

Article

On Ideals of Submonoids of Power Monoids

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Abstract: Let $\mathbb{S}$ be a numerical monoid, while a $\mathcal{P}_{\text{fin}}(\mathbb{S})$ -monoid S is a monoid generated by a finite number of finite non-empty subsets of $\mathbb{S}$. That is, S is a non-cancellative commutative monoid obtained from the sumset of finite non-negative integer sets. This work provides an algorithm for computing the ideals associated with some $\mathcal{P}_{\text{fin}}(\mathbb{S})$ -monoids. These are the key to studying some factorization properties of $\mathcal{P}_{\text{fin}}(\mathbb{S})$ -monoids and some additive properties of sumsets. This approach links computational commutative algebra with additive number theory.

Keywords: atomic monoid; elasticity; h -fold sumset; non-cancellative monoid; power monoid; monoid ideal; semigroup ideal; sumset

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1. Introduction

Additive number theory is the subfield of number theory that is concerned with the study of subsets of integers and their behaviour under addition. More abstractly, the field of additive number theory includes the analysis of abelian groups and commutative monoids with an operation of addition (see [1] and the references therein). Some of the main objects of this study are (i) the sumset of two subsets, A and B , in an abelian group G , $A + B = \{a + b \mid a \in A, b \in B\}$, and (ii) to determine the structure and properties of the h -fold sumset hA when the set A is known. There is a beautiful and straightforward solution to the direct problem of describing the structure of the h -fold sumset hA for any finite set A of integers and for all sufficiently large h (see [2] (Theorem 1.1)). This result has implications for the study of Weierstrass semigroups, such as shown in [3]. In an inverse problem, we start with the sumset hA and try to deduce information about the underlying set A . A reference for inverse problems can be found in [4] (Chapter 5).

A numerical monoid $\mathbb{S}$ (also called numerical semigroup) is an additive submonoid of $\mathbb{N}$ whose greatest common divisor is equal to 1, or equivalently, $\mathbb{N} \setminus \mathbb{S}$ is finite [5]. Here, we consider the commutative monoid whose elements are the finite non-empty subsets of $\mathbb{S}$, denoted by $(\mathcal{P}_{\text{fin}}(\mathbb{S}), +)$ with $+$ the operation defined as before. This monoid is the power monoid of $\mathbb{S}$ (see [6,7], and the references therein). A submonoid of the power monoid $(\mathcal{P}_{\text{fin}}(\mathbb{S})$ -monoid) is a monoid generated by a finite number of elements of $\mathcal{P}_{\text{fin}}(\mathbb{S})$. By [8], these monoids are atomic reduced monoids with finite elasticity.

In general, it is well known that finitely generated monoids are finitely presented (see [9]). That is, there exists $p \in \mathbb{N}$ and a congruence σ in $\mathbb{N}^p \times \mathbb{N}^p$ such that the monoid S

is isomorphic to $\mathbb{N}^p / \sigma$. Equivalently, there exists a binomial ideal I_S in the polynomial ring $\mathbf{k}[x_1, \dots, x_p]$ such that S is isomorphic to the set of monomials in $\mathbf{k}[x_1, \dots, x_p] / I_S$ with the product operation. The presentation of S (or a system of generators of the monoid ideal I_S) provides us with a way to obtain the expressions of an element of the monoid in terms of its generators. This computation can be performed with Gröbner bases and related techniques. For instance, particularizing these techniques on $\mathcal{P}_{\text{fin}}(\mathbb{S})$ -monoids, given $A \subset \mathbb{N}$ a finite subset, we could check whether the h -fold of any A can be expressed as a sumset of other elements. Hence, a bridge between computational commutative algebra and additive number theory is constructed by using these techniques. In this context, Theorem 3 and Theorem 4 are the first bricks of this link. In this work, we give the ideals of some families of $\mathcal{P}_{\text{fin}}(\mathbb{S})$ -monoids.

We also introduce a new Python library [10] that includes an implementation of our algorithms and the examples that illustrate it.

The work is organized as follows. In Section 2, we present some definitions and results on Gröbner bases. In Section 3, we introduce the $\mathcal{P}_{\text{fin}}(\mathbb{S})$ -monoids and study some of their properties. In Sections 4 and 5, by using algebraic commutative algebra tools, we study the ideals of some families of $\mathcal{P}_{\text{fin}}(\mathbb{S})$ -monoids, thereby allowing us to introduce Algorithm 1 and provide some examples.

2. Some Results on Commutative Algebra

For a field $\mathbf{k}$ and a set of indeterminates $X = \{x_1, \dots, x_t\}$, the polynomial ring $\mathbf{k}[X] = \mathbf{k}[x_1, \dots, x_t]$ is the set of polynomials in $\{x_1, \dots, x_t\}$ with coefficients in $\mathbf{k}$; that is, the set $\{\sum_{i=1}^m a_i x_1^{\alpha_1} \cdots x_t^{\alpha_t} \mid m \in \mathbb{N}, a_i \in \mathbf{k}, \alpha_1, \dots, \alpha_t \in \mathbb{N}\}$. We denote by X^α the monomial $x_1^{\alpha_1} \cdots x_t^{\alpha_t}$, with $\alpha = (\alpha_1, \dots, \alpha_t) \in \mathbb{N}^t$. In this work, some results use Gröbner basis theory and the elimination theorem. The necessary background can be found in [11] (§2 and §3) but is also provided here so that the present work is self-contained.

It is well known that any ideal in a polynomial ring is finitely generated. In particular, a special generating set is associated with the ideals, namely a Gröbner basis. This concept depends on the election of an order on the monomials. A monomial order $\prec$ on $\mathbf{k}[X]$ is a multiplicative total order on the set of monomials if for each two monomials X^α, X^β such that $X^\alpha \prec X^\beta$, then $X^\alpha X^\gamma \prec X^\beta X^\gamma$ for every monomial X^γ .

For a fixed monomial order $\prec$ on $\mathbf{k}[X]$, $\text{In}_\prec(I)$ denotes the set of leading terms of non-zero elements of I , and $\langle \text{In}_\prec(I) \rangle$ the monomial ideal generated by $\text{In}_\prec(I)$. A subset G of I is a Gröbner basis of I if $\langle \text{In}_\prec(I) \rangle = \langle \{\text{In}_\prec(g) \mid g \in G\} \rangle$, where $\text{In}_\prec(g)$ is the leading term of g . An algorithm for computing a Gröbner basis for I is given in [11] (Chapter 2, §7). It is also well known that Gröbner bases of binomial ideals are sets of binomials.

Given two polynomials f and g , their S -polynomial is defined by $S(f, g) = \frac{aX^\alpha}{\text{In}_\prec(f)}f - \frac{aX^\alpha}{\text{In}_\prec(g)}g$, where aX^α is the least common multiple of $\text{In}_\prec(f)$ and $\text{In}_\prec(g)$. Also, for k non-zero polynomials $f_1, \dots, f_k$, we say that $S(f_i, f_j) = \sum_{l=1}^k g_l f_l$ has an *lcm* representation if the least common multiple of the monomial leaders of f_i and f_j is bigger than $\text{In}_\prec(g_l f_l)$ (respect $\prec$) whenever $g_l f_l \neq 0$. Thus, we obtain another equivalent definition of a Gröbner basis.

Theorem 1 ([11] (Chapter 2, §9, Theorem 6)). *A basis $\{f_1, \dots, f_k\}$ of an ideal I is a Gröbner basis if and only if for every $i \neq j$, the S -polynomial $S(f_i, f_j)$ has an lcm representation.*

The above theorem allows us to prove the following result. We use this lemma in the following sections.

Lemma 1. Let $I \subset \mathbf{k}[X, Y]$ and $J \subset \mathbf{k}[Z, X, Y]$ be two binomial ideals with J generated by $G = \{z_1 - M_1, \dots, z_t - M_t\}$, where each M_i is a monomial in $\mathbf{k}[X, Y]$. Fix $\prec$ a monomial order on $\mathbf{k}[Z, X, Y]$ such that $x_j, y_k \prec z_i$, for every x_j, y_k and z_i . Then, the union of G and a Gröbner basis of I respect $\prec$ is a Gröbner basis of $I + J$ respect $\prec$. Moreover, if a binomial $L - T$ belongs to $(I + J) \cap \mathbf{k}[X, Y]$, then $L - T \in I$.

Proof. Note that G is a Gröbner basis of J respect $\prec$, and consider $G' = \{g_1, \dots, g_h\} \subset \mathbf{k}[X, Y]$ a Gröbner basis of I . Thus, $S(f, f')$ and $S(g, g')$ have an *lcm* representation for every $f, f' \in G$ and $g, g' \in G'$. Let $L - T$ be a binomial in G' , $z_i - M_i \in G$, and assume $L \succ T$. Hence, $S(z_i - M_i, L - T) = L(z_i - M_i) - z_i(L - T) = T(z_i - M_i) - M_i(L - T)$; that is to say, $S(z_i - M_i, L - T)$ has an *lcm* representation. Therefore, $G \cup G'$ is a Gröbner basis of $I + J$.

