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Reflexivity and Duplicability in Set Theory

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Abstract: Set reflexivity and duplicability are considered by showing, with different proofs, their equivalence with Dedekind's infinity. Then, an easy derivation of the Schröder–Bernstein theorem is presented, a fundamental result in the theory of cardinal numbers, usually based on arguments that are not very intuitive.

Keywords: cardinality; choice axiom; set duplicability; Dedekind infinity

MSC: 03E25

1. Introduction

In a search for a simple proof of a well-known result about infinite sets, I drew a diagram (see later on), which immediately suggested a striking analogy with the picture of an egg, a sort of contradiction of the logical principle: "The part cannot be equal to the whole". The egg is a part of an organism containing information about the whole organism. This reflexivity is a sort of Dedekind's infinity (a set bijective with a proper part of itself, Galileo Galilei was the first who noticed this property of natural numbers). Set reflexivity and infinity are strictly related to duplicability, the basis of potentially infinite reproductive lines. Cells are finite multisets of molecules, but it is worthwhile to remark that, even if cells are finite, they are dynamic and open structures (objects inside them change in time and can continuously enter from or exit the environment). Cell duplicability is strongly based on the reflexivity of DNA, the property that a proper part of the DNA structure contains the information for generating the whole structure (see Figure 1). In other words, reflexivity and duplicability are intertwined at the biochemistry level, providing the essence of reproducibility upon which life is based [1]. In the context of his speculation on non-standard models of set theory, Ennio De Giorgi developed a deep vision of the reflexivity in the whole of mathematics [2].

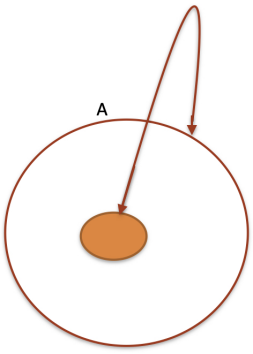


Figure 1. A visualization of reflexivity.

Reflexivity and duplicability are also strictly related in set theory. However, the nature of this relationship is even more general, involving neurological and philosophical argu-

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ments. Namely, reflection and speculation are two words expressing self-application and duality. Light rays are reflected, giving an image, and a "speculum" (Latin word for mirror) is a device reflecting light and providing a copy of the reflected figure.

The infinite is the essence of mathematics, and the various forms of mathematical infinities (ordinal and cardinal numbers, infinitesimals, real and hyperreal numbers, and asymptotic orders) are the fields where the power of mathematical imagination reaches the highest level of creativity with paradoxes opening windows that shock our common sense. Mathematics tells us that reality is not only what we think to be but a wider unimaginable painting of which we can reconstruct some pieces. However, contrarily to their apparent unpractical interest, infinite processes are driving forces of important applications of mathematics. As Gregory Chaitin claimed in many papers and conferences [3], computers are also conceptually a product of *Cantor's Paradise* (an expression used by Hilbert in [4]) because the Turing machine was invented for generating computable real numbers. Turing's discovery of the recursive enumerable set K is a version of Cantor's diagonal argument, which is a case of a refexive schema. Finally, deep results of set theory could be applied in the analysis of biological phenomena, as it is shown in [5].

A Glimpse of Set Theory

In 1878 [6], Cantor proved two instances of $A \leftrightarrow (A \times A)$, where $\leftrightarrow$ denotes a 1-to-1 (bijective) function, that is, $|A| = |A \times A|$, where $|A|$ is the cardinality of the set A, such that $A \leftrightarrow |A|$, and $|A|$ is a kind of ordinal number called cardinal number. In the first instance of Cantor's proofs $A = \mathbb{N}$, while in the second instance $A = \mathbb{R}$ (the naturals and the reals, respectively). These results about cardinal numbers were a shock. In a letter to Richard Dedekind, Cantor wrote: *I see it, but I cannot believe it*, as reported in Abraham Fränkel's book [7].

Axiomatic set theories, such as ZF (Zermelo–Fränkel) [2,7–12], establish nine axioms (specification, equality, empty, pair, union, power, infinity, regularity, replacement) expressing the essence of sets, but avoiding the paradoxes arising when a naive notion of class is assumed, such as Russell's paradox reported in 1902 [13]. The theory ZFC is the ZF theory plus the Axiom of Choice as tenth axiom. Informally, it says that, for any collection of nonempty sets, it is possible to construct a new set by choosing one element from each set (even for an infinite collection of infinite sets). An equivalent formulation says that the Cartesian product of a collection of non-empty sets is non-empty. Many important results in set theory need AC to be proved.

