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Second-Level Numerical Semigroups

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Abstract: Let S be a numerical semigroup with multiplicity $\mathrm{m}(S)$. Then, S is called a second-level numerical semigroup if $x+y+z-\mathrm{m}(S) \in S$ for every $\{x,y,z\} \subseteq S \setminus \{0\}$. In this paper, we present some algorithms to compute all the second-level numerical semigroups with multiplicity, genus, and a Frobenius fixed number. For m and r, which are positive integers, such that $m < r$ and $\gcd(m,r) = 1$, we show that there exists the minimal second-level numerical semigroup with multiplicity m containing r. We solve the Frobenius problem for these semigroups and show that they satisfy Wilf's conjecture.

Keywords: numerical semigroups; Frobenius number; genus; embedding dimension; Wilf's conjecture

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1. Introduction

Let $\mathbb{Z}$ be the set of integer numbers, and $\mathbb{N} = \{x \in \mathbb{Z} \mid x \geq 0\}$. A *submonoid* of $(\mathbb{N},+)$ is a subset of a closed $\mathbb{N}$ under addition and contains 0. A *numerical semigroup* S is a submonoid of $(\mathbb{N},+)$, such that $\mathbb{N} \setminus S = \{n \in \mathbb{N} \mid n \notin S\}$ is finite.

If S is a numerical semigroup, then $\mathrm{m}(S) = \min(S \setminus \{0\})$, $\mathrm{F}(S) = \max(\mathbb{Z} \setminus S)$, and $\mathrm{g}(S)$, the cardinal of the set $\mathbb{N} \setminus S$, are three important invariants of S called the *multiplicity*, the *Frobenius number*, and the *genus* of S, respectively.

If A is a non-empty subset of $\mathbb{N}$, then we denote by $\langle A \rangle$ the submonoid of $(\mathbb{N},+)$ generated by A, i.e., $\langle A \rangle = \{\lambda_1 a_1 + \cdots + \lambda_n a_n \mid n \in \mathbb{N} \setminus \{0\}, \{a_1,\ldots,a_n\} \subseteq A, \text{ and } \{\lambda_1,\ldots,\lambda_n\} \subseteq \mathbb{N}\}$. In ([1], Lemma 2.1), it is shown that $\langle A \rangle$ is a numerical semigroup if and only if $\gcd(A) = 1$.

If M is a submonoid of $(\mathbb{N},+)$ and $M = \langle A \rangle$, then we say that A is a *system of generators of* M. Furthermore, if $M \neq \langle B \rangle$ for all $B \subsetneq A$, then we say that A is a *minimal system of generators of* M. In ([1], Corollary 2.8), it is shown that for any M submonoid of $(\mathbb{N},+)$, there exists a unique minimal system of generators, and it is finite. We denote this minimal system of generators as $\mathrm{msg}(M)$. Its cardinality is called the *embedding dimension of* M; we will denote it as $\mathrm{e}(M)$.

The Frobenius problem for numerical semigroups (see [2]) is to find formulas to compute the Frobenius number and the genus of a numerical semigroup using its minimal system of generators. This problem was solved in [3] for numerical semigroups with an embedding dimension equal to 2. Today, the problem is still open for numerical semigroups with a higher embedding dimension.

An unfamiliar reader with the numerical semigroup can find these names a bit strange; however, there exists a large number of publications devoted to the study of analytically irreducible unidimensional local domains using the semigroups of values, which is a

numerical semigroup and where the reader can find the connection and the interpretation of these invariants (see, for example, [4–8]). Following this question, there also appear interesting classes of numerical semigroups as numerical semigroups having a maximal embedding dimension, called *MED-semigroups* (see [4,9,10]).

In ([1], Proposition 2.10), it is shown that for any numerical semigroup S, we have $\mathrm{e}(S) \leq \mathrm{m}(S)$. We say that S is a maximal embedding dimension semigroup if $\mathrm{e}(S) = \mathrm{m}(S)$. In ([1], Proposition 3.12), it is shown that a numerical semigroup has a maximal embedding dimension if and only if $x + y - \mathrm{m}(S) \in S$ for every $\{x, y\} \subseteq S \setminus \{0\}$.

Following the terminology used in [11], a numerical semigroup is *second-level* (or a *second-level semigroup*) if $x + y + z - \mathrm{m}(S) \in S$, for every $\{x, y, z\} \subseteq S \setminus \{0\}$. Note that every numerical semigroup with a maximal embedding dimension is a second-level numerical semigroup.

Let m be an integer greater than or equal to 2. We denote

$$\mathcal{L}_2(m) = \{S \mid S \text{ is a second-level semigroup with } \mathrm{m}(S) = m\}.$$

In Section 2, we show that $\mathcal{L}_2(m)$ is a Frobenius pseudo-variety. Using several results from [12], we deduce an algorithm to obtain all the elements in $\mathcal{L}_2(m)$ for a fixed genus. Let f be a positive integer. We denote $\mathcal{L}_2(f, m) = \{S \in \mathcal{L}_2(m) \mid \mathrm{F}(S) = f\}$. In Section 3, we show that $\mathcal{L}_2(f, m)$ is a ratio-covariety. The results in [13] allow us to present an algorithm to compute all the elements in $\mathcal{L}_2(f, m)$.

A $\mathcal{L}_2(m)$-*monoid* is a submonoid of $(\mathbb{N}, +)$, which can be written as the intersection of elements in $\mathcal{L}_2(m)$. For $A \subseteq \{m, \to\}$, we denote by $\mathcal{L}_2(m)[A]$ the intersection of all elements in $\mathcal{L}_2(m)$ containing A. It is clear that $\mathcal{L}_2(m)[A]$ is the smallest $\mathcal{L}_2(m)$-monoid containing A. In Section 4, we give an algorithm to compute $\mathcal{L}_2(m)[A]$. We also show that $\mathcal{L}_2(m)[A]$ belongs to $\mathcal{L}_2(m)$ if and only if $\gcd(A \cup \{m\}) = 1$.

If $M = \mathcal{L}_2(m)[A]$, then we say that A is a $\mathcal{L}_2(m)$-*system of generators* for M. Moreover, if $M \neq \mathcal{L}_2(m)[B]$ for every $B \subsetneq A$, then we say that A is a *minimal* $\mathcal{L}_2(m)$-*system of generators* for M. In Section 4, we show that every $\mathcal{L}_2(m)$-monoid M has a unique minimal $\mathcal{L}_2(m)$-system of generators, which will be denoted as $\mathcal{L}_2(m)$-$\mathrm{msg}(M)$. We define the $\mathcal{L}_2(m)$-*rank* of M as the cardinality of $\mathcal{L}_2(m)$-$\mathrm{msg}(M)$.

In Section 5, we focus on the elements of $\mathcal{L}_2(m)$ with the $\mathcal{L}_2(m)$-rank being equal to 1. These semigroups will be $\mathcal{L}_2(m)[\{r\}]$ for some $r \in \mathbb{N}$, such that $m < r$ and $\gcd(m, r) = 1$. We provide the minimal system of generators for $\mathcal{L}_2(m)[\{r\}]$ and solve the Frobenius problem for these semigroups.

Given S, a numerical semigroup, and x, an integer number, we say that x is a *pseudo-Frobenius number* of S if $x \notin S$ and $x + s \in S$ for every $s \in S \setminus \{0\}$. The set of all pseudo-Frobenius numbers will be denoted as $\mathrm{PF}(S)$. We define the *type* of S as the cardinality of $\mathrm{PF}(S)$, and we will denote it as $\mathrm{t}(S)$.

In Section 5, we show that $\mathrm{t}(S) + 1 = \mathrm{e}(S)$ for any $S \in \mathcal{L}_2(m)$ with the $\mathcal{L}_2(m)$-rank being equal to 1.

For a numerical semigroup S, we denote the set of *small numbers* by $\mathrm{N}(S) = \{s \in S \mid s < \mathrm{F}(S)\}$, and the cardinality of N(S) is denoted as $\mathrm{n}(S)$. H.S. Wilf conjectured in [14] that $\mathrm{F}(S) + 1 \leq \mathrm{e}(S)\,\mathrm{n}(S)$. Today, this question is still open, and it has become one of the most important challenges in the theory of numerical semigroups. This conjecture has been proved for a large number of families of numerical semigroups. Readers interested in this problem may refer to [15–20].

