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Mellin Transform of Weierstrass Zeta Function and Integral Representations of Some Lambert Series

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Abstract: We consider a series which combines two Dirichlet series constructed from the coefficients of a Laurent series and derive a general integral representation of the series as a Mellin transform. As an application, we obtain a family of Mellin integral identities involving the Weierstrass elliptic functions and some Lambert series. These identities are used to derive some of the properties of the Lambert series.

Keywords: Mellin transform; Weierstrass zeta function; Lambert series

MSC: 11M99; 30B50; 33E05

1. Introduction

For a power series $f(q) = \sum_{n=1}^{\infty} a_n q^n$, $a_n \in \mathbb{C}$, the Dirichlet series

$$L(f;s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

is called the L-series attached to f. Let us use the same notation in general for a series as well as the function defined by its limit. Under an appropriate condition on f, the function $\frac{\Gamma(s)}{(2\pi)^s} L(f;s)$ is given by the Mellin transform of $f(e^{-2\pi t})$; namely, the identity

$$\frac{\Gamma(s)}{(2\pi)^s} L(f;s) = \int_0^{\infty} f(e^{-2\pi t}) t^{s-1} dt \tag{1}$$

holds when s is in some right half-plane [1,2]. This relation simply follows from the integral

$$\Gamma(s) = \int_0^{\infty} e^{-t} t^{s-1} dt, \quad \operatorname{Re} s > 0,$$

provided that we can interchange the integral and the sum in (1). Such a representation of the Dirichlet series dates back to Riemann, and various properties of the power series f are generally reflected in the nature of its L-series and vice versa [3]. In particular, if the function $f(e^{-2\pi t})$ exhibits some additional symmetry under the modular transformation $t \mapsto 1/t$, this property is reflected in the existence of the corresponding functional equation of $L(f;s)$ [4,5].

In this article, we are interested in the following analogous situation. Instead of a power series, suppose we are given a Laurent series

$$f(z) = \sum_{n \in \mathbb{Z}} a_n z^n, \; a_n \in \mathbb{C}. \tag{2}$$

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Academic Editors: Shigeru Kanemitsu and Antanas Laurincikas

Received: 13 January 2025
Revised: 5 February 2025
Accepted: 8 February 2025
Published: 10 February 2025

Citation: Kim, N. Mellin Transform of Weierstrass Zeta Function and Integral Representations of Some Lambert Series. *Mathematics* **2025**, *13*, 582. https://doi.org/10.3390/math13040582

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Let us define a "two-sided" L-series, called the H-series attached to f in this article (to be given more precisely in Definition 3), as

$$H(f;s) = e^{\frac{\pi i s}{2}} \sum_{n=1}^{\infty} \frac{a_n}{n^s} + e^{-\frac{\pi i s}{2}} \sum_{n=1}^{\infty} \frac{a_{-n}}{n^s}. \tag{3}$$

Let us assume the Laurent series (2) converges in a neighborhood of the unit circle. Under this assumption, some immediate properties of $H(f;s)$ are listed in Proposition 1. For example, $H(f;s)$ converges for all $s \in \mathbb{C}$ to an entire function. Furthermore, it is not hard to show that $\frac{\Gamma(s)}{(2\pi)^s} H(f;s)$ is also given by a Mellin integral of the form

$$\frac{\Gamma(s)}{(2\pi)^s} H(f;s) = \int_0^{\infty} \left(f(e^{2\pi i u}) - \sum_{j=0}^{m} b_j u^j\right) u^{s-1} dt \tag{4}$$

where s is in some vertical strip, and the values of b_j are the Taylor coefficients of $f(e^{2\pi i u})$ at $u = 0$. The precise statement and a simple proof of this formula are given in Theorem 1.

One application of this result involves the following generating function of an arithmetic function. For $s \in \mathbb{C}$, let $\sigma_s(n) = \sum_{d|n} d^s$ be the sum of positive divisors function [6]. Let us use the notation

$$\mathfrak{H} = \{\tau \in \mathbb{C} \mid \operatorname{Im} \tau > 0\}$$

to denote the upper half-plane. For $\tau \in \mathfrak{H}$ and $s \in \mathbb{C}$, let

$$A(\tau, s) = \sum_{n=1}^{\infty} \sigma_{s-1}(n) q^n = \sum_{n=1}^{\infty} n^{s-1} \frac{q^n}{1-q^n}, \tag{5}$$

where $q = e^{2\pi i \tau}$. $A(\tau, s)$ is an entire function in s for every $\tau \in \mathfrak{H}$ and a Lambert series in q for every $s \in \mathbb{C}$, and it was studied in [7] in the context of the analytic continuation of the classical Eisenstein series to a complex weight s. It is closely related to the periodic function of the real analytic Eisenstein series given in [8]. The transformation property of $A(\tau, s)$ under

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}$$

for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{Z})$ is known to have many applications, which generalizes the modular transformation properties of the classical Eisenstein series. The transformation formula of the logarithm of the Dedekind eta function can also be deduced as a special case. A general formula expressing $A\left(\frac{a\tau+b}{c\tau+d}, s\right)$ in terms of $A(\tau, s)$ is somewhat complicated and is given in terms of some integral expressions in [9,10]. Another integral expression involving cotangent sums with a simpler proof is given in [11]. In [12], some modular-type transformation properties are expressed in terms of a generalization of the modified Bessel function of the second kind.

We now give an overview of our result on $A(\tau, s)$. We consider, for a fixed $\tau \in \mathfrak{H}$ with $q = e^{2\pi i \tau}$, the Laurent series $f(z) = -2\pi i \sum_{n=1}^{\infty} \frac{q^n}{1-q^n} (z^n - z^{-n})$ which converges for $|q| < |z| < |q|^{-1}$. The H-series (3) attached to f is

$$-2\pi i (e^{\frac{\pi i s}{2}} - e^{-\frac{\pi i s}{2}}) A(\tau, 1-s). \tag{6}$$

In fact, $f(z)$ can be expressed in terms of the Weierstrass zeta function (Proposition 2). By using (4), we obtain identities relating $A(\tau, s)$ and the Mellin integral of some expressions involving the Weierstrass elliptic functions (Corollaries 1 and 2). Some known results can also be derived from these identities, including a transformation formula of $A(\tau, s)$ under the modular transformation of τ (Corollaries 3 and 4).