According to John von Neumann [10,11], *an* **ordinal number** *is the set of numbers preceding it*. Therefore, the first ordinal number is $\emptyset$, the second one is $\{\emptyset\}$, the third is $\{\emptyset, \{\emptyset\}\}$, and in general the successor of an ordinal γ is $\gamma \cup \{\gamma\}$. The union of all finite ordinals is ω, the first infinite ordinal and the first **limit ordinal** (an ordinal which is not the successor of another ordinal). Then, the successor of ω is $\omega \cup \{\omega\}$; therefore, the usual counting process of numbers can be extended to all transfinite numbers. Surprisingly, Cantor's transfinite counting is strongly related to Archimedes' counting process based on orders and periods [14].

The set of all ordinal numbers equipotent to an infinite ordinal number α, the ordinal $\beta = \bigcup_{\gamma \leftrightarrow \alpha} \gamma$, is not equipotent to α and is also called an **initial ordinal** or a **cardinal** number (if $\beta \leftrightarrow \alpha$, then $\beta \in \beta$, which is impossible for ordinal numbers). The cardinality $|A|$ of a set A corresponds to the first cardinal number that is equipotent with A. Cardinal numbers are denoted by the symbol $\aleph$ with an index $i \in \mathbb{N}$, where $\aleph_0 = |\mathbb{N}| = \omega$. Cantors's Continuum Hypothesis CH claims that $\aleph_{i+1} = 2^{\aleph_i}$, and a crucial result of the last century, solving the 23° Hilbert's Problem presented in 1900, shows that there are models of ZF where CH holds

and models where CH does not hold (and analogously ZF models where AC holds and ZF models where AC does not hold) [5,15,16].

Set theory disclosed new perspectives for the whole of mathematics, from mathematical logic and abstract algebra, based on sets with operations acting on them, to spaces in the most general sense, going from topologies to hyperspaces of many, possibly infinite, dimensions. This foundational power of sets motivates the importance of a deep understanding of set-theoretical concepts.

2. Results

2.1. Set Duplicability

A set F is finite if there is a natural number n such that F is equipotent to the set $\bar{n}$ of natural numbers less than n. According to Cantor, a set is A **infinite** if it is not finite; that is, for any $n \in \mathbb{N}$, A is not equipotent to $\bar{n}$.

A set A is **reflexive** or **Dedekind infinite** when A is equipotent to a proper part of itself. The set $\mathbb{N}$ is 1-to-1 with the set $2\mathbb{N}$ of even numbers. Hence, it is **reflexive**. Any reflexive set cannot be finite; therefore, it is infinite.

A set is (sum) **duplicable** if $A \times \{0,1\} \leftrightarrow A$; that is, $A \leftrightarrow 2 \times A \leftrightarrow A_0 \oplus A_1$ with A_0, A_1 both bijective with A ($\oplus$ is the disjoint union of sets).

It is easy to show that an infinite set including a denumerable set A (bijective with $\mathbb{N}$) is reflexive (Lemma 2). Even the converse implication holds as the following lemma states it.

Lemma 1 (Reflexivity implies the inclusion of a denumerable set)**.** *If A is a reflexive set, it includes a denumerable set.*

Proof. If A is reflexive, a bijective F exists such that $A \supset F(A)$ and $F(A) \leftrightarrow A$; then, $F(F(A)) \leftrightarrow F(A)$. Therefore, iterating F, a "telescopic" denumerable chain $F^i(A)$ is obtained from reflexive, and hence, infinite, subsets of A. For any $i > 0$, let H_i be a finite subsets of $F^i(A)$ consisting of i elements. Then, $\bigcup_{i\in\mathbb{N}} H_i$ is the denumerable union of different finite sets, whence A includes a denumerable infinite set. □

Lemma 2 (Denumerable Inclusion implies Reflexivity)**.** *If a set A includes a denumerable set, it is reflexive.*

Proof. If A includes a denumerable set B; then, B is bijective with a denumerable subset $C \subset B$ with a bijection $F : B \leftrightarrow C$. Therefore, the function G that coincides with the identity over $A - B$ and F over B is bijective between A and its subset $(A - B) \cup C$, whence A is reflexive. □

In conclusion, Dedekind infinity is equivalent to the inclusion of a denumerable set; that is, A is reflexive if, and only if, $A \supseteq B$ and $B \leftrightarrow \mathbb{N}$.

