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σ-Martingales: Foundations, Properties, and a New Proof of the Ansel–Stricker Lemma

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Abstract: σ-martingales generalize local martingales through localizing sequences of predictable sets, which are essential in stochastic analysis and financial mathematics, particularly for arbitrage-free markets and portfolio theory. In this work, we present a new approach to σ-martingales that avoids using semimartingale characteristics. We develop all fundamental properties, provide illustrative examples, and establish the core structure of σ-martingales in a new, straightforward manner. This approach culminates in a new proof of the Ansel–Stricker lemma, which states that one-sided bounded σ-martingales are local martingales. This result, referenced in nearly every publication on mathematical finance, traditionally relies on the original French-language proof. We use this result to prove a generalization, which is essential for defining the general semimartingale model in mathematical finance.

Keywords: σ-martingales; stochastic processes; stochastic integration; localization; semimartingales; Ansel–Stricker lemma

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1. Introduction

σ-martingales are a class of stochastic processes that extend the concept of local martingales by broadening the localization procedure. While local martingales are processes that can be localized to uniformly integrable martingales using stopping times, σ-martingales generalize this idea through sequences of predictable sets. They exhibit distinctive properties, making them particularly valuable for addressing challenges in theoretical probability and financial mathematics. Notably, unlike the subset of local martingales, σ-martingales are closed under stochastic integration—a property that underscores their natural emergence in financial mathematics and is often used as an alternative definition for σ-martingales, distinct from the one adopted in this publication.

In finance, σ-martingales play a central role in markets with no arbitrage opportunities and in modeling self-financing trading strategies. The First Fundamental Theorem of Asset Pricing [1] guarantees that with "No Free Lunch with Vanishing Risk" (NFLVR) satisfied, there exists an Equivalent Sigma-Martingale Measure (ESMM), under which discounted price processes are σ-martingales. Even when price processes are modeled as local martingales (e.g., due to positivity constraints or continuity), stochastic integrals like $H \bullet M$, arising in self-financing trading strategies, are typically only σ-martingales and not local martingales.

The concept of σ-martingales was introduced as processes "de la classe (Σ_m)" in the pioneering work of Chou [2] and was further explored by Émery [3]. Despite their fundamental role in financial mathematics, σ-martingales have received relatively limited attention in the broader literature. For instance, the treatment in [4] is concise but somewhat superficial, whereas works such as [5,6] offer a more in-depth exploration. However, they heavily rely on the framework of semimartingale characteristics, which, while highly useful for establishing important properties, can also be mathematically demanding. The didactically excellent and concise presentation in [7] also discusses deeper results similar to those presented here. However, their approach relies on advanced results about special semimartingales and the decomposition of semimartingales.

This publication seeks to bridge the gap in the literature by developing a comprehensive, self-contained, and accessible framework for σ-martingales without relying on the useful but technical apparatus of semimartingale characteristics. Instead, we demonstrate that all fundamental results about σ-martingales can be derived from first principles. While all major results, except for some auxiliary lemmata, are already known, the approach presented here is novel, more straightforward, and avoids the complex concept of semimartingale characteristics. This streamlined method leads to all major properties of σ-martingales and a new proof of the crucial Ansel–Stricker lemma, which states that one-sided bounded σ-martingales are local martingales. We use this new approach to prove a generalization, which, to the best of our knowledge, is so far only mentioned in a slightly different version in [7]. In the context of mathematical finance, this can be used to show that, if a trading strategy is admissible (i.e., it satisfies specific boundedness conditions on the nominal price process), the discounted stochastic integral $H \bullet M$ is not merely a σ-martingale but also a local martingale, significantly simplifying the theoretical framework.

Additionally, this work elaborates on the major examples of σ-martingales that are widely known in the literature but are often only briefly mentioned without explicit demonstration of their important properties or making use of more complex concepts (such as the above-mentioned characteristics of semimartingales). The present approach avoids such complexity. By focusing on clear, intuitive arguments, we not only provide a new perspective on these examples, but also ensure a more accessible and didactic exposition of σ-martingales.

2. Definition and Properties

We start by repeating some basic definitions.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space, and let $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$ be a filtration of sub-σ-algebras of $\mathcal{F}$ satisfying the usual conditions:

1. $\mathcal{F}_0$ includes all $\mathbf{P}$-null sets from $\mathcal{F}$;
2. $\mathbb{F}$ is right continuous, i.e.,
$$\bigcap_{s>t} \mathcal{F}_s = \mathcal{F}_t,$$
for each $t \geq 0$.

Given a measure $\mathbf{Q}$ on $\mathcal{F}$, we say that $\mathbf{Q} \ll \mathbf{P}$ if $\mathbf{P}(E) = 0$ implies $\mathbf{Q}(E) = 0$ for $E \in \mathcal{F}$. Similarly, $\mathbf{P} \ll \mathbf{Q}$ is defined analogously. If both $\mathbf{P} \ll \mathbf{Q}$ and $\mathbf{Q} \ll \mathbf{P}$, we say that $\mathbf{P}$ and $\mathbf{Q}$ are equivalent, and we write $\mathbf{P} \sim \mathbf{Q}$.

A set $E \subseteq \Omega \times \mathbb{R}_+$ is called evanescent if

$$\{\omega \in \Omega : \exists t \geq 0 \text{ such that } (t, \omega) \in E\}$$

is a $\mathbf{P}$-null set. Here, we always consider stochastic processes as modulo evanescent sets.

A càdlàg function is a mapping $\xi : \mathbb{R}_+ \to \mathbb{R}$ that is right continuous and has left-hand limits at every point. From this point forward, we assume that all stochastic processes are càdlàg (at least up to evanescence).

For a càdlàg process X, the jump process ΔX is defined as $\Delta X_t := X_t - X_{t-}$ with $X_{t-} = \lim_{s \nearrow t} X_s$, and, for a stochastic process Y, the process Y_- is defined to be the process Z that satisfies $Z_t = Y_{t-}$. Hence, one obtains $Y_- = Y - \Delta Y$.

A sequence of processes $(H^n)_{n\geq 1}$ converges to a process H uniformly on compacts in probability (abbreviated to ucp) if, for each $t > 0$,

$$\sup_{0\leq s\leq t} |H^n_s - H_s| \text{ converges to 0 in probability.}$$

A càdlàg, adapted process $X = (X_t)_{t\geq 0}$ is called a semimartingale if it can be decomposed as $X_t = M_t + A_t$, where M is a local martingale and A is a process of finite variation on every finite interval. The space of d-dimensional semimartingales is denoted by $\mathcal{S}^d$.

Beyond this decomposition, semimartingales can also be defined equivalently through their properties as good integrators. In this sense, a semimartingale is a process for which the integral operator is continuous with respect to certain metrics. Finally, semimartingales can also be described as topological semimartingales, whose definition relies on certain convergence properties in the semimartingale topology (see, for example, [8]). These three characterizations—the classical decomposition, the good integrator perspective, and the topological semimartingale framework—are mathematically equivalent.

For a stochastic process X and a stopping time T, the stopped process X^T is defined as $X^T_t = X_{t\wedge T}$ for all $t \geq 0$. It is well known that, if X is a martingale and T is a stopping time, then the stopped process X^T remains a martingale (see Corollary A1). For a stochastic process X and a stopping time T, the stopped process X^T is defined as $X^T_t = X_{t\wedge T}$ for all $t \geq 0$. It is well known that, if X is a martingale and T is a stopping time, then the stopped process X^T remains a martingale (see Corollary A1).

The predictable σ-algebra, denoted by $\mathcal{P}$, is the smallest σ-algebra on $\Omega \times \mathbb{R}_+$ such that all left-continuous adapted processes are measurable with respect to $\mathcal{B}(\mathbb{R})$, the Borel σ-algebra on $\mathbb{R}$. This definition is equivalent to several other characterizations of the predictable σ-algebra on $\mathbb{R}_+ \times \Omega$. For example, the predictable σ-algebra can also be generated by simple or elementary predictable processes, continuous adapted processes, or sets of predictable stopping times (see the results in ([9], Section 7.2)).