2. Laurent Series and the *H*-Series

Definition 1. *For $\varepsilon > 0$, let $\mathcal{O}_\varepsilon$ be the $\mathbb{C}$-algebra of analytic functions on the annulus $(1+\varepsilon)^{-1} < |z| < 1+\varepsilon$. For $\varepsilon > \varepsilon' > 0$, we have the inclusion $\mathcal{O}_\varepsilon \to \mathcal{O}_{\varepsilon'}$ by restriction. We define the $\mathbb{C}$-algebra $\mathcal{O}$ to be the direct limit*

$$\mathcal{O} = \varinjlim \mathcal{O}_\varepsilon.$$

The notation $\mathcal{O}$ will be used throughout the article. By the above definition, $f \in \mathcal{O}$ can be represented by a function $f(z)$ which is analytic on $(1+\varepsilon)^{-1} < |z| < 1+\varepsilon$ for some $\varepsilon > 0$. As the unit circle is compact, this condition is the same as $f(z)$ being analytic in any neighborhood of the circle. Equivalently, $f(e^{2\pi i u})$ as a function of u is periodic with period 1 and is analytic in a neighborhood of $\mathbb{R}$.

Definition 2. *We define the derivation $D : \mathcal{O} \to \mathcal{O}$ by*

$$(Df)(z) = 2\pi i z \frac{d}{dz} f(z). \tag{7}$$

Note that for $f \in \mathcal{O}$, we have $(Df)(e^{2\pi i u}) = \frac{d}{du} f(e^{2\pi i u})$.

Lemma 1. *$f \in \mathcal{O}$ if and only if its representative function has the Laurent expansion*

$$f(z) = \sum_{n \in \mathbb{Z}} a_n z^n \tag{8}$$

such that $|a_n| = O(\rho^{|n|})$ for some $0 < \rho < 1$.

Proof. If $f(z)$ is an analytic function on $(1+\varepsilon)^{-1} < |z| < 1+\varepsilon$ for some $\varepsilon > 0$, the convergence of (8) at $z = \rho$ and $z = \rho^{-1}$ for any $(1+\varepsilon)^{-1} < \rho < 1$ shows $|a_n| = O(\rho^{|n|})$. Conversely, if $|a_n| = O(\rho^{|n|})$ for $0 < \rho < 1$, then the series (8) converges for $\rho < |z| < \rho^{-1}$. □

Let us use the same notation to denote $f \in \mathcal{O}$ and its representative function $f(z)$ as well as its Laurent series, and write

$$\sum_{n \in \mathbb{Z}} a_n z^n \in \mathcal{O}$$

to indicate that the series satisfies $|a_n| = O(\rho^{|n|})$ for some $0 < \rho < 1$. We regard $\mathcal{O}$ as a subset of $\mathbb{C}[[z, z^{-1}]]$, the set of all formal series of the form $\sum_{n \in \mathbb{Z}} a_n z^n$, $a_n \in \mathbb{C}$.

Definition 3. *Given a series $f(z) = \sum_{n \in \mathbb{Z}} a_n z^n \in \mathbb{C}[[z, z^{-1}]]$ and $s \in \mathbb{C}$, the H-series attached to f is*

$$H(f; s) = e^{\frac{\pi i s}{2}} \sum_{n=1}^{\infty} \frac{a_n}{n^s} + e^{-\frac{\pi i s}{2}} \sum_{n=1}^{\infty} \frac{a_{-n}}{n^s}. \tag{9}$$

If the series converges at $s \in \mathbb{C}$, we denote its limit again by $H(f; s)$.

We have

$$H(f; s) = e^{\frac{\pi i s}{2}} L_+(f; s) + e^{-\frac{\pi i s}{2}} L_-(f; s)$$

for two Dirichlet series

$$L_+(f; s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \quad L_-(f; s) = \sum_{n=1}^{\infty} \frac{a_{-n}}{n^s}. \tag{10}$$

Proposition 1. *For $f \in \mathcal{O}$, let $H(f;s)$ be the H-series attached to f. Then, we have the following:*

(*i*) *$H(f;s)$ converges for all $s \in \mathbb{C}$ to an entire function of s.*

(*ii*) *$H(f;s) = 0$ identically if and only if f is constant.*

(*iii*) *$H(Df;s) = -2\pi H(f;s-1)$, where D is the derivation given in Definition 2.*

Proof. Proposition part (i) follows from the bounds on the Laurent coefficients of $f(z)$ from Lemma 1.

If f is constant, then $H(f;s) = 0$ by definition. Conversely, if $H(f;s)$ vanishes identically but $f(z)$ is not constant, let $n > 0$ be the least integer such that either $a_n \neq 0$ or $a_{-n} \neq 0$. Taking the limit of $n^s H(f;s) = 0$ as $s \to \infty$ along the real line, we see that $e^{\frac{\pi i s}{2}} a_n + e^{-\frac{\pi i s}{2}} a_{-n} \to 0$, which is a contradiction, and (ii) follows.

If $f(z) = \sum_{n\in\mathbb{Z}} a_n z^n$, then $Df(z) = \sum_{n\in\mathbb{Z}} 2\pi i n a_n z^n$. Thus,

$$\begin{aligned} H(Df;s) &= e^{\frac{\pi i s}{2}} \sum_{n=1}^{\infty} \frac{2\pi i n a_n}{n^s} - e^{-\frac{\pi i s}{2}} \sum_{n=1}^{\infty} \frac{2\pi i n a_{-n}}{n^s} \\ &= -2\pi \big(e^{\frac{\pi i (s-1)}{2}} \sum_{n=1}^{\infty} \frac{a_n}{n^{s-1}} + e^{-\frac{\pi i (s-1)}{2}} \sum_{n=1}^{\infty} \frac{a_{-n}}{n^{s-1}} \big) \\ &= -2\pi H(f;s-1) \end{aligned}$$

and (iii) holds. □

3. Representation as Mellin Transform

Let $g : (0,\infty) \to \mathbb{C}$ be a locally integrable function such that

$$g(u) = \begin{cases} O(u^{\alpha}) & \text{as } u \to 0 \\ O(u^{\beta}) & \text{as } u \to \infty \end{cases} \tag{11}$$

for $\infty > \alpha > \beta > -\infty$. For $-\alpha < \operatorname{Re} s < -\beta$, we have $g(u)u^{s-1} \in L^1(0,\infty)$, and the Mellin transform of g is given by