If AC is assumed, set infinity is equivalent to Dedekind infinity [10,11,17–19]. Namely, for every finite set F_n of n elements of A, $A - F_n \neq \emptyset$. Therefore, if we chose an element a_n from $A - F_n$, for every $n \in \mathbb{N}$, we obtain a denumerable subset of A.

However, non-necessarily, Cantor infinity coincides with Dedekind infinity because there exist models of ZF with infinite sets that are not Dedekind infinite, called **iDf** (infinite Dedekind finite). However, the notion of iDf, even if it admits ZF models, exhibits paradoxical situations contrasting with the intuitive notion of set. Namely, according to the definition of reflexivity, in an iDf set A of a given cardinality, for any $a \in A$, the set $A - \{a\}$ is surely a set having a smaller cardinality than $|A|$. This means that, by removing only one element from A, we pass from an infinite cardinality to a smaller infinite cardinality. We

have seen that, using AC, the notions of infinite and reflexive set coincide; hence, this means that the counterintuitive situation raised by iDf sets confirms that the intuitive notion of set implicitly assumes the axiom of choice [20–23]. iDf sets can be constructed by using Fränkel–Mostowski models (sets of "atoms" that, when their elements are permuted, there is a finite set on which the permutation coincides with the identity function [10]).

DI hypothesis means that **any infinite set is Dedekind infinite**. This hypothesis is weaker than AC and even than CC (the Countable Choice axiom, which says that choice functions exist for countable families of non-empty sets).

The bijection $A \leftrightarrow (A \times A)$ means that the set A is **Cartesian Duplicable**. The general proof that, for every infinite set A, $(A \times A) \leftrightarrow A$ requires the Axion of Choice AC [7]. Moreover, Tarski showed in 1924 [18] that "$A \leftrightarrow (A \times A)$ for any infinite A" is equivalent to the Axiom of Choice **AC**.

It was proved that (sum) set duplicability is weaker than AC [17].

Lemma 3 (Cartesian Duplicability implies Sum Duplicability)**.** *If A is a Cartesian duplicable set, it is a sum duplicable set.*

Proof. If A is empty, $A \times A$ is empty, then A is Cartesian duplicable. Let us assume that A is not empty; by hypothesis, the following equation holds:

$$(A \times A) \leftrightarrow A \tag{1}$$

and the following chain of bijective functions holds:

$$A' \leftrightarrow A \leftrightarrow A \times A \leftrightarrow A' \times A. \tag{2}$$

where $a_0 \in A$ (A is not empty) and $A' = A - \{a_0\}$. Namely,

$$(A' \times A) = (A - \{a_0\}) \times A = (A \times A) - (\{a_0\} \times A)$$

if $A'' = \{a_0\} \times A$, then $A'' \leftrightarrow A$ and $A \times A' = (A \times A) - A''$ that is, $A \times A = (A' \times A) \oplus A''$; therefore, by (2) and $A'' \leftrightarrow A$, the thesis follows. □

The next lemma will prove the crucial relationship between set reflexivity and duplicability.

Lemma 4 (**Main Lemma**: Reflexivity implies Duplicability)**.** *Any reflexive set A is duplicable; that is, it is equipotent to the union of two disjoint sets both equipotent to A.*

Classical Proof ([10]). The set $\mathbb{N}$ is duplicable because $\mathbb{N}$ is the disjoint sum of even and odd numbers that are both bijective with all the numbers: $\mathbb{N} = 2\mathbb{N} \oplus 2\mathbb{N} + 1$. We will extend this idea to prove that any reflexive set A is duplicable by partitioning A into $2i$ and $2i + 1$ stripes. For any $i \in \mathbb{N}$, they are mutually bijective, and their disjoint sum is bijective with one of them.

