}{\ln|\ln\epsilon|}$; then,

$$\begin{aligned}I(\omega_0;K_2) &\le M^2e^{(4R+1)\omega_0}\epsilon^{\mu(\frac{1}{\omega_0})} \\ &\le M^2e^{(4R+1)\omega_0-\frac{c_2a}{a^2+c_3}e^{c_1(\frac{a}{2}-\frac{1}{\omega_0})}|\ln\epsilon|} \\ &\le M^2e^{-\frac{c_2a}{a^2+c_3}e^{c_1(\frac{a}{2}-\frac{1}{\omega_0})}|\ln\epsilon|(1-\frac{c_4\omega_0(a^2+c_3)}{a}e^{c_1(\frac{a}{2}-\frac{1}{\omega_0})}|\ln\epsilon|^{-1})} \\ &\le M^2e^{-b_0e^{-c_1\frac{1}{\omega_0}}|\ln\epsilon|(1-b_1\omega_0e^{c_1\omega_0}|\ln\epsilon|^{-1})} \\ &\le M^2e^{-\frac{1}{2}b_0e^{-c_1\frac{1}{\omega_0}}|\ln\epsilon|}.\end{aligned}$$

Using the inequality $e^{-t} \le \frac{(2n)!}{t^{2n}}$ for $t > 0$, we obtain

$$|I(\omega_0;K_2)| \le M^2|\ln\epsilon|^{-n}|\ln\epsilon|^{-2n}.$$

Noting that $|\ln\epsilon| \ge \ln|\ln\epsilon|$ and $K_1 > \frac{2c}{\ln|\ln\epsilon|}$, we have

$$|I(\omega;s_0)| \le C\frac{M^2}{(|\ln\epsilon|)^{2n}} \le C\frac{M^2}{(\ln|\ln\epsilon|)^n} \le C\frac{M^2}{K_1^{-\frac{n}{2}}(\ln|\ln\epsilon|)^{\frac{n}{2}}}. \tag{26}$$

Combining (25) and (26), we have

$$\int_{|\xi|\le\omega} |\widehat{f}(\xi)|^2 d\xi + I(K_1;K_2) \le C\frac{M^2}{K_1^{-\frac{n}{2}} \ln|\ln\epsilon|^{\frac{n}{2}}}.$$

Therefore,

$$\begin{aligned}(2\pi)^n ||f||^2_{L^2(\mathbb{R}^n)} &= ||\widehat{f}||^2_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 d\xi \\ &= \int_{|\xi|\le\omega} |\widehat{f}(\xi)|^2 d\xi + \int_{\omega\le|\xi|\le s} |\widehat{f}(\xi)|^2 d\xi + \int_{|\xi|>s} |\widehat{f}(\xi)|^2 d\xi \\ &\le C\frac{M^2}{K_1^{-\frac{n}{2}} \ln|\ln\epsilon|^{\frac{n}{2}}} + I(K_1;K_2) + C\frac{M^2}{K_2^{\frac{4d-n}{2}} (\ln|\ln\epsilon|)^{\frac{4d-n}{2}}}.\end{aligned}$$

This completes the proof of Equation (10).

4. Conclusions

In this paper, we presented an analysis of increasing stability in the inverse source problems with multi-frequency boundary data and demonstrated a deterioration of this stability. In future, we hope to provide some numerical examples in three spatial dimensions in support of the theoretical results.

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