By $\mathcal{M}^d$, we denote the space of d-dimensional martingales, by $\mathcal{M}^d_\infty$ the space of bounded d-dimensional martingales, and by $\mathcal{M}^d_{\text{loc}}$ the space of d-dimensional local martingales. A subscript 0, as in $\mathcal{M}_{\text{loc},0}$, further indicates that the process starts at 0, that is, $M_0 = 0$ almost surely for all $M \in \mathcal{M}_{\text{loc},0}$.

In the following, we use $H \bullet X$ to denote the stochastic integral of a d-dimensional predictable process H with respect to a d-dimensional semimartingale X, as defined, for example, in [9].

For a martingale M and $p \in [1, \infty)$, write

$$\|M\|_{\mathcal{H}^p} := \|M^*_\infty\|_p = \mathbf{E}\left[\sup_t |M_t|^p\right]^{\frac{1}{p}}.$$

Here, $\|\cdot\|_p$ denotes the norm in L^p. Then, $\mathcal{H}^p$ is the space of martingales such that

$$\|M\|_{\mathcal{H}^p} < \infty.$$

There are several equivalent definitions of σ-martingales in the literature. The definition adopted in this work was originally proposed by Goll et al. [10] and later refined by

other authors (e.g., ([5], Definition III.6.33)). This definition emphasizes how σ-martingales extend local martingales through a broader localization framework. In contrast, works such as [1,4] define σ-martingales as processes that can be represented as stochastic integrals with respect to martingales. While this perspective underscores their crucial role in mathematical finance, it makes the connection to their generalization of local martingales less immediately apparent. From a didactic perspective, we find that the earlier definition—adopted here—provides a clearer and more intuitive introduction to the concept. In Theorem 3, we establish the equivalence of our definition with that of [1,4], naturally concluding that σ-martingales hold significant importance in the study of mathematical finance.

Definition 1 (σ-martingale)**.** *A one-dimensional semimartingale S is called a σ-martingale if there exists a sequence of sets $D_n \subset \mathcal{P}$ such that*

(i) $D_n \subset D_{n+1}$ *for all n;*
(ii) $\bigcup_{n=1}^{\infty} D_n = \Omega \times \mathbb{R}_+$*;*
(iii) *For any $n \geq 1$, the process $\mathbb{1}_{D_n} \bullet S$ is a uniformly integrable martingale.*

Such a sequence $(D_n)_{n\in\mathbb{N}}$ is called a σ-localizing sequence. A d-dimensional semimartingale is called a σ-martingale if each of its components is a one-dimensional σ-martingale. By $\mathcal{M}_{\sigma}^{d}$, we denote the set of all d-dimensional σ-martingales.

First, observe that, by setting $D_n := [\![0, T^n]\!]$, all local martingales are σ-martingales.

Theorem 1. *Every local martingale is a σ-martingale.*

Proof. Let M be a local martingale and $(T^n)_{n\in\mathbb{N}}$ a localizing sequence. Define $D_n := [\![0, T^n]\!]$. Since $\mathbb{1}_{[\![0,T^n]\!]} \bullet M = M^{T^n}$, it follows that M is also a σ-martingale. □

In discrete time, any σ-martingale is a local martingale. This follows from the fact that, in discrete time, any predictable set D_n can be expressed as a finite union of intervals of the form $]\!]T_k, T_n]\!]$, combined with the property that the set of local martingales forms a vector space. Alternatively, this result can also be derived using Theorem 3 and the observation that, in discrete time, the predictable integrand can be assumed to be locally bounded. The conclusion then follows directly from results such as ([9], Theorem 12.3.3), which states that a stochastic integral with a locally bounded integrand and a local martingale integrator is again a local martingale, or ([11], Theorem 10.7), which states that, in discrete time, any stochastic integral with a bounded integrand and a martingale integrator is again a martingale.

It turns out that $\mathcal{M}_{\mathrm{loc}}^{d} \subsetneq \mathcal{M}_{\sigma}^{d}$, as we will illustrate in the following example, which can also be found in ([5], Example 6.40). We revisit the example with a detailed demonstration and elaboration on the claimed properties.

Example 1. *Let $(Y_n)_{n\in\mathbb{N}}$ be a sequence of independent random variables with*

$$\mathbf{P}(Y_n = n) = \frac{1}{2n^2}, \quad \mathbf{P}(Y_n = -n) = \frac{1}{2n^2}, \quad \mathbf{P}(Y_n = 0) = 1 - \frac{1}{n^2}.$$

We put

$$X_t := \sum_{\left\{n\,;\,1-t\leq\frac{1}{n}\right\}} Y_n.$$

Then, X is a well-defined σ-martingale but no local martingale with respect to the filtration created by X.

First, we have to show that X is well defined. Therefore, we define $A_n = \{\omega \in \Omega\,;\, Y_n \neq 0\}$. Clearly, we have $\mathbf{P}(A_n) = \frac{1}{n^2}$ and hence $\sum_{n=1}^{\infty} \mathbf{P}(A_n) < \infty$. By the Borel–Cantelli lemma, we conclude that $\mathbf{P}(\{Y_n \neq 0 \text{ for infinite many } n\}) = 0$ and thus X is well defined.

By setting

$$D_n := \Omega \times \left(\left[0, 1 - \frac{1}{n}\right] \cup [1, \infty) \right),$$

we obtain $\mathbb{1}_{D_n} \bullet X = \sum_{i=1}^{n} Y_n$ for each n. Since the sum is finite and all Y_n are symmetric and integrable, it is easy to see that D_n is a localizing sequence and X a σ-martingale.

Furthermore, X is not a local martingale. In order to show that, we assume that $X \in \mathcal{M}^1_{\mathrm{loc}}$. Since X is a process with independent increments, we even have $X \in \mathcal{M}^1$ (see, for example, (Medvegyev [12] Theorem 7.97)). We put

$$A_n := \{Y_n = n\} \cap \bigcap_{k \neq n} \{Y_k = 0\}$$

and, by the independence of the random variables Y_k, we obtain

$$\mathbf{P}(A_n) = \mathbf{P}(Y_n = n) \prod_{k \neq n} \mathbf{P}(Y_k = 0) \geq \mathbf{P}(Y_n = n) \prod_{k \in \mathbb{N}} \mathbf{P}(Y_k = 0) = c \frac{1}{2n^2}$$

with $c := \prod_{k \in \mathbb{N}} \left(1 - \frac{1}{k^2}\right)$. (We note that c is well defined, since $\sum_{k \in \mathbb{N}} \frac{1}{k^2}$ converges.)

By applying monotone convergence, and since the sets A_n are pairwise disjoint, we obtain

$$\mathbf{E}[|X_1|] \geq \mathbf{E}\left[|X_1| \sum_{n \in \mathbb{N}} \mathbb{1}_{A_n} \right] = \sum_{n \in \mathbb{N}} \mathbf{E}[|X_1| \mathbb{1}_{A_n}] = \sum_{n \in \mathbb{N}} n\mathbf{P}(A_n) \geq c \sum_{n \in \mathbb{N}} n \frac{1}{2n^2} = \infty.$$

This is a contradiction, and we conclude that X is not a local martingale.

The following example is a variant of the most prominent example for a σ-martingale that is not a local martingale. It is from Émery [3] and mentioned in most publications about σ-martingales (see, for example, ([13], Example 9.29), ([4], the example preceding Theorem IV.34), or ([14], Example 5.2)).

Example 2. *Let τ, ξ be independent random variables with $\mathbf{P}(\tau > t) = \exp(-2t)$ and $\mathbf{P}(\xi = 1) = \mathbf{P}(\xi = -1) = \frac{1}{2}$. We put $X_t = \mathbb{1}_{[\![\tau, \infty[\![} \frac{\xi}{\tau}$. Then, X is a σ-martingale but not a local martingale with respect to the filtration created by X.*

By putting $D_n = \{0\} \cup \left[\frac{1}{n}, \infty\right)$, we obtain

$$\mathbb{1}_{D_n} \bullet \left(\frac{\xi}{\tau} \mathbb{1}_{[\![\tau, \infty[\![} \right) = \begin{cases} \frac{\xi}{\tau} & \text{for } \tau \geq \frac{1}{n} \text{ and } t \geq \tau \\ 0 & \text{else} \end{cases}.$$

And it is easy to see that D_n is a localizing sequence.