If A is reflexive, there exists a proper subset of A that is 1-to-1 with A; therefore, for a bijective F, $A \leftrightarrow F(A)$. However, $F(A)$ is also reflexive because replacing A by $F(A)$ in the bijection $A \leftrightarrow F(A)$, we obtain $F(A) \leftrightarrow F(F(A))$, and so on, we can iteratively substitute A with $F(A)$. Let us define, for $i > 0$, $A_i = F^i(A)$, where F^i is the composition of F i times. Then, For $i \geq 0$:

$$A_i \leftrightarrow A.$$

Now, for $i \geq 0$, we define the stripes:

$$S_{i+1} = (A_i - A_{i+1})$$

where $A = A_0$ (see Figure 2). Being $A \leftrightarrow A_i \leftrightarrow A_{i+1}$ for any $i > 0$, the following bijection holds:

$$S_i \leftrightarrow S_{i+1}$$

namely,

$$S_i = A_{i-1} - A_i \leftrightarrow F(A_{i-1}) - F(A_i) = A_i - A_{i+1} = S_{i+1}$$

moreover,

$$(S_i \cup S_{i+1}) = (A_i - A_{i+2}) \leftrightarrow (A_i - A_{i+1}) = S_i$$

therefore, odd stripes are bijective with the contiguous even stripes, whence the union of all odd stripes is bijective with the union of all even stripes:

$$A \leftrightarrow \bigcup_{i\geq 0} S_{2i+1} \leftrightarrow \bigcup_{i\geq 0}(S_{2i+1} \cup S_{2i}) = \bigcup_{i\geq 0} S_{2i+1} \oplus \bigcup_{i\geq 0} S_{2i}$$

that is, A is duplicable. □

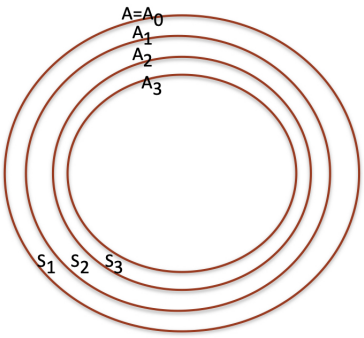


Figure 2. The partition of A into stripes (only the first three are visualized).

Fixpoint Proof (new). Let us define, for $i > 0$, $A_i = F^i(A)$, where F^i is the composition of i copies of F and $F^0(A) = A$.

$$A_\omega = \bigcap_{i>0} A_i \tag{3}$$

From $A_i \leftrightarrow A$, it follows that $A_\omega \leftrightarrow A$, and

$$F(A_\omega) = F(\bigcap_{i\geq 0} A_i) = \bigcap_{i\geq 0} F(A_i) = \bigcap_{i\geq 0} A_{i+1} = \bigcap_{i>0} A_i = A_\omega \tag{4}$$

meaning that A_ω is a non-empty subset of A (it is equipotent to A) that is a fixpoint for F (seen as function from sets to sets).

Now, $(A - A_\omega)$ and A_ω are bijective. F is surely injective between these sets, being bijective between A and A_ω. We will show that, if F is not surjective, it can be extended to a bijection between $(A - A_\omega)$ and A_ω. Let us assume that F is bijective only between $(A - A_\omega)$ and a proper part B of A_ω. If, for some $x \in A_\omega - B$, the image $F(x)$ belongs to B, then $F(A_\omega) \subset A_\omega$ against Equation (4).

Therefore, the impossibility of $F(x) \in B$ implies that all the F-images of the elements of $A_\omega - B$ belong to $A_\omega - B$. In this case, a bijection can be found between $(A - A_\omega)$ and A_ω by extending $F : A - A_\omega \to B$ into $G : (A - A_\omega) \leftrightarrow A_\omega$ such that $G(x) = F(x)$ if $F(x) \in B$, while $G(x) = x$ if $x \in (A_\omega - B)$ (G coincides with the identity function over $A_\omega - B$). The function G is a bijection between $(A - A_\omega)$ and A_ω (see Figure 3 where A_ω is A_∞).

Figure 4 visualizes the process of reflexive iteration and the 1-to-1 limit correspondence between A^ω and $A - A^\omega$ (A^ω is also 1-to-1 with A). Figure 5 visualizes the set duplication chain producing A_ω (equipotent to A and $A - A_\omega$).

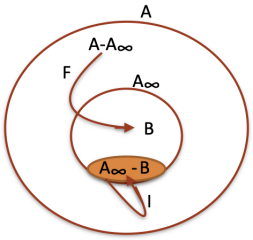


Figure 3. The 1-to-1 correspondence between $A - A_\omega$ and A_ω.