Consider any $L - T \in I + J$ with $L - T \in \mathbf{k}[X, Y]$. By [11] (Chapter 2, §3, Theorem 3) and [11] (Chapter 2, §6, Corollary 2), we have that $L - T = \sum_{i=1}^h f_i g_i$, with $\text{In}_{\prec}(L - T) \succeq \text{In}_{\prec}(f_i g_i)$, for $i = 1, \dots, h$. Hence, every f_i belongs to $\mathbf{k}[X, Y]$. $\square$

A method for computing the ideal $I \cap \mathbf{k}[x_{l+1}, \dots, x_t]$ (for $t > l \geq 1$) is called the elimination theorem.

Theorem 2 ([11] (Chapter 3)). Let $I \subset \mathbf{k}[x_1, \dots, x_t]$ be an ideal and G be a Gröbner basis of I with respect to lexicographical order where $x_1 > x_2 > \dots > x_t$. Then, for every $0 \leq l \leq t$, the set $G \cap \mathbf{k}[x_{l+1}, \dots, x_t]$ is a generating set of the ideal $I \cap \mathbf{k}[x_{l+1}, \dots, x_t]$. Furthermore, $G \cap \mathbf{k}[x_{l+1}, \dots, x_t]$ is a Gröbner basis of $I \cap \mathbf{k}[x_{l+1}, \dots, x_t]$.

We introduce the monoid ideal as an important object in this work. Recall that a monoid is a non-empty set equipped with an associative binary operation (denoted by $+$), and an identity element. In this work, we assume that all our monoids are commutative. A monoid S is finitely generated when there exists a finite set $A = \{a_1, \dots, a_t\} \subset S$ such that $S = \langle A \rangle := \{\alpha_1 a_1 + \dots + \alpha_t a_t \mid \alpha_1, \dots, \alpha_t \in \mathbb{N}\}$. For a field $\mathbf{k}$, S has associated the binomial ideal in $\mathbf{k}[x_1, \dots, x_t]$,

$$I_S = \left\langle \left\{ X^\alpha - X^\beta \mid \sum_{i=1}^t \alpha_i a_i = \sum_{i=1}^t \beta_i a_i, \text{ with } \alpha = (\alpha_1, \dots, \alpha_t), \beta = (\beta_1, \dots, \beta_t) \in \mathbb{N}^t \right\} \right\rangle.$$

This ideal is usually called the monoid ideal (or semigroup ideal) of S , and it plays an essential role in studying some properties of monoids [12]. Note that if $X^\alpha - X^\beta \in I_S$, then α and β are two factorizations of the same element of S . Therefore, I_S codifies the relationships among the elements of S . For example, the semigroup $\langle 3, 5, 7 \rangle$ is associated with the ideal $\langle y^7 - z^5, xz - y^2, xy^5 - z^4, x^2y^3 - z^3, x^3y - z^2, x^4 - yz \rangle$ which shows us one relationship for each binomial. The interpretation of this ideal is straightforward; for example, the fact that $xy^5 - z^4$ is in it means that the first generator plus five times the second one is the same as four times the third generator. In other words, the monomial z^4 is identified by xy^5 . Associated with these ideals, we have the lattice M of $\mathbb{Z}^t$ generated by the elements $\{\alpha - \beta \mid X^\alpha - X^\beta \in I_S\} \subseteq \mathbb{Z}^t$. In our previous example, the lattice is generated by the elements of $\{(0, 7, -5), (1, -2, 1), (1, 5, -4), (2, 3, -3), (3, 1, -2), (4, -1, -1)\}$. We say that I_S is strongly reduced whenever $M \cap \mathbb{N}^t = \{0\}$ (this concept was introduced in [13]). Define $\mathcal{I}_M$, the finitely generated cancellative submonoid of $\mathbb{N}^t \times \mathbb{N}^t$, all of whose elements (α, β) verify $\alpha - \beta \in M$. Note that $\mathcal{I}_M$ is a Krull monoid [14]. Let $\mathcal{A}(\mathcal{I}_M)$ be the minimal system of generators of $\mathcal{I}_M$. This minimal generating set can be computed by performing the following steps:

1. Compute a system of linear homogeneous equations $Ax = 0$ of M from its system of generators [15] (Chapter 2).
2. The Hilbert basis of $(A| - A)(x, y) = 0$ is the set $\mathcal{A}(\mathcal{I}_M)$. That basis can be computed by using [16].

In [5], readers can find some algorithms for computing the ideals of numerical monoids.

3. Some Properties of $\mathcal{P}_{\text{fin}}(\mathbb{S})$ -Monoids

We start this section by recalling some standard definitions in monoid theory. Assume S is a commutative monoid, S is cancellative if $x + z = y + z$ for some $x, y, z \in S$, implies $x = y$. An element $x \in S$ is a unit if $x + y = 0$ for some $y \in S$. The set of units of S is denoted by $\mathcal{U}(S)$. When $S \cap (-S) = \{0\}$, S is named reduced. An atom in S is any non-unit $x \in S$ such that there do not exist two non-units $y, z \in S$ with $x = y + z$. The monoid is atomic if $S \setminus \mathcal{U}(S)$ is generated by its atoms. The set of atoms of S is denoted by $\mathcal{A}(S)$.

From now on, $\mathbb{S}$ is a numerical monoid, and let $\mathcal{P}_{\text{fin}}(\mathbb{S})$ be the set whose elements are the finite non-empty subsets of $\mathbb{S}$; that is, the power monoid of $\mathbb{S}$. Recall that on $\mathcal{P}_{\text{fin}}(\mathbb{S})$, the binary operation $+$ is defined as $A + B = \{a + b \mid a \in A, b \in B\}$ for all $A, B \in \mathcal{P}_{\text{fin}}(\mathbb{S})$. Since $\mathbb{S}$ is commutative and associative, with 0 being its identity element, the pair $(\mathcal{P}_{\text{fin}}(\mathbb{S}), +)$ is a commutative monoid and $\{0\}$ is its identity element. Every finitely generated submonoid of $(\mathcal{P}_{\text{fin}}(\mathbb{S}), +)$ is called a $\mathcal{P}_{\text{fin}}(\mathbb{S})$ -monoid. If $A \in \mathcal{P}_{\text{fin}}(\mathbb{S})$ and $\alpha \in \mathbb{N}$, denote by αA the sumset $\sum_{i=1}^{\alpha} A$, and $\alpha \cdot A$ is defined as $\{\alpha a \mid a \in A\}$. For example, $\{1, 3\} + \{1, 2, 3\} = \{2, 3, 4, 5, 6\}$, and $2\{1, 2, 4, 5\} = \{2, 3, 4, 5, 6, 7, 8, 9, 10\}$, illustrate the results of the previous operations.

The power monoid $\mathcal{P}_{\text{fin}}(\mathbb{N})$ satisfies the following interesting properties studied in [17]:

- it is non-cancellative, for example $\{1, 3\} + \{1, 2, 3\} = \{1, 2, 3\} + \{1, 2, 3\}$;
- it is a reduced monoid;
- it is not torsion-free, for example $2\{1, 2, 4, 5\} = 2\{1, 2, 3, 4, 5\}$;
- $\mathcal{P}_{\text{fin}}(\mathbb{N})$ is atomic and BF-monoid. Moreover, its submonoids are also BF-monoids.

The operation αA has useful properties in this work. We show some of them in the following lemma.

Lemma 2. Let A, B be in $\mathcal{P}_{\text{fin}}(\mathbb{S})$ and $\alpha, \beta \in \mathbb{N}$; then

1. $\alpha(\beta A) = (\alpha\beta)A$;
2. $\alpha(A + B) = \alpha A + \alpha B$.

Let S be a $\mathcal{P}_{\text{fin}}(\mathbb{S})$ -monoid minimally generated by $\{A_1, \dots, A_t\}$. By definition, the elasticity of a non-unit $A \in S$ is $\rho(A) = \sup \{m/n \mid \exists a_1, \dots, a_m, b_1, \dots, b_n \in \mathcal{A}(S) \text{ with } A = \sum_{i=1}^m a_i = \sum_{i=1}^n b_i\}$; that is, it measures how far an element is from having all its factorizations be of the same length (see [18]). The elasticity of S is defined as $\rho(S) = \sup \{\rho(A) \mid A \in S \setminus \{0\}\}$. If there is an element in the monoid whose elasticity “reaches” that of the whole monoid, we say that the monoid has accepted elasticity.

Recall that the ideal associated with S is

$$I_S = \left\langle \left\{ X^\alpha - X^\beta \mid \sum_{i=1}^t \alpha_i A_i = \sum_{i=1}^t \beta_i A_i \right\} \right\rangle \subset \mathbf{k}[x_1, \dots, x_t].$$

This ideal is strongly reduced. In the other case, let $\alpha \in M \cap \mathbb{N}^t$ be a non-zero element; thus, there exists $\beta \in \mathbb{N}^t$ such that $X^{\alpha+\beta} - X^\beta = X^\beta(X^\alpha - 1) \in I_S$, and then $\sum_{i=1}^t (\alpha_i + \beta_i) A_i = \sum_{i=1}^t \beta_i A_i$. Therefore, $\max \sum_{i=1}^t (\alpha_i + \beta_i) A_i = \sum_{i=1}^t (\alpha_i + \beta_i) \max A_i > \sum_{i=1}^t \beta_i \max A_i = \max \sum_{i=1}^t \beta_i A_i$, which it is not possible. Hence, $M \cap \mathbb{N}^t$ is $\{0\}$.