However, X is not a local martingale as we encounter integrability problems. We assume $X \in \mathcal{M}^1_{\mathrm{loc}}$, and hence there exists a stopping time $T > 0$ such that X^T is a uniformly integrable martingale.

Since X is constant on $[\![0, \tau[\![$, we deduce that T is constant on $\{\omega \in \Omega\,;\, T < \tau\}$. There exists an $\varepsilon > 0$ such that $T \geq \tau$ on $\{\omega \in \Omega\,;\, \tau < \varepsilon\}$. Hence, we have $T \geq \tau \wedge \varepsilon$ and thus

$$\mathbf{E}\left[\left| X^T_\varepsilon \right| \right] = \int_0^\varepsilon \left| \frac{\xi}{t} \right| \mathrm{d}\mathbf{P}(\tau = t) = \int_0^\varepsilon \frac{1}{t} 2 \exp(-2t) \mathrm{d}t = \infty.$$

So X^T is not a martingale, and thus X is not a local martingale.

Remark 1. *It turns out that, for both of the above examples, there exists an equivalent probability measure* $\mathbf{Q}$ *such that* X *is a* $\mathbf{Q}$*-martingale.*

For the first example, assume a probability measure $\mathbf{Q}$ *such that* Y_n *are independent random variables with*

$$\mathbf{Q}(Y_n = n) = \frac{1}{2n^3}, \quad \mathbf{Q}(Y_n = -n) = \frac{1}{2n^3}, \quad \mathbf{Q}(Y_n = 0) = 1 - \frac{1}{n^3}.$$

Then, we have

$$\mathbf{E}|X_t| \leq \mathbf{E}|X_1| = \mathbf{E}\left|\sum_{n=1}^{\infty} Y_n\right| \leq \mathbf{E}\left[\sum_{n=1}^{\infty} |Y_n|\right] = \sum_{n=1}^{\infty} n \cdot \frac{1}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

Furthermore, it is easy to see that we have $\mathbf{E}[X_t \mid \mathcal{F}_s] = X_s$*. Hence,* X *is a martingale.*

The equivalence of the original probability measure $\mathbf{P}$ *and* $\mathbf{Q}$ *can be seen by constructing the Radon–Nikodym derivative. For each* $n \in \mathbb{N}$*, the laws of* Y_n *under* $\mathbf{P}$ *and* $\mathbf{Q}$ *are mutually absolutely continuous. In fact, defining*

$$Z_n := \frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}}(Y_n) = \begin{cases} \frac{\frac{1}{2n^3}}{\frac{1}{2n^2}} = \frac{1}{n}, & \text{if } Y_n \in \{n, -n\}, \\ \frac{1 - \frac{1}{n^3}}{1 - \frac{1}{n^2}}, & \text{if } Y_n = 0, \end{cases}$$

the overall Radon–Nikodym derivative is given by

$$\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}} = \prod_{n=1}^{\infty} Z_n.$$

Since, by the Borel–Cantelli lemma, only finitely many of the events $\{Y_n \neq 0\}$ *occur* $\mathbf{P}$*-almost surely, the infinite product involves only finitely many factors differing from* 1*, and hence converges to a strictly positive random variable. This shows that* $\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}}$ *is well defined and strictly positive* $\mathbf{P}$*-almost surely, implying that* $\mathbf{P}$ *and* $\mathbf{Q}$ *are equivalent.*

(Alternatively, one could verify the equivalence by applying Kakutani's theorem.)

For the second example, assume a probability measure $\mathbf{Q}$ *such that* τ *and* ξ *are independent random variables and* $\mathbf{P}(\xi \in A) = \mathbf{Q}(\xi \in A)$ *for all* $A \in \mathcal{B}$ *and* $\int_0^t \left|\frac{\xi}{s}\right| \mathrm{d}\mathbf{Q}(\tau = s) < \infty$ *(for example, you can choose* $\mathbf{Q}$ *with* $\mathbf{Q}(\tau > t) = \frac{1}{t+1}$ *for all* $t \geq 0$*). Then, we have* $\mathbf{E}_{\mathbf{Q}}|X_t| < \infty$ *and* $\mathbf{E}_{\mathbf{Q}}[X_t \mid \mathcal{F}_s] = X_s$ *and hence* X *is a martingale.*

However, in general, such a probability measure does not necessarily exist. We will illustrate that in Example 4.

In the definition of σ-martingales, we refer to the localizing sequence D_n as a σ-localizing sequence.

To establish properties of σ-martingales, we prefer to work with a definition that is relatively "strong". However, when proving that a given process is a σ-martingale, it is more convenient to use criteria that appear "weak" or less restrictive but are nonetheless equivalent to the definition of σ-martingales.

We achieve this by potentially relaxing the conditions that the sequence $\mathbb{1}_{D_n} \bullet S$ must satisfy for a σ-martingale S. Consequently, we extend the notion of a σ-localizing sequence.

Definition 2. *A sequence of sets* $D_n \subset \mathcal{P}$ *is called a* Σ*-localizing sequence if*

(i) $D_n \subset D_{n+1}$ *for all* n*;*

(ii) $\bigcup_{n=1}^{\infty} D_n = \Omega \times \mathbb{R}_+$*;*

(iii) *For any $n \geq 1$, the process $\mathbb{1}_{D_n} \bullet S$ is a local martingale.*

The notion of the σ– (or Σ–) localizing sequence is new in the literature but is inspired by the procedure of σ-localization, which was first described by Jacod and Shiryaev [5] and Kallsen [6]. It does simplify some proofs since the following theorem holds:

Theorem 2. *Let S be a semimartingale. The following are equivalent:*

(i) *The process S is a σ-martingale;*
(ii) *For S, there exists a σ-localizing sequence;*
(iii) *For S, there exists a Σ-localizing sequence;*
(iv) *For S, there exists a sequence $D_n \subset \mathcal{P}$, such that $D_n \subset D_{n+1}$ and $\bigcup_{n=1}^{\infty} D_n = \Omega \times \mathbb{R}_+$, and, for any $n \geq 1$, the process $\mathbb{1}_{D_n} \bullet S$ is a σ-martingale.*

In order to prove this theorem, we need the following lemma:

Lemma 1. *Let $S \in \mathcal{S}^1$ and $D_n \subset \mathcal{P}$ sets, which form a countable partition of $\Omega \times \mathbb{R}_+$, such that $\mathbb{1}_{D_n} \bullet S$ is a uniformly integrable martingale for any $n \geq 1$. Then, S is a σ-martingale.*

Proof. We put $\tilde{D}_n = \bigcup_{i=1}^{n} D_i$. Then, it is easy to see that $(\tilde{D}_n)_{n\in\mathbb{N}}$ is a σ-localizing sequence. □

Proof of Theorem 2. It suffices to prove that the theorem for $S \in \mathcal{S}^1$. $(i) \Leftrightarrow (ii) \Rightarrow (iii) \Rightarrow (iv)$ is clear, so we just have to show $(iv) \Rightarrow (i)$.

Let S be a semimartingale, for which a sequence $(D_n)_{n\in\mathbb{N}}$ of subsets of the predictable σ-algebra exists such that $D_n \subset D_{n+1}$ and $\bigcup_{n=1}^{\infty} D_n = \Omega \times \mathbb{R}_+$ and for which $S^n := \mathbb{1}_{D_n} \bullet S$ is a σ-martingale for any $n \geq 1$.

