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Metric Locations in Pseudotrees: A Survey and New Results

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Abstract: This paper presents a comprehensive review of the literature on the original concept of metric location, along with its various adaptations and extensions that have been developed over time. Given that determining a minimum location set is generally NP-hard, we focus on analyzing the behavior of these sets within specific graph families, including paths, cycles, trees and unicyclic graphs. In addition to synthesizing existing knowledge, we contribute new findings and insights to the field, advancing the understanding of metric location problems in these structured graph classes.

Keywords: metric dimension; metric locating sets; distance in graphs; unicyclic graphs

MSC: 05C05; 05C12; 05C35; 05C75

1. Introduction

Location problems consist of determining a reference set in a graph such that every vertex is unequivocally associated to a set of "coordinates" that localize it. Since the vertex set of a graph is enough for this task, the question is not to prove its existence, but to find the minimum one.

Metric locations in graphs officially came into being with the publication of two seminal papers by Slater [1] and Harary and Melter [2]. However, the field can be traced back to a paper by Silverman [3] about the problem of detecting a certain subset of a given set by comparing it with some other subsets. The approach to solve that problem was to define a distance between the different subsets of a set and to carefully choose the comparisons needed. To the best of our knowledge, in this work the concept of a metric basis is named for the first time.

The adjective "metric" is used through this paper to refer those locations in which the usual distance between graph vertices plays a crucial role, in the spirit of the original definition given in [1,2]. By contrast, there exist other types of locations in which the role of the graph distance is substituted by the relation of adjacency of vertices, given rise to "neighbor" locations, which are not treated here.

Since the publication of the works by Slater [1] and Harary and Melter [2], metric location theory has evolved at great velocity giving rise to hundreds of different variations, developing many new methods and deepening into several directions, although the first of those variations had to wait 20 years to appear in a paper published in 2003 by Brigham et al. [4] who imposed the condition of being dominating to the location sets. Almost immediately, in 2004, Sebő and Tannier [5] published their definition of strong metric dimension by using a certain notion of betweenness in geodetic paths. Just a year

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later, Cáceres et al. [6] defined what is currently known as the doubly metric dimension in which the difference between the vertices coordinates is not a constant number. Hernando et al. [7] asked themselves in 2008 what if an element of the metric location set "fails" or gives incorrect lectures, and gave rise to the fault-tolerant metric dimension. In 2010, Okamoto et al. showed in [8] a local metric dimension to distinguish only adjacent vertices instead of using a coloring. In 2015, Estrada-Moreno et al., in [9], generalized the fault-tolerant dimension into the k-metric dimension by considering the case in which more than one element of the location set might fail. In 2017, Kelenc et al. [10] distinguished vertices and edges with the mixed metric dimension, and a year later, Kelenc et al. [11] introduced the edge metric dimension for only locating edges.

One of the crucial facts of the field is that computing a minimum metric location set is typically NP-hard. Therefore, it becomes essential to identify graph families for which this computation can be accomplished efficiently in polynomial time. Among these families, trees are often considered, and various algorithms have been developed specifically for this purpose.

Since trees do not contain cycles, it is natural to explore the graph family characterized by a single cycle: unicyclic graphs. There exists an extensive volume of research papers that makes it exceedingly difficult to follow the advancements in this particular area. Consequently, the authors recognized the necessity of conducting a comprehensive survey on the topic of metric location within this graph family. Although there are many location parameters, including variations of variations, we opted to choose nine of them having in common that the corresponding dimensions for paths, cycles and trees are well known (see Table 1). However, a complete characterization of locations in unicyclic graphs has not been found in all cases and we were able to provide some new results in four of the dimensions studied.

Table 1. Metric locating parameters of paths, cycles and trees of order $n \geq 4$.

G	**dmd**	**dim**	**sdim**	**ddim**	$\mathbf{dim_2}$	$\mathbf{dim_k}$	**edim**	**mdim**	**ldim**
P_n	2	1	1	$\lceil \frac{n}{3} \rceil$	2	$k+1$	1	2	1
C_n	2,3	2	$\lceil \frac{n}{2} \rceil$	$\lceil \frac{n}{3} \rceil$	3	$k+1$, $k+2$	2	3	1,2
T_n	ℓ	$\ell - \lambda$	$\ell - 1$	$\gamma + \ell - s$	ℓ_s	[12]	$\ell - \lambda$	ℓ	1

This paper is organized as follows: after concluding this section with a description of the notation that will be followed in the rest of the work, Section 3 is dedicated to doubly metric locating sets where the work has been completed. This metric location parameter is chosen to be first because of the application it has for the rest of the metric locations. Section 4 is devoted to the original and first metric location that gave rise to the entire field. Here, we provide significant new results for the strong metric dimension, which is studied in Section 5. We finish with the dominating metric dimension in Section 6 where again the goal has been achieved.

In two of the remaining five parameters (mixed and local metric dimensions) other authors have proved bounds and exact values (see Table 2); however, for the edge, fault-tolerant and k-metric dimensions (see Table 3) there exist partial results but not a complete characterization of those dimensions for unicyclic graphs. In Section 7, we offer a glimpse of the actual landscape for those cases and this paper is finished with a section of conclusions and further work.

Table 2. Metric locating parameters of proper unicyclic graphs (no open problems).

Bounds	dmd	ddim	mdim	ldim
Lower	ℓ	$\gamma + \ell - s$	$\ell + \hat{t}$	1
Upper	$\ell + 2$	$\gamma + \ell - s + 1$	$\ell + \hat{t} + 1$	2
Exact values	Theorem 3	Theorem 13	Theorem 23	Theorem 25

Table 3. Metric locating parameters of proper unicyclic graphs (with open problems).