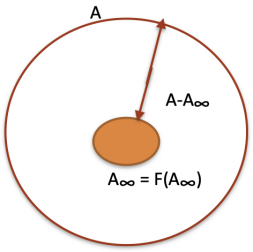


Figure 4. If A_ω is a fixpoint of F, there is a 1-to-1 correspondence between A_ω and $A - A_\omega$.

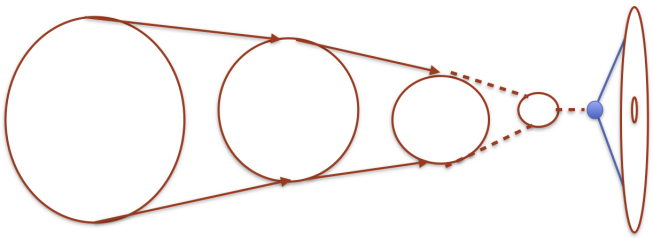


Figure 5. The chain of subsets providing the duplication of a reflexive set. The last full circle is A_ω, which results 1-to-1 with $A - A_\omega$ (the orthogonal circle on the right).

In conclusion, we have:

$$A = A_\omega \oplus (A - A_\omega)$$

where $A \leftrightarrow A_\omega$ and $(A - A_\omega) \leftrightarrow A_\omega \leftrightarrow A$; that is, A is duplicable. □

Let us reconsider the duplicability of a reflexive set A from a different perspective, which will help us better understand the relationship between reflexivity and duplicability. Now, we will sketch an informal argument showing that joining duplications of parts of A can obtain the duplication of a reflexive set A.

Another Proof (informal). According to Lemma 1, A includes a denumerable set B; then, a bijection G exits such that $G : B \leftrightarrow \mathbb{N}$. Let $B_0 = G(2\mathbb{N})$ and $B_1 = B - G(2\mathbb{N})$; then, B is duplicable because $B = B_0 \oplus B_1$ and both B_0, B_1 are equipotent with B. Therefore, a part of A is duplicable. If $A - B$ includes a denumerable set, we can apply the previous argument to extend the partial duplication of A with the duplication of another infinite part B' of A. If this process goes indefinitely, then the duplication of A is obtained as the union of the duplications of its parts $B, B', B'', \ldots$. Otherwise, at some step, for example with $A - (B \oplus B' \oplus B'')$, one of two possibilities can occur: (i) either $A - (B \oplus B' \oplus B'')$ is finite, then the proof is concluded because $A \leftrightarrow (B \oplus B' \oplus B'')$; or (ii) $A - (B \oplus B' \oplus B'')$ is not finite but does not include a denumerable set (meaning that it is not reflexive). In this case, $A - (B \oplus B' \oplus B'')$ includes a finite set of even cardinality; hence, we can (finitely) extend the partial duplication of B, B', B'' into B, B', B'', F, and surely this extension goes

indefinitely with other finite sets $B, B', B'', F, F', F'', \ldots$ (otherwise, $A - (B \oplus B' \oplus B'')$ would be finite, against the hypothesis). Also, in this case, the union of all partial duplications provides a duplication of A. □

Lemma 5 (Sum Duplicability implies Set Reflexivity)**.** *A duplicable set is reflexive.*

Proof. If a set is duplicable, it is bijective with a proper part of itself. Therefore, it is reflexive. □

2.2. Cantor–Schröder–Bernstein Theorem

The following proposition will prove the Cantor–Schröder–Bernstein Theorem, shortly CSBT, by set reflexivity. A classical proof of CSBT can be found in [8], and historical accounts of CSBT are given in [24,25]. Given two sets A and B, then $|A| \leq |B|$ means that there is an injective function from A to B, while $|A| = |B|$ means that $A \leftrightarrow B$.

CSBT is a famous result in set theory enunciated, without proof, by Georg Cantor in 1887. The first proof of CSBT was given in 1887 by Dedekind [25,26], who did not publish it. This proof and a second proof, given by Dedekind in 1897, do not use AC, but both use the **Dedekind Lemma** [24]: $A \supset B \supset C$ and $|A| = |C|$ implies $\restriction A| = |B|$. Dedekind Lemma is easily proven as a consequence of set reflexivity.

Lemma 6 (Dedekind)**.** $A \supset B \supset C$ *and* $|A| = |C|$ *implies* $\restriction A| = |B|$.