By assumption, for every $n \geq 1$, there exists a sequence $(D_{n,m})_{m\in\mathbb{N}}$, such that

$$S^{n,m} := \mathbb{1}_{D_{n,m}} \bullet S^n$$

is a uniformly integrable martingale for all m. By defining $\tilde{D}_{n,m} := D_{n,m} \cap (D_n \setminus D_{n-1})$, we obtain $\tilde{D}_{n,m} = D_{n,m} \cap D_n \cap (D_n \setminus D_{n-1})$, and

$$\begin{aligned} \tilde{S}^{n,m} &:= \mathbb{1}_{\tilde{D}_{n,m}} \bullet S = \left(\mathbb{1}_{D_{n,m}} \mathbb{1}_{D_n \setminus D_{n-1}} \mathbb{1}_{D_n}\right) \bullet S \\ &= \mathbb{1}_{D_n \setminus D_{n-1}} \bullet \left(\mathbb{1}_{D_{n,m}} \bullet S^n\right) = \mathbb{1}_{D_n \setminus D_{n-1}} \bullet S^{n,m} \end{aligned}$$

is a local martingale because stochastic integrals with bounded integrands and local martingale integrators are again local martingales. Thus, for every pair (n, m), there exists a fundamental sequence $(T^{n,m,k})_{k\in\mathbb{N}\setminus 0}$. We put

$$\overline{D}_{n,m,1} = \tilde{D}_{n,m} \cap [\![0, T^{n,m,1}]\!], \quad \text{and} \quad \overline{D}_{n,m,k} = \tilde{D}_{n,m} \cap]\!]T^{n,m,k-1}, T^{n,m,k}]\!]$$

for $k = 2, 3, \ldots$ and all $(m, n) \in \mathbb{N}^2$. Now

$$\mathbb{1}_{\overline{D}_{n,m,1}} \bullet S = \left(\mathbb{1}_{\tilde{D}_{n,m}} \bullet S\right)^{T^{n,m,1}} = \left(\tilde{S}^{n,m}\right)^{T^{n,m,1}}$$

is a uniformly integrable martingale and so is

$$\begin{aligned} \mathbb{1}_{\overline{D}_{n,m,k}} \bullet S &= \left(\mathbb{1}_{\tilde{D}_{n,m}} \bullet S\right)^{T^{n,m,k}} - \left(\mathbb{1}_{\tilde{D}_{n,m}} \bullet S\right)^{T^{n,m,k-1}} \\ &= \left(\tilde{S}^{n,m}\right)^{T^{n,m,k}} - \left(\tilde{S}^{n,m}\right)^{T^{n,m,k-1}} \end{aligned}$$

The sets $\overline{D}_{n,m,k}$ are subsets of the predictable σ-algebra and form a countable partition of $\Omega \times \mathbb{R}_+$. Thus, by Lemma 1, S is a σ-martingale. □

The following corollary is immediate.

Corollary 1. *Every local σ-martingale X is a σ-martingale.*

The following result shows that the set of σ-martingales is closed under stochastic integration, as opposed to the set of local martingales.

Corollary 2. *Let $S \in \mathcal{M}_\sigma^d$ and $H \in L(S)$. Then, $H \bullet S$ is also a σ-martingale.*

Proof. Let $\tilde{S}^1, \ldots, \tilde{S}^d$ be the components of S. Consider a Σ-localizing sequence $(D_n)_{n\in\mathbb{N}}$ and define $\tilde{D}_n = \{(\omega, t) \in \Omega \times \mathbb{R}_+ \,;\, \|H_t(\omega)\|_u \leq n\}$.

Since H is a predictable process, all of the $\tilde{D}_n$ lie in the predictable σ-algebra. Therefore, the sets $\overline{D}_n = D_n \cap \tilde{D}_n$ are predictable and we have $\overline{D}_n \subseteq \overline{D}_{n+1}$ and $\bigcup_{n=1}^\infty \overline{D}_n = \Omega \times \mathbb{R}_+$. By putting $H^n := H\mathbb{1}_{\tilde{D}_n}$ and $S^n := (\mathbb{1}_{D_n} \bullet \tilde{S}^1, \ldots, \mathbb{1}_{D_n} \bullet \tilde{S}^d)^\top$, the process H^n is bounded, and, by the linearity of the integral, we obtain

$$\mathbb{1}_{\overline{D}_n} \bullet (H \bullet S) = (\mathbb{1}_{D_n}\mathbb{1}_{\tilde{D}_n}) \bullet (H \bullet S) = \left(\mathbb{1}_{\tilde{D}_n} H \mathbb{1}_{D_n}\right) \bullet S = H^n \bullet S^n$$

Hence, since S^n is a local martingale, $\mathbb{1}_{\overline{D}_n} \bullet (H \bullet S)$ is also a local martingale and thus $\overline{D}_n$ a Σ-localizing sequence. By Theorem 2, $H \bullet S$ is a σ-martingale. □

Corollary 3. *For a σ-martingale S with Σ-localizing sequence $(D_n)_{n\in\mathbb{N}}$ and a sequence of subsets of the predictable σ-algebra $(\tilde{D}_n)_{n\in\mathbb{N}}$, which satisfies $D_n \subset D_{n+1}$ and $\bigcup_{n=1}^\infty D_n = \Omega \times \mathbb{R}_+$, $(\overline{D}_n)_{n\in\mathbb{N}} := (D_n \cap \tilde{D}_n)_{n\in\mathbb{N}}$ is also a Σ-localizing sequence.*

Proof. We have

$$\mathbb{1}_{\overline{D}_n} \bullet S = \left(\mathbb{1}_{D_n}\mathbb{1}_{\tilde{D}_n}\right) \bullet S = \left(\mathbb{1}_{\tilde{D}_n} \bullet (\mathbb{1}_{D_n} \bullet S)\right).$$

Since $(D_n)_{n\in\mathbb{N}}$ is a Σ-localizing sequence, $\mathbb{1}_{D_n} \bullet S$ is, by definition, a local martingale, and, since $\mathbb{1}_{\tilde{D}_n}$ is bounded, $\mathbb{1}_{\overline{D}_n} \bullet S$ is a local martingale for all n and $(\overline{D}_n)_{n\in\mathbb{N}}$ is a Σ-localizing sequence. □

Corollary 4. *The set of σ-martingales forms a vector space.*

Proof. Without loss of generality, we assume $d = 1$. Consider $\alpha, \beta \in \mathbb{R}$ and $X, Y \in \mathcal{M}_\sigma^1$ with Σ-localizing sequences $(D_n)_{n\in\mathbb{N}}$ for X and $(\tilde{D}_n)_{n\in\mathbb{N}}$ for Y. By Corollary 3, $(D_n \cap \tilde{D}_n)_{n\in\mathbb{N}}$ is a Σ-localizing sequence for both X and Y and we have

$$\mathbb{1}_{D_n\cap\tilde{D}_n} \bullet (\alpha X + \beta Y) = \alpha\left(\mathbb{1}_{D_n\cap\tilde{D}_n} \bullet X\right) + \beta\left(\mathbb{1}_{D_n\cap\tilde{D}_n} \bullet Y\right).$$

Since $\mathcal{M}_{\text{loc}}^1$ is a vector space, $\mathbb{1}_{D_n\cap\tilde{D}_n} \bullet (\alpha X + \beta Y)$ is a local martingale and thus $D_n \cap \tilde{D}_n$ is a Σ-localizing sequence for $\alpha X + \beta Y$. Hence, $\alpha X + \beta Y$ is a σ-martingale. □

We now come to one of the main statements about σ-martingales. As mentioned earlier, $H \bullet M$ for $M \in \mathcal{M}_{\text{loc}}$ is not necessarily a local martingale. Hence, we have a closer look at the class

$$\{H \bullet M \,;\, H \in L(M), M \in \mathcal{M}_{\text{loc}}\}$$

and it turns out that this class corresponds exactly to the vector space of the σ-martingales. Furthermore, by proving this theorem, we also show that our definition of a σ-martingale is equivalent to the definition used in [1,4].

The theorem is mentioned in almost every publication about σ-martingales (for example, in ([4], Theorem IV.Theorem 89) or ([5], Theorem 6.4.1)). Because of our different approach, the proof given here differs slightly from the one given in the above-mentioned literature.