Bounds	dim	sdim	$\dim_2$	$\dim_k$	edim
Lower	$\ell - \lambda + \hat{\rho}$	$\mu(g, \ell)$	–	–	$\ell - \lambda + \hat{\rho}$
Upper	$\ell - \lambda + \hat{\rho} + 1$	$\ell + \lfloor \frac{c_2}{2} \rfloor$	–	–	$\ell - \lambda + \hat{\rho} + 1$
Exact values	Cor. 1, 2 and [13]	Prop. 3, 4 and Th. 11	–	–	[13]

2. Preliminaries

All the graphs considered are undirected, simple, finite and (unless otherwise stated) connected. Unless otherwise specified, $G = (V, E)$ stands for a graph of order n and size $m = 1$, where $|V| = n$, $V = [n] = \{1, \ldots, n\}$ and $|E| = m$. Let v be a vertex of G. The *open neighborhood* of v is $N_G(v) = \{w \in V(G) : vw \in E\}$ and the *closed neighborhood* of v is $N_G[v] = N_G(v) \cup \{v\}$ (we will write $N(v)$ and $N[v]$ if the graph G is clear from the context). The *degree* of v is $\deg(v) = |N(v)|$. The minimum degree (resp.,maximum degree) of G is $\delta(G) = \min\{\deg(u) : u \in V(G)\}$ (resp.,$\Delta(G) = \max\{\deg(u) : u \in V(G)\}$).

For any two vertices $u, v \in V(G)$ of a connected graph G, a $u - v$ *geodesic* is a $u - v$ shortest path, i.e., a $u - v$ path of minimum order. The length of a $u - v$ geodesic is called the *distance* $d_G(u, v)$ between u and v, or simply $d(u, v)$, when the graph G is clear from the context. The diameter of G is $\operatorname{diam}(G) = \max\{d(v, w) : v, w \in V(G)\}$.

Let $S = \{w_1, w_2, \ldots, w_k\}$ be a subset of vertices of a graph G. The distance between a vertex $v \in V(G)$ and S, denoted by $d(v, S)$, is the minimum of the distances between v and the vertices of S, that is, $d(v, S) = \min\{d(v, w) : w \in S\}$. Given a vertex $v \in V(G)$ and an edge $e = xy \in E(G)$, the distance between v and e is $d_G(v, e) = \min\{d_G(v, x), d_G(v, y)\}$. The *metric representation* $r(v|S)$ of a vertex v with respect to S is defined as the k-vector $r(v|S) = (d_G(v, w_1), d_G(v, w_2), \ldots, d_G(v, w_k))$.

Let $u, v \in V(G)$ be a pair of vertices such that $d(u, w) = d(v, w)$ for all $w \in V(G) \setminus \{u, v\}$, i.e., such that either $N(u) = N(v)$ or $N[u] = N[v]$. In both cases, u and v are said to be *twins*. Let W be a set of vertices of G. If the vertices of W are pairwise twins, then W is called a *twin set* of G.

Let $W \subseteq V(G)$ be a subset of vertices of G. The *closed neighborhood* of W is $N[W] = \cup_{v \in W} N[v]$. The subgraph of G induced by W, denoted by $G[W]$, has W as vertex set and $E(G[W]) = \{vw \in E(G) : v \in W, w \in W\}$.

For a graph G of minimum degree $\delta(G) = 1$, end-vertices are called *leaves*. The set and the number of leaves of G are denoted by $\mathcal{L}(G)$ and $\ell(G)$, respectively. A *support vertex* (resp.,*strong support vertex*) is a vertex adjacent to a leaf (resp., to at least two leaves). A *major vertex* of G is any vertex of degree at least 3. A *terminal vertex* of a major vertex v of G is a leaf u such that $d(u, v) < d(u, w)$ for every other major vertex w of G. An *exterior major vertex* (resp., *strong exterior major vertex*) is a major vertex with at least one terminal vertex (resp., at least two terminal vertices) (see Figure 1).

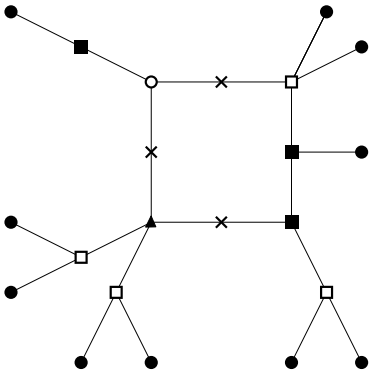


Figure 1. (White) square vertices: (Strong) support vertices. The white circle is an exterior major vertex but it is neither strong nor a support vertex. The triangle vertex is a major vertex but it is neither an exterior major vertex nor a support vertex. Check that $g = 8$, $\ell = 10$, $\lambda = 6$, $\ell_s = 8$, $\lambda_s = 4$, $c_2 = 3$, $c_3 = 5$ and $\rho = 3$.

The number of exterior major vertices (resp., strong exterior major vertices) of G is denoted by $\lambda(G)$ (resp., $\lambda_s(G)$). A leaf of G is called *strong* if its exterior major vertex is strong. The number of strong leaves of G is denoted by $\ell_s(G)$. Clearly, $\ell_s(G) - \lambda_s(G) = \ell(G) - \lambda(G)$.

A graph G of order n and size m is called *unicyclic* if $n = m$. From now on, and unless stated otherwise, if g is the girth of G, then its unique cycle will be denoted by C_g. A unicyclic graph G of order n and girth g is said to be *proper* if $n > g$, i.e., if it contains at least a leaf. If g is odd (resp., even), then G is said to be an *odd (resp., even) unicyclic graph*. The connected component of $G - E(C_g)$ containing a vertex $v \in V(C_g)$ is denoted by T_v and is called the *branching tree* of v. The tree T_v is said to be *trivial* if $V(T_v) = \{v\}$. A branching tree is called a *thread* if both T_v is a path and $\deg(v) = 3$. A vertex $v \in V(G)$ is a *branching vertex* if either $v \notin V(C_g)$ and $\deg_G(v) \geq 3$ or $v \in V(C_g)$ and $\deg_G(v) \geq 4$. A vertex $v \in V(C_g)$ is *branch-active* if T_v contains a branching vertex. The number of branch-active vertices of C_g is denoted by $\rho(G)$.