Proof. According to the hypothesis, A is a reflexive set. The fixpoint proof of Lemma 5 tells us that if $H : A \to A$ is a proper injection, then, for any $x \in A$, there exists some $i \in \mathbb{N}$ such that $H^i(x) \in A_\omega$. For any $x \in A$, let

$$H^\omega(x) = H^{max\{j | H^j(x) \in A^\omega\}}(x). \tag{5}$$

Therefore, for any subset B of A including A_ω, $H^\omega(B) = A_\omega$ and H^ω is 1-to-1 between B and A_ω, whence the thesis follows. □

Proposition 1 (Cantor–Schröder–Bernstein Theorem)**.** *Given two sets A and B, if* $|A| \leq |B|$ *and* $|B| \leq |A|$*, then* $|A| = |B|$.

Proof. From the hypothesis, two injective functions $F : A \to B$ and $G : B \to A$ exist. If one of them is also surjective, then the proof is concluded. Therefore, let us assume that both are not surjective. Then,

$$G(F(A)) \leftrightarrow A \tag{6}$$

$$A \supset G(B) \supset G(F(A)) \tag{7}$$

$$G(B) \leftrightarrow A \tag{8}$$

From (6) and (7), according to Lemma 6, the 1-to-1 correspondence $A \leftrightarrow G(B)$ follows, whence from (8) the thesis $A \leftrightarrow B$ follows (see Figure 6). □

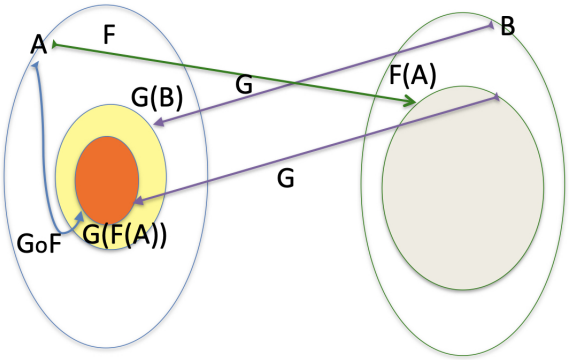


Figure 6. A visualization of Cantor–Scröder–Bernstein theorem.

3. Conclusions

This paper shows the equivalence between set reflexivity (Dedekind infinity) and (sum) set duplicability, giving a new simple proof of the Dedekind Lemma in set cardinalities, which provides an illuminating proof of the famous Cantor–Schröder–Bernstein theorem.

Reflexivity and duplicability are two crucial aspects of life at any biological level, from DNA structure to the neurological and psychological level [1,27]. Namely, mind consciousness results in a particular kind of reflexivity where a part exhibits a form of control on the whole mental activity [28].

Therefore, biological concepts have clear counterparts in set theory, a general mathematical theory on which almost all mathematical theories can be formally expressed. The general lesson of this perspective is the cruciality of mathematics for life, as it is confirmed again by recent papers going from biological codes to the mathematical analysis of transformer large language models (LLMs), used in the last-generation artificial intelligence chatbots [5,29,30]. The biological relevance of reflexivity occurs in one of its most typical characteristics of life, that is, the reproducibility of cells, based on DNA duplication. It is a link with infinity because biological germinal lines can, in principle, endlessly reproduce. We know that physical resources such as matter and energy are finite, indeed. But mathematics prescinds from real possibilities.

On the other hand, reflexivity is an old subject always recurrent in logic and mathematics [9]. It determined the discovery of many antinomies of mathematical logic and pushed toward the axiomatic set theory, and the famous Gödel's incompleteness theorems and Turing's undecidability results [12,13,31]. This shows that a basilar logical and theoretical principle could be the fuel of dynamics driving the mathematical logic and computability theory of the last century, on which the present technological revolution is based.

Mathematics is the art of the infinite [32], and the previous section shows that infinite is intrinsically and directly related to reflexivity and duplicability. The well-known Banach–Tarski paradox [33] uses AC to prove that a solid ball in three-dimensional space can be decomposed into a finite number of disjoint subsets, which can be put together in a different way to yield two identical copies of the original ball. Other formulations of this result, and other related results, put in evidence the counterintuitive effects of the notions related to infinity. But, probably, these are the price for the conceptual power of infinite mathematical objects. Analogously, reflexivity may produce inconsistent self-reference, as in Russell's paradox, but it is also present in the powerful recurrence, at the basis of numbers and computability. The double-face character of these concepts is the basis of their foundational power and centrality in mathematics.

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