$$G(s) = \int_0^{\infty} g(u) u^{s-1} du.$$

For any $\alpha > \alpha_0 > \beta_0 > \beta$, when s is in the set $-\alpha_0 \leq \operatorname{Re} s \leq -\beta_0$, we have

$$g(u)u^{s-1} = \begin{cases} O(u^{\alpha-\alpha_0-1}) & \text{as } u \to 0 \\ O(u^{\beta-\beta_0-1}) & \text{as } u \to \infty \end{cases}$$

and $|g(u)u^{s-1}| \leq |g(u)|u^{-\alpha_0-1}$ on $(0,1)$ and $|g(u)u^{s-1}| \leq |g(u)|u^{-\beta_0-1}$ on $(1,\infty)$, independently of s. It follows from Morera's and Fubini's theorems that $G(s)$ is analytic in the strip $-\alpha < \operatorname{Re} s < -\beta$. Hence, if $\infty \geq \overline{\alpha} > \overline{\beta} \geq -\infty$ are the supremum and the infimum of α and β satisfying (11), respectively, then $G(s)$ is analytic in the region $-\overline{\alpha} < \operatorname{Re} s < -\overline{\beta}$.

Now, suppose $f \in \mathcal{O}$ is not constant. Since $f(e^{2\pi i u})$ cannot be a polynomial in u, the set of integers $n \geq 0$ such that $(D^n f)(1) \neq 0$ is infinite. Let $0 < n_1 < n_2 < \cdots$ be the sequence of all positive integers such that $(D^{n_j} f)(1) \neq 0$, and let $n_0 = 0$. Hence, $f(e^{2\pi i u})$ has the following Taylor expansion

$$f(e^{2\pi i u}) = \sum_{j\geq 0} \frac{(D^{n_j} f)(1)}{n_j!} u^{n_j} \tag{12}$$

at $u = 0$, which is valid in a neighborhood of 0. Here, $\frac{(D^{n_j}f)(1)}{n_j!} \neq 0$ for all $j > 0$, but there is no such requirement for $f(1)$ for $j = 0$ in (12).

Theorem 1. *Suppose $f(z) = \sum_{n\in\mathbb{Z}} a_n z^n \in \mathcal{O}$ is not constant, and let $H(f;s)$ be the H-series attached to f. Let $0 < n_1 < n_2 < \cdots$ be the sequence of all positive integers such that $(D^{n_j}f)(1) \neq 0$ for $j > 0$, and let $n_0 = 0$. Let $\ell \geq 0$ be an integer. Then, we have*

$$\frac{\Gamma(s)}{(2\pi)^s} H(f;s) = \int_0^\infty \Big(f(e^{2\pi i u}) - \sum_{j=0}^{\ell} \frac{(D^{n_j}f)(1)}{n_j!} u^{n_j} \Big) u^{s-1} du \tag{13}$$

in the strip $-n_{\ell+1} < \operatorname{Re} s < -n_\ell$.

Proof. Let $\ell \geq 0$ be given, and let m be an integer such that $n_\ell \leq m < n_{\ell+1}$. Let $L_+(f;s)$ and $L_-(f;s)$ be the entire functions defined by (10). Let us write

$$f_+(z) = \sum_{n=1}^{\infty} a_n z^n, \quad f_-(z) = \sum_{n=1}^{\infty} a_{-n} z^{-n}$$

so that $f(z) - a_0 = f_+(z) + f_-(z)$. Let $b_j = \frac{D^j f(1)}{j!}$ and $b_j^{\pm} = \frac{D^j f_{\pm}(1)}{j!}$ for $j \geq 0$, and let $\sigma = \operatorname{Re} s$. We have Taylor expansions

$$f_+(e^{-2\pi t}) = \sum_{j\geq 0} b_j^+ (it)^j,$$
$$f_-(e^{2\pi t}) = \sum_{j\geq 0} b_j^- (-it)^j$$

at $t = 0$. Since the partial sums of $f_+(e^{-2\pi t}) = \sum_{n=1}^{\infty} a_n e^{-2\pi n t}$ are dominated by $\big(\sum_{n=1}^{\infty} |a_n|\big) e^{-2\pi t}$ on $(0,\infty)$, we have

$$\frac{\Gamma(s)}{(2\pi)^s} L_+(f;s) = \int_0^\infty f_+(e^{-2\pi t}) t^{s-1} dt \tag{14}$$

for $\sigma > 0$, and it follows that

$$\frac{\Gamma(s)}{(2\pi)^s} L_+(f;s) = \int_0^\infty \Big(f_+(e^{-2\pi t}) - \sum_{j=0}^{m} b_j^+ (it)^j \Big) t^{s-1} dt \tag{15}$$

for $-(m+1) < \sigma < -m$. Indeed, the expression

$$\int_0^1 \Big(f_+(e^{-2\pi t}) - \sum_{j=0}^{m} b_j^+ (it)^j \Big) t^{s-1} dt + \sum_{j=0}^{m} \frac{b_j^+ i^j}{s+j} + \int_1^\infty f_+(e^{-2\pi t}) t^{s-1} dt,$$

which is meromorphic for $\sigma > -(m+1)$, reduces to (14) for $\sigma > 0$ and to (15) for $-(m+1) < \sigma < -m$.

Similarly, we have

$$\frac{\Gamma(s)}{(2\pi)^s} L_-(f;s) = \int_0^\infty \Big(f_-(e^{2\pi t}) - \sum_{j=0}^{m} b_j^- (-it)^j \Big) t^{s-1} dt$$

for $-(m+1) < \sigma < -m$, and thus,

$$\begin{aligned}\frac{\Gamma(s)}{(2\pi)^s} H(f;s) = e^{\frac{\pi i s}{2}} \int_0^\infty \Big(f_+(e^{-2\pi t}) - \sum_{j=0}^m b_j^+ (it)^j\Big) t^{s-1} dt \\ + e^{-\frac{\pi i s}{2}} \int_0^\infty \Big(f_-(e^{2\pi t}) - \sum_{j=0}^m b_j^- (-it)^j\Big) t^{s-1} dt\end{aligned} \tag{16}$$

for $-(m+1) < \sigma < -m$. The functions $f_+(e^{2\pi i u})$, $f_-(e^{2\pi i u})$ are analytic in u in some neighborhoods of the upper and the lower half-planes, respectively. Letting $u = it$ and $u = -it$ in the two integrals in (16), respectively, we have

$$\frac{\Gamma(s)}{(2\pi)^s} H(f;s) = \int_0^{i\infty} \Big(f_+(e^{2\pi i u}) - \sum_{j=0}^m b_j^+ u^j\Big) u^{s-1} du + \int_0^{-i\infty} \Big(f_-(e^{2\pi i u}) - \sum_{j=0}^m b_j^- u^j\Big) u^{s-1} du.$$