A triple of vertices $u, v, w \in V(C_g)$ is called a *geodesic triple* of G if $d(u,v) + d(v,w) + d(w,u) = g$. A vertex v from C_g is called a *root vertex* on C_g if T_v is non-trivial and otherwise, v is said to be a *trivial vertex* of C_g. The set and number of all root vertices of C_g is denoted by $C_3(C_g)$ and $c_3(C_g)$, while the set and the number of all trivial vertices of C_g are denoted by $C_2(C_g)$ and $c_2(C_g)$, respectively. Certainly, every vertex of C_g of degree at least 3 is a root vertex, while all vertices of C_g that are of degree 2 are trivial.

A set U of vertices of a graph G is called *independent* if no two vertices in U are adjacent. The *independence number* of G, denoted by $\alpha(G)$, is the cardinality of a maximum independent set of G.

A vertex v of a graph G is said to be a *boundary vertex* of a vertex u if no neighbor of v is further away from u than v, i.e., if for every vertex $w \in N(v)$, $d(u,w) \leq d(u,v)$. The set of boundary vertices of a vertex u is denoted by $\partial(u)$. The *boundary* of G, denoted by $\partial(G)$, is the set of all of its boundary vertices, i.e., $\partial(G) = \cup_{u \in V(G)} \partial(u)$. Given a pair of vertices $u, v \in V(G)$, if $v \in \partial(u)$, then v is also said to be *maximally distant* from u. Moreover, a pair of vertices $u, v \in V(G)$ are said to be *mutually maximally distant*, or simply *MMD*, if both $v \in \partial(u)$ and $u \in \partial(v)$. Notice that, as was pointed out in [14], the boundary of G can also be defined as the set of *MMD* vertices of G, i.e., $\partial(G) = \{v \in V(G) : \text{there exists } u \in V(G) \text{ such that } u, v \text{ are } MMD\}$. A pair of *MMD* vertices of a cycle are also called *antipodal*.

For additional details and information on basic graph theory, we refer the reader to [15].

3. Doubly Metric Dimension

This parameter was formally introduced by Cáceres et al. in [6] as a means to study the metric dimension in grids, cylinders and tori.

In this paper, it will also be a powerful tool to study unicyclic graphs, so we commence with this section.

Let $G = (V, E)$ be a graph. A pair of vertices $u, v \in V$ is said to *doubly resolve* two vertices $x, y \in V$ if

$$d_G(x,u) - d_G(x,v) \neq d_G(y,u) - d_G(y,v).$$

Definition 1. A set of vertices S of G is called *doubly locating* if every pair of vertices $x, y \in V$ is doubly resolved by some pair of vertices of S.

A doubly locating set of minimum cardinality is called a *doubly metric basis* of G. The *doubly metric dimension* of G, denoted by $\mathrm{dmd(G)}$, is the cardinality of a doubly metric basis. To know more about this parameter, see mainly [6,16] and also [17–19].

The problem of computing the doubly metric dimension of trees was approached and completely solved in [6].

Theorem 1 ([6])**.** *Let T be a tree having $\ell(T)$ leaves. Then,* $\mathrm{dmd(T)} = \ell(\mathrm{T})$.

In particular, $\mathrm{dmd}(\mathrm{P_n}) = 2$. The problem of characterizing all minimum doubly locating sets of a cycle C_g was implicitly approached and partially proved in [6]. Here, we present the result completed.

Theorem 2. *Consider the cycle C_g and assume that $V(C_g) = [g]$. Then, the following hold:*

1. *If g is odd, then* $\mathrm{dmd}(\mathrm{C_g}) = 2$.
   Moreover, a pair of vertices is a doubly metric basis if and only if they are antipodal.
2. *If g is even, then* $\mathrm{dmd}(\mathrm{C_g}) = 3$.
   Moreover, a triple of vertices is a doubly metric basis if and only if it is a geodesic triple.

Proof.

1. Suppose that $g = 2k+1$ and take the antipodal pair $\{1, k+1\}$. Notice that for every $i \in [g]$

$$d(i,k+1) - d(i,1) = \begin{cases} k-2i & \text{if } 1 \leq i \leq k+1, \\ 2i-3k-3 & \text{if } k+2 \leq i \leq 2k+1. \end{cases}$$

   Let $\{i, j\}$ such that $1 \leq i < j \leq 2k+1$. We distinguish the following cases:
   Case (a): If $1 \leq i < j \leq k+1$, then $d(i,k+1) - d(i,1) = k-2i \neq k-2j = d(j,k+1) - d(j,1)$.
   Case (b): If $k+2 \leq i < j \leq 2k+1$, then $d(i,k+1) - d(i,1) = 2i-3k-3 \neq 2j-3k-3 = d(j,k+1) - d(j,1)$.
   Case (c): If $1 \leq i \leq k+1$ and $k+2 \leq j \leq 2k+1$, then $d(i,k+1) - d(i,1) = k-2i \neq 2j-3k-3 = d(j,k+1) - d(j,1)$, since $k-2i = 2j-3k-3$ if and only if $2j-2i = -4k-3$.
   To end the proof of this item, consider a pair of non-antipodal vertices $\{1, h\}$. W.l.o.g., we can assume that $1 < h < k+1$. Notice that the pair $\{h, h+1\}$ is not doubly resolved by $\{1, h\}$, since $d(h,h) - d(h,1) = 1-h = d(h+1,h) - d(h+1,1)$.
2. Suppose that $g = 2k$ and take a pair of vertices $\{1, h\}$. W.l.o.g., we can assume that $1 < h \leq k$. If $h < k$, then the pair $\{k, k+1\}$ is not doubly resolved by $\{1, h\}$, since $d(k,h) - d(k,1) = 1-h = d(k+1,h) - d(k+1,1)$. If $h = k$, then the pair

$\{k-1, k+1\}$ is not doubly resolved by $\{1, k\}$, since $d(k-1, k) - d(k-1, 1) = 3 - k = d(k+1, k) - d(k+1, 1)$. Hence, $\mathrm{dmd}(\mathrm{C_g}) \geq 3$.

