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Iterative Kneser-Type Criteria for Oscillation of Half-Linear Second-Order Advanced Dynamic Equations

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Abstract: This work aims to develop new iterative Kneser-type criteria for determining the oscillatory behaviour of half-linear second-order advanced dynamic equations on arbitrary unbounded-above time scales $\mathbb{T}$. The results extend and refine previously established criteria for these equations while also generalising classical criteria for corresponding ordinary dynamic equations. This study provides a broader and more flexible approach to analysing such systems by introducing iterative methods. Several examples are included to demonstrate the accuracy, usefulness, and adaptability.

Keywords: oscillation; second order; half-linear; differential equations; dynamic equations; time scales

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1. Introduction

The theory of time scales, introduced by Stefan Hilger [1], is a powerful tool that unifies discrete and continuous analyses. Providing a general framework allows for exploring and applying various time scales in a wide range of disciplines. This theory seamlessly integrates classical theories of difference and differential equations while extending beyond these conventional boundaries to accommodate intermediate cases. For instance, q-difference equations, which play a significant role in quantum theory (see [2]), can be derived when the time scale is $\mathbb{T} = q^{\mathbb{N}_0} := \{q^s : s \in \mathbb{N}_0 \text{ for } q > 1\}$. This is just one example of how time scales can be defined. Other time scales, such as $\mathbb{T} = h\mathbb{N}$, $\mathbb{T} = \mathbb{N}^2$, and $\mathbb{T} = \mathbb{T}_n$, where $\mathbb{T}_n$ denotes the set of harmonic numbers, can also be utilised to model various dynamic phenomena. The flexibility of the theory allows for the application of dynamic equations to a broader range of problems, whether they involve continuous processes, discrete steps, or hybrid systems that lie in-between. The theory's applicability to multiple time scales makes it a versatile tool in fields ranging from quantum mechanics to biology,

economics, and engineering. Providing a unifying approach simplifies the transition between continuous and discrete models, making it easier to study complex systems with varying degrees of time granularity. For a deeper understanding of the calculus of time scales and its numerous applications, see [3–5].

Advanced dynamic equations have been developed for various practical domains, where the rates of change depend on both current and future conditions. An advanced term must be included to incorporate the impact of potential future factors into the decision-making process. For instance, fields like population dynamics, economics, and mechanical control engineering commonly exhibit scenarios where future factors affect the growth of dynamic components (see [6–8]).

Oscillation has received much attention from applied researchers because of its origins in mechanical vibrations and widespread development in the engineering and sciences. To account for the dependence of solutions on future or past timeframes, oscillation models often include advanced or delayed terms. Although there has been extensive research on oscillation in delay equations (see [9–18]), studies on advanced oscillation are relatively scarce (see [19–22]).

This paper investigates the phenomenon of advanced oscillation, particularly emphasising the half-linear case. Half-linear dynamic equations, which serve as an extension of the classical Laplace equation, play a crucial role in modelling complex systems. These equations have wide-ranging applications in various scientific and engineering fields. Notably, they are used in the study of non-Newtonian fluid dynamics, where they help describe the behaviour of fluids with non-constant viscosity and the analysis of turbulent flow in polytropic gases within porous media. Additionally, these equations are integral to specific mathematical models in biology, such as those describing population dynamics or disease spread. The importance of half-linear dynamic equations in these areas highlights their versatility and significance. For further details on these applications and the theoretical foundations, see [23–35]. Therefore, we will examine half-linear second-order advanced dynamic equations

$$\left[a(r)\varphi(x^{\Delta}(r))\right]^{\Delta} + b(r)\varphi(x(g(r))) = 0, \tag{1}$$

on an arbitrary unbounded-above time scale $\mathbb{T}$, where $r \in [r_0,\infty)_{\mathbb{T}} := [r_0,\infty) \cap \mathbb{T}$, $r_0 \geq 0$, $r_0 \in \mathbb{T}$; $\varphi(u) := |u|^{\alpha-1}u$, $\alpha > 0$; $b \in \mathrm{C}_{\mathrm{rd}}([r_0,\infty)_{\mathbb{T}},[0,\infty))$ such that $b \not\equiv 0$; $g \in \mathrm{C}_{\mathrm{rd}}(\mathbb{T},\mathbb{T})$ such that $\lim_{r\to\infty} g(r) = \infty$; and $a \in \mathrm{C}_{\mathrm{rd}}([r_0,\infty)_{\mathbb{T}},(0,\infty))$ such that

$$K(r) := \int_{r_0}^{r} \frac{\Delta\rho}{a^{1/\alpha}(\rho)} \to \infty \text{ as } r \to \infty, \tag{2}$$

where C_{rd} is the space of right-dense continuous functions. By a solution of Equation (1) we mean a nontrivial real-valued function $x \in \mathrm{C}^1_{\mathrm{rd}}[r_x,\infty)_{\mathbb{T}}$ for some $r_x \geq r_0$ with $r_0 \in \mathbb{T}$ such that $x^{\Delta}, a(r)\varphi\left(x^{\Delta}(r)\right) \in \mathrm{C}^1_{\mathrm{rd}}[r_x,\infty)_{\mathbb{T}}$ and $x(r)$ satisfies Equation (1) on $[r_x,\infty)_{\mathbb{T}}$, where $\mathrm{C}^1_{\mathrm{rd}}$ is the set of functions in C_{rd} with right-dense continuous Δ-derivatives. We consider only the solutions $x(r)$ of Equation (1) which satisfy $\sup\{|x(r)| : r \geq T\} > 0$ for all $T \in [r_0,\infty)_{\mathbb{T}}$. We will exclude solutions that vanish near infinity from our investigation.