We see that the ray of integration in the first integral can be deformed to have any angle $0 \le \theta \le \pi$ when $-(m+1) < \sigma < -m$, since $(f_+(e^{2\pi i u}) - \sum_{j=0}^m b_j^+ u^j) u^{s-1} = O(|u|^{m+\sigma})$ as $u \to 0$ and $(f_+(e^{2\pi i u}) - \sum_{j=0}^m b_j^+ u^j) u^{s-1} = O(|u|^{m+\sigma-1})$ as $u \to \infty$ in the sector $0 \le \arg u \le \pi$. Hence, the contour in the first integral can be deformed to the positive real line. The same bounds hold for the second integrand in the lower half-plane, and the contour there can also be deformed to the positive real line. Hence,

$$\begin{aligned}&\frac{\Gamma(s)}{(2\pi)^s} H(f;s) \\ &\quad = \int_0^\infty \Big(f_+(e^{2\pi i u}) - \sum_{j=0}^m b_j^+ u^j\Big) u^{s-1} du + \int_0^\infty \Big(f_-(e^{2\pi i u}) - \sum_{j=0}^m b_j^- u^j\Big) u^{s-1} du \\ &\quad = \int_0^\infty \Big(f(e^{2\pi i u}) - \sum_{j=0}^m b_j u^j\Big) u^{s-1} du\end{aligned} \tag{17}$$

for $-(m+1) < \sigma < -m$. The expression (17) is in fact

$$\frac{\Gamma(s)}{(2\pi)^s} H(f;s) = \int_0^\infty \Big(f(e^{2\pi i u}) - \sum_{j=0}^\ell b_{n_j} u^{n_j}\Big) u^{s-1} du \tag{18}$$

due to the vanishing of b_j for $n_\ell < j < n_{\ell+1}$, as we have chosen m such that $n_\ell \le m < n_{\ell+1}$. The equality (18) is valid for $-(m+1) < \sigma < -m$, but for $u \in (0,\infty)$, $f(e^{2\pi i u}) - \sum_{j=0}^\ell b_{n_j} u^{n_j} = O(u^{n_{\ell+1}})$ as $u \to 0$, while it is $O(u^{n_\ell})$ as $u \to \infty$, so the right-hand side of (18) is analytic for $-n_{\ell+1} < \sigma < -n_\ell$. Hence, the left-hand side of (18) is also analytic in this region, and the equality holds for $-n_{\ell+1} < \sigma < -n_\ell$. □

4. Weierstrass Zeta Function and a Lambert Series

Let τ be in the upper half-plane $\mathfrak{H}$. Let us use the notations

$$\mathbb{Z} + \mathbb{Z}\tau = \{m + n\tau \mid m, n \in \mathbb{Z}\}$$

and $(\mathbb{Z} + \mathbb{Z}\tau)^\times = (\mathbb{Z} + \mathbb{Z}\tau) \backslash \{0\}$. For integers $k \ge 2$, the Eisenstein series of weight $2k$ is

$$G_{2k}(\tau) = \sum_{\omega \in (\mathbb{Z}+\mathbb{Z}\tau)^\times} \omega^{-2k} = \sum_{(m,n) \ne (0,0)} \frac{1}{(m+n\tau)^{2k}}. \tag{19}$$

The Eisenstein series of weight 2 is defined by summing over m first in (19) so that

$$G_2(\tau) = \sum_{m\neq 0} \frac{1}{m^2} + \sum_{n\neq 0} \Big(\sum_{m\in\mathbb{Z}} \frac{1}{(m+n\tau)^2} \Big). \tag{20}$$

The Weierstrass zeta function is given by

$$\zeta(u,\tau) = \frac{1}{u} + \sum_{(m,n)\neq(0,0)} \left(\frac{1}{u-(m+n\tau)} + \frac{1}{m+n\tau} + \frac{u}{(m+n\tau)^2} \right) \tag{21}$$

where $\tau \in \mathfrak{H}$ and $u \in \mathbb{C}\backslash(\mathbb{Z}+\mathbb{Z}\tau)$. Our normalization of the Eisenstein series is such that they appear as the Laurent coefficients of $\zeta(u,\tau)$ as

$$\zeta(u,\tau) = \frac{1}{u} - G_4(\tau)u^3 - G_6(\tau)u^5 - \cdots = \frac{1}{u} - \sum_{j\geq 2} G_{2j}(\tau)u^{2j-1} \tag{22}$$

which can be obtained by expanding (21).

Proposition 2. *Let $\tau \in \mathfrak{H}$. Let $\zeta(u,\tau)$ be the Weierstrass zeta function, and let $G_2(\tau)$ be the Eisenstein series of weight* 2. *For $|q| < |z| < |q|^{-1}$,*

$$\zeta(u,\tau) - G_2(\tau)u - \pi\cot(\pi u) = -2\pi i \sum_{k=1}^{\infty} \frac{q^k}{1-q^k}(z^k - z^{-k}) \in \mathcal{O} \tag{23}$$

where $q = e^{2\pi i\tau}$ and $z = e^{2\pi iu}$.