Take a geodesic triple of vertices S. W.l.o.g., we can assume that $S = \{1, \alpha, \beta\}$ and $1 < \alpha < k \leq \beta \leq 2k+1$. Take a pair pf vertices $\{i, j\}$. W.l.o.g., we can assume that $1 < i < \alpha$. We distinguish the following cases:

Case (a): If $1 < i < j < \alpha$, then $\{1, \alpha\}$ doubly resolves $\{i, j\}$, since $d(i, \alpha) - d(i, 1) = \alpha - 2i + 1 \neq \alpha - 2j + 1 = d(j, \alpha) - d(j, 1)$.

Case (b): If $\alpha < j \leq k$, then $\{1, \alpha\}$ doubly resolves $\{i, j\}$, since $d(i, \alpha) - d(i, 1) = \alpha - 2i + 1 \neq -\alpha + 1 = d(j, \alpha) - d(j, 1)$.

Case (c): If $k < j$, then $\{1, \beta\}$ doubly resolves $\{i, j\}$. To prove this, we distinguish the following cases:

Case (c.1): If $i + k - 1 \leq \beta$ and $j < \beta$, then $d(i, \beta) - d(i, 1) = (i - 1 + 2k - \beta) - (i - 1) = 2k - \beta$ and $d(j, \beta) - d(j, 1) = (\beta - j) - (2k - j) = \beta - 2k$.

Case (c.2): If $i + k - 1 \leq \beta$ and $\beta < j$, then $d(i, \beta) - d(i, 1) = (i - 1 + 2k - \beta) - (i - 1) = 2k - \beta$ and $d(j, \beta) - d(j, 1) = (j - \beta) - (2k - j) = 2j - \beta - 2k$. Notice that if $2k - \beta = 2j - \beta - 2k$, then $j = 2k$, which is a contradiction.

Case (c.3): If $\beta < i + k - 1$ and $j < \beta$, then $d(i, \beta) - d(i, 1) = (\beta - i) - (i - 1) = \beta - 2i + 1$ and $d(j, \beta) - d(j, 1) = (\beta - j) - (2k - j) = \beta - 2k$. Notice that if $\beta - 2i + 1 = \beta - 2k$, then $2i - 1 = 2k$, a contradiction.

Case (c.4): If $\beta < i + k - 1$ and $\beta < j$, then $d(i, \beta) - d(i, 1) = (\beta - i) - (i - 1) = \beta - 2i + 1$ and $d(j, \beta) - d(j, 1) = (j - \beta) - (2k - j) = 2j - \beta - 2k$. Notice that if $\beta - 2i + 1 = 2j - \beta - 2k$, then $2i + 2j = 2\beta + 2k + 1$, which is a contradiction.

Hence, we have proved both that $\mathrm{dmd}(\mathrm{C_g}) = 3$ and that every geodesic triple is a doubly metric basis. To end the proof of this item, consider a non-geodesic triple of vertices $S = \{1, \alpha, \beta\}$. W.l.o.g., we can assume that $1 < \alpha < \beta < k$. Notice that the pair $\{\beta, \beta + 1\}$ is neither doubly resolved by $\{1, \alpha\}$, nor by $\{1, \beta\}$, nor by $\{\alpha, \beta\}$, since for every $x \in S$, $d(\beta + 1, x) = d(\beta, x) + 1$.

□

In [16], after noticing that every doubly locating set of a unicyclic graph G must contain all its leaves, it was proved that if G is a proper unicyclic graph of girth g with ℓ leaves, then $\ell \leq \mathrm{dmd}(\mathrm{G}) \leq \ell + 1$ if g is odd, and $\ell \leq \mathrm{dmd}(\mathrm{G}) \leq \ell + 2$ if g is even. Starting both from this result and from Theorem 2, we have characterized the different families of unicyclic graphs achieving each of the possible values for its doubly metric dimension.

Lemma 1. *Let u, v be a pair of vertices of the cycle C_g in a unicyclic graph G. Let ℓ_u (resp., ℓ_v) be either a leaf of the branching tree T_u (resp., the branching tree T_v) or the vertex u (resp., v). If $\{x, y\} \subsetneq V(C_g)$ is doubly resolved by $\{u, v\}$, then it is also doubly resolved by $\{\ell_u, \ell_v\}$.*

Proof. Suppose that $d(u, \ell_u) = k_u$ and $d(v, \ell_v) = k_v$. If $d(x, u) - d(x, v) = h_u$ and $d(y, u) - d(y, v) = h_v$, then $d(x, \ell_u) - d(x, \ell_v) = h_u + k_u - k_v \neq h_v + k_u - k_v = d(y, \ell_u) - d(y, \ell_v)$, since $h_u \neq h_v$. □

Theorem 3. *Let G be a proper unicyclic graph with ℓ leaves. Then, the following hold:*

1. *If g is odd, then* $\mathrm{dmd}(\mathrm{G}) = \begin{cases} \ell & \text{if } C_3(C_g) \text{ contains an antipodal pair,} \\ \ell + 1 & \text{otherwise.} \end{cases}$
2. *If g is even, then* $\mathrm{dmd}(\mathrm{G}) = \begin{cases} \ell & \text{if } C_3(C_g) \text{ contains a geodesic triple,} \\ \ell + 2 & \text{if } c_3(C_g) = 1, \\ \ell + 1 & \text{otherwise.} \end{cases}$

Proof.