Since I_S is strongly reduced, we have that S is an atomic reduced monoid with finite elasticity (see Theorem 15 in [8]). Moreover,

$$\rho(S) = \max \left\{ \frac{\sum_{i=1}^t \alpha_i}{\sum_{i=1}^t \beta_i} \mid (\alpha, \beta) \in \mathcal{A}(\mathcal{I}_M) \right\}. \quad (1)$$

Any finite subset $A = \{a_1 < \dots < a_n\} \subset \mathbb{N}$ can be expressed as $A = \{a_1\} + \{0, a_2 - a_1, \dots, a_n - a_1\}$, where $\tilde{A}$ denotes the set $\{0, a_2 - a_1, \dots, a_n - a_1\}$. Note that the monoid $S = \langle A_1, \dots, A_t \rangle$, with $A_i = \{a_{i1} < \dots < a_{it_i}\}$, is a submonoid of $\langle \{a_{11}\}, \dots, \{a_{1t}\}, \tilde{A}_1, \dots, \tilde{A}_t \rangle$. Trivially, the monoid $\langle \{a_{11}\}, \dots, \{a_{1t}\} \rangle$ is isomorphic to the monoid $\langle a_{11}, \dots, a_{1t} \rangle$; thus $I_{\langle a_1, \dots, a_s \rangle} = I_{\langle \{a_{11}\}, \dots, \{a_{1t}\} \rangle}$.

Proposition 1. For every $\tilde{S} = \{\{a_1\}, \dots, \{a_s\}, \tilde{A}_1, \dots, \tilde{A}_t\}$, with $a_i \neq 0$ and $\min \tilde{A}_i = 0$, we have $I_{\tilde{S}} = I_{\langle a_1, \dots, a_s \rangle} + I_{\langle \tilde{A}_1, \dots, \tilde{A}_t \rangle}$, where $I_{\langle a_1, \dots, a_s \rangle} \subset \mathbf{k}[x_1, \dots, x_s]$ and $I_{\langle \tilde{A}_1, \dots, \tilde{A}_t \rangle} \subset \mathbf{k}[y_1, \dots, y_t]$.

Proof. Trivially, $I_{\langle a_1, \dots, a_s \rangle}, I_{\langle \tilde{A}_1, \dots, \tilde{A}_t \rangle} \subset I_{\tilde{S}}$.

Let $X^\alpha Y^\beta - X^\gamma Y^\delta \in I_{\tilde{S}}$, then $\sum_{i=1}^s \alpha_i \{a_i\} + \sum_{i=1}^t \beta_i \tilde{A}_i = \sum_{i=1}^s \gamma_i \{a_i\} + \sum_{i=1}^t \delta_i \tilde{A}_i$. Since $\min \tilde{A}_i = 0$, we then have $\sum_{i=1}^s \alpha_i \{a_i\} = \sum_{i=1}^s \gamma_i \{a_i\}$, and $\sum_{i=1}^t \beta_i \tilde{A}_i = \sum_{i=1}^t \delta_i \tilde{A}_i$. That is to say, $X^\alpha - X^\gamma \in I_{\langle a_1, \dots, a_s \rangle}$, and $Y^\beta - Y^\delta \in I_{\langle \tilde{A}_1, \dots, \tilde{A}_t \rangle}$. Note that $X^\alpha Y^\beta - X^\gamma Y^\delta = Y^\beta (X^\alpha - X^\gamma) + X^\gamma (Y^\beta - Y^\delta) \in I_{\langle a_1, \dots, a_s \rangle} + I_{\langle \tilde{A}_1, \dots, \tilde{A}_t \rangle} \subset \mathbf{k}[x_1, \dots, x_s, y_1, \dots, y_t]$. $\square$

Since the monoid $\langle a_1, \dots, a_s \rangle$ is isomorphic to a numerical monoid, there exist algorithms for computing $I_{\langle a_1, \dots, a_s \rangle}$. Thus, to compute a presentation of $\tilde{S}$, we need an algorithm to calculate $I_{\langle \tilde{A}_1, \dots, \tilde{A}_t \rangle}$. In the next sections, we provide results for computing the ideals of some families of $\mathcal{P}_{\text{fin}}(\mathbb{S})$ -monoids.

4. Monoid Ideals of a Fundamental Family of $\mathcal{P}_{\text{fin}}(\mathbb{S})$ -Monoids

In this section, we explicitly give the ideal associated with the $\mathcal{P}_{\text{fin}}(\mathbb{S})$ -monoid generated by the elements $\{0, k\mathbf{a}\}$ and $\{0, k\mathbf{b}\}$, where $\mathbf{a} < \mathbf{b}$ are two positive co-prime integers, and $k \in \mathbb{N} \setminus \{0\}$. These monoids are key to provide an algorithm to compute the monoid ideals of more types of $\mathcal{P}_{\text{fin}}(\mathbb{S})$ -monoids. Consider that $\mathbb{S}$ is a numerical monoid with $\mathbf{a}, \mathbf{b} \in \mathbb{S}$.

Fix $\mathbf{a} < \mathbf{b}$ as two positive co-prime integers and $k \in \mathbb{N} \setminus \{0\}$, and consider the monoid $\tilde{S} = \langle k\mathbf{a}, k\mathbf{b} \rangle \subset \mathbb{N}$ and the $\mathcal{P}_{\text{fin}}(\mathbb{S})$ -monoid $\tilde{S}$ minimally generated by $\{0, k\mathbf{a}\}$ and $\{0, k\mathbf{b}\}$. We prove that $I_{\tilde{S}} \subset \mathbf{k}[x, y]$ is a principal ideal providing its generator. Note that $I_{\tilde{S}} \subset I_{\mathbb{S}} = \langle x^{\mathbf{b}} - y^{\mathbf{a}} \rangle$.

Lemma 3. Set $x^\alpha y^\beta - x^\gamma y^\delta \in I_{\tilde{S}} \setminus \{0\}$. Then, $\alpha \neq \gamma$, $\beta \neq \delta$ and $\alpha\beta\gamma\delta \geq 1$.

Proof. Note that, since $f = x^\alpha y^\beta - x^\gamma y^\delta \in I_{\tilde{S}}$, $\alpha\{0, k\mathbf{a}\} + \beta\{0, k\mathbf{b}\} = \gamma\{0, k\mathbf{a}\} + \delta\{0, k\mathbf{b}\}$.

Suppose $\alpha = \gamma$, then, we have $\alpha k\mathbf{a} + \beta k\mathbf{b} = \max\{\alpha\{0, k\mathbf{a}\} + \beta\{0, k\mathbf{b}\}\} = \max\{\alpha\{0, k\mathbf{a}\} + \delta\{0, k\mathbf{b}\}\} = \alpha k\mathbf{a} + \delta k\mathbf{b}$, and $\beta = \delta$. Since $f \neq 0$, this is not possible, and therefore, $\alpha \neq \gamma$. Analogously, it can be proved that $\beta \neq \delta$.

Suppose $\alpha = 0$; then, we have $k\mathbf{b} = \min \{\beta\{0, k\mathbf{b}\} \setminus \{0\}\} = \min \{(\gamma\{0, k\mathbf{a}\} + \delta\{0, k\mathbf{b}\}) \setminus \{0\}\}$. If γ is non-zero, then $k\mathbf{b} = \min \{(\gamma\{0, k\mathbf{a}\} + \delta\{0, k\mathbf{b}\}) \setminus \{0\}\} = k\mathbf{a}$. Therefore, the integers γ , β , and δ are zero and $f = 0$. Similarly, $\beta, \gamma, \delta \geq 1$ can be proved. $\square$

From the previous lemma, in the sequel, we assume $\alpha > \gamma \geq 1$, and $x^\alpha y^\beta - x^\gamma y^\delta \in I_{\tilde{S}} \setminus \{0\}$. Since $I_{\tilde{S}} \subset \langle x^{\mathbf{b}} - y^{\mathbf{a}} \rangle$, $\alpha \geq \mathbf{b}$ and $\delta \geq \mathbf{a}$.