In Section 5, we will show that the elements in $\mathcal{L}_2(m)$ with the $\mathcal{L}_2(m)$-rank being equal to 1 verify Wilf's conjecture.

2. Elements in $\mathcal{L}_2(m)$ for a Fixed Genus

In this section, m will be an integer greater than or equal to 2.

Let g be a positive integer. We denote $\mathcal{L}_2(m,g) = \{S \in \mathcal{L}_2(m) \mid \mathrm{g}(S) = g\}$. The aim of this section is to obtain an algorithm to compute $\mathcal{L}_2(m,g)$.

The following result is easy to check:

Lemma 1.

- $\Delta(m) = \{0, m, \to\}$ *is the maximum of* $\mathcal{L}_2(m)$ *with respect to the inclusion order, where* $\{a_1, \ldots, a_n, \to\}$ *means that every integer bigger than* a_n *belongs to the set.*
- *For* $k \in \mathbb{N}$ *such that* $k \geq m$, *we have* $\langle m \rangle \cup \{k, \to\} \in \mathcal{L}_2(m)$.

If $q \in \mathbb{Q}$, then $\lfloor q \rfloor = \max\{z \in \mathbb{Z} \mid z \leq q\}$. For $\{a,b\} \subseteq \mathbb{Z}$ with $b \neq 0$, we denote $a \bmod b = a - \left\lfloor \frac{a}{b} \right\rfloor b$.

Proposition 1. *Take* $g \in \mathbb{N}$. *Then,* $\mathcal{L}_2(m,g) \neq \emptyset$ *if and only if* $g \geq m-1$.

Proof. *Necessity*. If $S \in \mathcal{L}_2(m,g)$, then $S \in \mathcal{L}_2(m)$ and $\mathrm{g}(S) = g$. Applying Lemma 1, we have $S \subseteq \Delta(m)$ so $g = \mathrm{g}(S) \geq \mathrm{g}(\Delta(m)) = m-1$.

Sufficiency. If $g \geq m-1$, it is clear that $\left\lfloor \frac{g}{m-1} \right\rfloor + g + 1 \geq m$. Applying Lemma 1, we have $S = \langle m \rangle \cup \left\{ \left\lfloor \frac{g}{m-1} \right\rfloor + g + 1, \to \right\} \in \mathcal{L}_2(m)$. On the other hand, the number of positive elements in $\langle m \rangle$ smaller than $\left\lfloor \frac{g}{m-1} \right\rfloor + g$ is $\left\lfloor \frac{\left\lfloor \frac{g}{m-1} \right\rfloor + g}{m} \right\rfloor = \left\lfloor \frac{\frac{g}{m-1} + g}{m} \right\rfloor = \left\lfloor \frac{g + (m-1)g}{(m-1)m} \right\rfloor = \left\lfloor \frac{mg}{m(m-1)} \right\rfloor = \left\lfloor \frac{g}{m-1} \right\rfloor$, so $\mathrm{g}(S) = g$, and $S \in \mathcal{L}_2(m,g)$. □

A *Frobenius pseudo-variety* $\mathcal{P}$ is a non-empty family of numerical semigroups satisfying the following properties:

1. The family $\mathcal{P}$ has a maximum element (with respect to the inclusion order);
2. If $\{S,T\} \subseteq \mathcal{P}$, then $S \cap T \in \mathcal{P}$;
3. If $S \in \mathcal{P}$ and $S \neq \max(\mathcal{P})$, then $S \cup \{\mathrm{F}(S)\} \in \mathcal{P}$.

The following result can be deduced from [11], Proposition 2.6 and Corollary 3.4.

Proposition 2. $\mathcal{L}_2(m)$ *is a Frobenius pseudo-variety and* $\Delta(m)$ *is its maximum.*

A *graph* G is a pair (V,E), where V is a non-empty set and $E \subseteq \{(u,v) \in V \times V \mid u \neq v\}$. The elements in V are called the *vertices* of G, and the elements in E are called the *edges* of G. A *path of length* n connecting the vertices x and y of G is a sequence of different elements in E, $(v_0,v_1),(v_1,v_2),\ldots,(v_{n-1},v_n)$, where $v_0 = x$ and $v_n = y$.

A graph G is called a *tree* if there exists a vertex r named the *root* of G, such that for every x of G there exists a unique path joining x and r. Finally, if (x,y) is an edge of a tree, we say that x is a *child* of y.

We define the graph $\mathrm{G}(\mathcal{L}_2(m))$ as follows: The vertices of this graph are the elements of $\mathcal{L}_2(m)$, and $(S,T) \in \mathcal{L}_2(m) \times \mathcal{L}_2(m)$ is an edge if and only if $T = S \cup \{\mathrm{F}(S)\}$.

As a consequence of Proposition 2, ([12], Lemmas 11 and 12, and Theorem 3), we can obtain the following result:

Theorem 1. $\mathrm{G}(\mathcal{L}_2(m))$ *is a tree whose root is* $\Delta(m)$. *Moreover, the set of children of a vertex* S *of* $\mathrm{G}(\mathcal{L}_2(m))$ *is* $\{S \setminus \{x\} \mid x \in \mathrm{msg}(S)$ *with* $x > \mathrm{F}(S)$ *and* $S \setminus \{x\} \in \mathcal{L}_2(m)\}$.

The following result is easy to check:

Lemma 2. *Let* S *be a numerical semigroup, and let* $x \in S$. *Then,* $S \setminus \{x\}$ *is a numerical semigroup if and only if* $x \in \mathrm{msg}(S)$. *Moreover,* $\mathrm{g}(S \setminus \{x\}) = \mathrm{g}(S) + 1$.

Let S be a numerical semigroup. We consider $\mathrm{msg}(S) = \{n_1, \ldots, n_e\}$. If $s \in S$, then we denote $L_S(s) = \max\{a_1 + \cdots + a_e \mid (a_1, \ldots, a_e) \in \mathbb{N}^e \text{ and } a_1 n_1 + \cdots + a_e n_e = s\}$.

Lemma 3 ([11], Proposition 3.5)**.** *Let $S \in \mathcal{L}_2(m)$, and let $x \in \mathrm{msg}(S) \setminus \{m\}$. Then, we have $S \setminus \{x\} \in \mathcal{L}_2(m)$ if and only if $L_{S\setminus\{x\}}(x+m) \leq 2$.*

A tree can be recurrently built starting from the root and by connecting each vertex with its children using an edge. Also, note that elements in $\mathcal{L}_2(m)$ with genus g are children of elements in $\mathcal{L}_2(m)$ with genus $g-1$. Following this idea and the above results, we present Algorithm 1 to compute $\mathcal{L}_2(m, g)$.

Algorithm 1

Input: Integers m and g, such that $1 \leq m - 1 \leq g$.
Output: $\mathcal{L}_2(m, g)$.
1. $A = \{\Delta(m)\}$, $i = m - 1$;
2. If $i = g$, then return A, and the algorithm is finished;
3. For every $S \in A$ compute $B_S = \{x \in \mathrm{msg}(S) \setminus \{m\} \mid x > \mathrm{F}(S) \text{ and } L_{S\setminus\{x\}}(x+m) \leq 2\}$;
4. $A := \cup_{S\in A}\{S \setminus \{x\} \mid x \in B_S\}$; $i := i + 1$ and go back to (2).

Example 1. *Using Algorithm 1, we are going to compute $\mathcal{L}_2(4, 5)$.*

- $A = \{\langle 4, 5, 6, 7\rangle\}$, $i = 3$;
- $B_{\langle 4,5,6,7\rangle} = \{5, 6, 7\}$;
- $A = \{\langle 4, 6, 7, 9\rangle, \langle 4, 5, 7\rangle, \langle 4, 5, 6\rangle\}$, $i = 4$;
- $B_{\langle 4,6,7,9\rangle} = \{6, 7, 9\}$; $B_{\langle 4,5,7\rangle} = \{7\}$; $B_{\langle 4,5,6\rangle} = \emptyset$;
- $A = \{\langle 4, 7, 9, 10\rangle, \langle 4, 6, 9, 11\rangle, \langle 4, 6, 7\rangle, \langle 4, 5, 11\rangle\}$, $i = 5 = g$ *and the algorithm is finished.*

Algorithm 1 returns $\mathcal{L}_2(4, 5) = \{\langle 4, 7, 9, 10\rangle, \langle 4, 6, 9, 11\rangle, \langle 4, 6, 7\rangle, \langle 4, 5, 11\rangle\}$.