1. If $C_3(C_g)$ contains an antipodal pair, then, according to Theorem 2 **(1)**, the set $\mathcal{L}(G)$ of leaves of G is a doubly locating set of G.
   Conversely, if $C_3(C_g)$ contains no antipodal pairs, then to obtain a doubly locating set of G, in addition to $\mathcal{L}(G)$, a trivial vertex h of C_g must be added, to be more precise, a vertex h of C_g antipodal to one of the vertices of $C_3(C_g)$. Then, according to Theorem 2 **(1)**, the set $\mathcal{L}(G) \cup \{h\}$ is a doubly locating set of G.
2. If $C_3(C_g)$ contains a geodesic triple then, according to Theorem 2 **(2)**, the set $\mathcal{L}(G)$ of leaves of G is a doubly locating set of G.
   Conversely, suppose that $C_3(C_g)$ contains no geodesic triples. If $c_3(C_g) = 1$ and $C_3(C_g) = \{h_1\}$, take a pair of trivial vertices $h_2, h_3 \in [g]$ such that that $\{h_1, h_2, h_3\}$ is a geodesic triple of C_g. Then, according to Theorem 2 **(2)**, the set $\mathcal{L}(G) \cup \{h_2, h_3\}$ is a minimum doubly locating set of G. If $c_3(C_g) \geq 2$ and $\{h_1, h_2\} \subseteq C_3(C_g)$, take a trivial vertex $h_3 \in [g]$ such that $\{h_1, h_2, h_3\}$ is a geodesic triple of C_g. Then, according to Theorem 2 **(2)**, the set $\mathcal{L}(G) \cup \{h_3\}$ is a minimum doubly locating set of G.

□

Hence, in this section we have completed previous works to obtain a total characterization of the doubly metric dimension of unicyclic graphs.

4. Metric Dimension

Being introduced primarily in [1,2], a large part of the literature regarding graph locating has to do with metric dimension, the first and more important way of locating vertices in a graph.

Let $G = (V, E)$ be a graph. A vertex $v \in V$ is said to *resolve* two vertices $x, y \in V$ if $d_G(x, v) \neq d_G(y, v)$.

Definition 2. A set of vertices S of G is called *metric-locating* if every pair of distinct vertices $x, y \in V$ is resolved by a vertex of S.

Metric-locating sets are also known as locating sets and resolving sets. Moreover, if S is a metric-locating set of a graph G, then it is usually said that S resolves the set of vertices $V(G)$.

A metric-locating set of minimum cardinality is called a *metric basis* of G. The *metric dimension* of G, denoted by $\dim(G)$, is the cardinality of a metric basis. To know more about this parameter, see mainly [1,2,6,20] and also [5,13,21–34].

It is straightforward to check, first, that $\dim(P_n) = 1$, being a vertex v of P_n isa metric basis if and only if v is a leaf, and second, that $\dim(C_g) = 2$, being any pair of vertices (resp., of non-antipodal vertices) of C_g is a metric basis if $g = 2k + 1$ is odd (resp., $g = 2k$ is even).

Lemma 2 ([20]). *If G is a graph, then* $\dim(G) \geq \ell(G) - \lambda(G)$.

Moreover, it was also proved in [20] that any locating set of a graph G must contain, at least, all but one of the terminal vertices of every exterior major vertex of G.

Theorem 4 ([1,2,20]). *Let T be a tree having $\ell(T)$ leaves and $\lambda(T) \geq 1$ exterior major vertices. Then,*

$$\dim(T) = \ell(T) - \lambda(T).$$

At this point, we are interested in finding the metric dimension of unicyclic graphs. In [13,32], the authors gave a characterization of unicyclic graphs G based on forbidden configurations. Here, we introduce a different approach that relies on the parameters of the graph and using doubly resolving sets. Naturally, our results could be derived from those presented in [13,32] and vice versa. In our opinion, this parameter approach has some value, and provides a different line of attack to the same problem, which could enrich the discussion.

Theorem 5 ([32]). *Let G be a unicyclic graph with ℓ leaves, λ exterior major vertices and ρ branch-active vertices. If $\hat{\rho} = max\{2-\rho, 0\}$, then*

$$\ell - \lambda + \hat{\rho} \leq \ dim(G) \leq \ell - \lambda + \hat{\rho} + 1.$$

In the previous section, the use of doubly locating sets for obtaining results in metric dimension was discussed. In the case of unicyclic graphs, the relationship is given by the next result.

Lemma 3. *Let G be a proper unicyclic graph with C_g being its unique cycle. If S is a doubly locating set of $V(C_g)$, then S resolves $V(C_g)$ together with the vertices of all threads in G.*

Proof. Let $v, v' \in S$. Assume that there exists one thread beginning in a certain vertex $u_i \in V(C_g)$. Let $\{x_0, x_1, \ldots, x_r\}$ be the vertices of the thread in consecutive order where $x_0 = u_i$. Clearly, $r(x_k|S) = r(u_i|S) + (k, k, \ldots, k)$ and so, two vertices of the same thread cannot have the same representation and thus, they are resolved by S.

However, it is possible that $r(u_j|S) = r(x_k|S)$ for $u_j \in V(C_g)$ and x_k in the thread. In this case, $r(u_j|S) = r(x_k|S) = r(u_i|S) + (k, k, \ldots, k)$ and that means that $d(v, u_j) - d(v, u_i) = d(v', u_j) - d(v', u_i) = k$, contradicting the fact that S is a doubly locating set for the vertices of the cycle.