A solution $x(r)$ of Equation (1) is called oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is called nonoscillatory. Equation (1) is considered oscillatory if every solution to the equation exhibits oscillatory behaviour.

We begin by mentioning some existing oscillation findings for differential equations related to Equation (1). Following Sturm's significant contribution to the field, oscillation

theory has long been grounded in Euler differential equations and their generalisations. Among the most notable and utilised is the second-order Euler equation

$$x''(r) + \frac{\beta}{r^2}x(r) = 0, \tag{3}$$

which oscillates if and only if

$$\beta > \frac{1}{4}.$$

Some of the essential oscillation criteria for second-order differential equations include Kneser-type criteria (see [36]), which utilise Sturmian comparison methods, as well as the oscillatory behaviour of Equation (3). These methods demonstrate that the linear differential equation

$$x''(r) + b(r)x(r) = 0,$$

oscillates if

$$\liminf_{r\to\infty} r^2 b(r) > \frac{1}{4}. \tag{4}$$

Since then, numerous results have been developed using a similar approach to establish Kneser-type criteria for various types of differential equations. Some of these works are as follows (see [37–39]):

(I) The linear differential equation

$$\left[a(r)x'(r)\right]' + b(r)x(r) = 0,$$

oscillates if

$$\liminf_{r\to\infty} a(r)K^2(r)b(r) > \frac{1}{4}. \tag{5}$$

(II) The half-linear differential equation

$$\left[\varphi(x'(r))\right]' + b(r)\varphi(x(r)) = 0$$

oscillates if

$$\liminf_{r\to\infty} r^{\alpha+1}b(r) > \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}. \tag{6}$$

(III) The half-linear differential equation

$$\left[a(r)\varphi(x'(r))\right]' + b(r)\varphi(x(r)) = 0,$$

oscillates if

$$\liminf_{r\to\infty} a^{\frac{1}{\alpha}}(r)K^{\alpha+1}(r)b(r) > \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}. \tag{7}$$

Recently, Hassan et al. [40] discovered some noteworthy Kneser-type oscillation criteria for Equation (1), as follows:

Theorem 1 (see [40]). *If* $\lambda = \liminf_{r\to\infty} \dfrac{K(r)}{K(\sigma(r))} > 0$ *and*

$$\liminf_{r\to\infty} a^{1/\alpha}(r)K^{\alpha+1}(r)b(r) > \frac{1}{\lambda^{\alpha(\alpha+1)}}\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}, \tag{8}$$

then Equation (1) *oscillates.*

The advanced argument $g(r)$ is missing from the Kneser-type criterion (8), which aligns better with the ordinary dynamic equation

$$\left[a(r)\varphi(x^{\Delta}(r))\right]^{\Delta} + b(r)\varphi(x(r)) = 0,$$

and does not show how the advanced argument impacts oscillation. Additionally, these findings do not address the cases that fall under the condition

$$\liminf_{r\to\infty} a^{1/\alpha}(r)K^{\alpha+1}(r)b(r) \leq \frac{1}{\lambda^{\alpha(\alpha+1)}}\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}. \tag{9}$$

This means that, in more detail, Theorem 1 does not apply when condition (9) is satisfied.

This goal of work is to develop new Kneser-type criteria that can be practically and readily applied to (1), incorporating the advanced function $g(r)$, and that remain effective when Condition (9) is satisfied. The reader is referred to the papers on oscillation criteria [41–44] and the references listed therein.

2. Preliminaries

Throughout this section, we define some definitions on time scales. A time scale is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$ denoted by the symbol $\mathbb{T}$. It has the topology that it inherits from the real numbers with the standard topology.

Definition 1. *For $r \in \mathbb{T}$, the forward operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by*

$$\sigma(r) = \inf\{s \in \mathbb{R} : s > r\},$$

and the backward operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(r) = \sup\{s \in \mathbb{T} : s < r\}.$$

If $\sigma(r) > r$, we say that r is right-scattered, while if $\rho(r) < r$, we say that r is left-scattered. Points r such that

$$\rho(r) < r < \sigma(r),\ \rho(r) < r = \sup\mathbb{T} \text{ or } \inf\mathbb{T} = r < \sigma(r),$$

are called isolated points. If a time scale consists of only isolated points, then it is an isolated (discrete) time scale. Also, if $r < \sup\mathbb{T}$ and $\sigma(r) = r$, then r is called right-dense, and if $r > \inf\mathbb{T}$ and $\rho(r) = r$, then r is called left-dense. Points r that are either left-dense or right-dense are called dense.