Lemma 4. *If $\alpha > \gamma$, then $\delta > \beta$. Furthermore, there exists a positive integer n such that $\alpha = n\mathbf{b} + \gamma$ and $\delta = n\mathbf{a} + \beta$.*

Proof. Since $x^\alpha y^\beta - x^\gamma y^\delta \in I_{\tilde{S}}$, $\alpha\{0, k\mathbf{a}\} + \beta\{0, k\mathbf{b}\} = \gamma\{0, k\mathbf{a}\} + \delta\{0, k\mathbf{b}\}$, and $\alpha k\mathbf{a} + \beta k\mathbf{b} = \max \{\alpha\{0, k\mathbf{a}\} + \beta\{0, k\mathbf{b}\}\} = \max \{\gamma\{0, k\mathbf{a}\} + \delta\{0, k\mathbf{b}\}\} = \gamma k\mathbf{a} + \delta k\mathbf{b}$. So, $(\delta - \beta)k\mathbf{b} = (\alpha - \gamma)k\mathbf{a} > 0$. Furthermore, $(\alpha - \gamma)/(\delta - \beta) = \mathbf{b}/\mathbf{a}$. Since $\gcd(\mathbf{a}, \mathbf{b}) = 1$, there exist two positive integers n and m such that $\alpha = n\mathbf{b} + \gamma$ and $\delta = m\mathbf{a} + \beta$. From the equality $\alpha k\mathbf{a} + \beta k\mathbf{b} = \gamma k\mathbf{a} + \delta k\mathbf{b}$, we deduce that $n = m$. $\square$

Lemma 5. *Let $x^\alpha y^\beta - x^\gamma y^\delta \in I_{\tilde{S}} \setminus \{0\}$. Then, $\gamma \geq \mathbf{b} - 1$, $\beta \geq \mathbf{a} - 1$, and there is a positive integer $n \in \mathbb{N}$ such that $x^\alpha y^\beta - x^\gamma y^\delta = x^\gamma y^\beta (x^{n\mathbf{b}} - y^{n\mathbf{a}})$.*

Proof. Assume $\gamma < \mathbf{b} - 1$. Take $(\gamma + 1)k\mathbf{a} + \beta k\mathbf{b}$ in $\alpha\{0, k\mathbf{a}\} + \beta\{0, k\mathbf{b}\}$ (recall $\alpha > \gamma$). For that element, there exist two integers $i \in [0, \gamma]$ and $j \in [0, \delta]$ such that $(\gamma + 1)k\mathbf{a} + \beta k\mathbf{b} = (\gamma - i)k\mathbf{a} + (\delta - j)k\mathbf{b} \in \gamma\{0, k\mathbf{a}\} + \delta\{0, k\mathbf{b}\}$, hence $(1 + i)k\mathbf{a} = (\delta - \beta - j)k\mathbf{b}$. Since $\gcd(\mathbf{a}, \mathbf{b}) = 1$, $i + 1 \geq \mathbf{b}$ and $i \geq \mathbf{b} - 1 > \gamma$, but $i \leq \gamma$, which is a contradiction. Analogously, the fact $\beta \geq \mathbf{a} - 1$ can be proved.

By Lemma 4, there exists $n \in \mathbb{N} \setminus \{0\}$ such that

$$x^\alpha y^\beta - x^\gamma y^\delta = x^{n\mathbf{b} + \gamma} y^\beta - x^\gamma y^{n\mathbf{a} + \beta} = x^\gamma y^\beta (x^{n\mathbf{b}} - y^{n\mathbf{a}}).$$

$\square$

Theorem 3. *Let $1 \leq \mathbf{a} < \mathbf{b}$ be two co-prime integers, $k \in \mathbb{N} \setminus \{0\}$, and $\tilde{S}$ be the $\mathcal{P}_{\text{fin}}(\mathbb{S})$ -monoid $\langle \{0, k\mathbf{a}\}, \{0, k\mathbf{b}\} \rangle$. The ideal $I_{\tilde{S}} \subset \mathbf{k}[x, y]$ is principal and is generated by $x^{\mathbf{b}-1} y^{\mathbf{a}-1} (x^{\mathbf{b}} - y^{\mathbf{a}})$.*

Proof. Observe that $x^{\mathbf{b}-1} y^{\mathbf{a}-1} (x^{\mathbf{b}} - y^{\mathbf{a}}) = x^{2\mathbf{b}-1} y^{\mathbf{a}-1} - x^{\mathbf{b}-1} y^{2\mathbf{a}-1}$. To prove this theorem, we describe explicitly the sets $A = (2\mathbf{b} - 1)\{0, k\mathbf{a}\} + (\mathbf{a} - 1)\{0, k\mathbf{b}\}$ and $B = (\mathbf{b} - 1)\{0, k\mathbf{a}\} + (2\mathbf{a} - 1)\{0, k\mathbf{b}\}$ associated with the monomials $x^{2\mathbf{b}-1} y^{\mathbf{a}-1}$ and $x^{\mathbf{b}-1} y^{2\mathbf{a}-1}$ (respectively), to achieve $A = B$.

Note that the first set $A = (2\mathbf{b} - 1)\{0, k\mathbf{a}\} + (\mathbf{a} - 1)\{0, k\mathbf{b}\}$ is equal to

$$\begin{aligned} & \{0, k\mathbf{a}, \dots, (2\mathbf{b} - 1)k\mathbf{a}\} + \{0, k\mathbf{b}, \dots, (\mathbf{a} - 1)k\mathbf{b}\} = \\ & = k \cdot \left(\{0, \mathbf{a}, \dots, (2\mathbf{b} - 1)\mathbf{a}\} \cup \{0, \mathbf{b}, \dots, (\mathbf{a} - 1)\mathbf{b}\} \cup \right. \\ & \quad \left. \{\mathbf{a} + \mathbf{b}, \dots, \mathbf{a} + (\mathbf{a} - 1)\mathbf{b}\} \cup \{2\mathbf{a} + \mathbf{b}, \dots, 2\mathbf{a} + (\mathbf{a} - 1)\mathbf{b}\} \cup \dots \right. \\ & \quad \left. \cup \{(2\mathbf{b} - 1)\mathbf{a} + \mathbf{b}, \dots, (2\mathbf{b} - 1)\mathbf{a} + (\mathbf{a} - 1)\mathbf{b}\} \right) \\ & = k \cdot \left(\{0, \mathbf{a}, \dots, (\mathbf{b} - 1)\mathbf{a}, \mathbf{b}\mathbf{a}, (\mathbf{b} + 1)\mathbf{a}, \dots, (2\mathbf{b} - 1)\mathbf{a}\} \cup \right. \\ & \quad \left. \{0, \mathbf{b}, \dots, (\mathbf{a} - 1)\mathbf{b}\} \cup \{\mathbf{a} + \mathbf{b}, \dots, \mathbf{a} + (\mathbf{a} - 1)\mathbf{b}\} \cup \dots \right. \\ & \quad \left. \cup \{(\mathbf{b} - 1)\mathbf{a} + \mathbf{b}, \dots, (\mathbf{b} - 1)\mathbf{a} + (\mathbf{a} - 1)\mathbf{b}\} \cup \right. \\ & \quad \left. \{\mathbf{b}\mathbf{a} + \mathbf{b}, \dots, \mathbf{b}\mathbf{a} + (\mathbf{a} - 1)\mathbf{b}\} \cup \right. \\ & \quad \left. \{(\mathbf{b} + 1)\mathbf{a} + \mathbf{b}, \dots, (\mathbf{b} + 1)\mathbf{a} + (\mathbf{a} - 1)\mathbf{b}\} \cup \dots \right. \\ & \quad \left. \cup \{(2\mathbf{b} - 1)\mathbf{a} + \mathbf{b}, \dots, (2\mathbf{b} - 1)\mathbf{a} + (\mathbf{a} - 1)\mathbf{b}\} \right). \end{aligned}$$

We denote $C_1 = \{0, \mathbf{a}, \dots, (\mathbf{b} - 1)\mathbf{a}\}$, $C_2 = \{0, \mathbf{b}, \dots, (\mathbf{a} - 1)\mathbf{b}\}$, $C_3 = \cup_{i=1}^{\mathbf{b}-1} \{\mathbf{ia} + \mathbf{b}, \dots, \mathbf{ia} + (\mathbf{a} - 1)\mathbf{b}\}$, $C_4 = \{\mathbf{ab}\} \cup \{\mathbf{ba} + \mathbf{b}, \dots, \mathbf{ba} + (\mathbf{a} - 1)\mathbf{b}\}$, $C_5 = \{(\mathbf{b} + 1)\mathbf{a}, \dots, (2\mathbf{b} - 1)\mathbf{a}\}$, and $C_6 = \cup_{i=\mathbf{b}+1}^{2\mathbf{b}-1} \{\mathbf{ia} + \mathbf{b}, \dots, \mathbf{ia} + (\mathbf{a} - 1)\mathbf{b}\}$. The set A is the union $\cup_{i=1}^6 k \cdot C_i$.

The set $B = (\mathbf{b} - 1)\{0, k\mathbf{a}\} + (2\mathbf{a} - 1)\{0, k\mathbf{b}\}$ is

$$\begin{aligned} & \{0, k\mathbf{a}, \dots, (\mathbf{b} - 1)k\mathbf{a}\} + \{0, k\mathbf{b}, \dots, (2\mathbf{a} - 1)k\mathbf{b}\} = \\ & = k \cdot \left(\{0, \mathbf{a}, \dots, (\mathbf{b} - 1)\mathbf{a}\} \cup \{0, \mathbf{b}, \dots, (\mathbf{a} - 1)\mathbf{b}, \mathbf{ab}, \dots, (2\mathbf{a} - 1)\mathbf{b}\} \cup \right. \\ & \quad \left. \{\mathbf{a} + \mathbf{b}, \dots, \mathbf{a} + (\mathbf{a} - 1)\mathbf{b}, \mathbf{a} + \mathbf{ab}, \dots, \mathbf{a} + (2\mathbf{a} - 1)\mathbf{b}\} \cup \dots \right. \\ & \quad \left. \cup \{(\mathbf{b} - 1)\mathbf{a} + \mathbf{b}, \dots, (\mathbf{b} - 1)\mathbf{a} + (\mathbf{a} - 1)\mathbf{b}, (\mathbf{b} - 1)\mathbf{a} + \mathbf{ab}, \dots, (\mathbf{b} - 1)\mathbf{a} + (2\mathbf{a} - 1)\mathbf{b}\} \right) \\ & = \cup_{i=1}^6 k \cdot C_i. \end{aligned}$$

Thus, $A = B$, and $x^{\mathbf{b}-1}y^{\mathbf{a}-1}(x^{\mathbf{b}} - y^{\mathbf{a}}) \in I_S$.