Finally, suppose that there are at least two threads $\{x_0, x_1, \ldots, x_r\}$ and $\{y_0, y_1, \ldots, y_s\}$ where $u_i = x_0$ and $u_j = y_0$ and such that $r(x_k|S) = r(y_l|S)$ (w.l.o.g. we can assume that $k \geq l$). Thus, $r(u_i|S) + (k, k, \ldots, k) = r(x|S) = r(y_l|S) = r(u_j|S) + (l, l, \ldots, l)$ and this implies that $d(v, u_i) - d(v, u_j) = d(v', u_i) - d(v', u_j) = k - l$, and again S would not doubly resolve u_i and u_j. So, S resolves the vertices of the cycle together with the vertices of any thread. □

In the rest of this section, we divide the study into two parts depending on the parity of the unique cycle.

4.1. Odd Unicyclic Graphs

Let us start with the easiest case, although a complete explicit characterization seems to be very complicated and obscure. The cases in which $\rho(G) = 0$ or 1 have been solved with the following two results.

Lemma 4. *Let G be an odd proper unicyclic graph, with C_g being its unique cycle. If $\rho(G) = 0$, then $\dim(G) = 2$.*

Proof. Let v and v' be a pair of antipodal vertices of C_g. By Theorem 2, they doubly resolve all the vertices of the cycle C_g, and hence, according to Lemma 3, since $\rho(G) = 0$, they resolve all the vertices of G. □

Lemma 5. *Let G be an odd proper unicyclic graph, with C_g being its unique cycle. If $\rho(G) = 1$, then $\dim(G) = \ell(G) - \lambda(G) + 1$.*

Proof. Under the condition of the hypothesis, there is exactly one branch-active vertex v in the cycle C_g, with T_v as its branching tree.

Given a metric basis S of G and according to Lemma 2, all the terminal vertices except one of each exterior major vertex with positive terminal degree of T_v should belong to S. However, since v is a cut-point of G, not all the vertices of S lie in T_v, otherwise S will not resolve two vertices of C_g at the same distance from v. Thus, a lower bound for a metric dimension is $\dim(G) \geq \ell(G) - \lambda(G) + 1$.

Now, let S be a set with $\ell(G) - \lambda(G)$ leaves together with a vertex v' antipodal of v. If $u \in S$ is a leaf, then it is clear that $\{u, v'\}$ doubly resolves all the vertices of the cycle, and hence by Lemma 3, S resolves all the vertices of the cycle and the threads.

Moreover, S resolves all the vertices in T_v except perhaps v and other vertex u' with the same exterior major vertex as v. But in this case, they are resolved by v' since $d(v, v') < d(u', v')$. Finally, a vertex in T_v and any other vertex in $V(G) \setminus V(T_v)$ is resolved by a leaf in S.

Thus, S is a metric-locating set achieving the lower bound $\ell(G) - \lambda(G) + 1$, i.e., S is a metric basis of G. □

For the rest of the cases, i.e., when $\rho(G) \geq 2$, we can give a partial result of the value of the metric dimension.

Lemma 6. *Let G be an odd proper unicyclic graph, with C_g being its unique cycle. If $g \geq 3$, $\rho(G) \geq 2$ and there are, at least, two antipodal branch-active vertices, then* $\dim(G) = \ell(G) - \lambda(G)$. *Particularly, if $g = 3$ and $\rho(G) \geq 2$, then* $\dim(G) = \ell(G) - \lambda(G)$.

Proof. Let S be a vertex subset formed by $\ell - \lambda$ leaves for each branching tree. By Lemma 2, $\dim(G) \geq |S|$, and thus it remains to be proven that S is a metric-locating set. We first claim that S doubly resolves the vertices of $V(C_g)$.

Let T_1, T_2 be the branching trees whose branch-active vertices u_1, u_2 are antipodal, and let v_1, v_2 be two leaves such that $v_i \in V(T_i) \cap S$ for $i = \{1, 2\}$. By Lemma 2, $\{u_1, u_2\}$ doubly resolves the vertices of cycle. However, for any $x \in V(C_g)$, we have that $r(x|\{v_1, v_2\}) = r(x|\{u_1, u_2\}) + (d(u_1, v_1), d(u_2, v_2))$, and hence $\{v_1, v_2\}$ also doubly resolves the vertices of the cycle, and so does S since $\{v_1, v_2\} \subseteq S$.

By Lemma 3, S resolves all the vertices of the cycle and the threads of G. A similar reasoning can be used to prove that S distinguishes between a vertex in a branching tree, and a vertex either in a thread or in the cycle C_g.

Two vertices in the same branching tree T different from its branch-active vertex are resolved by the leaves in $V(T) \cap S$. A vertex in T and its branch-active vertex u are resolved by a leaf of another branching tree in S. So, S distinguishes two vertices of the same branching tree.

Finally, let us consider two vertices x and y in different branching trees. Its branching-active vertices are joined by a shortest path in the cycle. If we delete the rest of the cycle and all the vertices hanging out, we build a subtree T of G, which is also an isometric subgraph. By Theorem 4, x and y are resolved in T by the vertices in $V(T) \cap S$. However, since T is an isometric subgraph, x and y are also resolved in G by the same vertices in S. This completes the proof.

For the particular case of $g = 3$, note that if $\rho(G) \geq 2$, there are always two branching-active vertices that are antipodal. □

The metric dimension of an odd unicyclic graph G with at least two branch-active vertices without antipodal branch-active vertices may be, according to Theorem 5, either $\dim(G) = \ell$ or $\dim(G) = \ell - \lambda + 1$.