Finally, the graininess operator $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(r) = \sigma(r) - r$ and if $f : \mathbb{T} \to \mathbb{R}$ is a function, then the function $f^{\sigma} : \mathbb{T} \to \mathbb{R}$ is defined by

$$f^{\sigma}(r) = f(\sigma(r)) \text{ for all } r \in \mathbb{T}.$$

Now, define the so-called delta (or Hilger) derivative of f at a point $r \in \mathbb{T}^k$.

Definition 2. *Assume that $f : \mathbb{T} \to \mathbb{R}$ is a function and let $r \in \mathbb{T}^k$. Then, we define*

$$f^{\Delta}(r) = \lim_{s\to r}\frac{f(\sigma(r)) - f(s)}{\sigma(r) - s}.$$

We will use the next theorem in the derivative of the sum $f + g$, product fg and the quotient $\frac{f}{g}$ (where $gg^{\sigma} \neq 0$) of two differential functions f and g.

Theorem 2 ([4] (Theorem 1.20)). *Assume that $f, g : \mathbb{T} \to \mathbb{R}$ are differentiable at $r \in \mathbb{T}^k$. Then,*

(i) The sum $f + g : \mathbb{T} \to \mathbb{R}$ is differentiable at r with

$$(f+g)^{\Delta}(r) = f^{\Delta}(r) + g^{\Delta}(r).$$

(ii) For any constant α, $\alpha f : \mathbb{T} \to \mathbb{R}$ is differentiable at r with

$$(\alpha f)^{\Delta}(r) = \alpha f^{\Delta}(r).$$

(iii) The product $fg : \mathbb{T} \to \mathbb{R}$ is differentiable at r with

$$\begin{aligned}(f(r)g(r))^{\Delta} &= f^{\Delta}(r)g(r) + f(\sigma(r))g^{\Delta}(r),\\ &= f(r)g^{\Delta}(r) + f^{\Delta}(r)g(\sigma(r)).\end{aligned}$$

(iv) The quotient $\frac{f}{g} : \mathbb{T} \to \mathbb{R}$ is differentiable at r with

$$\left(\frac{f(r)}{g(r)}\right)^{\Delta} = \frac{f^{\Delta}(r)g(r) - f(r)g^{\Delta}(r)}{g(r)g(\sigma(r))}.$$

Theorem 3 (Pötzsche chain rule (see [4] (Theorem 1.90))). *Let $g : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and suppose that $f : \mathbb{T} \to \mathbb{R}$ is delta-differentiable. Then, $g \circ f : \mathbb{T} \to \mathbb{R}$ is delta-differentiable and satisfies*

$$(g \circ f)^{\Delta}(r) = \left(\int_0^1 g'\Big(f(r) + h\mu(r)f^{\Delta}(r)\Big)dh\right) f^{\Delta}(r). \tag{10}$$

3. Main Results

For simplicity, we define a sequence $\{\gamma_m\}_{m\in\mathbb{N}_0}$ as follows:

$$\gamma_{m+1} := \frac{1}{\alpha}\liminf_{r\to\infty} a^{1/\alpha}(r)K^{\alpha\left(1-\sqrt[\alpha]{\gamma_m}\right)+1}(r)K^{\alpha\sqrt[\alpha]{\gamma_m}}(g(r))b(r),\ m \in \mathbb{N}_0, \tag{11}$$

with $\gamma_0 := 0$. It is important to observe that the sequence $\{\gamma_m\}_{m\in\mathbb{N}_0}$ is nondecreasing.

First, we begin this section by the following preliminary lemmas:

Lemma 1 (see [45,46]). *Suppose that $x(r)$ is an eventually positive solution of Equation (1). Then,*

$$x^{\Delta}(r) > 0 \quad \text{and} \quad \Big[a(r)\varphi(x^{\Delta}(r))\Big]^{\Delta} < 0, \tag{12}$$

eventually.

Lemma 2. *Suppose that $x(r)$ is an eventually positive solution of Equation (1). Then, for any $m \in \mathbb{N}_0$,*

$$a^{1/\alpha}(r)x^{\Delta}(r)K(r) > \sqrt[\alpha]{\gamma_m}x(r), \tag{13}$$

eventually.

Proof. According to Lemma 1, $x^{\Delta}(r) > 0$ and $\left[a(r)\varphi(x^{\Delta}(r))\right]^{\Delta} < 0$ eventually. It is easy to show that (13) is achieved if there is a $m \in \mathbb{N}_0$ such that $\gamma_l = 0$. Assume that $\gamma_1 > 0$, so

$\gamma_m > 0$ for any $m \in \mathbb{N}$ since $\{\gamma_m\}_{m\in\mathbb{N}}$ is a nondecreasing sequence. Now, for arbitrary but fixed $\varepsilon_m \in (0,1)$, we will demonstrate using induction that

$$a^{1/\alpha}(r)x^{\Delta}(r)K(r) \geq \varepsilon_m \sqrt[\alpha]{\gamma_m}x(r), \tag{14}$$

eventually. Since $\left[a(r)\varphi(x^{\Delta}(r))\right]^{\Delta} < 0$ eventually, we obtain

$$x(r) > \varphi^{-1}\left[a(r)\varphi(x^{\Delta}(r))\right]\int_{r_0}^{r}\frac{\Delta\rho}{a^{1/\alpha}(\rho)} = \varphi^{-1}\left[a(r)\varphi(x^{\Delta}(r))\right]K(r),$$