3. Elements in $\mathcal{L}_2(m)$ for a Fixed Frobenius Number

In this section, m will be an integer greater than or equal to 2.

Let f be a positive integer. We consider $\mathcal{L}_2(f, m) = \{S \in \mathcal{L}_2(m) \mid \mathrm{F}(S) = f\}$. The objective of this section is to obtain an algorithm to compute $\mathcal{L}_2(f, m)$.

Proposition 3. *Take $f \in \mathbb{N} \setminus \{0\}$. Then, $\mathcal{L}_2(f, m) \neq \emptyset$ if and only if $m - 1 \leq f$ and $m \nmid f$.*

Proof.

Necessity. If $S \in \mathcal{L}_2(f, m)$, then $m - 1 \notin S$, so $m - 1 \leq f$. Moreover, since $f \notin S$, then $f \notin \langle m\rangle$. Hence, $m \nmid f$.

Sufficiency. Observe that $\langle m\rangle \cup \{f + 1, \to\} \in \mathcal{L}_2(f, m)$. □

If $f = m - 1$ and $S \in \mathcal{L}_2(f, m)$, then $S = \{0, f + 1, \to\}$. In the rest of this section, we assume $\{m, f\} \subset \mathbb{N}$, such that $2 \leq m < f$ and $m \nmid f$.

Let S be a numerical semigroup, such that $S \neq \mathbb{N}$. The *ratio* of S is $\mathrm{r}(S) = \min\{s \in S \mid \mathrm{m}(S) \nmid s\}$. Note that $\mathrm{r}(S) = \min(\mathrm{msg}(S) \setminus \{\mathrm{m}(S)\})$.

A *ratio-covariety* is a family $\mathcal{R}$ of numerical semigroups satisfying the following properties:

1. The family $\mathcal{R}$ has a minimum element (with respect to the inclusion order);
2. If $\{S, T\} \subseteq \mathcal{R}$, then $S \cap T \in \mathcal{R}$;
3. If $S \in \mathcal{R}$ and $S \neq \min(\mathcal{R})$, then $S \setminus \{\mathrm{r}(S)\} \in \mathcal{R}$.

The following result is easy to prove:

Lemma 4. *Let S, T be two numerical semigroups. Then, $S \cap T$ is also a numerical semigroup, and* $\mathrm{F}(S \cap T) = \max\{\mathrm{F}(S), \mathrm{F}(T)\}$.

We denote $\Delta(f,m) = \langle m \rangle \cup \{f+1, \to\}$.

Proposition 4. *$\mathcal{L}_2(f,m)$ is a ratio-covariety and $\Delta(f,m)$ is its minimum.*

Proof. It is clear that $\Delta(f,m)$ is the minimum of $\mathcal{L}_2(f,m)$. If $\{S,T\} \subseteq \mathcal{L}_2(f,m)$, then using Lemma 4, we deduce $S \cap T \in \mathcal{L}_2(f,m)$. To finish the proof, we see that if $S \in \mathcal{L}_2(f,m)$ such that $S \neq \Delta(f,m)$, then $S \setminus \{\mathrm{r}(S)\} \in \mathcal{L}_2(f,m)$. As $S \neq \Delta(f,m)$, then $\mathrm{r}(S) < f$, and we obtain that $S \setminus \{\mathrm{r}(S)\}$ is a numerical semigroup with multiplicity m and a Frobenius number f using Lemma 2. On the other hand, if $\{x,y,z\} \subseteq S \setminus \{\mathrm{r}(S), 0\}$, then $x+y+z-m \in S$. Furthermore, $x+y+z-m \neq \mathrm{r}(S)$, so $x+y+z-m \in S \setminus \{\mathrm{r}(S)\}$. From this, we have $S \setminus \{\mathrm{r}(S)\} \in \mathcal{L}_2(f,m)$. □

Let S be a numerical semigroup, and $x \in \mathbb{Z}$. We say that x is a *special gap* of S if $x \notin S$ and $S \cup \{x\}$ is a numerical semigroup. We denote by $\mathrm{SG}(S)$ the set of special gaps of S.

We define the graph $\mathrm{G}(\mathcal{L}_2(f,m))$ as follows: The set of vertices is $\mathcal{L}_2(f,m)$, and $(S,T) \in \mathcal{L}_2(f,m) \times \mathcal{L}_2(f,m)$ is an edge if and only if $T = S \setminus \{\mathrm{r}(S)\}$.

Using Proposition 4, [13], and Propositions 3 and 4, we obtain the following result:

Theorem 2. *$\mathrm{G}(\mathcal{L}_2(f,m))$ is a tree whose root is $\Delta(f,m)$. Moreover, the set of children of a vertex S is $\{S \cup \{x\} \mid x \in \mathrm{SG}(S), m < x < \mathrm{r}(S),$ and $S \cup \{x\} \in \mathcal{L}_2(f,m)\}$.*

The following result is Proposition 3.1 of [11].

Proposition 5. *Let S be a numerical semigroup with multiplicity m. Then, $S \in \mathcal{L}_2(m)$ if and only if $a+b+c-m \in S$ for every $\{a,b,c\} \subseteq \mathrm{msg}(S) \setminus \{m\}$.*

Corollary 1. *Consider $S \in \mathcal{L}_2(f,m)$, and $x \in \mathrm{SG}(S)$, such that $m < x < f$. Then, $S \cup \{x\} \in \mathcal{L}_2(f,m)$ if and only if $\{3x-m, 2x+a-m, x+a+b-m\} \subseteq S$ for every $\{a,b\} \subseteq \mathrm{msg}(S) \setminus \{m\}$.*

Proof. To apply Proposition 5, we need to observe that $\mathrm{msg}(S \cup \{x\}) \subseteq \mathrm{msg}(S) \cup \{x\}$. □

Let S be a numerical semigroup, and $n \in S \setminus \{0\}$, we define (in honor of [21]) the Apéry set of n in S as $\mathrm{Ap}(S,n) = \{s \in S \mid s-n \notin S\}$. The following result appears in [1], Lemma 2.4:

Lemma 5. *Let S be a numerical semigroup and $n \in S \setminus \{0\}$. Then, $\mathrm{Ap}(S,n) = \{w(0) = 0, w(1), \ldots, w(n-1)\}$, where $w(i)$ is the smallest element in S congruent with i modulo n.*

Remark 1. *Let S be a numerical semigroup and assume that $\mathrm{Ap}(S,\mathrm{m}(S)) = \{w(0), w(1), \ldots, w(\mathrm{m}(S)-1)\}$ is known; then, we have the following:*

1. *We can solve the problem of belonging to S as follows: Any integer x belongs to S if and only if $x \geq w(x \bmod \mathrm{m}(S))$.*
2. *We can obtain the special gaps of S. From ([22], Remark 1), we have an easy way to compute them as $\mathrm{SG}(S) = \{x \in \mathrm{PF}(S) \mid 2x \notin \mathrm{PF}(S)\}$ [22], Lemma 3.4, where $\mathrm{PF}(S) = \{w - \mathrm{m}(S) \mid w \in \mathrm{Maximals}_{\leq_S} \mathrm{Ap}(S,\mathrm{m}(S))\}$ [23], Lemma 10. Here, $\leq_S$ is the partial order defined on $\mathbb{Z}$ as $a \leq_S b$, if and only if $b-a \in S$.*
3. *From ([22], Lemma 3.5), we have $\mathrm{Ap}(S \cup \{x\}, \mathrm{m}(S)) = (\mathrm{Ap}(S,\mathrm{m}(S)) \setminus \{x + \mathrm{m}(S)\}) \cup \{x\}$, for every $x \in \mathrm{SG}(S)$.*
4. *Finally, as $\mathrm{msg}(S) \setminus \{\mathrm{m}(S)\} = \{w \in \mathrm{Ap}(S,\mathrm{m}(S)) \setminus \{0\} \mid w - w' \notin \mathrm{Ap}(S,\mathrm{m}(S)) \forall w' \in \mathrm{Ap}(S,\mathrm{m}(S)) \setminus \{0,w\}\}$, we can obtain $\mathrm{msg}(S \cup \{x\})$ from (3).*

Using the above ideas, we present Algorithm 2. It allow to compute $\mathcal{L}_2(f,m)$ being f and m integers under certain conditions.