To finish the proof, we use Lemma 5. So, $\gamma \geq \mathbf{b} - 1$, $\beta \geq \mathbf{a} - 1$, and there is $n \in \mathbb{N} \setminus \{0\}$ such that $x^\alpha y^\beta - x^\gamma y^\delta = x^\gamma y^\beta (x^{n\mathbf{b}} - y^{n\mathbf{a}})$. If $n = 1$, then $x^\alpha y^\beta - x^\gamma y^\delta = x^{\gamma-\mathbf{b}+1}y^{\beta-\mathbf{a}+1}x^{\mathbf{b}-1}y^{\mathbf{a}-1}(x^{\mathbf{b}} - y^{\mathbf{a}})$. In case $n > 1$, by factorizing the binomial $x^{n\mathbf{b}} - y^{n\mathbf{a}}$, we obtain

$$\begin{aligned} x^\alpha y^\beta - x^\gamma y^\delta &= \\ & x^{\gamma-\mathbf{b}+1}y^{\beta-\mathbf{a}+1}(x^{(n-1)\mathbf{b}} + x^{(n-2)\mathbf{b}}y + \dots + xy^{(n-2)\mathbf{a}} + y^{(n-1)\mathbf{a}})x^{\mathbf{b}-1}y^{\mathbf{a}-1}(x^{\mathbf{b}} - y^{\mathbf{a}}). \end{aligned}$$

In any case, $I_{\tilde{S}} = \langle x^{\mathbf{b}-1}y^{\mathbf{a}-1}(x^{\mathbf{b}} - y^{\mathbf{a}}) \rangle$. $\square$

Corollary 1. Let $a_1, \dots, a_s$ be positive integers, and S be the $\mathcal{P}_{\text{fin}}(\mathbb{N})$ -monoid generated by $\{ \{a_1\}, \dots, \{a_s\}, \{0, k\mathbf{a}\}, \{0, k\mathbf{b}\} \}$. Then,

$$I_S = I_{\langle a_1, \dots, a_s \rangle} + \langle x^{\mathbf{b}-1}y^{\mathbf{a}-1}(x^{\mathbf{b}} - y^{\mathbf{a}}) \rangle \subset \mathbf{k}[x_1, \dots, x_s, x, y].$$

Proof. The proof comes from Proposition 1 and Theorem 3. $\square$

5. Computing the Monoid Ideals

The aim of this section is to determine an algorithm for computing the ideals associated with some families of submonoids of power monoids. As in the previous section, we consider two positive co-prime integers $\mathbf{a} < \mathbf{b}$, $k \in \mathbb{N} \setminus \{0\}$, the monoid $\bar{S} = \langle k\mathbf{a}, k\mathbf{b} \rangle$, and the power monoid $\tilde{S} = \langle \{0, k\mathbf{a}\}, \{0, k\mathbf{b}\} \rangle$.

For any two non-negative integers n and m , $A_{(n,m)}$ denotes $\{\alpha k\mathbf{a} + \beta k\mathbf{b} \mid \alpha \in \{0, \dots, n\}, \beta \in \{0, \dots, m\}\} = nk\{0, \mathbf{a}\} + mk\{0, \mathbf{b}\}$.

Theorem 4. Let $\{(n_i, m_i) \mid n_i, m_i \in \mathbb{N}, n_i + m_i > 0, i = 1, \dots, t\}$ be a non-empty subset of $\mathbb{N}^2$, $b_1, \dots, b_p, a_1, \dots, a_s \in \mathbb{N} \setminus \{0\}$ with $s \leq t$, and consider S the $\mathcal{P}_{\text{fin}}(\mathbb{N})$ -monoid generated by

$$\{ \{b_1\}, \dots, \{b_p\}, \{a_1\} + A_{(n_1, m_1)}, \dots, \{a_s\} + A_{(n_s, m_s)}, A_{(n_{s+1}, m_{s+1})}, \dots, A_{(n_t, m_t)} \},$$

and the $\mathcal{P}_{\text{fin}}(\mathbb{N})$ -monoid $S' = \langle \{b_1\}, \dots, \{b_p\}, \{a_1\}, \dots, \{a_s\}, \{0, k\mathbf{a}\}, \{0, k\mathbf{b}\} \rangle$.

Then, the ideal $I_S \in \mathbf{k}[x_1, \dots, x_p, z_1, \dots, z_t]$ is

$$\begin{aligned} & \left(I_{S'} + \langle z_1 - w_1 x^{n_1} y^{m_1}, \dots, z_s - w_s x^{n_s} y^{m_s}, z_{s+1} - x^{n_{s+1}} y^{m_{s+1}}, \dots, z_t - x^{n_t} y^{m_t} \rangle \right) \\ & \quad \bigcap \mathbf{k}[x_1, \dots, x_p, z_1, \dots, z_t], \end{aligned}$$

with $I_{S'} \subset \mathbf{k}[x_1, \dots, x_p, w_1, \dots, w_s, x, y]$.

Proof. We prove that ideal I_S is an elimination ideal of $I_{S'} + J$, where J is the ideal

$$\langle z_1 - w_1 x^{n_1} y^{m_1}, \dots, z_s - w_s x^{n_s} y^{m_s}, z_{s+1} - x^{n_{s+1}} y^{m_{s+1}}, \dots, z_t - x^{n_t} y^{m_t} \rangle.$$

Firstly, we show that any monomials Z^β and Z^δ can be rewritten as follows. Denote $f_i = \prod_{k=i}^t z_k^{\beta_k}$, $\hat{f}_i = \prod_{k=i}^t z_k^{\delta_k}$, $g_i^j = \prod_{k=i}^j (w_k x^{n_k} y^{m_k})^{\beta_k}$, $\hat{g}_i^j = \prod_{k=i}^j (w_k x^{n_k} y^{m_k})^{\delta_k}$, $h_i^j = \prod_{k=i}^j (x^{n_k} y^{m_k})^{\beta_k}$, $\hat{h}_i^j = \prod_{k=i}^j (x^{n_k} y^{m_k})^{\delta_k}$, and suppose $\beta_0 = \delta_0 = 0$, then we have that

$$Z^\beta = z_1^{\beta_1} \cdots z_t^{\beta_t} = \sum_{i=0}^{s-1} f_{i+2} g_0^i (z_{i+1}^{\beta_{i+1}} - g_{i+1}^{i+1}) + \sum_{i=s+1}^t f_{i+1} g_1^s h_{s+1}^{i-1} (z_i^{\beta_i} - h_i^i) + g_1^s h_{s+1}^t$$

and

$$Z^\delta = z_1^{\delta_1} \cdots z_t^{\delta_t} = \sum_{i=0}^{s-1} \hat{f}_{i+2} \hat{g}_0^i (z_{i+1}^{\delta_{i+1}} - \hat{g}_{i+1}^{i+1}) + \sum_{i=s+1}^t \hat{f}_{i+1} \hat{g}_1^s \hat{h}_{s+1}^{i-1} (z_i^{\delta_i} - \hat{h}_i^i) + \hat{g}_1^s \hat{h}_{s+1}^t.$$

Thus, consider F the binomial $X^\alpha g_1^s h_{s+1}^t - X^\gamma \hat{g}_1^s \hat{h}_{s+1}^t$.

Since any binomial $u^n - v^n$ is equal to $(u - v)(u^{n-1} + u^{n-2}v + \cdots + uv^{n-2} + v^{n-1})$, the binomials $z_i^n - (w_i x^{n_i} y^{m_i})^n$ and $z_j^n - (x^{n_j} y^{m_j})^n$ belong to J for every non-negative integer n and all $i = 1, \dots, s$ and $j = s+1, \dots, t$.

Let $X^\alpha Z^\beta - X^\gamma Z^\delta$ be a binomial belonging to $I_{S'} + J$, so $F = X^\alpha Z^\beta - X^\gamma Z^\delta + q \in I_{S'} + J$, with $q \in J$. By Lemma 1, we have $F \in I_{S'}$, hence

$$\begin{aligned} \sum_{i=1}^p \alpha_i \{b_i\} + \sum_{i=1}^s \beta_i \{a_i\} + \sum_{i=1}^t \left(\beta_i (n_i \{0, k\mathbf{a}\}) + \beta_i (m_i \{0, k\mathbf{b}\}) \right) = \\ \sum_{i=1}^p \gamma_i \{b_i\} + \sum_{i=1}^s \delta_i \{a_i\} + \sum_{i=1}^t \left(\delta_i (n_i \{0, k\mathbf{a}\}) + \delta_i (m_i \{0, k\mathbf{b}\}) \right). \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{i=1}^p \alpha_i \{b_i\} + \sum_{i=1}^s \beta_i (\{a_i\} + A_{(n_i, m_i)}) + \sum_{i=s+1}^t \beta_i A_{(n_i, m_i)} = \\ \sum_{i=1}^p \gamma_i \{b_i\} + \sum_{i=1}^s \delta_i (\{a_i\} + A_{(n_i, m_i)}) + \sum_{i=1}^t \delta_i A_{(n_i, m_i)}, \end{aligned}$$

and $X^\alpha Z^\beta - X^\gamma Z^\delta \in I_S$.