that is,

$$a^{1/\alpha}(r)x^{\Delta}(r)K(r) < x(r). \tag{15}$$

Let $\varepsilon_0 \in (0,1)$ be arbitrary but fixed. From (15) and the definition (11) of γ_1, for sufficiently large r, we obtain

$$a^{1/\alpha}(r)K^{\alpha+1}(r)b(r) \geq \alpha\varepsilon_0\gamma_1 \tag{16}$$

Given that $x^{\Delta}(r) > 0$ eventually, from Equation (1), we have

$$\begin{aligned} a(r)\varphi(x^{\Delta}(r)) &\geq \int_r^{\infty} b(\rho)\varphi(x(g(\rho)))\Delta\rho \geq \int_r^{\infty} b(\rho)\varphi(x(\rho))\Delta\rho, \\ &\geq \varphi(x(r))\int_r^{\infty} b(\rho)\Delta\rho, \\ &\overset{(16)}{\geq} \varepsilon_0\gamma_1\varphi(x(r))\int_r^{\infty}\frac{\alpha}{a^{1/\alpha}(\rho)K^{\alpha+1}(\rho)}\Delta\rho. \end{aligned} \tag{17}$$

for sufficiently large r. Using the Pötzsche chain rule (10), we obtain

$$\begin{aligned} \left(\frac{-1}{K^{\alpha}(\rho)}\right)^{\Delta} &= \frac{\alpha}{a^{1/\alpha}(\rho)}\int_0^1 \frac{\mathrm{d}h}{[(1-h)K(\rho)+hK^{\sigma}(\rho)]^{\alpha+1}}, \\ &\leq \frac{\alpha}{a^{1/\alpha}(\rho)K^{\alpha+1}(\rho)}. \end{aligned} \tag{18}$$

By substituting (18) into (17), we obtain

$$\begin{aligned} a(r)\varphi(x^{\Delta}(r)) &\geq \varepsilon_0\gamma_1\varphi(x(r))\int_r^{\infty}\left(\frac{-1}{K^{\alpha}(\rho)}\right)^{\Delta}\Delta\rho, \\ &= \varepsilon_0\gamma_1\varphi(x(r))\frac{1}{K^{\alpha}(r)}, \end{aligned}$$

Hence,

$$a^{1/\alpha}(r)x^{\Delta}(r)K(r) \geq \varepsilon_1\sqrt[\alpha]{\gamma_1}x(r), \tag{19}$$

where $\varepsilon_1 = \sqrt[\alpha]{\varepsilon_0}$. This shows that, for $m = 1$,(14) holds. Assume that, for $m = l$, (14) holds, i.e.,

$$a^{1/\alpha}(r)x^{\Delta}(r)K(r) \geq \varepsilon_l\sqrt[\alpha]{\gamma_l}x(r),$$

and from (15), we have $\varepsilon_l\sqrt[\alpha]{\gamma_l} < 1$. By the quotient rule and (10), it follows that

$$\begin{aligned} \left(\frac{x(r)}{K^{\varepsilon_l\sqrt[\alpha]{\gamma_l}}(r)}\right)^{\Delta} &= \frac{K^{\varepsilon_l\sqrt[\alpha]{\gamma_l}}(r)x^{\Delta}(r) - \left(K^{\varepsilon_l\sqrt[\alpha]{\gamma_l}}(r)\right)^{\Delta}x(r)}{K^{\varepsilon_l\sqrt[\alpha]{\gamma_l}}(r)K^{\varepsilon_l\sqrt[\alpha]{\gamma_l}}(\sigma(r))}, \\ &\geq \frac{a^{1/\alpha}(r)x^{\Delta}(r)K(r) - \varepsilon_l\sqrt[\alpha]{\gamma_l}x(r)}{a^{1/\alpha}(r)K(r)K^{\varepsilon_l\sqrt[\alpha]{\gamma_l}}(\sigma(r))} \geq 0, \end{aligned} \tag{20}$$

eventually. This together with (1) and (20) shows that

$$\begin{aligned} a(r)\varphi(x^{\Delta}(r)) &\geq \int_r^{\infty} b(\rho)\varphi(x(g(\rho)))\Delta\rho, \\ &\geq \int_r^{\infty} \left(\frac{K(g(\rho))}{K(\rho)}\right)^{\alpha\varepsilon_l \sqrt[\alpha]{\gamma_l}} \varphi(x(\rho))b(\rho)\Delta\rho, \\ &\overset{(12)}{\geq} \varphi(x(r)) \int_r^{\infty} \left(\frac{K(g(\rho))}{K(\rho)}\right)^{\alpha\varepsilon_l \sqrt[\alpha]{\gamma_l}} b(\rho)\Delta\rho, \\ &\geq \varphi(x(r)) \int_r^{\infty} \left(\frac{K(\rho)}{K(g(\rho))}\right)^{\alpha \sqrt[\alpha]{\gamma_l}(1-\varepsilon_l)} \left(\frac{K(g(\rho))}{K(\rho)}\right)^{\alpha \sqrt[\alpha]{\gamma_l}} b(\rho)\Delta\rho. \end{aligned}$$