Theorem 3. *Let $X = (X^1, \dots, X^d)$ be a d-dimensional semimartingale. The following are equivalent:*

(i) *The process X is a σ-martingale.*

(ii) *There exists a strictly positive process $H \in \mathcal{P}$ and an $\mathcal{H}^1$-martingale $M = (M^1, \dots, M^d)$ with*
$$X^i = H \bullet M^i \quad \text{for all } i \in \{1, \dots, d\}.$$

(iii) *There exists a strictly positive process $H \in \mathcal{P}$ and a martingale $M = (M^1, \dots, M^d)$ with*
$$X^i = H \bullet M^i \quad \text{for all } i \in \{1, \dots, d\}.$$

(iv) *There exists a strictly positive process $H \in \mathcal{P}$ and a local martingale $M = (M^1, \dots, M^d)$ with*
$$X^i = H \bullet M^i \quad \text{for all } i \in \{1, \dots, d\}.$$

(v) *There exists a local martingale $M = (M^1, \dots, M^d)$ and a predictable process $H = (H^1, \dots, H^d)$ with*
$$H^i \in L(M^i) \quad \text{and} \quad X^i = H^i \bullet M^i \quad \text{for all } i \in \{1, \dots, d\}.$$

Proof. The implications $(ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (v)$ are clear.

$(i) \Rightarrow (ii)$. By assumption, there exists a σ-localizing sequence $(D_n)_{n\in\mathbb{N}}$. By Theorem A4, each martingale is locally in $\mathcal{H}^1$. Hence, for each $n \in \mathbb{N}$ and each $i \in \{1, \dots, d\}$, there exists an increasing sequence of stopping times $(T_i^{n,m})_{m\in\mathbb{N}}$ tending to infinity, such that $\left(\mathbb{1}_{D_n} \bullet X^i\right)^{T_i^{n,m}} \in \mathcal{H}^1$. Therefore, we can construct a sequence $(T^{n,m})_{m\in\mathbb{N}}$ of stopping times, such that $\left(\mathbb{1}_{D_n} \bullet X^i\right)^{T^{n,m}} \in \mathcal{H}^1$ for all $n, m \in \mathbb{N}$ and all $i \in \{1, \dots, d\}$.

We choose appropriate $\alpha_{m,n} > 0$ such that
$$\max_{i\in\{1,\dots,d\}} \left\| \alpha_{n,m} \left(M^{n,i}\right)^{T^{n,m}} \right\|_{\mathcal{H}^1} \leq 2^{-(m+n)}, \quad \text{for all } m, n \in \mathbb{N}$$
and put $T^{0,n} := 0$ as well as
$$\tilde{K} := \mathbb{1}_0 + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \alpha_{m,n} \mathbb{1}_{]\!]T^{n,m-1}, T^{m,n}]\!] \cap D_n}$$
$$N^i := \tilde{K} \bullet X^i, \quad i = 1, \dots, d.$$

Because of
$$\mathbb{1}_{]\!]T^{n,m-1}, T^{n,m}]\!] \cap D_n} \bullet X = \left(\mathbb{1}_{D_n} \bullet X\right)^{T^{n,m}} \in \mathcal{H}^1,$$
N^i is the limit of a sequence of $\mathcal{H}^1$-martingales which is convergent in $\mathcal{H}^1$. Since, by Lemma A1, $\mathcal{H}^1$ is a Banach space, N is also an $\mathcal{H}^1$-martingale. Furthermore, we have $\tilde{K} > 0$ and hence $K = \tilde{K}^{-1}$ exists and we obtain $X^i = K \bullet N^i$ for all $i \in \{1, \dots, d\}$.

$(v) \Rightarrow (i)$. As every martingale is locally in $\mathcal{H}^1$, for each i, there exists an increasing sequence of stopping times $(T_i^n)_{n\in\mathbb{N}}$ tending to infinity, such that $(M^i)^{T_i^n} \in \mathcal{H}^1$. Hence, we

can construct an increasing sequence of stopping times $(T^n)_{n\in\mathbb{N}}$ tending to infinity, such that $(M^i)^{T^n} \in \mathcal{H}^1$ for all $n \in \mathbb{N}$ and for all $i \in \{1, \dots, d\}$. We define

$$D_n := \bigcap_{i=1}^{d} \left\{ \left| H^i \right| \leq n \right\} \cap [\![0, T^n]\!] \subset \Omega \times \mathbb{R}_+$$

and hence, we have

$$\mathbb{1}_{D_n} \bullet X^i = \left(\mathbb{1}_{D_n} H^i \right) \bullet M^i = \left(\mathbb{1}_{D_n} H^i \right) \bullet (M^i)^{T^n}.$$

Because of $\left(\mathbb{1}_{D_n} H^i \right) \leq n$ and because the process $(M^i)^{T^n}$, stopped at T^n, is an $\mathcal{H}^1$-martingale, $\mathbb{1}_{\tilde{D}_n} \bullet X^i$ is for all n a local martingale, as a stochastic integral with a locally bounded integrand and a local martingale integrator is again a local martingale by Theorem A5. Therefore, there exists a Σ-localizing sequence for X, and thus X is a σ-martingale. □

The following lemma is a simple yet useful result about general stochastic integration. To the best of our knowledge, it has only been explicitly mentioned in the unpublished work [8]. Alternatively, it can be derived as a corollary from [15], although the latter uses a different approach and slightly different terminology. Since this result is helpful for our purposes, we provide a proof here.

Lemma 2. *Let $(X^n)_{n\in\mathbb{N}}$ be a sequence of local martingales, which converges to a process X in ucp. If $(\sup_n (X^n)^*_t)_{t\in\mathbb{R}_+}$ is locally integrable, then X is a local martingale.*

Proof. Without loss of generality, we assume all processes to be one dimensional. Because of the ucp convergence, we can conclude that X is also càdlàg and adapted. By assuming a suitable subsequence, we can, with Theorem A1, also assume that the convergence is almost surely on compact subsets, and thus

$$M_t := \sup_n (X^n)^*_t$$

is also càdlàg and adapted. Furthermore, X^* is increasing, and we have

$$|\Delta M_t| \leq 2 \sup_n (X^n)^*_t.$$

By assumption, the right-hand side is locally integrable; thus, M is also locally integrable.

We now want to show that X is a local martingale. In order to prove that, we have to find a sequence T^k such that X^{T^k} is a martingale for all k. For that, it suffices to show that, for every stopping time τ, we have $\mathbf{E}[X_0^{T^k}] = \mathbf{E}\left[X_\tau^{T^k}\right]$ $(= \mathbf{E}[X_{\tau\wedge T^k}])$ by the martingale criterion Theorem A3.

First, note that we can find a sequence $(T^k)_{k\in\mathbb{N}}$ such that $(X^n)^{T^k}$ is a martingale for all n and k and M^{T^k} is integrable for all k. And, because of $X^{T^k} \leq M$, we can apply the dominated convergence theorem and obtain with Corollary A1 for every bounded stopping time τ

$$\mathbf{E}\left[X_\tau^{T^k}\right] = \mathbf{E}\left[\lim_{n\to\infty} (X^n)_\tau^{T^k}\right] = \lim_{n\to\infty} \mathbf{E}\left[(X^n)_\tau^{T^k}\right] = \lim_{n\to\infty} \mathbf{E}\left[X_0^{T^k}\right] = \mathbf{E}\left[X_0^{T^k}\right].$$

Hence, X^{T^k} is a martingale for all k; thus, we conclude that X is a local martingale. □

σ-martingales are processes that behave "like" local martingales. It can even be shown that σ-martingales are semimartingales with vanishing drift ([6], Lemma 2.1). It therefore raises the questions of why they are not local martingales and what additional assumption must be made so that they are. We have a criterion available with Lemma 2, which allows us to prove the following simple criterion for this question. Despite this simplicity, to the best of our knowledge, it is not explicitly mentioned in the σ-martingale literature. However, it will be enormously helpful for this new approach.

Theorem 4. *A σ-martingale X is a local martingale if and only if it is locally integrable.*

Proof. Since every local martingale is a σ-martingale and locally integrable, it is enough to prove the converse.

Let X be, without loss of generality, a locally integrable one-dimensional σ-martingale. By Theorem 3, there exists a representation $X = H \bullet M$ with $M \in \mathcal{M}^1_{\text{loc}}$ and $H \in L(X)$. We define $H^n := H\mathbb{1}_{\{|H| \leq n\}}$. Clearly, we have $|H^n| \leq |H|$ and H^n is a bounded predictable process. We obtain

$$|\Delta(H^n \bullet M)| = |H^n||\Delta M| \leq |H||\Delta M| = |\Delta(H \bullet M)|.$$

Since each H^n is bounded for all n, we apply Theorem A5 and obtain $H^n \bullet M \in \mathcal{M}_{\text{loc}}$ for all $n \in \mathbb{N}$, and, with the Dominated Convergence Theorem, we obtain $H^n \bullet M \xrightarrow{\text{ucp}} H \bullet M$. Choosing a subsequence, we can assume by Theorem A1 that $H^n \bullet M \to H \bullet M$ converges almost surely on compact subsets.