Algorithm 2

Input: Integers f and m such that $2 \leq m < f$ and $m \not| f$.
Output: $\mathcal{L}_2(f,m)$.

1. $\mathcal{L}_2(f,m) = \{\Delta(f,m)\}$ and $B = \{\Delta(f,m)\}$.
2. Compute $\mathrm{Ap}(\Delta(f,m),m)$.
3. For every $S \in B$, compute $\theta(S) = \{x \in \mathrm{SG}(S) \mid m < x < \mathrm{r}(S), x \neq f, \{3x-m, 2x+a-m, x+a+b-m\} \subseteq S \text{ for every } \{a,b\} \subseteq \mathrm{msg}(S) \setminus \{m\}\}$.
4. If $\cup_{S\in B}\theta(S) = \emptyset$, then return $\mathcal{L}_2(f,m)$, and the algorithm is finished.
5. $C = \cup_{S\in B}\{S \cup \{x\} \mid x \in \theta(S)\}$.
6. $\mathcal{L}_2(f,m) := \mathcal{L}_2(f,m) \cup C$ and $B := C$.
7. For every $S \in B$, compute $\mathrm{Ap}(S,m)$ and go back to (3).

Note that the computation of $\mathrm{Ap}(S,m)$ in line 7 is used to compute the set $\mathrm{SG}(S)$ in a new loop, as we point out in Remark 1.

Example 2. *Using Algorithm 2, we compute* $\mathcal{L}_2(7,4)$.

First loop of the algorithm is:

(1) $\mathcal{L}_2(7,4) = \{\Delta(7,4)\}$ *and* $B = \{\Delta(7,4)\}$.

(2) $\mathrm{Ap}(\Delta(7,4),4) = \{0,9,10,11\}$.
Therefore, $\mathrm{Maximals}(\mathrm{Ap}(\Delta(7,4),4)) = \{9,10,11\}$. $\mathrm{PF}(\Delta(7,4)) = \{5,6,7\}$.
Moreover, $\mathrm{SG}(\boldsymbol{\Delta(7,4)}) = \{\mathbf{5},\mathbf{6},\mathbf{7}\}$.
Meanwhile, $\mathrm{msg}(\Delta(7,4)) = \{4,9,10,11\}$ *and* $\mathrm{r}(\Delta(7,4)) = 9$.
Then, we have

(3) $\theta(\Delta(7,4)) = \{5,6\}$, *as* $7 = f$.

(5) $C = \{\Delta(7,4) \cup \{5\}, \Delta(7,4) \cup \{6\}\}$.

(6) $\mathcal{L}_2(7,4) = \{\Delta(7,4), \Delta(7,4) \cup \{5\}, \Delta(7,4) \cup \{6\}\}$ *and* $B = \{\Delta(7,4) \cup \{5\}, \Delta(7,4) \cup \{6\}\}$.

(7) $\mathrm{Ap}(\Delta(7,4) \cup \{5\},4) = \{0,5,10,11\}$ *and* $\mathrm{Ap}(\Delta(7,4) \cup \{6\},4) = \{0,6,9,11\}$.

Therefore,

(a) $\mathrm{Maximals}(\mathrm{Ap}(\Delta(7,4) \cup \{5\},4)) = \{10,11\}$. $\mathrm{PF}(\Delta(7,4) \cup \{5\}) = \{6,7\}$.
Moreover, $\mathrm{SG}(\boldsymbol{\Delta(7,4)} \cup \{\mathbf{5}\}) = \{\mathbf{6},\mathbf{7}\}$.
Meanwhile, $\mathrm{msg}(\Delta(7,4) \cup \{5\}) = \{4,5,11\}$ *and* $\mathrm{r}(\Delta(7,4) \cup \{5\}) = 5$.

(b) $\mathrm{Maximals}(\mathrm{Ap}(\Delta(7,4) \cup \{6\},4)) = \{6,9,11\}$. $\mathrm{PF}(\Delta(7,4) \cup \{6\}) = \{2,5,7\}$.
Moreover, $\mathrm{SG}(\boldsymbol{\Delta(7,4)} \cup \{\mathbf{6}\}) = \{\mathbf{2},\mathbf{5},\mathbf{7}\}$.
Meanwhile, $\mathrm{msg}(\Delta(7,4) \cup \{6\}) = \{4,6,9,11\}$ *and* $\mathrm{r}(\Delta(7,4) \cup \{6\}) = 6$.

Then, second loop is

(3) $\theta(\Delta(7,4) \cup \{5\}) = \emptyset$, *as* $\mathrm{r}(\Delta(7,4) \cup \{5\}) < 6 < 7$
and $\theta(\Delta(7,4) \cup \{6\}) = \{5\}$, *as* $2 < m < 5 < \mathrm{r}(\Delta(7,4) \cup \{6\}) < 7$.

(5) $C = \{\Delta(7,4) \cup \{5,6\}\}$.

(6) $\mathcal{L}_2(7,4) = \{\Delta(7,4), \Delta(7,4) \cup \{5\}, \Delta(7,4) \cup \{6\}, \Delta(7,4) \cup \{5,6\}\}$ *and* $B = \{\Delta(7,4) \cup \{5,6\}\}$.

(7) $\mathrm{Ap}(\Delta(7,4) \cup \{5,6\}),4) = \{0,5,6,11\}$.

Therefore, $\mathrm{Maximals}(\mathrm{Ap}(\Delta(7,4) \cup \{5,6\},4)) = \{11\}$. $\mathrm{PF}(\Delta(7,4) \cup \{5,6\}) = \{7\}$.
Moreover, $\mathrm{SG}(\boldsymbol{\Delta(7,4)} \cup \{\mathbf{5},\mathbf{6}\}) = \{\mathbf{7}\}$.
Meanwhile, $\mathrm{msg}(\Delta(7,4) \cup \{5,6\}) = \{4,5,6\}$ *and* $\mathrm{r}(\Delta(7,4) \cup \{5,6\}) = 5$.
Then, the third loop is

(3) $\theta(\Delta(7,4) \cup \{5,6\}) = \emptyset$, *as* $7 = f$.

(4) $\mathcal{L}_2(7,4) = \{\Delta(7,4), \Delta(7,4) \cup \{5\}, \Delta(7,4) \cup \{6\}, \Delta(7,4) \cup \{5,6\}\}$ *and the algorithm is finished.*

Algorithm 2 returns $\mathcal{L}_2(7,4) = \{\Delta(7,4), \Delta(7,4) \cup \{5\}, \Delta(7,4) \cup \{6\}, \Delta(7,4) \cup \{5,6\}\}$.

4. $\mathcal{L}_2(m)$-Systems of Generators

It is clear that $\mathcal{L}_2(1) = \{\mathbb{N}\}$ and $\mathcal{L}_2(2) = \{\langle 2, 2k+1\rangle \mid k \in \mathbb{N} \setminus \{0\}\}$. In this section, m will be an integer greater than or equal to 3.

From Proposition 2, we know that the intersection of a finite number of elements in $\mathcal{L}_2(m)$ is again an element in $\mathcal{L}_2(m)$. However, when we consider an infinite number of them, the intersection is generally not an element in $\mathcal{L}_2(m)$, as shown in the following example:

Example 3. *For any* $k \in \{m, \rightarrow\}$, *we consider* $S(k) = \langle m\rangle \cup \{k, \rightarrow\}$. *Clearly,* $S(k) \in \mathcal{L}_2(m)$. *However,* $\cap_{k\in\{m,\rightarrow\}} S(k) = \langle m\rangle \notin \mathcal{L}_2(m)$.

The finite or infinite intersection of elements in $\mathcal{L}_2(m)$ is always a submonoid of $(\mathbb{N}, +)$. This fact allows us to give the following definition. A $\mathcal{L}_2(m)$-*monoid* is a submonoid of $(\mathbb{N}, +)$ that can be written as an intersection of elements in $\mathcal{L}_2(m)$.