Define

$$\bar{\lambda} := \liminf_{r\to\infty} \frac{K(r)}{K(g(r))}, \quad 0 \leq \bar{\lambda} \leq 1.$$

Hence, for sufficiently large r,

$$\frac{K(r)}{K(g(r))} \geq \varepsilon\bar{\lambda},$$

where $\varepsilon \in (0,1)$ is arbitrary but fixed. Consequently,

$$a(r)\varphi(x^{\Delta}(r)) \geq \left(\varepsilon\bar{\lambda}\right)^{\alpha \sqrt[\alpha]{\gamma_l}(1-\varepsilon_l)} \varphi(x(r)) \int_r^{\infty} \left(\frac{K(g(\rho))}{K(\rho)}\right)^{\alpha \sqrt[\alpha]{\gamma_l}} b(\rho)\Delta\rho.$$

From the definition (11) of γ_{l+1}, for a sufficiently large r, we have

$$a^{1/\alpha}(r)K^{\alpha\left(1-\sqrt[\alpha]{\gamma_l}\right)+1}(r)K^{\alpha \sqrt[\alpha]{\gamma_l}}(g(r))b(r) \geq \alpha\varepsilon_l\gamma_{l+1}.$$

Therefore,

$$\begin{aligned} a(r)\varphi(x^{\Delta}(r)) &\geq \varepsilon_l\left(\varepsilon\bar{\lambda}\right)^{\alpha \sqrt[\alpha]{\gamma_l}(1-\varepsilon_l)} \gamma_{l+1}\varphi(x(r)) \int_r^{\infty} \frac{\alpha}{a^{1/\alpha}(\rho)K^{\alpha+1}(\rho)}\Delta\rho \\ &\overset{(18)}{\geq} \varepsilon_l\left(\varepsilon\bar{\lambda}\right)^{\alpha \sqrt[\alpha]{\gamma_l}(1-\varepsilon_l)} \gamma_{l+1}\varphi(x(r)) \int_r^{\infty} \left(\frac{-1}{K^{\alpha}(\rho)}\right)^{\Delta}\Delta\rho \\ &= \varepsilon_l\left(\varepsilon\bar{\lambda}\right)^{\alpha \sqrt[\alpha]{\gamma_l}(1-\varepsilon_l)} \gamma_{l+1}\varphi(x(r))\frac{1}{K^{\alpha}(r)}; \end{aligned}$$

hence,

$$\begin{aligned} a^{1/\alpha}(r)x^{\Delta}(r)K(r) &\geq \sqrt[\alpha]{\varepsilon_l}\left(\varepsilon\bar{\lambda}\right)^{\sqrt[\alpha]{\gamma_l}(1-\varepsilon_l)} \sqrt[\alpha]{\gamma_{l+1}}x(r), \\ &= \varepsilon_{l+1}\sqrt[\alpha]{\gamma_{l+1}}x(r), \end{aligned} \tag{21}$$

where ε_{l+1} satisfies

$$0 < \varepsilon_{l+1} := \sqrt[\alpha]{\varepsilon_l}\left(\varepsilon\bar{\lambda}\right)^{\sqrt[\alpha]{\gamma_l}(1-\varepsilon_l)} < 1.$$

This demonstrates that (14) holds when $m = l+1$. By (11), there is an $\kappa \geq 1$ such that

$$\gamma_{l+1} = \gamma_l\left(\frac{K(g(r))}{K(r)}\right)^{\alpha\left(\sqrt[\alpha]{\gamma_l}-\sqrt[\alpha]{\gamma_{l-1}}\right)} \geq \gamma_l\kappa^{\alpha\left(\sqrt[\alpha]{\gamma_l}-\sqrt[\alpha]{\gamma_{l-1}}\right)} \tag{22}$$

Therefore, from (11) and (22), we obtain

$$K(r)a^{1/\gamma}(r)x^{\Delta}(r) \geq \varepsilon_{l+1}\sqrt[\alpha]{\gamma_{l+1}}x(r) \geq \varepsilon_{l+1}\kappa^{\sqrt[\alpha]{\gamma_l}-\sqrt[\alpha]{\gamma_l}}\sqrt[\alpha]{\gamma_l}x(r).$$

Given that $0 < \varepsilon_{l+1} < 1$ is arbitrary, we can take $\varepsilon_{l+1} > \frac{1}{\kappa \sqrt[\alpha]{\gamma_l} - \sqrt[\alpha]{\gamma_l}}$. Thus,

$$K(r)a^{1/\gamma}(r)x^{\Delta}(r) > \sqrt[\alpha]{\gamma_l}x(r).$$

Then, for all $m \in \mathbb{N}_0$, (13) holds. □

Theorem 4. *If there exist $m \in \mathbb{N}_0$ such that the ordinary dynamic equation*

$$\left[a(r)\varphi(x^{\Delta}(r))\right]^{\Delta} + \left(\frac{K(g(r))}{K(r)}\right)^{\alpha \sqrt[\alpha]{\gamma_m}} b(r)\varphi(x(r)) = 0, \tag{23}$$

oscillates, then (1) *oscillates.*

Proof. Assume (1) has a nonoscillatory solution x on $[r_0, \infty)_{\mathbb{T}}$. Without loss of generality, let $x(r) > 0$ on $[r_0, \infty)_{\mathbb{T}}$. By using Lemma 2, we obtain