Proof. In the expression (21) for $\zeta(u,\tau)$, we can sum over m first by absolute convergence. Thus, from (20), we have

$$\zeta(u,\tau) - G_2(\tau)u = \frac{1}{u} + \sum_{m\neq 0}\left(\frac{1}{u-m} + \frac{1}{m}\right) + \sum_{n\neq 0}\sum_{m\in\mathbb{Z}}\left(\frac{1}{u-(m+n\tau)} + \frac{1}{m+n\tau}\right).$$

Subtracting $\pi\cot(\pi u) = \frac{1}{u} + \sum_{m\neq 0}\left(\frac{1}{u-m} + \frac{1}{m}\right)$,

$$\begin{aligned}
\zeta(u,\tau) - G_2(\tau)u - \pi\cot(\pi u) &= \sum_{n\neq 0}\sum_{m\in\mathbb{Z}}\left(\frac{1}{u-(m+n\tau)} + \frac{1}{m+n\tau}\right) \\
&= \sum_{n>0}\sum_{m\in\mathbb{Z}}\left(\frac{1}{u-(m+n\tau)} + \frac{1}{u+(m+n\tau)}\right) \\
&= \sum_{n>0}\Big(\pi\cot\big(\pi(u-n\tau)\big) + \pi\cot\big(\pi(u+n\tau)\big)\Big).
\end{aligned}$$

We see that $\zeta(u,\tau) - G_2(\tau)u - \pi\cot(\pi u)$ is periodic in u with period 1 and analytic on the horizontal strip $|\operatorname{Im} u| < \operatorname{Im}\tau$ containing the real line, and thus, it is analytic in $z = e^{2\pi iu}$ in the annulus $|q| < |z| < |q|^{-1}$ with $q = e^{2\pi i\tau}$. For each $n > 0$, we have expansions

$$\begin{aligned}
\pi\cot(\pi(u+n\tau)) &= -\pi i\frac{1+q^n z}{1-q^n z} = -\pi i\Big(1 + 2\sum_{k>0} q^{kn}z^k\Big) \quad \text{for } |z| < 1/|q|^n, \\
\pi\cot(\pi(u-n\tau)) &= -\pi i\frac{q^n/z+1}{q^n/z-1} = \pi i\Big(1 + 2\sum_{k>0} q^{kn}z^{-k}\Big) \quad \text{for } |z| > |q|^n.
\end{aligned}$$

Summing over $n > 0$ gives $\zeta(u,\tau) - G_2(\tau)u - \pi\cot(\pi u) = -2\pi i\sum_{k>0}\frac{q^k}{1-q^k}(z^k - z^{-k})$ for $|q| < |z| < |q|^{-1}$. □

Corollary 1. *For $\tau \in \mathfrak{H}$, let*

$$f(z) = -2\pi i \sum_{n=1}^{\infty} \frac{q^n}{1-q^n}(z^n - z^{-n}) \in \mathcal{O},$$

where $q = e^{2\pi i \tau}$. Then, the H-series attached to f is

$$H(f;s) = 4\pi \sin\left(\frac{\pi s}{2}\right) A(\tau, 1-s),$$

where

$$A(\tau, s) = \sum_{n=1}^{\infty} n^{s-1} \frac{q^n}{1-q^n}$$

is the Lambert series given in (5), and

$$\begin{aligned} &\frac{\Gamma(s)}{(2\pi)^s} 4\pi \sin\left(\frac{\pi s}{2}\right) A(\tau, 1-s) \\ &\quad = \begin{cases} \int_0^\infty \big(\zeta(u,\tau) - G_2(\tau)u - \pi\cot(\pi u)\big)u^{s-1}du & \text{if } -1 < \operatorname{Re} s < 0, \\ \int_0^\infty \big(\zeta(u,\tau) - \pi\cot(\pi u) - 2\zeta(2)u\big)u^{s-1}du & \text{if } -3 < \operatorname{Re} s < -1, \\ \int_0^\infty \big(\zeta(u,\tau) - \pi\cot(\pi u) - 2\zeta(2)u & \\ \qquad -(2\zeta(4) - G_4(\tau))u^3\big)u^{s-1}du & \text{if } -5 < \operatorname{Re} s < -3, \\ \quad \vdots & \quad \vdots \end{cases} \end{aligned} \tag{24}$$

where $\zeta(s)$ is the Riemann zeta function, and in general, for $\ell \geq 1$,

$$\begin{aligned} \frac{\Gamma(s)}{(2\pi)^s} 4\pi \sin\left(\frac{\pi s}{2}\right) A(\tau, 1-s) = \int_0^\infty \Big(&\zeta(u,\tau) - G_2(\tau)u - \pi\cot(\pi u) \\ &- \sum_{j=1}^{\ell} (2\zeta(2j) - G_{2j}(\tau))u^{2j-1}\Big)u^{s-1}du \end{aligned} \tag{25}$$

for $-(2\ell+1) < \operatorname{Re} s < -(2\ell-1)$.

Proof. Since $f(z) = -2\pi i \sum_{n>0} \frac{q^n}{1-q^n}(z^n - z^{-n})$, we have

$$H(f;s) = -2\pi i (e^{\frac{\pi i s}{2}} - e^{-\frac{\pi i s}{2}}) \sum_{n>0} n^{-s} \frac{q^n}{1-q^n} = 4\pi \sin\left(\frac{\pi s}{2}\right) A(\tau, 1-s).$$

From the Laurent expansions

$$\zeta(u,\tau) = \frac{1}{u} - G_4(\tau)u^3 - G_6(\tau)u^5 - \cdots$$

and

$$\pi\cot(\pi u) = \frac{1}{u} - 2\zeta(2)u - 2\zeta(4)u^3 - 2\zeta(6)u^5 - \cdots,$$

we have the Taylor expansion

$$\begin{aligned} &\zeta(u,\tau) - G_2(\tau)u - \pi\cot(\pi u) \\ &\quad = (2\zeta(2) - G_2(\tau))u + (2\zeta(4) - G_4(\tau))u^3 + (2\zeta(6) - G_6(\tau))u^5 + \cdots \\ &\quad = \sum_{j\geq 1} (2\zeta(2j) - G_{2j}(\tau))u^{2j-1} \end{aligned} \tag{26}$$

and (24) and (25) follow from Proposition 2 and Theorem 1. □

Corollary 2. *For $\tau \in \mathfrak{H}$, let*

$$f(z) = -2\pi i \sum_{n=1}^{\infty} \frac{q^n}{1-q^n}(z^n - z^{-n}) \in \mathcal{O},$$

where $q = e^{2\pi i\tau}$. For $\varphi(D) = c_0 + c_1 D + \cdots + c_k D^k \in \mathbb{C}[D]$, we have $\varphi(D)f(z) \in \mathcal{O}$, and

$$H(\varphi(D)f; s) = \sum_{j=0}^{k} c_j(-2\pi)^j 4\pi \sin\Big(\frac{\pi(s-j)}{2}\Big) A(\tau, 1+j-s), \tag{27}$$

and

$$(\varphi(D)f)(e^{2\pi i u}) = \varphi(\partial_u)(\zeta(u,\tau) - G_2(\tau)u - \pi\cot(\pi u)).$$