The following result is easy to check:

Lemma 6. *The intersection of* $\mathcal{L}_2(m)$*-monoids is again a* $\mathcal{L}_2(m)$*-monoid.*

Consider $A \subseteq \Delta(m)$. We denote by $\mathcal{L}_2(m)[A]$ the intersection of all the $\mathcal{L}_2(m)$-monoids containing A. As a consequence of Lemma 6, we find that $\mathcal{L}_2(m)[A]$ is the smallest (with respect to the inclusion) $\mathcal{L}_2(m)$-monoid containing A.

We can easily reach the following result:

Lemma 7. *Let* A *be a subset of* $\Delta(m)$. *Then,* $\mathcal{L}_2(m)[A]$ *is equal to the intersection of all the elements in* $\mathcal{L}_2(m)$ *containing* A.

If $M = \mathcal{L}_2(m)[A]$, then we will say that A is a $\mathcal{L}_2(m)$-*system of generators of* M. Furthermore, if $M \neq \mathcal{L}_2(m)[B]$ for any $B \subsetneq A$, then A is called a *minimal* $\mathcal{L}_2(m)$-*system of generators* of M.

The following result is easy to deduce from Proposition 2 and [12], Corollary 1.

Proposition 6. *Every* $\mathcal{L}_2(m)$*-monoid has a unique minimal* $\mathcal{L}_2(m)$*-system of generators. Furthermore, such a minimal* $\mathcal{L}_2(m)$*-system of generators is finite.*

Let M be a $\mathcal{L}_2(m)$-monoid. We denote $\mathcal{L}_2(m)$-msg(M) as the minimal $\mathcal{L}_2(m)$-system of generators of M. We define the $\mathcal{L}_2(m)$-*rank* of M as the cardinality of this minimal $\mathcal{L}_2(m)$-system of generators.

Proposition 7. *Let* M *be a submonoid of* $(\mathbb{N}, +)$. *Then, the following statements are equivalent.*

1. M *is a* $\mathcal{L}_2(m)$*-monoid.*
2. $m = \min(M \setminus \{0\})$ *and* $x + y + z - m \in M$ *for every* $\{x, y, z\} \subseteq M \setminus \{0\}$.
3. $m = \min(M \setminus \{0\})$ *and* $x + y + z - m \in M$ *for every* $\{x, y, z\} \subseteq \text{msg}(M) \setminus \{m\}$.

Proof.

(1) $\Rightarrow$ (2). If M is a $\mathcal{L}_2(M)$-monoid, then there exists a family $\{S_i\}_{i\in I}$ of elements in $\mathcal{L}_2(m)$, such that $M = \cap_{i\in I} S_i$. Hence, $\min(M \setminus \{0\}) = m$. On the other hand, if $\{x, y, z\} \subseteq M \setminus \{0\}$, then $\{x, y, z\} \subseteq S_i \setminus \{0\}$ for every $i \in I$, so $x + y + z - m \in S_i$ for every $i \in I$, and then $x + y + z - m \in M$.

(2) $\Rightarrow$ (3). Trivial.

(3) $\Rightarrow$ (1). For every $k \in \{m, \rightarrow\}$, we consider $S(k) = M \cup \{k, \rightarrow\}$. From Proposition 5, we easily deduce that $S(k) \in \mathcal{L}_2(m)$ for every $k \in \{m, \rightarrow\}$. It is clear that $M = \cap_{k\in\{m,\rightarrow\}} S(k)$, so M is a $\mathcal{L}_2(m)$-monoid. □

Proposition 7 allows us to present Algorithm 3 which compute $\mathcal{L}_2(m)[A]$ for any $A \subseteq \Delta(m)$.

Algorithm 3

Input: A finite subset A of $\Delta(m)$.
Output: $\mathcal{L}_2(m)[A]$

1. $M = \langle A \cup \{m\}\rangle$.
2. $B = \{x + y + z - m \mid \{x, y, z\} \subseteq \text{msg}(M) \setminus \{m\}$ and $x + y + z - m \notin M\}$.
3. If $B = \emptyset$, then return M, and the algorithm is finished.
4. $M := \langle \text{msg}(M) \cup B\rangle$ and go back to (2).

Now, we present two different examples to show how Algorithm 3 works. In the first example, we calculate $\mathcal{L}_2(5)[\{7\}]$, and we will see that $\mathcal{L}_2(5)[\{7\}]$ belongs to $\mathcal{L}_2(5)$. In the second example, we calculate $\mathcal{L}_2(8)[\{10\}]$, and we will show that $\mathcal{L}_2(8)[\{10\}]$ does not belong to $\mathcal{L}_2(8)$. This raises the following question: What is the condition for $\mathcal{L}_2(m)[A]$ to belong to $\mathcal{L}_2(m)$?

Example 4. *Let us calculate* $\mathcal{L}_2(5)[\{7\}]$ *using Algorithm 3.*

- $M = \langle 5, 7\rangle$.
- $B = \{16\}$.
- $M = \langle 5, 7, 16\rangle$.
- $B = \emptyset$, *and the algorithm is finished.*

Algorithm 3 returns $\mathcal{L}_2(5)[\{7\}] = \langle 5, 7, 16\rangle$.

Example 5. *Let us calculate* $\mathcal{L}_2(8)[\{10\}]$ *using Algorithm 3.*

- $M = \langle 8, 10\rangle$.
- $B = \{22\}$.
- $M = \langle 8, 10, 22\rangle$.
- $B = \emptyset$, *and the algorithm is finished.*

Algorithm 3 returns $\mathcal{L}_2(8)[\{10\}] = \langle 8, 10, 22\rangle$. Since $\gcd\{8, 10, 22\} \neq 1$, then $\langle 8, 10, 12\rangle$ is not a numerical semigroup; therefore, $\langle 8, 10, 22\rangle \notin \mathcal{L}_2(8)$.

Proposition 8. *Consider* $A \subseteq \Delta(m)$. *Then,* $\mathcal{L}_2(m)[A] \in \mathcal{L}_2(m)$ *if and only if* $\gcd(A \cup \{m\}) = 1$.

Proof.

Necessity. If $\gcd(A \cup \{m\}) = d \neq 1$, then using Proposition 7, we deduce $M = \{0\} \cup \{m + kd \mid k \in \mathbb{N}\}$ is a $\mathcal{L}_2(m)$-monoid. It is clear that $A \subseteq M$ and $\mathcal{L}_2(m)[A] \subseteq M$. From this, we have that $\mathcal{L}_2(m)[A]$ is not a numerical semigroup. Consequently, $\mathcal{L}_2(m)[A] \notin \mathcal{L}_2(m)$.

Sufficiency. We have that $\mathcal{L}_2(m)[A]$ is a submonoid of $(\mathbb{N}, +)$ and $A \cup \{m\} \subseteq \mathcal{L}_2(m)[A]$. As $\gcd(A \cup \{m\}) = 1$, we deduce $\gcd(\mathcal{L}_2(m)[A]) = 1$, and then $\mathcal{L}_2(m)[A]$ is a numerical semigroup. Consequently, $\mathcal{L}_2(m)[A]$ belongs to $\mathcal{L}_2(m)$. □

5. Elements in $\mathcal{L}_2(m)$ with $\mathcal{L}_2(m)$-Rank Equal to 1

In this section, m denotes an integer greater than or equal to 3.

Proposition 9. *Consider $S \in \mathcal{L}_2(m)$. Then, the following statements are equivalent:*

1. *The $\mathcal{L}_2(m)$-rank of S is equal to 1.*
2. *$S = \mathcal{L}_2(m)[\{r\}]$ for some integer r, such that $m < r$ and $\gcd(m, r) = 1$.*

Proof.

(1) $\Rightarrow$ (2). If $\mathcal{L}_2(m)$-rank$(S) = 1$, then there exists $r \in \Delta(m)$ such that $\mathcal{L}_2(m)$-msg$(S) = \{r\}$. Hence, $r \geq m$ and $S = \mathcal{L}_2(m)[\{r\}]$. Using Proposition 8, we have $\gcd(m, r) = 1$, which implies $m < r$.

(2) $\Rightarrow$ (1). It is clear that $\{r\} = \mathcal{L}_2(m)$-msg(S). Then, $\mathcal{L}_2(m)$-rank$(S) = 1$. □

In the rest of this section, r will be an integer greater than m such that $\gcd(m, r) = 1$. We also denote $A(m, r) = \{(2k+1)r - km \mid k \in \mathbb{N}\}$.