$$\begin{aligned}\left(\frac{x(r)}{K^{\sqrt[\alpha]{\gamma_m}}(r)}\right)^{\Delta} &= \frac{K^{\sqrt[\alpha]{\gamma_m}}(r)x^{\Delta}(r) - x(r)\left(K^{\sqrt[\alpha]{\gamma_m}}(r)\right)^{\Delta}}{K^{\sqrt[\alpha]{\gamma_m}}(r)K^{\sqrt[\alpha]{\gamma_l}}(\sigma(r))} \\ &\geq \frac{K(r)a^{1/\alpha}(r)x^{\Delta}(r) - \sqrt[\alpha]{\gamma_m}x(r)}{a^{1/\alpha}(r)K(r)K^{\sqrt[\alpha]{\gamma_m}}(\sigma(r))} > 0,\end{aligned}$$

since $\sqrt[\alpha]{\gamma_m} < 1$. Then, by (10), we obtain

$$\begin{aligned}\left(K^{\sqrt[\alpha]{\gamma_m}}(r)\right)^{\Delta} &= \sqrt[\alpha]{\gamma_l}\int_0^1 [(1-h)K(r) + hK(\sigma(r))]^{\sqrt[\alpha]{\gamma_m}-1}\mathrm{d}h a^{-1/\alpha}(r), \\ &\leq \sqrt[\alpha]{\gamma_l}K^{\sqrt[\alpha]{\gamma_m}-1}(r)a^{-1/\alpha}(r).\end{aligned}$$

Therefore,

$$x(g(r)) \geq \left(\frac{K(g(r))}{K(r)}\right)^{\sqrt[\alpha]{\gamma_m}} x(r).$$

Hence, (1) becomes

$$\left[a(r)\varphi(x^{\Delta}(r))\right]^{\Delta} + \left(\frac{R(g(r))}{K(r)}\right)^{\alpha \sqrt[\alpha]{\gamma_m}} b(r)\varphi(x(r)) \leq 0.$$

As demonstrated in the proof of [47] (Theorem 6), we find that (23) has a nonoscillatory solution that contradicts the assumption that Equation (23) oscillates. □

Theorem 5. *If $\lambda = \liminf_{r\to\infty} \frac{K(r)}{K(\sigma(r))} > 0$ and there exists $m \in \mathbb{N}_0$ such that*

$$\gamma_{m+1} > \frac{\alpha^{\alpha}}{\lambda^{\alpha(\alpha+1)}(\alpha+1)^{\alpha+1}}, \tag{24}$$

then Equation (1) *oscillates.*

Proof. Without loss of generality, assume that $m \in \mathbb{N}$ is the least number such that (24) holds. Hence,

$$\liminf_{r\to\infty} a^{1/\alpha}(r)K^{\alpha\left(1-\sqrt[\alpha]{\gamma_m}\right)+1}(r)K^{\alpha\sqrt[\alpha]{\gamma_m}}(g(r))b(r) > \frac{1}{\lambda^{\alpha(\alpha+1)}}\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}.$$

Applying Theorem 1 with $b(r)$ replaced by $\left(\frac{K(g(r))}{K(r)}\right)^{\alpha \sqrt[\alpha]{\gamma_l}}$ to Equation (23), we see that Equation (23) oscillates. Consequently, by Theorem 4, Equation (1) oscillates. □

Theorem 6. *If there exists $m \in \mathbb{N}_0$ such that*

$$\gamma_{m+1} \geq 1, \tag{25}$$

then Equation (1) oscillates.

Proof. Let Equation (1) have a nonoscillatory solution $x(r)$ on $[r_0, \infty)_{\mathbb{T}}$. Without loss of generality, let $x(r) > 0$ on $[r_0, \infty)_{\mathbb{T}}$. By integrating Equation (1) from r to $v \in [r_0, \infty)_{\mathbb{T}}$, we obtain

$$\int_r^v b(\rho)x^\alpha(g(\rho))\Delta\rho = a(r)(x^\Delta(r))^\alpha - a(v)(x^\Delta(v))^\alpha \leq a(r)(x^\Delta(r))^\alpha.$$

Letting $v \to \infty$, we obtain

$$a(r)(x^\Delta(r))^\alpha \geq \int_r^\infty b(\rho)x^\alpha(g(\rho))\Delta\rho. \tag{26}$$

By using the facts that $\left(\frac{x(r)}{K\sqrt[\alpha]{\gamma_m}(r)}\right)^\Delta > 0$ and $x^\Delta(r) > 0$, for $\rho \in [r, \infty)_{\mathbb{T}}$, we obtain

$$b(\rho)x^\alpha(g(\rho)) \geq b(\rho)\left(\frac{K(g(\rho))}{K(\rho)}\right)^{\alpha \sqrt[\alpha]{\gamma_m}} x^\alpha(\rho) \geq b(\rho)\left(\frac{R(g(\rho))}{R(\rho)}\right)^{\alpha \sqrt[\alpha]{\gamma_m}} x^\alpha(r),$$

and by the definition (11) of γ_{l+1}, we obtain

$$b(\rho)x^\alpha(g(\rho)) \geq b(\rho)\left(\frac{K(g(\rho))}{K(\rho)}\right)^{\alpha \sqrt[\alpha]{\gamma_m}} x^\alpha(r) \geq \frac{\alpha\varepsilon\gamma_{m+1}}{a^{1/\alpha}(\rho)K^{\alpha+1}(\rho)}x^\alpha(r), \tag{27}$$

for arbitrary $\varepsilon \in (0, 1)$. By substituting (27) into (26), we obtain

$$a(r)(x^\Delta(r))^\alpha \geq \varepsilon\gamma_{m+1}x^\alpha(r)\int_r^\infty \frac{\alpha}{a^{1/\alpha}(\rho)K^{\alpha+1}(\rho)}\Delta\rho.$$