As an example, the unicyclic graph G displayed in Figure 2 (left) has $\dim(G) = \ell(G) - \lambda(G)$ (a metric basis is $\{v_1, v_2\}$). Nevertheless, if G' is the graph obtained from G by adding a vertex w as shown in Figure 2 (right), then $\dim(G') = \ell(G') - \lambda(G') + 1$, being a metric basis the set $\{v_1, v_2, w\}$.

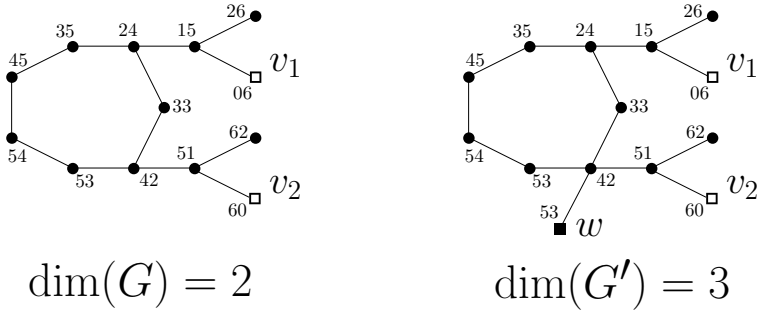


Figure 2. (**Left**): Odd unicyclic graph G with $\ell(G) = 4$, $\lambda(G) = 2$, $\rho(G) = 2$, without antipodal branch-active vertices. (**Right**): Odd unicyclic graph G' with $\ell(G) = 5$, $\lambda(G) = 3$, $\rho(G) = 2$, without antipodal branch-active vertices.

As an immediate consequence of the previous three lemmas, the following result is obtained.

Corollary 1. *Let G be an odd unicyclic graph, with C_g being its unique cycle, with ℓ leaves, λ exterior major vertices and ρ branch-active vertices If $g \geq 3$, then $\dim(G) = \ell - \lambda + \max\{2 - \rho, 0\}$ whenever any of the following conditions hold:*

1. $0 \leq \rho \leq 1$.
2. $\rho \geq 2$ *and there are, at least, two antipodal branch-active vertices.*
3. $g = 3$.

4.2. Even Unicyclic Graphs

The case in which the unicyclic graph has an even cycle turns out to be much more difficult than the previous case. Hence, the results are not so satisfactory.

Proposition 1. *Let G be an even proper unicyclic graph, with C_g being its unique cycle. If $g \geq 8$, $\rho(G) = 0$ and $c_2(C_g) \leq 1$, then $\dim(G) = 3$.*

Proof. On the contrary, suppose that $\{u, v\}$ is a metric basis of G in the conditions of the hypothesis. Since u, v or both may be a vertex in a thread or in the cycle, we will use the following notation for treating all the cases at once: If u is a vertex in the interior of a thread, u' will be the vertex of that thread that also belongs to the cycle, otherwise $u = u'$. Analogously, we define v'.

Note that u' and v' cannot be antipodal in the cycle since, otherwise $\{u, v\}$ will not be resolving. Thus, there is a unique shortest path between u' and v' and $d(u', v') \leq k - 1$. We organize the proof by studying different distances between u' and v'.

Firstly, suppose that $d(u', v') = 1$ (see Figure 3 left). Since $g \geq 8$, we can add $w_1, w_2, w_3 \in V(C_g)$ to obtain the shortest path $u - v - w_1 - w_2 - w_3$. The minimum number of threads is $2k - 1$, so w_1 or w_2 is the beginning of a thread and if we denote w_i' ($i = 1, 2$) as the adjacent to w_i in the thread, then we have that $r(w_i'|\{u, v\}) = r(w_{i+1}|\{u, v\})$ and hence $\{u, v\}$ is not a locating set.

Consider now the case $2 \leq d(u', v') < k - 1$ (see Figure 3 center). Again, since $g \geq 8$, we can take $u_1, u_2, v_1, v_2 \in V(C_g)$ to construct the path $u_2 - u_1 - u' - \ldots - v' - v_1 - v_2$. This path might not be a shortest path but the subpaths $u' - v_2$ and $u_2 - v'$ are shortest paths and that is enough for our reasoning. Then, either u_1 or v_1 is the beginning of a

thread. Without loss of generality, we can assume that it is v_1 and its adjacent in the thread is v_1'. Then, $r(v_1'|\{u,v\}) = r(v_2|\{u,v\})$ and again $\{u,v\}$ is not a locating set.

The final case occurs when $d(u',v') = k-1$ (see Figure 3 right). Let u_1 and v_1 be the adjacent vertices to u' and v', respectively, in the cycle, which are not in the shortest path $u'-v'$ (see Figure 3). Then, the shortest path $u_1 - v_1$ has the same length as $u'-v'$, and every interior vertex x of $u'-v'$ corresponds with x_1 in u_1-v_1 such that $d(u',x) = d(u_1,x_1)$. Moreover, $r(x_1|\{u,v\}) = r(x|\{u,v\}) + (1,1)$. Since there are $k-2$ internal vertices in the path $u'-v'$, at least one of them, say x, has a thread and its adjacent vertex in that thread x' and x_1 has the same coordinates with respect to $\{u,v\}$. So, again $\{u,v\}$ cannot be a locating set and $\dim(G) = 3$. □

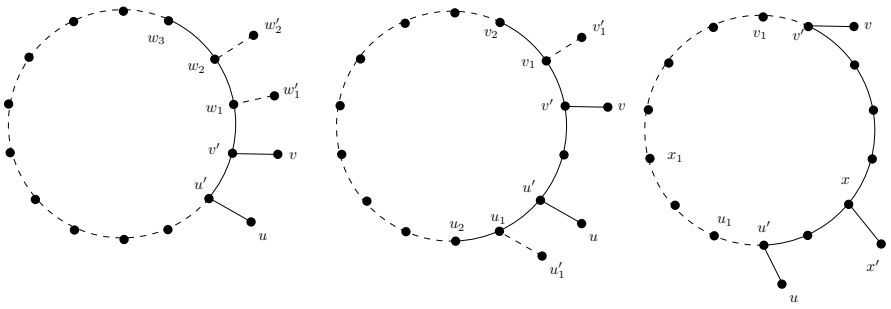


Figure 3. Several situations of the proof of Proposition 1.