Proof. This follows from Corollary 1 and Proposition 1. □

For the cases $\varphi(D) = -D$ and $\varphi(D) = -D^2$ in Corollary 2, we obtain, since

$$\partial_u \zeta(u,\tau) = -\wp(u,\tau)$$

where $\wp(u,\tau)$ is the Weierstrass $\wp$-function,

$$-Df(z) = -4\pi^2 \sum_{n=1}^{\infty} \frac{nq^n(z^n + z^{-n})}{1-q^n} = \wp(u,\tau) + G_2(\tau) - \pi^2\csc^2(\pi u),$$
$$-D^2 f(z) = -8\pi^3 i \sum_{n=1}^{\infty} \frac{n^2 q^n(z^n - z^{-n})}{1-q^n} = \wp'(u,\tau) + 2\pi^3\cot(\pi u)\csc^2(\pi u)$$

and

$$H(-Df; s) = -8\pi^2 \cos\big(\tfrac{\pi s}{2}\big) A(\tau, 2-s),$$
$$H(-D^2 f; s) = 16\pi^3 \sin\big(\tfrac{\pi s}{2}\big) A(\tau, 3-s).$$

We have Taylor expansions, by (26),

$$\begin{aligned}\wp(u,\tau) + G_2(\tau) - \pi^2\csc^2(\pi u) &= -(2\zeta(2) - G_2(\tau)) - (6\zeta(4) - 3G_4(\tau))u^2 \\ &\quad - (10\zeta(6) - 5G_6(\tau))u^4 - \cdots \\ \wp'(u,\tau) + 2\pi^3\cot(\pi u)\csc^2(\pi u) &= -(12\zeta(4) - 6G_4(\tau))u - (40\zeta(6) - 20G_6(\tau))u^3 \\ &\quad - (84\zeta(8) - 42G_8(\tau))u^5 - \cdots.\end{aligned}$$

Hence, we also obtain the identities

$$-\frac{\Gamma(s)}{(2\pi)^s} 8\pi^2 \cos\big(\tfrac{\pi s}{2}\big) A(\tau, 2-s)$$
$$= \begin{cases} \int_0^\infty \big(\wp(u,\tau) - \pi^2\csc^2(\pi u) + 2\zeta(2)\big)u^{s-1}du & \text{if } -2 < \operatorname{Re} s < 0, \\ \int_0^\infty \big(\wp(u,\tau) - \pi^2\csc^2(\pi u) + 2\zeta(2) & \\ \qquad + (6\zeta(4) - 3G_4(\tau))u^2\big)u^{s-1}du & \text{if } -4 < \operatorname{Re} s < -2, \\ \int_0^\infty \Big(\wp(u,\tau) - \pi^2\csc^2(\pi u) + 2\zeta(2) & \\ + (6\zeta(4) - 3G_4(\tau))u^2 + (10\zeta(6) - 5G_6(\tau))u^4\Big)u^{s-1}du & \text{if } -6 < \operatorname{Re} s < -4, \\ \quad \vdots & \quad \vdots \end{cases}$$

and

$$\frac{\Gamma(s)}{(2\pi)^s}16\pi^3\sin\left(\frac{\pi s}{2}\right)A(\tau,3-s)$$
$$=\begin{cases}\int_0^\infty \left(\wp'(u,\tau)+2\pi^3\cot(\pi u)\csc^2(\pi u)\right)u^{s-1}du & \text{if } -1<\operatorname{Re}s<0,\\ \int_0^\infty \left(\wp'(u,\tau)+2\pi^3\cot(\pi u)\csc^2(\pi u)\right. & \\ \qquad\left.+(12\zeta(4)-6G_4(\tau))u\right)u^{s-1}du & \text{if } -3<\operatorname{Re}s<-1,\\ \int_0^\infty \Big(\wp'(u,\tau)+2\pi^3\cot(\pi u)\csc^2(\pi u) & \\ \quad+(12\zeta(4)-6G_4(\tau))u+(40\zeta(6)-20G_6(\tau))u^3\Big)u^{s-1}du & \text{if } -5<\operatorname{Re}s<-3,\\ \quad\vdots & \quad\vdots\quad .\end{cases}$$

5. Some Properties of the Lambert Series

We now derive some implications of the above identities. For example, from the second equation in (24), we have

$$A(\tau,1-s)=\frac{(2\pi)^{s-1}\csc\left(\frac{\pi s}{2}\right)}{2\,\Gamma(s)}\int_0^\infty\left(\zeta(u,\tau)-\pi\cot(\pi u)-\frac{\pi^2u}{3}\right)u^{s-1}du \tag{28}$$

for $-3<\operatorname{Re}s<-1$. Replacing s with $1-s$,

$$A(\tau,s)=\frac{\csc(\pi s)\sin\left(\frac{\pi s}{2}\right)}{(2\pi)^s\Gamma(1-s)}\int_0^\infty\left(\zeta(u,\tau)-\pi\cot(\pi u)-\frac{\pi^2u}{3}\right)u^{-s}du \tag{29}$$

for $2<\operatorname{Re}s<4$. Using the function equation of the Riemann zeta function

$$\zeta(s)=2(2\pi)^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s),$$

we can rewrite the front factor in (29) as

$$A(\tau,s)=\frac{\tan\left(\frac{\pi s}{2}\right)\zeta(1-s)}{2\pi\,\zeta(s)}\int_0^\infty\left(\zeta(u,\tau)-\pi\cot(\pi u)-\frac{\pi^2u}{3}\right)u^{-s}du \tag{30}$$

for $2<\operatorname{Re}s<4$. The following relation of $A(\tau,s)$ given in [7] can be derived from this Mellin integral representation.

Corollary 3. *For* $\operatorname{Re}s>2$, *we have*

$$A(\tau,s)=e^{\frac{\pi i s}{2}}\frac{\Gamma(s)}{(2\pi)^s}\sum_{\omega\in\Omega_+}\omega^{-s} \tag{31}$$

where $\Omega_+=\{\omega\in\mathbb{Z}+\mathbb{Z}\tau\mid\operatorname{Im}\omega>0\}$.