From (18), we obtain

$$\begin{aligned} a(r)(x^\Delta(r))^\alpha &\geq \varepsilon\gamma_{m+1}x^\alpha(r)\int_r^\infty \left(\frac{-1}{R^\alpha(\rho)}\right)^\Delta \Delta\rho \\ &= \varepsilon\gamma_{m+1}\left(\frac{x(r)}{K(r)}\right)^\alpha \\ &\overset{(15)}{>} \varepsilon\gamma_{m+1}a(r)(x^\Delta(r))^\alpha. \end{aligned}$$

Which implies that

$$\varepsilon\gamma_{m+1} < 1.$$

Since $\varepsilon > 0$ is arbitrary, we obtain a contradiction with (25). □

Example 1. *Consider the advanced half-linear dynamic equation*

$$\left[\frac{1}{\sqrt{r^5}}\sqrt{|x^\Delta(r)|^3}\mathrm{sgn}x^\Delta(r)\right]^\Delta + \frac{1}{r^5}\sqrt{|x(3r)|^3}\mathrm{sgn}x(3r) = 0, \quad r \in [r_0, \infty), \tag{28}$$

where $a(r)=\dfrac{1}{\sqrt{r^5}}$, $\alpha=\dfrac{3}{2}$, $b(r)=\dfrac{1}{r^5}$, *and* $g(r)=3r$. *Now,*

$$K(r)=\int_{r_0}^{r}\frac{\Delta\rho}{a^{1/\alpha}(\rho)}=\int_{r_0}^{r}\rho^{5/3}\mathrm{d}\rho=\frac{3}{8}(r),$$

$$\begin{aligned}\gamma_{m+1} &= \frac{1}{\alpha}\liminf_{r\to\infty}a^{1/\alpha}(r)K^{\alpha\left(1-\sqrt[\alpha]{\gamma m}\right)+1}(r)K^{\alpha\sqrt[\alpha]{\gamma m}}(g(r))b(r),\\ &= \frac{2}{3}\liminf_{r\to\infty}\frac{1}{r^6\sqrt[3]{r^2}}\sqrt{\left[\frac{3}{8}(r)\right]^5}\sqrt{\left[\frac{(3r)^{8/3}-r_0}{r^{8/3}-r_0}\right]^{3\sqrt[3]{\gamma_l}}},\\ &= \frac{1}{32}\sqrt{\left(\frac{3}{2}\right)^3}\liminf_{r\to\infty}\sqrt{\left(\frac{r^{8/3}-r_0}{r^{8/3}}\right)^5}\sqrt{\left[\frac{(3r)^{8/3}-r_0}{r^{8/3}-r_0}\right]^{3\sqrt[3]{\gamma_l}}},\\ &= \frac{1}{32}\sqrt{\left(\frac{3}{2}\right)^3}3^{4\sqrt[3]{\gamma_l}},\end{aligned}$$

and

$$\frac{\alpha^\alpha}{\lambda^{\alpha(\alpha+1)}(\alpha+1)^{\alpha+1}}=0.185903201.$$

Therefore,

$$\gamma_1=\frac{1}{32}\sqrt{\left(\frac{3}{2}\right)^3}3^{4\sqrt[3]{\gamma_0^2}}=0.057409916<\frac{\alpha^\alpha}{\lambda^{\alpha(\alpha+1)}(\alpha+1)^{\alpha+1}};$$

$$\gamma_2=\frac{1}{32}\sqrt{\left(\frac{3}{2}\right)^3}3^{4\sqrt[3]{\gamma_1^2}}=0.110409248<\frac{\alpha^\alpha}{\lambda^{\alpha(\alpha+1)}(\alpha+1)^{\alpha+1}};$$

$$\gamma_3=\frac{1}{32}\sqrt{\left(\frac{3}{2}\right)^3}3^{4\sqrt[3]{\gamma_2^2}}=0.157840086<\frac{\alpha^\alpha}{\lambda^{\alpha(\alpha+1)}(\alpha+1)^{\alpha+1}};$$

and

$$\gamma_4=\frac{1}{32}\sqrt{\left(\frac{3}{2}\right)^3}3^{4\sqrt[3]{\gamma_3^2}}=0.207198643>\frac{\alpha^\alpha}{\lambda^{\alpha(\alpha+1)}(\alpha+1)^{\alpha+1}}.$$