Proposition 10. $\mathcal{L}_2(m)[\{r\}] = \langle A(m, r) \cup \{m\}\rangle$.

Proof. First of all, we will show, by induction on k, that $(2k+1)r - km \in \mathcal{L}_2(m)[\{r\}]$. For $k = 0$, the result is clear because $(2 \cdot 0 + 1)r - 0 \cdot m = r \in \mathcal{L}_2(m)[\{r\}]$. We assume the statement to be true for k; then, we prove it for $k+1$. As $\{(2k+1)r - km, r\} \subseteq \mathcal{L}_2(m)[\{r\}] \setminus \{0\}$ and $\mathcal{L}_2(m)[\{r\}] \in \mathcal{L}_2(m)$, then $(2k+1)r - km + r + r - m \in \mathcal{L}_2(m)[\{r\}]$ and so $(2(k+1)+1)r - (k+1)m \in \mathcal{L}_2(m)[\{r\}]$.

Therefore, we have that $A(m, r) \cup \{m\} \subseteq \mathcal{L}_2(m)[\{r\}]$ and consequently $\langle A(m, r) \cup \{m\}\rangle \subseteq \mathcal{L}_2(m)[\{r\}]$.

Now, we will see that $\mathcal{L}_2(m)[\{r\}] \subseteq \langle A(m, r) \cup \{m\}\rangle$. For this, we only need to prove $\langle A(m, r) \cup \{m\}\rangle \in \mathcal{L}_2(m)$.

Consider $\{i, j, k\} \subseteq \mathbb{N}$. Then, $(2i+1)r - im + (2j+1)r - jm + (2k+1)r - km - m \in \langle A(m, r) \cup \{m\}\rangle$ as $(2i+1)r - im + (2j+1)r - jm + (2k+1)r - km - m = (2(i+j+k+1)+1)r - (i+j+k+1)m \in A(m, r)$. Then, Proposition 5 ensures that $\langle A(m, r) \cup \{m\}\rangle \in \mathcal{L}_2(m)$.

Consequently, we have $\mathcal{L}_2(m)[\{r\}] = \langle A(m, r) \cup \{m\}\rangle$. □

As a consequence of Proposition 10, we have $\mathrm{Ap}(\mathcal{L}_2(m)[\{r\}], m) \subseteq \langle A(m, r)\rangle$. We also have $\mathcal{L}_2(m)[\{r\}] \in \mathcal{L}_2(m)$. Then, $x + y + z - m \in \mathcal{L}_2(m)[\{r\}]$ for every $\{x, y, z\} \subseteq A(m, r)$. Then, we have the following result.

Lemma 8. *If $w \in \mathrm{Ap}(\mathcal{L}_2(m)[\{r\}], m)$, then we have one of the following asserts:*

1. *$w = (2i+1)r - im$ for some $i \in \mathbb{N}$.*
2. *$w = (2i+1)r - im + (2j+1)r - jm$ for some $\{i, j\} \subseteq \mathbb{N}$.*

As $\gcd(m, r) = 1$, we have

$$\{(1 \cdot r) \bmod m, (2 \cdot r) \bmod m, \ldots, ((m-1) \cdot r) \bmod m\} = \{1, 2, \ldots, m-1\}.$$

Theorem 3. *If $\mathrm{Ap}(\mathcal{L}_2(m)[\{r\}], m) = \{w(0), w(1), \ldots, w(m-1)\}$, then*

$$w(kr \bmod m) = \begin{cases} kr - \frac{k-1}{2}m & \text{if } k \text{ is odd} \\ kr - \frac{k-2}{2}m & \text{if } k \text{ is even} \end{cases}$$

for every $k \in \{1, \ldots, m-1\}$.

Proof. If k is odd, then there exists $s \in \mathbb{N}$ such that $k = 2s+1$, so $kr - \frac{k-1}{2}m = (2s+1)r - sm \in A(m, r) \subseteq \mathcal{L}_2(m)[\{r\}]$. However, if k is even, then $k = 2s+2$ for some $s \in \mathbb{N}$, and we have $kr - \frac{k-2}{2}m = (2s+2)r - sm = (2s+1)r - sm + r \in \mathcal{L}_2(m)[\{r\}]$ because $\{(sk+1)r - sm, r\} \subseteq A(m, r)$.

Now, we will first see that in the case in which k is odd, we have $kr - \frac{k-1}{2}m - m \notin \mathcal{L}_2(m)[\{r\}]$. Assume the contrary. By Lemma 8, we consider two cases:

1. There exist $t \in \mathbb{N} \setminus \{0\}$ and $i \in \mathbb{N}$ such that $kr - \frac{k-1}{2}m - tm = (2i+1)r - im$. Then, $kr \equiv (2i+1)r (\mathrm{mod}\, m)$. However, since $\gcd(m, r) = 1$, we have $k \equiv 2i + 1 (\mathrm{mod}\, m)$. Hence, there exists $l \in \mathbb{N}$ such that $2i + 1 = k + lm$. Consequently, $kr - \frac{k-1}{2}m - tm = (k+lm)r - \frac{k+lm-1}{2}m$ and then $\left(\frac{k+lm-1}{2} - \frac{k-1}{2} - t\right)m = lmr$. Therefore, $\frac{lm}{2} - t = lr$. Hence, $\frac{lm}{2} > lr$, and $\frac{m}{2} > r$, which is a contradiction, since $r > m$. Observe that $l \neq 0$. Otherwise, we have $t = 0$, which is impossible.
2. There exist $t \in \mathbb{N} \setminus \{0\}$ and $\{i, j\} \subseteq \mathbb{N}$, such that $kr - \frac{k-1}{2}m - tm = (2i+1)r - im + (2j+1)r - jm$. Then, $k \equiv (2i + 2j + 2) (\mathrm{mod}\, m)$. Hence, $2i + 2j + 2 = k + lm$ for some $l \in \mathbb{N}$. Therefore, $kr - \frac{k-1}{2}m - tm = (2i + 2j + 2)r - (i + j)m = (k + lm)r - \frac{k+lm-2}{2}m$. Then, following the same reasoning as in (1), we have $\frac{lm-1}{2} - t = lr$. Hence, $\frac{lm}{2} > lr$. Consequently, $\frac{m}{2} > r$, which is again a contradiction. Observe that $l \neq 0$. Otherwise, we have $t = \frac{1}{2}$, which is impossible.

We then focus on the case where k is even and try to prove that $kr - \frac{k-2}{2}m - m \notin \mathcal{L}_2(m)[\{r\}]$. As above, we assume the contrary, and using Lemma 8 we consider two cases:

1. There exist $t \in \mathbb{N} \setminus \{0\}$ and $i \in \mathbb{N}$ such that $kr - \frac{k-2}{2}m - tm = (2i+1)r - im$. Then, $kr \equiv (2i+1)r (\mathrm{mod}\, m)$, since $\gcd(m, r) = 1$ we have $k \equiv 2i + 1 (\mathrm{mod}\, m)$, and so there exists $l \in \mathbb{N}$ such that $2i + 1 = k + lm$. Then, $kr - \frac{k-2}{2}m - tm = (k + lm)r - \frac{k+lm-1}{2}m$ and then $\left(\frac{k+lm-1}{2} - \frac{k-2}{2} - t\right)m = lmr$. Therefore, $\frac{lm+1}{2} - t = lr$, which implies that $\frac{lm}{2} > lr$ and $\frac{m}{2} > r$, and we obtain the same contradiction. As we have commented in the above paragraph, l must be different from 0. Otherwise, $t = \frac{1}{2}$.
2. There exist $t \in \mathbb{N} \setminus \{0\}$ and $\{i, j\} \subseteq \mathbb{N}$ such that $kr - \frac{k-2}{2}m - tm = (2i+1)r - im + (2j+1)r - jm$. Then, $k \equiv (2i + 2j + 2) (\mathrm{mod}\, m)$, and we can write $2i + 2j + 2 = k + lm$ for some $l \in \mathbb{N}$. Therefore, $kr - \frac{k-2}{2}m - tm = (2i + 2j + 2)r - (i + j)m = (k + lm)r - \frac{k+lm-2}{2}m$. Consequently, $\frac{lm}{2} - t = lr$. Then, $\frac{lm}{2} > lr$, allowing us to obtain $\frac{m}{2} > r$, which is a contradiction. Again, we can ensure $l \neq 0$ since $t \neq 0$. □

We give the following example to illustrate Theorem 3:

Example 6. *Using Theorem 3, we calculate* $\mathrm{Ap}(\mathcal{L}_2(10)[\{13\}], 10)$.