We put $N_t := \sup_{n \in \mathbb{N}} (H^n \bullet M)_t$ and N is an adapted càdlàg process. Since $H \bullet M_-$ is left continuous and hence locally bounded, it is also locally integrable. Since $H \bullet M$ is locally integrable by assumption, $\Delta(H \bullet M) = H \bullet M - H \bullet M_-$ is also locally integrable.

Furthermore, we have

$$\Delta N = \Delta\left(\sup_{n \in \mathbb{N}} (H^n \bullet M)\right) \leq \sup_{n \in \mathbb{N}} (\Delta(H^n \bullet M)) \leq \Delta(H \bullet M).$$

Hence, ΔN and therefore also $\Delta(N^*)$ are locally integrable. Since any càdlàg process is locally integrable if its jump process is locally integrable, N^* and thus $\sup_n (H^n \bullet M)^*$ are also locally integrable. Now the result follows from Lemma 2. □

As every continuous semimartingale is locally integrable, the following corollary is immediate:

Corollary 5. *Every continuous σ-martingale is a local martingale.*

Remark 2. *As opposed to the criterion above, it is well known that any σ-martingale that is also a special semimartingale (see, for example, ([9], Definition 11.6.9) for the definition of a special semimartingale) is a local martingale (([9], Corollary 12.3.20) or ([4], Theorem IV.91)) and it can be shown that a semimartingale is a special semimartingale if and only if its supremum process is locally integrable (see, for example, ([9], Theorem 11.6.10) or ([16], Theorem 8.6)). Hence, we obtain that a local martingale is locally integrable if and only if its supremum process is locally integrable.*

The following theorem is of principal importance in financial mathematics. It can be found in many publications on financial mathematics using the semimartingale terminology (not only is it mentioned in almost all of the publications we mentioned frequently in this work, such as [1,14,17,18], but it also mentioned in many textbooks dealing with the different aspects of financial mathematics such as [19–22]). However, to our knowledge,

the only published proofs are the French-language original publication [23], Corollaire 3.5, and the more recent [24]. Theorem 4 enables us to give an alternative proof.

Theorem 5 (Ansel–Stricker)**.** *A one-sided bounded σ-martingale X is a local martingale. If X is bounded from below (resp. above), it is also a supermartingale (resp. submartingale).*

Proof. Assume, without loss of generality, $X \geq 0$ and X to be one dimensional. By Theorem 3, there exists a representation $X = H \bullet M$ with $M \in \mathcal{M}^1_{\text{loc}}$ and $H \in L(M)$.

Proceeding analogously to the proof of Theorem 4, we find a sequence $(H^n)_{n\in\mathbb{N}}$ of locally bounded predictable processes from $L(M)$ such that $H^n \bullet M \xrightarrow{\text{ucp}} H \bullet M$. Furthermore, we can assume that $H^n \bullet M \geq 0$ and $H^n \bullet M \to H \bullet M$ almost surely on compact sets (we can always find a modification of a subsequence for which these properties hold). Since the H^n are locally bounded, $H^n \bullet M$ is a local martingale for all n. Hence, we can find a sequence of stopping times $(T^k)_{k\in\mathbb{N}}$, such that $H^n \bullet M^{T^k}$ is a martingale for all $n, k \in \mathbb{N}$.

By Fatou's lemma, we know that

$$\begin{aligned}\mathbf{E}\left[H \bullet M_t^{T^k}\right] &= \mathbf{E}\left[\liminf_{n\to\infty} H^n \bullet M_t^{T^k}\right] \\ &\leq \liminf_{n\to\infty} \mathbf{E}\left[H^n \bullet M_t^{T^k}\right] \\ &= \liminf_{n\to\infty} \mathbf{E}[H_0^n M_0] = \mathbf{E}[H_0 M_0] < \infty.\end{aligned}$$

Hence, X^{T^k} is integrable and X locally integrable. By Theorem 4, it follows that $X \in \mathcal{M}_{\text{loc}}$.

We still have to show the supermartingale property. Therefore, still assuming $X \geq 0$, let $0 \leq s \leq t$, $A \in \mathcal{F}_s$, and let $\{T^k\}_{k\in\mathbb{N}}$ be the localizing times chosen above. Observe that, for any ω with $T^k(\omega) \geq s$,

$$\mathbb{1}_A \left(X_t^{T^k} - X_s^{T^k}\right) \;\geq\; \mathbb{1}_A \left(-X_s^{T^k}\right) \;=\; -\,\mathbb{1}_A\, X_s^{T^k} \;=\; -\,X_s,$$

where the last equality holds since on $\{T^k(\omega) \geq s\}$, we indeed have $X_s^{T^k} = X_s$. (The case $T^k < s$ is even simpler to handle, as then the difference is $X_t^{T^k} - X_s^{T^k} = 0$.) Hence,

$$Z_k \;:=\; \mathbb{1}_A \left(X_t^{T^k} - X_s^{T^k}\right) \;+\; X_s \;\geq\; 0.$$

Thus, Z_k is non-negative and, by local integrability of X, we can again use Fatou's lemma as $k \to \infty$. In the limit, we obtain

$$\begin{aligned}\mathbf{E}[\mathbb{1}_A(X_t - X_s)] &= \mathbf{E}\left[\lim_{k\to\infty}\left(\mathbb{1}_A\left(X_t^{T^k} - X_s^{T^k}\right) + X_s\right)\right] - \mathbf{E}[X_s] \\ &\leq \liminf_{k\to\infty} \mathbf{E}\left[\mathbb{1}_A\left(X_t^{T^k} - X_s^{T^k}\right) + X_s\right] - \mathbf{E}[X_s] \\ &\leq \liminf_{k\to\infty} \mathbf{E}\left[\mathbb{1}_A\left(X_t^{T^k} - X_s^{T^k}\right)\right] \\ &= 0.\end{aligned}$$

Hence, $\mathbf{E}[\mathbb{1}_A(X_t - X_s)] \leq 0$, which is precisely the supermartingale property. (In the case X is instead bounded above by 0, a symmetric argument shows that X is a submartingale.) □

In finite, discrete time, any non-negative local martingale that is bounded from below is a martingale and not just a supermartingale. The difference in continuous time is that

$\mathbf{E}[Y_t] < \infty$ for all t does not imply $\mathbf{E}[\sup_t Y_t] < \infty$ (not even on compacts). Thus, there is no integrable pointwise majorant which would be needed to prove the martingale property.

Remark 3. *There are three other approaches to proving the Ansel–Stricker lemma in the literature.*

The original proof by Ansel and Stricker [23]: This approach relies on the classification and control of jumps and the theory of special semimartingales. It involves intricate technical calculations and constructs a sequence that converges to the integral $H \bullet M$, ensuring that this sequence is bounded from below by integrable random variables.

The proof by De Donno and Pratelli (2007) [24]: This method categorizes jumps into positive and negative components to manage the stochastic integral's behavior. It leverages Fatou's lemma and Lebesgue's Theorem to control approximation sequences, thereby circumventing the complexities associated with special semimartingales.

The proof by Gushchin (2015) [7]: Here, the lemma by Ansel–Stricker appears as a corollary, a more general result that is very close to Theorem 6. This approach relies extensively on the theory of special semimartingales, utilizing their structural properties to establish the result.

As mentioned earlier, the Ansel–Stricker lemma is widely referenced in the literature on mathematical finance, particularly in the context of semimartingale models. However, our proof shows that with minimal additional effort, the result can be further generalized. This generalization is crucial for the analysis of general semimartingale market models. Surprisingly, this result does not seem to be widely known. To the best of our knowledge, only in ([7], Theorem 3.21), is a similar result mentioned.