Then, according to Theorem 5, Equation (28) oscillates. Observe that Theorem 1 fails to apply to Equation (28) since

$$\liminf_{r\to\infty}a^{1/\alpha}(r)K^{\alpha+1}(r)b(r)<\frac{1}{\lambda^{\alpha(\alpha+1)}}\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}.$$

Example 2. *Let* $\mathbb{T}=q^{\mathbb{N}_0}$, $q\in\{2,3,\ldots\}$, *and consider the advanced q-dynamic equation*

$$\Delta_q\left(a(r)\Delta_q x(r)\right)+b(r)x(q^q r)=0,\ \textit{for}\ r=q^s\in\mathbb{T}. \tag{29}$$

where

$$a(r)=\frac{q-1}{q^2\sqrt[3]{q^2}-1}\frac{1}{r\sqrt[3]{r^2}},\ b(r)=\frac{1}{r^4\sqrt[3]{r^2}},\ \textit{and}\ g(r)=q^q r.$$

Therefore,

$$\begin{aligned}K(r) &= \int_{r_0}^{r}\frac{\Delta\rho}{a^{1/\alpha}(\rho)}=\left(q^2\sqrt[3]{q^2}-1\right)\sum_{k=0}^{s-1}q^{2k}\sqrt[3]{q^{2k}}\\ &= \left(q^{2s}\sqrt[3]{q^{2s}}-1\right)\to\infty\ \textit{as}\ s\to\infty,\end{aligned}$$

and

$$\begin{aligned}\lambda &= \liminf_{r\to\infty}\frac{K(r)}{K(\sigma(r))},\\ &= \liminf_{s\to\infty}\left(\frac{\sum_{k=0}^{s-1} q^{2k}\sqrt[3]{q^{2k}}}{\sum_{k=0}^{s} q^{2k}\sqrt[3]{q^{2k}}}\right),\\ &= \liminf_{s\to\infty}\left(\frac{q^{2s}\sqrt[3]{q^{2s}}-1}{q^{2(s+1)}\sqrt[3]{q^{2(s+1)}}-1}\right)=\frac{1}{q^2\sqrt[3]{q^2}}>0.\end{aligned}$$

Also,

$$\begin{aligned}\gamma_{l+1} &= \frac{1}{\alpha}\liminf_{r\to\infty} a^{1/\alpha}(r)K^{\alpha\left(1-\sqrt[\alpha]{\gamma_l}\right)+1}(r)K^{\alpha\sqrt[\alpha]{\gamma_l}}(g(r))b(r)\\ &= \liminf_{s\to\infty}\frac{1}{q^s\sqrt[3]{q^{2s}}}\left(q^{2s}\sqrt[3]{q^{2s}}-1\right)^2\left(\frac{q^{2(s+q)}\sqrt[3]{q^{2(s+q)}}-1}{q^{2s}\sqrt[3]{q^{2s}}-1}\right)^{\gamma_l}\frac{1}{q^{4s}\sqrt[3]{q^{2s}}}\\ &= \frac{1}{q}\left(q^{2q}\sqrt[3]{q^{2q}}\right)^{\gamma_l-1},\end{aligned}$$

which implies

$$\gamma_1=\frac{1}{q}<1 \text{ and } \gamma_2=\frac{1}{q}\left(q^{2q}\sqrt[3]{q^{2q}}\right)^{\gamma_1}=q\sqrt[3]{q^2}>1.$$

According to Theorem 6, Equation (29) oscillates.

4. Discussion and Conclusions

(1) Throughout this study, Kneser-type iterative criteria are given, including a role for $g(r)$ that can be applied to Equation (1) , and they are applicable for different time scales, such as $\mathbb{T}=\mathbb{R}$, $\mathbb{T}=\mathbb{Z}$, $\mathbb{T}=h\mathbb{Z}$, $h>0$, $\mathbb{T}=q^{\mathbb{N}_0}$, $q>1$, etc. (see [4]). Furthermore, Theorem 5 improves Theorem 1 due to

$$\begin{aligned}\gamma_{l+1} &= \frac{1}{\alpha}\liminf_{r\to\infty} a^{1/\alpha}(r)K^{\alpha\left(1-\sqrt[\alpha]{\gamma_l}\right)+1}(r)K^{\alpha\sqrt[\alpha]{\gamma_l}}(g(r))b(r)\\ &\geq \frac{1}{\alpha}\liminf_{r\to\infty} a^{1/\alpha}(r)K^{\alpha+1}(r)b(r).\end{aligned}$$

(2) The result of Theorem 5 improves and expands Theorem 1. Specifically, if (24) is satisfied for $l\geq 1$ and

$$0<\gamma_{i+1}\leq\frac{\alpha^\alpha}{\lambda^{\alpha(\alpha+1)}(\alpha+1)^{\alpha+1}},\ i=0,1,\ldots,l-1,$$

and

$$\gamma_{l+1}>\frac{\alpha^\alpha}{\lambda^{\alpha(\alpha+1)}(\alpha+1)^{\alpha+1}}.$$

According to Theorem 5, Equation (1) oscillates, but Theorem 1 fails to be applied.

(3) It is interesting to find a Kneser-type condition for Equation (1) under the condition $\int_{r_0}^{\infty}\frac{\Delta\rho}{a^{1/\alpha}(\rho)}<\infty$.

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