Proof. Consider the integral

$$I(\tau,s)=\int_0^\infty\left(\zeta(u,\tau)-\pi\cot(\pi u)-\frac{\pi^2u}{3}\right)u^{-s}du \tag{32}$$

for $2<\operatorname{Re}s<4$. The integrand is $O(|u|^{3-\operatorname{Re}s})$ as $u\to0$, and away from the poles, it is $O(|u|^{1-\operatorname{Re}s})$ for large $|u|$. Hence, (32) can be evaluated using an appropriate branch of u^{-s} and a keyhole contour. Equivalently, by moving the ray of integration in (32) by an angle of

2π along the Riemann surface of u^{-s}, the integral changes to $e^{-2\pi is}I(\tau,s)$ while picking up the residues of the poles in between. Hence,

$$I(\tau,s) = e^{-2\pi is}I(\tau,s) + 2\pi i(1+e^{-\pi is})\sum_{\omega\in\Omega_+}\omega^{-s},$$

from which it follows that

$$I(\tau,s) = \pi\Big(\cot\Big(\frac{\pi s}{2}\Big)+i\Big)\sum_{\omega\in\Omega_+}\omega^{-s}. \tag{33}$$

From (30), we have

$$A(\tau,s) = \frac{1}{2}\Big(1+i\tan\Big(\frac{\pi s}{2}\Big)\Big)\frac{\zeta(1-s)}{\zeta(s)}\sum_{\omega\in\Omega_+}\omega^{-s} \tag{34}$$

and since the right-hand side is analytic for $\operatorname{Re} s > 2$, (34) holds for $\operatorname{Re} s > 2$. From the functional equation, one can show

$$\frac{1}{2}\Big(1+i\tan\Big(\frac{\pi s}{2}\Big)\Big)\zeta(1-s) = e^{\frac{\pi is}{2}}\frac{\Gamma(s)}{(2\pi)^s}\zeta(s),$$

and (34) can be written as (31). □

The Equations (34) and (31) can be written as

$$A(\tau,s) = \frac{1}{2}\Big(1+i\tan\Big(\frac{\pi s}{2}\Big)\Big)\zeta(1-s)\sum_{\omega\in\overline{\Omega}_+}\omega^{-s} = e^{\frac{\pi is}{2}}\frac{\Gamma(s)}{(2\pi)^s}\zeta(s)\sum_{\omega\in\overline{\Omega}_+}\omega^{-s}$$

where $\overline{\Omega}_+ = \{m+n\tau\in\Omega_+ \mid \gcd(m,n)=1\}$.

For $\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)\in \mathrm{SL}(2,\mathbb{Z})$, $\tau\in\mathfrak{H}$, and $s\in\mathbb{C}$, transformation formulas relating $A\big(\frac{a\tau+b}{c\tau+d},s\big)$ and $A(\tau,s)$ are given in [9–11]. Since $A(\tau,s)$ is clearly invariant under $\tau\mapsto\tau+b$ for $b\in\mathbb{Z}$, we only need to consider $\tau\mapsto\frac{a\tau+b}{c\tau+d}$ with $c>0$. The Mellin integral representation (30) gives a convenient way of deriving such a formula for $\operatorname{Re} s>2$, from which a formula for $s\in\mathbb{C}$ may be found by analytic continuation.

Corollary 4. *Let* $\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)\in\mathrm{SL}(2,\mathbb{Z})$ *such that* $c>0$. *For* $\operatorname{Re} s>2$, *we have*

$$\begin{aligned}(c\tau+d)^{-s}A\Big(\frac{a\tau+b}{c\tau+d},s\Big) = A(\tau,s) &+ \frac{1}{2}\big(1-(c\tau+d)^{-s}\big)\zeta(1-s)\\ &- i\tan\Big(\frac{\pi s}{2}\Big)\frac{\zeta(1-s)}{\zeta(s)}{\sum_{\omega\in\Omega}}'\omega^{-s}\end{aligned} \tag{35}$$

where $\Omega=\{\omega\in(\mathbb{Z}+\mathbb{Z}\tau)^\times \mid 0\le\arg\omega\le\arg(c\tau+d)\}$, *and the summation* $\sum'$ *means that we add* $\frac{1}{2}\omega^{-s}$ *instead of* ω^{-s} *if* $\arg\omega=0$ *or* $\arg\omega=\arg(c\tau+d)$.

Proof. The right-hand side of (35) is analytic in s for $\operatorname{Re} s>2$, and it suffices to show the equality for $2<\operatorname{Re} s<4$. In this region, from (30), we have

$$(c\tau+d)^{-s}A\Big(\frac{a\tau+b}{c\tau+d},s\Big) = \frac{\tan\big(\frac{\pi s}{2}\big)\zeta(1-s)}{2\pi\,\zeta(s)}(c\tau+d)^{-s}I\Big(\frac{a\tau+b}{c\tau+d},s\Big) \tag{36}$$

where

$$I\Big(\frac{a\tau+b}{c\tau+d},s\Big) = \int_0^\infty\Big(\zeta\Big(u,\frac{a\tau+b}{c\tau+d}\Big)-\pi\cot(\pi u)-\frac{\pi^2u}{3}\Big)u^{-s}du.$$

From (21) or (22), we have $\zeta(u, \frac{a\tau+b}{c\tau+d}) = (c\tau+d)\zeta((c\tau+d)u, \tau)$. By letting $t = (c\tau+d)u$,

$$\begin{aligned}(c\tau+d)^{-s} I\left(\frac{a\tau+b}{c\tau+d}, s\right) &\\ = (c\tau+d)^{-s}\int_0^\infty \left((c\tau+d)\zeta((c\tau+d)u,\tau) - \pi\cot(\pi u) - \frac{\pi^2 u}{3}\right)u^{-s}du &\\ = \int_0^{(c\tau+d)\infty}\left(\zeta(t,\tau) - \frac{\pi\cot(\frac{\pi t}{c\tau+d})}{c\tau+d} - \frac{\pi^2}{3}\frac{t}{(c\tau+d)^2}\right)t^{-s}dt &\end{aligned}$$

where the ray of integration in t has the angle $0 < \arg(c\tau+d) < \pi$. By the bounds of the integrand discussed previously, we can move the ray of integration clockwise to a contour C from 0 to ∞ traversing slightly above the positive real axis, picking up the residues of the poles in between, so that