Theorem 6. *Let X be a (one-dimensional) σ-martingale. The following statements are equivalent:*

(i) *X is a local martingale.*

(ii) *There exists a local martingale M and a càdlàg process of locally integrable variation A such that*
$$\Delta X \geq \Delta(M + A).$$

(iii) *There exists a local martingale M and a càdlàg process A with $\sup_{s \leq t} A_s$ locally integrable, such that*
$$\Delta X \geq \Delta(M + A).$$

(iv) *There exists a local martingale M and a càdlàg process of locally integrable variation A such that*
$$X \geq M + A.$$

(v) *There exists a local martingale M and a càdlàg process A with $\sup_{s \leq t} A_s$ locally integrable, such that*
$$X \geq M + A.$$

Proof. $(i) \Rightarrow (ii), (iii), (iv), (v)$. If X is a local martingale, then one may choose $M = X$ and $A \equiv 0$; hence, all the asserted inequalities are satisfied trivially.

$(v) \Rightarrow (i)$. Assume (v), i.e., there exists a local martingale M and a càdlàg process A with $\sup_{s \leq t} A_s$ locally integrable such that

$$X \geq M + A.$$

Since the family of local martingales is a vector space, X is a local martingale if and only if the difference

$$X - M = (X - M - A) + A$$

is a local martingale. In view of the inequality, we may, without loss of generality, assume that

$$X \geq A.$$

Then, one may follow the strategy used in the proof of Theorem 5 (the Ansel–Stricker result): Represent X as a stochastic integral $X = H \bullet N$ (with predictable H and a local L^1 martingale N) and approximate H by a sequence $(H^n)_{n\in\mathbb{N}}$ of bounded predictable processes so that the stochastic integrals $H^n \bullet N$ are local martingales. Using the generalized Fatou's Lemma (Theorem A2) to pass to the limit shows that X is locally integrable and hence a local martingale.

$(iii) \Rightarrow (i)$. We have $\Delta X \geq \Delta(M + A)$ for some local martingale M and a càdlàg A with locally integrable supremum. Set

$$Y := X_- - \left|\Delta(M + A)\right|.$$

Since X_- is a left-continuous (càglàd) adapted process, $\sup_{s\leq t} X_{s-}$ is locally integrable, and so is $\left|\Delta(M + A)\right|$. Hence, Y serves as an integrable lower bound. Repeating the Fatou-type limit argument from the $(v) \Rightarrow (i)$ case shows that X is locally integrable and therefore a local martingale. □

As mentioned above, Theorem 6 is not commonly mentioned in the mathematical finance literature. Nevertheless, it turns out to be essential for defining a general semimartingale market model.

Example 3. *Consider a financial market consisting of $d + 1$ assets, whose prices are modeled by the $(d + 1)$-dimensional semimartingale*

$$S = \left(S^0, S^1, \ldots, S^d\right).$$

A trading strategy is represented by a $(d + 1)$-dimensional predictable process

$$\varphi = \left(\varphi^0, \varphi^1, \ldots, \varphi^d\right) \in L(S),$$

meaning that φ is integrable with respect to S. The investor's wealth process is given by

$$V_t = \varphi_t^\top S_t,$$

and, under the self-financing condition (i.e., no capital inflows or outflows, all gains are reinvested, and losses are not compensated), we have

$$V_t = \varphi \bullet S_t.$$

In many such markets, riskless profits can be obtained via the popular doubling strategy. However, such a strategy requires an infinite line of credit, which is unrealistic. Consequently, a trading strategy is typically deemed admissible if the wealth process satisfies

$$V_t \geq \alpha \quad \textit{for all } t,$$

for some constant α.

Since it is not sufficient to consider only nominal values, one introduces a numeraire—namely, a strictly positive semimartingale N satisfying

$$\inf_{0\leq s\leq t} N_s > 0 \quad \textit{for all } t.$$

Let us assume S^0 is a numeraire, and let us consider the discounted price process

$$\hat{S} = \left(\frac{S^1}{S^0}, \frac{S^2}{S^0}, \ldots, \frac{S^d}{S^0} \right).$$

Then, the First Fundamental Theorem of Asset Pricing [1] asserts that, under an appropriate no-arbitrage condition, there exists an equivalent probability measure $\mathbf{Q}$ such that $\hat{S}$ is a d-dimensional σ-martingale.

For the analysis of the financial market, it is desirable to conclude that the discounted wealth process

$$\hat{V} := \varphi \bullet \hat{S} = \frac{V}{S^0}$$

is a local martingale rather than merely a σ-martingale. The Ansel–Stricker lemma guarantees that a one-sided bounded σ-martingale is a local martingale. Although this lemma would apply directly if $\varphi \bullet \hat{S}$ were bounded from below, in practice, we only assume that the original wealth process V is bounded from below—not necessarily the discounted wealth $\hat{V}$. However, since

$$\hat{V} = \frac{V}{S^0} \geq \frac{\alpha}{S^0},$$

and because S^0 is a numeraire (so that α / S^0 serves as an integrable lower bound), we can use Theorem 6 to conclude that $\hat{V}$ is a local martingale.

To conclude the study about σ-martingales, let us illustrate that there are σ-martingales where no equivalent probability measure exists under which it is a local martingale. The example is inspired by ([1], Example 2.3) and ([14], Example 5.3).

Example 4. *Assume a two-dimensional σ-martingale $Z = (X, Y)$, with X being the process from Example 2 and*

$$Y_t = \begin{cases} -2t & \text{for } t < \tau, \\ 1 - 2\tau & \text{else.} \end{cases}$$

Let $\mathcal{F}$ be the σ-algebra created by τ and ξ (from Example 2) and, furthermore, let $\mathcal{F}_t$ be the σ-algebra created by X_t. We examine the processes with respect to the according filtered probability space. Obviously, Y is a stopped compensated Poisson Process. As the compensated Poisson Process is a martingale (see, for example, ([9], Theorem 5.5.18)) and a stopped martingale is again a martingale by the Optional Stopping Theorem, Y is also a martingale. Hence, Z is a σ-martingale that is not a local martingale, and we are going to show that no probability measure $\mathbf{Q} \sim \mathbf{P}$ exists such that Z is a local martingale under $\mathbf{Q}$.

To that end, let $\mathbf{Q}$ be an equivalent probability such that Y is a σ-martingale under $\mathbf{Q}$. For each $t > 0$, the stopped process Y^t is bounded and since, by Theorem 5, any bounded σ-martingale is a martingale, we conclude that Y^t is a martingale. Thus, we have $\mathbf{E}_\mathbf{Q}[Y_t] = 0$ for all $t \geq 0$. With f being the density function of τ under $\mathbf{Q}$ and F being the cumulative distribution function of τ (that means $F(t) = \mathbf{Q}(\tau \leq t) = \int_0^t f(s)\mathrm{d}s$), we obtain

$$\begin{aligned} 0 = \mathbf{E}_\mathbf{Q}[Y_t] &= -2t\mathbf{Q}(\tau > t) + \int_0^t (1 - 2s) f(s)\mathrm{d}s \\ &= -2t\mathbf{Q}(\tau > t) + \int_0^t f(s)\mathrm{d}s - 2\int_0^t -s f(s)\mathrm{d}s \\ &= -2t\mathbf{Q}(\tau > t) + F(t) - 2\left([sF(s)]_0^t - \int_0^t F(s)\mathrm{d}s \right) \\ &= -2t\mathbf{Q}(\tau > t) - 2t\mathbf{Q}(\tau \leq t) + F(t) + 2\int_0^t F(s)\mathrm{d}s \end{aligned}$$

$$= -2t + F(t) + 2\int_0^t F(s)\mathrm{d}s$$

We derive with respect to t and obtain $0 = -2 + F'(t) + 2F(t)$. *By putting* $G(t) := 1 - F(t)$, *we obtain* $G'(t) = -F'(t)$ *and* $F'(t) = 2G(t)$. *Thus,*

$$G'(t) = -2G(t) \text{ and } G(0) = 1.$$

Hence, we obtain $G(t) = \exp(-2t)$ *and thus*

$$\mathbf{Q}(\tau > t) = 1 - \mathbf{Q}(\tau \leq t) = 1 - F(t) = G(t) = \exp(-2t) = \mathbf{P}(\tau > t).$$