Analogously, if $X^\alpha Z^\beta - X^\gamma Z^\delta \in I_S$, then $F \in I_{S'}$, and $X^\alpha Z^\beta - X^\gamma Z^\delta \in I_{S'} + J$. This completes the proof. $\square$

The above proof can also be carried out using [19] (Proposition 4). In our proof, we employ the language of polynomials, ideals, and Gröbner bases, avoiding congruences.

From Theorem 4, we obtain an algorithm (Algorithm 1) for computing the ideal of the $\mathcal{P}_{\text{fin}}(\mathbb{N})$ -monoid generated by

$$\{\{b_1\}, \dots, \{b_p\}, \{a_1\} + A_{(n_1, m_1)}, \dots, \{a_s\} + A_{(n_s, m_s)}, A_{(n_{s+1}, m_{s+1})}, \dots, A_{(n_t, m_t)}\}.$$

Algorithm 1: Computation of I_S .

Data: $\{\{b_1\}, \dots, \{b_p\}, \{a_1\} + A_{(n_1, m_1)}, \dots, \{a_s\} + A_{(n_s, m_s)}, A_{(n_{s+1}, m_{s+1})}, \dots, A_{(n_t, m_t)}\}$, the generating set of S .

Result: $\mathcal{G}$, a generating set of the monoid ideal of S .

begin

$S' \leftarrow \{\{b_1\}, \dots, \{b_p\}, \{a_1\}, \dots, \{a_s\}, \{0, \mathbf{ka}\}, \{0, \mathbf{kb}\}\};$

$S_1 \leftarrow \{\{b_1\}, \dots, \{b_p\}, \{a_1\}, \dots, \{a_s\}\};$

$G_1 \leftarrow$ a generating set of $I_{\langle S_1 \rangle} \subset \mathbf{k}[X, W];$

$G' \leftarrow G_1 \sqcup \{x^{\mathbf{b}-1}y^{\mathbf{a}-1}(x^{\mathbf{b}} - y^{\mathbf{a}})\} \subset \mathbf{k}[X, W, x, y]$, it is a generating set of $I_{\langle S' \rangle}$ (Corollary 1);

$G_2 \leftarrow G' \sqcup \{z_1 - w_1x^{n_1}y^{m_1}, \dots, z_s - w_sx^{n_s}y^{m_s}, z_{s+1} - x^{n_{s+1}}y^{m_{s+1}}, \dots, z_t - x^{n_t}y^{m_t}\} \subset \mathbf{k}[X, W, x, y, Z];$

$G_3 \leftarrow$ a Gröbner basis of G_2 respect to a monomial order with $x, y, w_q > x_i$ and $x, y, w_q > z_j$ for every $i = 1, \dots, p, j = 1, \dots, t$, and $q = 1, \dots, s;$

$\mathcal{G} \leftarrow \{X^\alpha Z^\beta - X^\gamma Z^\delta \mid X^\alpha Z^\beta - X^\gamma Z^\delta \in G_3\}$, it is a generating set (Gröbner basis) of $I_S;$

return $\mathcal{G};$

We show how this algorithm works with an example.

Example 1. Let $\mathbb{S}$ be the numerical monoid generated by $\{3, 4\}$, and S be the $\mathcal{P}_{\text{fin}}(\mathbb{S})$ -monoid generated by

$$\{\{3\}, \{4\}, \{6, 12\}, \{7, 10, 13\}, \{0, 3, 6, 9\}\}.$$

Then, from the first steps of Algorithm 1,

- $S' = \{\{3\}, \{4\}, \{6\}, \{7\}, \{0, 3\}, \{0, 6\}\},$
- $S_1 = \{\{3\}, \{4\}, \{6\}, \{7\}\}.$

If we compute a generating set of the ideal of S_1 , then we get the following one, $G_1 = \{w_1^7 - w_2^6, w_2^2x_2 - w_1^3, w_1^4x_2 - w_2^4, w_1x_2^2 - w_2^2, x_2^3 - w_1^2, w_2x_1 - w_1x_2, w_1^2x_1 - w_2x_2^2, x_1x_2 - w_2, x_1^2 - w_1\}$. Therefore, $G' = G_1 \cup \{x(x^2 - y)\}$ and $G_2 = G' \cup \{z_1 - w_1y, z_2 - w_2x^2, z_3 - x^3\}.$

Now, we compute a Gröbner basis of G_2 with respect to the lexicographical order where $x > y > w_i > x_j > z_k$ for all i, j , and k , and we obtain

$$\begin{aligned} G_3 = \{ & z_1^{21}z_3 - z_2^{18}z_3^3, z_1^{21}z_2 - z_2^{19}z_3^2, x_2z_2^{14}z_3^3 - z_1^{17}z_3, x_2z_2^{15}z_3^2 - z_1^{17}z_2, \\ & x_2z_1^4z_3 - z_2^4z_3, x_2z_1^4z_2 - z_2^5, x_2^2z_2^{10}z_3^3 - z_1^{13}z_3, x_2^2z_2^{11}z_3^2 - z_1^{13}z_2, \\ & x_2^3z_2^6z_3^3 - z_1^9z_3, x_2^3z_2^7z_3^2 - z_1^9z_2, x_2^4z_2^2z_3^3 - z_1^5z_3, x_2^4z_2^3z_3^2 - z_1^5z_2, \\ & x_2^5z_2^2 - z_1z_2^2, x_1z_2z_3 - x_2z_1z_3, x_1z_2^2 - x_2z_1z_2, x_1z_1^3z_3 - z_2^3z_3, \\ & x_1z_1^3z_2 - z_2^4, x_1x_2^4z_3^2 - z_1^2z_2, x_1^2z_1^2z_3 - x_2z_2^2z_3, x_1^2z_1^2z_2 - x_2z_2^3, \\ & x_1^2x_2^3z_3^3 - z_1^3z_3, x_1^3z_1z_3 - x_2^2z_2z_3, x_1^3z_1z_2 - x_2^2z_2^2, x_1^3x_2^3z_3^2 - z_2^3, \\ & x_1^4 - x_2^3, w_2 - x_1x_2, w_1 - x_1^2, yz_2^2 - x_1^2x_2^2z_3^2, yz_1z_2 - x_1^3x_2z_2^2, \\ & yz_1^2z_3 - x_2^3z_3^3, yx_2^2z_2z_3 - x_1z_1^2z_3, yx_2^2z_1z_3 - z_2^2z_3, yx_2^3 - x_1^2z_1, \\ & yx_1x_2z_3 - z_2z_3, yx_1x_2z_2 - z_2^2, yx_1^2 - z_1, y^2z_2 - x_1x_2z_2^3, y^2z_1z_3 - x_1^2z_3^3, \\ & y^3z_3 - z_3^3, xz_3^2 - y^2z_3, xz_2 - x_1x_2z_3, xz_1 - x_1^2z_3, xx_2^4z_3 - x_1z_1z_2, \\ & xx_1x_2z_3 - yz_2, xy - z_3, x^2z_3 - yz_3, x^2x_2^4 - x_1^3z_2, x^2x_1x_2 - z_2, x^3 - z_3\}. \end{aligned}$$

Finally, the output of the algorithm is

$$\begin{aligned} & \{z_1^{21}z_3 - z_2^{18}z_3^3, z_1^{21}z_2 - z_2^{19}z_3^2, x_2z_2^{14}z_3^3 - z_1^{17}z_3, x_2z_2^{15}z_3^2 - z_1^{17}z_2, \\ & x_2z_1^4z_3 - z_2^4z_3, x_2z_1^4z_2 - z_2^5, x_2^2z_2^{10}z_3^3 - z_1^{13}z_3, x_2^2z_2^{11}z_3^2 - z_1^{13}z_2, \\ & x_2^3z_2^6z_3^3 - z_1^9z_3, x_2^3z_2^7z_3^2 - z_1^9z_2, x_2^4z_2^2z_3^3 - z_1^5z_3, x_2^4z_2^3z_3^2 - z_1^5z_2, \\ & x_2^5z_2^2 - z_1z_2^2, x_1z_2z_3 - x_2z_1z_3, x_1z_2^2 - x_2z_1z_2, x_1z_1^3z_3 - z_2^3z_3, x_1z_1^3z_2 - z_2^4, \\ & x_1x_2^4z_3^2 - z_1^2z_2, x_1^2z_1^2z_3 - x_2z_2^2z_3, x_1^2z_1^2z_2 - x_2z_2^3, x_1^2x_2^3z_3^3 - z_1^3z_3, \\ & x_1^3z_1z_3 - x_2^2z_2z_3, x_1^3z_1z_2 - x_2^2z_2^2, x_1^3x_2^3z_3^2 - z_2^3, x_1^4 - x_2^3\}. \end{aligned}$$

Note that, since the binomial $x_1z_2z_3 - x_2z_1z_3 \in I_S$, $\{3\} + \{7, 10, 13\} + \{0, 3, 6, 9\} = \{4\} + \{6, 12\} + \{0, 3, 6, 9\}$, but $\{3\} + \{7, 10, 13\} \neq \{4\} + \{6, 12\}$, the $\mathcal{P}_{\text{fin}}(\mathbb{S})$ -monoid S is non-cancellative.