$w(1 \cdot 13 \bmod 10) = w(3) = 1 \cdot 13 - \frac{1-1}{2} \cdot 10 = 13$	$w(2 \cdot 13 \bmod 10) = w(6) = 2 \cdot 13 - \frac{2-2}{2} \cdot 10 = 26$
$w(3 \cdot 13 \bmod 10) = w(9) = 3 \cdot 13 - \frac{3-1}{2} \cdot 10 = 29$	$w(4 \cdot 13 \bmod 10) = w(2) = 4 \cdot 13 - \frac{4-2}{2} \cdot 10 = 42$
$w(5 \cdot 13 \bmod 10) = w(5) = 5 \cdot 13 - \frac{5-1}{2} \cdot 10 = 45$	$w(6 \cdot 13 \bmod 10) = w(8) = 6 \cdot 13 - \frac{6-2}{2} \cdot 10 = 58$
$w(7 \cdot 13 \bmod 10) = w(1) = 7 \cdot 13 - \frac{7-1}{2} \cdot 10 = 61$	$w(8 \cdot 13 \bmod 10) = w(4) = 8 \cdot 13 - \frac{8-2}{2} \cdot 10 = 74$
$w(9 \cdot 13 \bmod 10) = w(7) = 9 \cdot 13 - \frac{9-1}{2} \cdot 10 = 77$	

The following result is easy to prove:

Lemma 9. *Let S be a numerical semigroup, and $n \in S \setminus \{0\}$. Then, $S = \langle (\mathrm{Ap}(S, n) \setminus \{0\}) \cup \{n\} \rangle$.*

Lemma 10. $\mathcal{L}_2(m)[\{r\}] = \langle \{m\} \cup \{(2k+1)r - km \mid k \in \{0, 1, \ldots, \lfloor \frac{m-2}{2} \rfloor\}\} \rangle$.

Proof. Applying Lemma 9 and Theorem 3, we deduce that $\mathcal{L}_2(m)[\{r\}] = \langle \{m\} \cup \{jr - \frac{j-1}{2}m \mid j \in \{1, \ldots, m-1\} \text{ and } j \text{ odd}\} \cup \{jr - \frac{j-2}{2}m \mid j \in \{1, \ldots, m-1\} \text{ and } j \text{ even}\} \rangle$. However, when j is even, we have $jr - \frac{j-2}{2}m = (j-1)r - \frac{(j-1)-1}{2}m + r$, so $jr - \frac{j-2}{2}m \notin \mathrm{msg}(\mathcal{L}_2(m)[\{r\}])$. Therefore, $\mathcal{L}_2(m)[\{r\}] = \langle \{m\} \cup \{jr - \frac{j-1}{2}m \mid j \in \{1, \ldots, m-1\} \text{ and } j \text{ odd}\} \rangle = \langle \{m\} \cup \{(2k+1)r - km \mid k \in \{0, 1, \ldots, \lfloor \frac{m-2}{2} \rfloor\}\} \rangle$. □

Proposition 11. *The embedding dimension of $\mathcal{L}_2(m)[\{r\}]$ is $\lfloor \frac{m-2}{2} \rfloor + 2$. Moreover, $\mathrm{msg}(\mathcal{L}_2(m)[\{r\}]) = \{m\} \cup \{(2k+1)r - km \mid k \in \{0, 1, \ldots, \lfloor \frac{m-2}{2} \rfloor\}\}$.*

Proof. Suppose, on the contrary, that $(2k+1)r - km \notin \mathrm{msg}(\mathcal{L}_2(m)[\{r\}])$ for some $k \in \{0,1,\ldots,\lfloor\frac{m-2}{2}\rfloor\}$. Then, there exists some $g \in \mathrm{msg}(\mathcal{L}_2(m)[\{r\}])$ such that $(2k+1)r - km - g \in \mathcal{L}_2(m)[\{r\}] \setminus \{0\}$. Theorem 3 ensures that $(2k+1)r - km \in \mathrm{Ap}(\mathcal{L}_2(m)[\{r\}], m)$ for our k, which ensures that $g \neq m$. Then, using Lemma 10, we can assume $g = (2i+1)r - im$ for some $i < k$. Therefore, we have the following: $(2k+1)r - km - ((2i+1)r - im) = 2(k-i)r - (k-i)m \in \mathcal{L}_2(m)[\{r\}]$ and we also have $2(k-i)r - (k-i)m \equiv 2(k-i)r (\mathrm{mod} m)$. Furthermore, we have $2(k-i) < 2(k+1) \leq m$, since $k \leq \lfloor\frac{m-2}{2}\rfloor$. So, by Theorem 3, we can take $w(2(k-i)r \bmod m) \in \mathrm{Ap}(\mathcal{L}_2(m)[\{r\}], m)$. Thus, using Theorem 3, we have $2(k-i)r - (k-i)m \geq w(2(k-i)r \bmod m) = 2(k-i)r - \frac{2(k-i)-2}{2}m$. However, this implies $2(k-i) - 2 \geq 2(k-i)$, which is absurd, and the proof is finished. □

Example 7. *Using Proposition 11, we can compute* $\mathrm{msg}(\mathcal{L}_2(10)[\{13\}]) = \{10\} \cup \{(2k+1)13 - k10 \mid k \in \{0,1,2,3,4\}\} = \{10, 13, 29, 45, 61, 77\}$.

Now, we are going to solve the Frobenius problem for elements in $\mathcal{L}_2(m)$ with the $\mathcal{L}_2(m)$-rank being equal to 1. We will use Theorem 3 and the following result from [24]:

Lemma 11. *Let S be a numerical semigroup and $n \in S \setminus \{0\}$. Then, we have the following:*

1. $F(S) = \max(\mathrm{Ap}(S,n)) - n$.
2. $g(S) = \frac{1}{n}(\sum_{w \in \mathrm{Ap}(S,n)} w) - \frac{n-1}{2}$.

Proposition 12.

$$\mathrm{F}(\mathcal{L}_2(m)[\{r\}]) = \begin{cases} (m-1)r - \frac{m^2}{2} & \text{if } m \text{ is even} \\ (m-1)r - \frac{m(m-1)}{2} & \text{if } m \text{ is odd} \end{cases}$$

Proof. Using Theorem 3, we deduce the following:

$$\max(\mathrm{Ap}(\mathcal{L}_2(m)[\{r\}]), m) = \begin{cases} (m-1)r - \frac{(m-2)m}{2} & \text{if } m-1 \text{ is odd} \\ (m-1)r - \frac{(m-3)m}{2} & \text{if } m-1 \text{ is even} \end{cases}$$

Moreover, with Lemma 11, we finish the proof. □

Example 8.

1. $\mathrm{F}(\mathcal{L}_2(10)[\{11\}]) = 9 \cdot 11 - \frac{10 \cdot 10}{2} = 49$.
2. $\mathrm{F}(\mathcal{L}_2(9)[\{13\}]) = 8 \cdot 13 - \frac{9 \cdot 8}{2} = 68$.