Now, we turn to X. Our first goal is to show that we have $\mathbf{E_Q}[\xi \mid \tau] = 0$ *or, equivalently,* $\int_A \xi = 0$ *for all* $A \in \sigma(\tau)$. *We proceed in a sequence of steps.*

1. *We have* $\mathbf{E_Q}\left[\frac{\xi}{\tau}\mathbb{1}_{\{s<\tau\leq t\}}\right] = 0$.
   For any $s > 0$, *we define* $\tilde{X}_t^{(s)} := X_t - X_t^s$ *where* $X_t^s = X_{s\wedge t}$. *Then,* $\tilde{X}^{(s)}$ *is clearly a* σ*-martingale. Furthermore, we have*

$$\tilde{X}_t^{(s)} = \begin{cases} 0 & \text{for } t \leq s \\ X_t - X_s, & \text{for } t > s, \end{cases} = \begin{cases} 0 & \text{for } t \leq s, \\ 0 & \text{for } t > s \text{ and } t < \tau, \\ \frac{\xi}{\tau} & \text{for } t > s \text{ and } t \geq \tau \text{ and } s < \tau, \\ 0 & \text{for } t > s \text{ and } s \geq \tau. \end{cases}$$

   And, since $\left|\frac{\xi}{\tau}\right| < \left|\frac{\xi}{s}\right|$ *for* $s < \tau$ *and* $|\xi| \leq 1$, *we conclude that* $\tilde{X}_t^{(s)}$ *is a bounded* σ*-martingale for all* $s > 0$. *Hence, by Theorem 5, it is a martingale and we obtain*

$$0 = \mathbf{E_Q}\left[\tilde{X}_t^{(s)}\right] = \mathbf{E_Q}\left[X_t\mathbb{1}_{\{s<\tau\leq t\}}\right] = \mathbf{E_Q}\left[\frac{\xi}{\tau}\mathbb{1}_{\{s<\tau\leq t\}}\right].$$

2. *We have* $\int_A \xi \mathrm{d}\mathbf{Q} = 0$ *for all* $A \in \{\{\tau \in (s,t]\}\,;\, 0 < s < t\}$.
   Let $A \in \{\{\tau \in (s,t]\}\,;\, 0 < s < t\}$. *Then, we have* $\mathbf{E_Q}\left|\frac{\xi}{\tau}\mathbb{1}_A\right| < \infty$ *and we obtain*

$$\begin{aligned} \frac{\mathbb{1}_A}{\tau}\int_A \xi \mathrm{d}\mathbf{Q} = \frac{\mathbb{1}_A}{\tau}\int_A \mathbf{E_Q}[\xi \mid \tau]\mathrm{d}\mathbf{Q} &= \int_A \mathbf{E_Q}\left[\frac{\mathbb{1}_A}{\tau}\xi \mid \tau\right]\mathrm{d}\mathbf{Q} \\ &= \int_A \frac{\mathbb{1}_A}{\tau}\xi \mathrm{d}\mathbf{Q} = 0. \end{aligned}$$

   As $\frac{\mathbb{1}_A}{\tau}$ *is not constantly zero, we conclude that* $\int_A \xi \mathrm{d}\mathbf{Q} = 0$.
3. *We have* $\int_A \xi \mathrm{d}\mathbf{Q} = 0$ *for all* $A \in \sigma(\tau)$.
   By definition, we have $\int_A \xi \mathrm{d}\mathbf{Q} = \int_A \xi^+ \mathrm{d}\mathbf{Q} - \int_A \xi^- \mathrm{d}\mathbf{Q}$. *Both summands can be seen as measures on* Ω, *and, since the half-open intervals are an* $\cap$*-closed generator of the Borel* σ*-algebra (see, for example, ([25], Theorem 1.23)), these measures are uniquely determined by its values on* $\{\{\tau \in (s,t]\}\,;\, 0 < s < t\}$ *(see, for example, ([25], Lemma 1.42)) and we conclude that* $\int_A \xi \mathrm{d}\mathbf{Q} = 0$ *for all* $A \in \sigma(\tau)$ *and hence*

$$\mathbf{E_Q}[\xi \mid \tau] = 0. \tag{1}$$

Since $\mathbf{Q}$ *and* $\mathbf{P}$ *are equivalent probability measures, we have*

$$0 < \mathbf{Q}(\xi = 1 \mid \tau), \mathbf{Q}(\xi = -1 \mid \tau) < 1 \quad \Longleftrightarrow \quad 0 < \mathbf{P}(\xi = 1 \mid \tau), \mathbf{P}(\xi = -1 \mid \tau) < 1.$$

Thus, with (1), we obtain

$$\mathbf{Q}(\xi = -1 \mid \tau) = \mathbf{Q}(\xi = 1 \mid \tau) = \frac{1}{2}$$

almost surely and hence ξ and τ are $\mathbf{Q}$ independent. We conclude that $\mathbf{Q} = \mathbf{P}$.

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Appendix A. Some Cited Results

The following theorem can be found in ([9], Lemma 1.3.38).

Theorem A1. *Suppose that $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of measurable functions, and $\{f_n\}_{n\in\mathbb{N}}$ converges in measure. Then, there exists a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ which converges almost everywhere.*

The following theorem is a trivial extension of Fatou's lemma. A version of it can, for example, be found in ([26], Theorem 9.19).

Theorem A2 (Generalized Fatou's Lemma)**.** *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions on a measure space $(\Omega, \mathcal{F}, \mu)$. Suppose there exists an integrable function $g \in L^1(\mu)$ such that*

$$f_n \geq g \quad \text{for all } n.$$

Then,

$$\int \liminf_{n\to\infty} f_n \, d\mu \leq \liminf_{n\to\infty} \int f_n \, d\mu.$$

The following criterion for a process X to be a martingale will be useful. A proof can be found in ([9], Theorem 5.4.6).

Theorem A3. *An adapted càdlàg process X with closure X_∞ is a uniformly integrable martingale if and only if we have*

1. *$X_T \in L^1$ for each stopping time T;*
2. *$\mathbf{E}X_T = \mathbf{E}X_0$.*

An immediate consequence of the optional sampling theorem is the following corollary (see, for example, ([4], Theorem I.18)).

Corollary A1. *Let X be a right continuous martingale, and let T be a stopping time. Then, X^T is also a right continuous martingale. If T is bounded or X is uniformly integrable, then we have $X^T \in L^1$.*

Theorem A4 ([4], Theorem IV.51)**.** *Let M be a local martingale. Then, M is locally in $\mathcal{H}^1$.*

The next lemma is taken from ([9], Lemma 10.1.5).

Lemma A1. *Identifying indistinguishable martingales, $\mathcal{H}^1$ is a Banach space under the $\mathcal{H}^1$ norm.*

We used the following criterion from ([4], Theorem IV.29) multiple times.

Theorem A5. *Let X be a local martingale and let $H \in \mathcal{P}$ be locally bounded. Then, the stochastic integral $H \bullet X$ is also a local martingale.*

The Dominated Convergence Theorem for Stochastic Integration is a fundamental result in stochastic analysis and can be found in almost every textbook on the subject. For instance, it is presented in ([4], Theorem IV.32), ([9], Theorem 12.4.10), and the more recent ([8], Theorem 9.16).

Theorem A6 (Dominated Convergence Theorem)**.** *Let $X \in \mathcal{S}^d$ be a semimartingale, and let $H^m = (\tilde{H}^{1,m}, \dots \tilde{H}^{d,m}) \in \mathcal{P}$ be a sequence converging a.s. to a limit H. If there exists a process $G = (\tilde{G}^1, \dots, \tilde{G}^d)$ with $\tilde{G}^i \in L(X)$ for all $i \in \{1, \dots, d\}$ such that $\left|\tilde{H}^{i,m}\right| \leq G$ for all $i \in \{1, \dots, d\}$ and all $m \in \mathbb{N}$, then H^m, H are in $L(X)$ and $H^m \bullet X$ converges to $H \bullet X$ in the Émery topology (and hence also in ucp).*

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