The following example introduces an algorithm to obtain an expression for an integer set as a sum of other given integer sets, if possible. In particular, the i -fold sumset of a set is studied.

Example 2. We now use the above presentation of S to check whether the element $i\{7, 10, 13\}$ can be expressed in terms of the other generators of the $\mathcal{P}_{\text{fin}}(\mathbb{S})$ -monoid S . We compute the Gröbner basis with respect to the order given by the matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

and we obtain the set

$$\begin{aligned} G_A = \{ & x_1^4 - x_2^3, x_1^2x_2^3z_3^3 - z_1^3z_3, x_2^6z_3^3 - x_1^2z_1^3z_3, z_1^2z_2 - x_1x_2^4z_3^2, \\ & x_1z_2z_3 - x_2z_1z_3, x_2^2z_2z_3 - x_1^3z_1z_3, z_1z_2^2 - x_2^5z_3^2, x_1z_2^2 - x_2z_1z_2, \\ & x_2z_2^2z_3 - x_1^2z_1^2z_3, x_2^2z_2^2 - x_1^3z_1z_2, z_2^3 - x_1^3x_2^3z_3^2\}. \end{aligned}$$

In Table 1, we show some elements z_2^i that, when reduced with respect to the basis G_A , are expressed by using only the variables x_1 , x_2 , z_1 , and z_3 , and the expression of $i\{7, 10, 13\}$ in terms of the elements of the set $\{\{3\}, \{4\}, \{6, 12\}, \{0, 3, 6, 9\}\}$.

Table 1. Expressions of some elements $i\{7, 10, 13\}$ in terms of the subset of generators $\{\{3\}, \{4\}, \{6, 12\}, \{0, 3, 6, 9\}\}$.

i	Reduction of z_2^i	$i\{7, 10, 13\} =$
3	$x_1^3x_2^3z_3^2$	$3\{3\} + 3\{4\} + 2\{0, 3, 6, 9\}$
4	$x_1^4x_2^4z_3^2$	$2\{3\} + 4\{4\} + 1\{0, 6, 12\} + 2\{0, 3, 6, 9\}$
5	$x_1^5x_2^5z_3^2$	$1\{3\} + 5\{4\} + 2\{0, 6, 12\} + 2\{0, 3, 6, 9\}$
6	$x_2^6z_1^3z_3^2$	$6\{4\} + 3\{6, 12\} + 2\{0, 3, 6, 9\}$
7	$x_1^3x_2^4z_1^4z_3^2$	$3\{3\} + 4\{4\} + 4\{0, 6, 12\} + 2\{0, 3, 6, 9\}$
8	$x_1^2x_2^5z_1^5z_3^2$	$2\{3\} + 5\{4\} + 5\{0, 6, 12\} + 2\{0, 3, 6, 9\}$

In general, the reduction of z_2^i with respect to G_A is

$$x_1^{(2-i) \bmod 4} x_2^{\lfloor \frac{i+1}{4} \rfloor + 2 + ((i-3) \bmod 4)} z_1^{i-3} z_3^2.$$

Therefore, for every $i \geq 3$,

$$i\{7, 10, 13\} = ((2-i) \bmod 4)\{3\} + \left(\left\lfloor \frac{i+1}{4} \right\rfloor + 2 + ((i-3) \bmod 4)\right)\{4\} + (i-3)\{6, 12\} + 2\{0, 3, 6, 9\}.$$

The last examples are dedicated to study the elasticity of a $\mathcal{P}_{\text{fin}}(\mathbb{S})$ -monoid.

Example 3. Again, consider the $\mathcal{P}_{\text{fin}}(\mathbb{S})$ -monoid S given in Example 1. From its ideal, we compute a generating set of its associated lattice M ,

$$\begin{aligned} &\{ \{0, 0, 21, -18, -2\}, \{0, 1, -17, 14, 2\}, \{0, 1, 4, -4, 0\}, \{0, 2, -13, 10, 2\}, \\ &\{0, 3, -9, 6, 2\}, \{0, 4, -5, 2, 2\}, \{0, 5, -1, -2, 2\}, \{1, -1, -1, 1, 0\}, \\ &\{1, 0, 3, -3, 0\}, \{1, 4, -2, -1, 2\}, \{2, -1, 2, -2, 0\}, \{2, 3, -3, 0, 2\}, \\ &\{3, -2, 1, -1, 0\}, \{3, 3, 0, -3, 2\}, \{4, -3, 0, 0, 0\} \}, \end{aligned}$$

and its system of linear homogeneous equations,

$$Ax = \begin{pmatrix} -3 & -4 & 2 & 1 & 12 \\ -6 & -8 & 2 & 0 & 21 \end{pmatrix} x = 0.$$

We already know that S is strongly reduced, but this fact is far from being clear from the above equations. We can check it by computing its Hilbert basis with Normaliz [20]:

```
>>> c1=Cone(equations=[[-3,-4,2,1,12],[-6,-8,2,0,21]])
>>> c1.HilbertBasis()
[]
```