$$\begin{aligned}(c\tau+d)^{-s} I\left(\frac{a\tau+b}{c\tau+d}, s\right) = &-2\pi i \sum_{\omega\in\Omega^\circ}\omega^{-s} \\ &+ \int_C \left(\zeta(t,\tau) - \frac{\pi\cot(\frac{\pi t}{c\tau+d})}{c\tau+d} - \frac{\pi^2}{3}\frac{t}{(c\tau+d)^2}\right)t^{-s}dt\end{aligned} \tag{37}$$

where $\Omega^\circ = \{\omega \in (\mathbb{Z}+\mathbb{Z}\tau)^\times \mid 0 < \arg\omega < \arg(c\tau+d)\}$. Now, we write $\zeta(t,\tau) = (\zeta(t,\tau) - \pi\cot(\pi t) - \frac{\pi^2 t}{3}) + \pi\cot(\pi t) + \frac{\pi^2 t}{3}$ in (37), and since $\int_C(\zeta(t,\tau) - \pi\cot(\pi t) - \frac{\pi^2 t}{3})t^{-s}dt = I(\tau, s)$,

$$(c\tau+d)^{-s} I\left(\frac{a\tau+b}{c\tau+d}, s\right) = -2\pi i \sum_{\omega\in\Omega^\circ}\omega^{-s} + I(\tau,s) + J(\tau,s) \tag{38}$$

where

$$J(\tau,s) = \int_C \left(\pi\cot(\pi t) + \frac{\pi^2 t}{3} - \frac{\pi\cot(\frac{\pi t}{c\tau+d})}{c\tau+d} - \frac{\pi^2}{3}\frac{t}{(c\tau+d)^2}\right)t^{-s}dt. \tag{39}$$

The integrand of (39) has poles at $t = m$ and $t = (c\tau+d)m$ for integers $m \neq 0$, and by the same argument as before,

$$J(\tau,s) = e^{-2\pi i s}J(\tau,s) + 2\pi i\left((e^{-\pi i s} + e^{-2\pi i s})\sum_{m>0} m^{-s} - (1+e^{-\pi i s})\sum_{m>0}((c\tau+d)m)^{-s}\right)$$

and we obtain

$$\begin{aligned}J(\tau,s) &= \pi\left(\cot\left(\frac{\pi s}{2}\right) - i\right)\zeta(s) - \pi\left(\cot\left(\frac{\pi s}{2}\right) + i\right)(c\tau+d)^{-s}\zeta(s) \\ &= \pi\cot\left(\frac{\pi s}{2}\right)\left(1 - (c\tau+d)^{-s}\right)\zeta(s) - \pi i\left(1 + (c\tau+d)^{-s}\right)\zeta(s),\end{aligned} \tag{40}$$

and combining (38) and (40) with (36), we have

$$\begin{aligned}(c\tau+d)^{-s} A\left(\frac{a\tau+b}{c\tau+d}, s\right) = A(\tau,s) &+ \frac{1}{2}\left(1 - (c\tau+d)^{-s}\right)\zeta(1-s) \\ &- \frac{i}{2}\tan\left(\frac{\pi s}{2}\right)\zeta(1-s)\left(1 + (c\tau+d)^{-s}\right) \\ &- i\tan\left(\frac{\pi s}{2}\right)\frac{\zeta(1-s)}{\zeta(s)}\sum_{\omega\in\Omega^\circ}\omega^{-s}\end{aligned}$$

which equals (35). □

6. Conclusions

We have applied the Mellin integral formula (13) to the Laurent series (23) and obtained the identities (24) between the Weierstrass zeta function $\zeta(u,\tau)$ and the Lambert series $A(\tau,s)$. Corollaries 3 and 4 follow from the Mellin integral representation (28) of $A(\tau,s)$. Taking the branch of the log so that $-\pi \le \arg z < \pi$, the relation (31) in Corollary 3 can be equivalently written as

$$\sum_{(m,n)\neq(0,0)} (m+n\tau)^{-s} = (1+e^{\pi i s})\Big(\zeta(s) + e^{-\frac{\pi i s}{2}}\frac{(2\pi)^s}{\Gamma(s)}A(\tau,s)\Big), \quad \operatorname{Re} s > 2. \tag{41}$$

When $s = 2k$ for an integer $k \ge 2$, this formula reduces to

$$G_{2k}(\tau) = 2\zeta(2k) + \frac{2(2\pi i)^{2k}}{(2k-1)!}\sum_{n=1}^{\infty}\sigma_{2k-1}(n)q^n, \tag{42}$$

which is the usual Fourier expansion of the classical Eisenstein series. Since the right-hand side of (41) is entire in s, it gives an analytic continuation of the Eisenstein series to any complex weight s. In other words, the Mellin integral representation (28) provides a different proof of these results. On the other hand, one can show (31) or (41) by finding the Fourier expansion of the sum using the Poisson summation formula, as was done in [7]. Hence, we can use the identity (33) to give another proof of the integral representation (28), while we have derived it as a special case of Theorem 1.

Computation of the Mellin transforms of special functions is itself an interesting topic, and the formulas (27) give the Mellin transforms of the Weierstrass elliptic functions in the following sense. Although the straightforward integrals

$$\int_0^\infty \zeta(u,\tau)u^{s-1}du, \quad \int_0^\infty \wp(u,\tau)u^{s-1}du, \quad \int_0^\infty \wp'(u,\tau)u^{s-1}du, \quad \cdots$$

are not defined partly due to the poles of the integrand along the real line, considering the "regularized" functions

$$\overline{\zeta^{(k)}}(u,\tau) = \Big(\frac{\partial}{\partial u}\Big)^k\big(\zeta(u,\tau) - G_2(\tau)u - \pi\cot(\pi u)\big), \quad k \ge 0,$$

we have shown that

$$\int_0^\infty \Big(\overline{\zeta^{(k)}}(u,\tau) - \overline{\zeta^{(k)}}(0,\tau)\Big)u^{s-1}du = \frac{\Gamma(s)}{(2\pi)^s}(-2\pi)^k 4\pi\sin\Big(\frac{\pi(s-k)}{2}\Big)A(\tau,1+k-s)$$

for $-1 < \operatorname{Re} s < 0$. We expect the algebraic relations among the Weierstrass elliptic functions to be reflected in the corresponding Mellin transforms as some convolution relations. Finding the detailed implications of this correspondence is left for further study.

Funding: This work was supported by the Hongik University Research Fund.

Data Availability Statement: The original contributions presented in this study are included in the article. Further inquiries can be directed to the author.

Conflicts of Interest: The author declares no conflicts of interest.

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