Lemma 12. *If $a_j = \left\lfloor \frac{j-1}{2} \right\rfloor$ for all $j \in \{1,\ldots,m-1\}$, then we have*

$$(a_1,\ldots,a_{m-1}) = \begin{cases} (0,0,1,1,\ldots,\frac{m-3}{2},\frac{m-3}{2}) & \text{if } m \text{ is odd} \\ (0,0,1,1,\ldots,\frac{m-4}{2},\frac{m-4}{2},\frac{m-2}{2}) & \text{if } m \text{ is even} \end{cases}$$

Proposition 13.

$$\mathrm{g}(\mathcal{L}_2(m)[\{r\}]) = \begin{cases} \frac{(m-1)r}{2} - \frac{(m-1)^2}{4} & \text{if } m \text{ is odd} \\ \frac{(m-1)r}{2} - \frac{(m-1)^2+1}{4} & \text{if } m \text{ is even} \end{cases}$$

Proof. By Theorem 3, we can write the elements in $\mathrm{Ap}(\mathcal{L}_2(m)[\{r\}])$ as $w(jr \bmod m) = jr - \left\lfloor \frac{j-1}{2} \right\rfloor m$ for all $j \in \{1,\ldots,m-1\}$. Moreover, from Lemmas 11 and 12, we have

1. If m is odd: $\mathrm{g}(\mathcal{L}_2(m)[\{r\}]) = \frac{1}{m}\left(r\sum_{j=1}^{m-1} j - 2m\sum_{i=1}^{\frac{m-3}{2}} i\right) - \frac{m-1}{2} = \frac{(m-1)r}{2} - \frac{(m-1)^2}{4}$.

2. If m is even: $\mathrm{g}(\mathcal{L}_2(m)[\{r\}]) = \frac{1}{m}\left(r\sum_{j=1}^{m-1} j - 2m\sum_{i=1}^{\frac{m-4}{2}} i - \frac{m-2}{2}m\right) - \frac{m-1}{2} = \frac{(m-1)r}{2} - \frac{(m-1)^2+1}{4}$.

□

Example 9.

1. $\mathrm{g}(\mathcal{L}_2(10)[\{11\}]) = \frac{9\cdot 11}{2} - \frac{9\cdot 9+1}{4} = \frac{99}{2} - \frac{41}{2} = \frac{58}{2} = 29$.
2. $\mathrm{g}(\mathcal{L}_2(9)[\{13\}]) = \frac{8\cdot 13}{2} - \frac{8\cdot 8}{4} = 52 - 16 = 36$.

As we pointed out in Remark 1, the next result appears in [23], Lemma 10.

Lemma 13. *Let S be a numerical semigroup, and $n \in S \setminus \{0\}$. Then,*

$$\mathrm{PF}(S) = \{w - n \mid w \in \mathrm{Maximals}_{\leq_S}(\mathrm{Ap}(S,n))\}$$

Proposition 14.
$\mathrm{Maximals}_{\leq_{\mathcal{L}_2(m)[\{r\}]}}(\mathrm{Ap}(\mathcal{L}_2(m)[\{r\}],m)) = \{w(rj \bmod m) \mid j \text{ is even}\} \cup \{\mathrm{F}(\mathcal{L}_2(m)[\{r\}]) + m\}$. *For $j \in \{1,\ldots,m-1\}$.*

Proof. If $j < m$ is odd, then $w(jr \bmod m) + r = jr - \frac{j-1}{2}m + r = (j+1)r - \frac{(j+1)-2}{2}m = w((j+1)r \bmod m)$, so $w(jr \bmod m)$ is not a maximal element in $\mathrm{Ap}(\mathcal{L}_2(m)[\{r\}],m)$.

On the other hand, if $j < m$ is even, from Proposition 11, we have that every element in $\mathrm{msg}(\mathcal{L}_2(m)[\{r\}]) \setminus \{m\}$ can be written as $w(ir \bmod m)$ for any i odd number, such that $i < m$. Therefore, using Theorem 3, we have $w(jr \bmod m) + w(ir \bmod m) = jr - \frac{j-2}{2}m + ir - \frac{i-1}{2}m = (i+j)r - \frac{(j-2)+(i-1)}{2}m = (i+j)r - \frac{(i+j-3)}{2}m \notin \mathrm{Ap}(\mathcal{L}_2(m)[\{r\}],m)$, which is also stated in Theorem 3. Furthermore, $w(jr \bmod m) + m = jr - \frac{j-2}{2}m + m = jr - \frac{j-4}{2}m \notin \mathrm{Ap}(\mathcal{L}_2(m)[\{r\}],m)$ for every $j < m$, such that j is even. Therefore, $w(jr \bmod m)$ is maximal in $\mathrm{Ap}(\mathcal{L}_2(m)[\{r\}],m)$ when j is even. Clearly, $F(\mathcal{L}_2(m)[\{r\}]) + m$ is also maximal in $\mathrm{Ap}(\mathcal{L}_2(m)[\{r\}],m)$. □

Corollary 2.
$\mathrm{PF}(\mathcal{L}_2(m)[\{r\}]) = \{k(2r-m) \mid k \in \{1,\ldots,\lfloor\frac{m-2}{2}\rfloor\}\} \cup \{\mathrm{F}(\mathcal{L}_2(m)[\{r\}])\}$.

Example 10. *Using Corollary 2, we can compute* $\mathrm{PF}(\mathcal{L}_2(10)[\{13\}]) = \{16k \mid k \in \{1,2,3,4\}\} \cup \{\mathrm{F}(\mathcal{L}_2(10)[\{13\}])\} = \{16,32,48,64,67\}$.

We have obtained in Examples 6, 7 and 10 the Apéry set of the multiplicity, the minimal system of generators, and the pseudo-Frobenius numbers of $\mathcal{L}_2(10)[\{13\}]$. Now, we are going to change the parity of m to show an example where m is odd.

Example 11. *Using Theorem 3, Proposition 11, and Corollary 2, we are going to compute the Apéry set of the multiplicity, the minimal system of generators, and the pseudo-Frobenius numbers of* $\mathcal{L}_2(11)[\{13\}]$.

The Apéry set $\mathrm{Ap}(\mathcal{L}_2(11)[\{13\}],11)$ *is*

$w(1\cdot 13 \bmod 11) = w(3) = 1\cdot 13 - \frac{1-1}{2}\cdot 11 = 13$	$w(2\cdot 13 \bmod 11) = w(6) = 2\cdot 13 - \frac{2-2}{2}\cdot 11 = 26$
$w(3\cdot 13 \bmod 11) = w(9) = 3\cdot 13 - \frac{3-1}{2}\cdot 11 = 28$	$w(4\cdot 13 \bmod 11) = w(2) = 4\cdot 13 - \frac{4-2}{2}\cdot 11 = 41$
$w(5\cdot 13 \bmod 11) = w(5) = 5\cdot 13 - \frac{5-1}{2}\cdot 11 = 43$	$w(6\cdot 13 \bmod 11) = w(8) = 6\cdot 13 - \frac{6-2}{2}\cdot 11 = 56$
$w(7\cdot 13 \bmod 11) = w(1) = 7\cdot 13 - \frac{7-1}{2}\cdot 11 = 58$	$w(8\cdot 13 \bmod 11) = w(4) = 8\cdot 13 - \frac{8-2}{2}\cdot 11 = 71$
$w(9\cdot 13 \bmod 11) = w(7) = 9\cdot 13 - \frac{9-1}{2}\cdot 11 = 73$	$w(10\cdot 13 \bmod 11) = w(4) = 10\cdot 13 - \frac{10-2}{2}\cdot 11 = 86$

From the above, we have $\mathrm{msg}(\mathcal{L}_2(11)[\{13\}]) = \{11,13,28,43,58,73\}$, *and* $\mathrm{PF}(\mathcal{L}_2(11)[\{13\}]) = \{15,30,45,60,75\}$.

Corollary 3. $\mathrm{t}(\mathcal{L}_2(m)[\{r\}]) = \frac{m}{2}$ *if* m *is even or* $\mathrm{t}(\mathcal{L}_2(m)[\{r\}]) = \frac{m-1}{2}$ *if* m *is odd. Summarizing* $\mathrm{t}(\mathcal{L}_2(m)[\{r\}]) = \lfloor \frac{m}{2} \rfloor$.

The following result comes from Proposition 11 and Corollary 3:

Corollary 4. *We have* $\mathrm{e}(S) = \mathrm{t}(S) + 1$ *for every* $S \in \mathcal{L}_2(m)$ *with the* $\mathcal{L}_2(m)$*-rank being equal to 1.*

On page 15 of [1], it is shown that $\mathrm{F}(S) + 1 \leq (\mathrm{t}(S) + 1)\,\mathrm{n}(S)$ for any numerical semigroup S (it was earlier obtained in [7]). Then, using Corollary 4, we achieve the following result:

Proposition 15. *Every element in* $\mathcal{L}_2(m)$ *with the* $\mathcal{L}_2(m)$*-rank being equal to 1 satisfies Wilf's conjecture.*

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