Since the above output is the empty list, we have $M \cap \mathbb{N}^5 = \{0\}$. The Hilbert basis of $(A \mid -A)(x, y) = 0$ has 109 elements:

```
HB = { {0, 1, 4, 0, 0, 0, 0, 4, 0}, {1, 0, 3, 0, 0, 0, 0, 3, 0}, {0, 0, 0, 1, 0, 0, 0, 0, 1, 0}, {0, 0, 21, 0, 0, 0, 0, 18, 2},
{0, 21, 0, 0, 8, 0, 0, 12, 0}, {0, 0, 0, 12, 0, 0, 21, 0, 8}, {0, 0, 1, 0, 0, 0, 0, 1, 0, 0}, {0, 0, 0, 0, 1, 0, 0, 0, 0, 1},
{0, 0, 0, 18, 2, 0, 0, 21, 0, 0}, {1, 0, 0, 15, 2, 0, 0, 18, 0, 0}, {0, 1, 0, 14, 2, 0, 0, 17, 0, 0}, {1, 0, 0, 0, 0, 1, 0, 0, 0, 0},
{0, 0, 0, 10, 0, 0, 16, 1, 0, 6}, {0, 1, 1, 0, 0, 1, 0, 0, 1, 0}, {0, 1, 0, 0, 0, 0, 1, 0, 0, 0}, {0, 16, 1, 0, 6, 0, 0, 0, 10, 0},
{1, 15, 0, 0, 6, 0, 0, 9, 0}, {0, 0, 0, 9, 0, 1, 15, 0, 0, 6}, {2, 0, 0, 12, 2, 0, 0, 15, 0, 0}, {2, 0, 2, 0, 0, 0, 1, 0, 2, 0},
{0, 0, 6, 0, 0, 9, 0, 0, 4}, {0, 0, 7, 0, 0, 1, 8, 0, 1, 4}, {1, 1, 0, 11, 2, 0, 0, 14, 0, 0}, {0, 0, 18, 0, 0, 1, 0, 0, 15, 2},
{0, 0, 17, 0, 0, 0, 1, 0, 14, 2}, {0, 2, 0, 10, 2, 0, 0, 13, 0, 0}, {3, 0, 0, 9, 2, 0, 0, 12, 0, 0}, {0, 0, 0, 8, 0, 0, 11, 2, 0, 4},
{2, 1, 0, 8, 2, 0, 0, 11, 0, 0}, {0, 11, 2, 0, 4, 0, 0, 8, 0}, {2, 0, 3, 0, 0, 0, 6, 0, 0, 2}, {0, 0, 0, 7, 0, 1, 10, 1, 0, 4},
{0, 0, 1, 2, 0, 4, 2, 0, 0, 2}, {1, 10, 1, 0, 4, 0, 0, 0, 7, 0}, {0, 0, 14, 0, 0, 1, 1, 0, 11, 2}, {0, 0, 15, 0, 2, 0, 0, 12, 2},
{0, 0, 13, 0, 0, 2, 0, 10, 2}, {1, 2, 0, 7, 2, 0, 0, 10, 0, 0}, {2, 9, 0, 0, 4, 0, 0, 6, 0}, {0, 0, 0, 6, 0, 2, 9, 0, 0, 4},
{0, 0, 2, 1, 0, 5, 1, 0, 0, 2}, {0, 10, 0, 0, 4, 1, 0, 5, 1, 0}, {0, 0, 1, 2, 0, 0, 5, 0, 0, 2}, {4, 0, 0, 6, 2, 0, 0, 9, 0, 0},
{1, 8, 0, 1, 4, 0, 0, 7, 0, 0}, {0, 0, 3, 0, 0, 6, 0, 0, 0, 2}, {0, 3, 0, 6, 2, 0, 0, 9, 0, 0}, {1, 0, 4, 0, 0, 5, 0, 1, 2},
{3, 1, 0, 5, 2, 0, 0, 8, 0, 0}, {0, 0, 10, 0, 0, 1, 2, 0, 7, 2}, {0, 0, 11, 0, 0, 2, 1, 0, 8, 2}, {0, 0, 12, 0, 0, 3, 0, 9, 2},
{0, 0, 9, 0, 0, 0, 3, 0, 6, 2}, {0, 9, 0, 0, 4, 0, 0, 6, 0, 0}, {3, 0, 1, 0, 0, 0, 2, 0, 1, 0}, {0, 0, 0, 6, 0, 0, 6, 3, 0, 2},
{0, 0, 2, 1, 0, 1, 4, 0, 0, 2}, {2, 2, 0, 4, 2, 0, 0, 7, 0, 0}, {1, 0, 5, 1, 0, 0, 10, 0, 0, 4}, {4, 0, 0, 0, 0, 3, 0, 0, 0, 0},
{0, 3, 0, 0, 0, 4, 0, 0, 0, 0}, {0, 0, 6, 0, 0, 5, 0, 0, 3, 2}, {0, 0, 5, 0, 0, 4, 1, 0, 2, 2}, {0, 0, 4, 0, 0, 3, 2, 0, 1, 2},
{0, 0, 3, 0, 0, 2, 3, 0, 0, 2}, {9, 0, 0, 0, 2, 0, 2, 2, 1, 0}, {0, 0, 0, 5, 0, 5, 2, 2, 0, 2}, {5, 0, 0, 3, 2, 0, 0, 6, 0, 0},
{0, 0, 5, 0, 0, 4, 0, 2, 2}, {0, 0, 9, 0, 0, 4, 0, 0, 6, 2}, {0, 0, 8, 0, 0, 3, 1, 0, 5, 2}, {0, 0, 6, 0, 0, 1, 3, 0, 3, 2},
{0, 0, 7, 0, 0, 2, 2, 0, 4, 2}, {1, 0, 0, 1, 0, 0, 1, 1, 0, 0}, {0, 0, 0, 5, 0, 1, 5, 2, 0, 2}, {0, 7, 0, 0, 2, 3, 0, 2, 1, 0},
{1, 3, 0, 3, 2, 0, 0, 6, 0, 0}, {0, 2, 2, 1, 0, 9, 0, 0, 0, 2}, {0, 2, 0, 1, 0, 3, 0, 1, 0, 0}, {0, 1, 0, 2, 0, 2, 0, 2, 0, 0},
{0, 0, 0, 3, 0, 1, 0, 3, 0, 0}, {0, 6, 3, 0, 2, 0, 0, 0, 6, 0}, {5, 2, 2, 0, 2, 0, 0, 0, 5, 0}, {8, 0, 0, 0, 2, 0, 1, 1, 2, 0},
{0, 0, 0, 4, 0, 0, 1, 4, 0, 0}, {0, 0, 0, 4, 0, 6, 1, 1, 0, 2}, {4, 1, 0, 2, 2, 0, 0, 5, 0, 0}, {0, 1, 1, 2, 0, 8, 0, 0, 0, 2},
{6, 1, 1, 0, 2, 0, 0, 0, 4, 0}, {1, 5, 2, 0, 2, 0, 0, 0, 5, 0}, {0, 5, 0, 2, 0, 0, 1, 2, 0}, {1, 4, 0, 0, 2, 0, 0, 2, 1, 0},
{2, 3, 0, 0, 2, 0, 0, 3, 0, 0}, {3, 0, 2, 1, 0, 0, 7, 0, 0, 2}, {0, 0, 0, 4, 0, 2, 4, 1, 0, 2}, {0, 4, 0, 2, 2, 0, 0, 5, 0, 0},
{0, 5, 0, 1, 2, 1, 0, 4, 0, 0}, {0, 6, 0, 0, 2, 2, 0, 3, 0, 0}, {0, 0, 0, 3, 0, 7, 0, 0, 0, 2}, {7, 0, 0, 0, 2, 0, 0, 0, 3, 0},
{4, 2, 0, 0, 2, 0, 0, 1, 2, 0}, {5, 1, 0, 0, 2, 0, 0, 2, 1, 0}, {6, 0, 0, 0, 2, 0, 0, 3, 0, 0}, {2, 4, 1, 0, 2, 0, 0, 0, 4, 0},
{1, 0, 0, 3, 0, 0, 6, 0, 0, 2}, {0, 6, 0, 0, 2, 1, 0, 0, 3, 0}, {3, 2, 0, 1, 2, 0, 0, 4, 0, 0}, {0, 0, 0, 3, 0, 3, 0, 0, 2},
{3, 3, 0, 0, 2, 0, 0, 0, 3, 0} }
```

Using the formula (1), we conclude that the elasticity of S is 3.

To know if S has accepted elasticity, we use Algorithm 28 of [8], which is implemented in <https://github.com/D-marina/CommutativeMonoids/blob/master/Sumsetssemigroups/sumsetSemigroups.ipynb> (accessed on 16 December 2024) of [10]. Running the commands,

```
>>> gb14=computationIS([3],[4],[6,12],[7,10,13],[0,3,6,9])
>>> hasAcceptedElasticity(gb14, debug=True)
...
We compute a Groebner basis with the lexicographical ordering of
the variables [x2, z1, x1, z2, z3]
GroebnerBasis([-x1**4 + x2**3, x1**4*x2**2*z3**2 - z1*z2**2,
-x1**3*z1*z2 + x2**2*z2**2, -x1**3*z1*z3 + x2**2*z2*z3,
-x1*z2**2 + x2*z1*z2, -x1*z2*z3 + x2*z1*z3,
x1**5*x2*z3**2 - z1**2*z2, -x1**2*z1**2*z2 + x2*z2**3,
-x1**2*z1**2*z3 + x2*z2**2*z3, -x1**6*z2*z3**2 + z1**3*z2,
-x1**6*z3**3 + z1**3*z3, x1**7*z3**2 - z2**3],
x2, z1, x1, z2, z3, domain='ZZ', order='lex')
Once removed the variables [x2, z1], we obtain
GroebnerBasis([x1**7*z3**2 - z2**3], x1, z2, z3, domain='ZZ',
order='lex')
...
True
```

we obtain that the monoid has accepted elasticity. Moreover, from the above output, we see that the binomial $x_1^7 x_3^2 - z_2^3$ belongs to I_S . Since the quotient of the addition of the exponents of these two monomials is $(7+2)/3 = 3 = \rho(S)$, the element $7\{3\} + 2\{0,3,6,9\} = 3\{7,10,13\}$ reaches the elasticity.

We see now an example of $\mathcal{P}_{\text{fin}}(\mathbb{S})$ -monoid without accepted elasticity.

Example 4. Let S be the 3-generated $\mathcal{P}_{\text{fin}}(\mathbb{S})$ -monoid

$$\langle \{0,3\}, \{0,4\}, \{7,10,11,13,14,15,17,18,19,1,22,25\} \rangle.$$

Analogously to the preceding example, we use function `hasAcceptedElasticity` to check if S has accepted elasticity. The display output of this function shows some steps of Algorithm 28 of [8] and returns `False`; that is, the monoid has not accepted elasticity.

```
>>> gb15=computationIS([0,3],[0,4],
[7,10,11,13,14,15,17,18,19,1,22,25])
>>> hasAcceptedElasticity(gb15, debug=True)
Positive cone of M (the semigroup is strongly reduced): []
Equations of M
[[3, 4, 0], [0, 0, 1]]
Matrix (A|-A) (equations of M \cap N^n, n=5):
[[3, 4, 0, -3, -4, 0], [0, 0, 1, 0, 0, -1]]
Generator system of M \cap N^n (number of generators 5):
[[0, 0, 1, 0, 0, 1], [0, 1, 0, 0, 1, 0], [0, 3, 0, 4, 0, 0],
[1, 0, 0, 1, 0, 0], [4, 0, 0, 0, 3, 0]]
Elasticity of S: 4/3
Atoms of A(I_M) that reach the elasticity:
[[4, 0, 0, 0, 3, 0]]
S has not accepted elasticity (the monoid C is empty,
```

see step 6 of algorithm in Algorithm 28 in
<http://doi.org/10.1007/s00233-002-0022-4>
 False

6. Discussion

In this work, we use some tools from computational commutative algebra to study some problems related to additive number theory. For example, the relationship between these two theories allows us to check when a subset of positive integers can be expressed as the sum of other subsets, or to study the structure of hA , where A is a finite subset of $\mathbb{N}$. The primary keys to construct the bridge between these two theories are $\mathcal{P}_{\text{fin}}(\mathbb{N})$ -monoids and their associated polynomial ideals. To use this computational approach, we need to know a system of generators of the ideal associated with a given $\mathcal{P}_{\text{fin}}(\mathbb{N})$ -monoid. The results in this work characterize a system of generators for the ideals associated with a family of $\mathcal{P}_{\text{fin}}(\mathbb{N})$ -monoids, starting a way to provide more computational tools to solve some problems in additive number theory.

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