The above result is the best bound we can obtain, since there are unicyclic graphs such that the number of threads is $2k-2$ and the metric dimension equals two. Consider for example, a unicyclic graph G with girth $g = 2k$ with $k \geq 3$ and such that every vertex has a thread of length one except two antipodal vertices. In Figure 4, it is shown the case $k = 7$, where the pair $\{u,v\}$ of square vertices is a metric basis. The construction can be repeated for any integer $k \geq 3$.

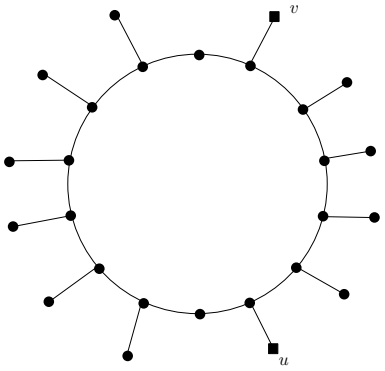


Figure 4. Even unicyclic graph G with $g-2$ threads and $\dim(G) = 2$.

Now, let us check the cases we left out int he previous result.

Lemma 7. *Let G be an unicyclic graph such that $\rho(G) = 0$. Then the following hold:*

1. *If $g = 4$, then $\dim(G) = 2$.*
2. *Let $g = 6$. Then, $c_2(C_g) \neq 0$ if and only if $\dim(G) = 2$.*

Proof.

1. Consider first a unicyclic graph G with $g = 4$ and $\rho(G) = 0$. Whenever you have two threads whose initial vertices are adjacent, say u_1, u_2, in the cycle then pick the leaves

of those two threads as a metric basis, say v_1, v_2. If the only two threads of G have as initial vertices u_1 and u_3, two non-adjacent vertices of the cycle, then pick $\{v_1, u_2\}$ as a metric basis.
If G contains only one thread in, say u_1, then $\{v_1, u_2\}$ will be a metric basis. Finally, if G does not have threads, then $G = C_4$ and two consecutive vertices as u_1, u_2 are enough to resolve the graph.

2. Suppose now that $g = 6$. If $c_2(C_g) = 0$, then there exists a leaf for every vertex in the cycle. The reader can easily prove that $\dim(K_1 \odot C_g) = 3$ and so $\dim(G) = 3$.
If $c_2(G) \neq 0$, then there exists at least one vertex in the cycle without a thread. If there is only one u_1, pick the leaves hanging from the adjacent vertices in the cycle to u_1. It is not difficult to check that the pair of vertices form a metric basis. The same applies if there is a vertex in the cycle adjacent to two vertices with threads.
Otherwise, we have at least two adjacent vertices $u_1, u_2 \in V(C_g)$ such that another adjacent to one of them, say u_3, has a thread. Let v_3 be the leaf in that thread. Again, it is not difficult to prove that $\{u_1, v_3\}$ is a basis for G and therefore $\dim(G) = 2$.

□

The next result is very similar to Lemma 6 with the particularities of dealing with an even cycle.

Proposition 2. *Let G be an even unicyclic graph, with C_g as its unique cycle. If $\rho(G) \geq 3$ and there are three branching-active vertices forming a geodesic triple, then* $\dim(G) = \ell(G) - \lambda(G)$.

Proof. Let S be a vertex subset with $\ell(G) - \lambda(G)$ leaves. By Lemma 2, we know that any locating set must contain S, hence $\dim(G) \geq |S|$. We next show that S is a locating set.

First, we claim that a subset of S, and consequently S, doubly resolves the vertices of the cycle. Let T_1, T_2, T_3 be the trees whose branching-active vertices u_1, u_2, u_3 are a geodesic triple. We construct $S' = \{v_1, v_2, v_3\}$ where $v_i \in V(T_i) \cap S$, so $S' \subseteq S$. As $\{u_1, u_2, u_3\}$ is a geodesic triple, then it doubly resolves the vertices of the cycle, by Theorem 2. That means

$$\begin{aligned}
\forall x, y \in V(C_g), \exists i, j \in \{1,2,3\} : d(x,u_i) - d(x,u_j) \neq d(y,u_i) - d(y,u_j) \Rightarrow \\
\Rightarrow d(x,u_i) - d(x,u_j) + (d(u_i,v_i) - d(u_j,v_j)) \neq d(y,u_i) - d(y,u_j) + (d(u_i,v_i) - d(u_j,v_j)) \Rightarrow \\
\Rightarrow d(x,u_i) + d(u_i,v_i) - (d(x,u_j) + d(u_j,v_j)) \neq d(y,u_i) + d(u_i,v_i) - (d(y,u_j) + d(u_j,v_j)) \Rightarrow \\
\Rightarrow d(x,v_i) - d(x,v_j) \neq d(y,v_i) - d(y,v_j). \quad (1)
\end{aligned}$$

And consequently, S' also doubly resolves the vertices of the cycle. The rest of the proof is similar to the one of Lemma 6. □

Figure 5 (left) shows an example of a graph G with $\dim(G) = \ell(G) - \lambda(G)$ (a metric basis is $\{v_1, v_2\}$). When a new vertex w is added to get the graph G' (see Figure 5 (right)), then $\dim(G') = \ell(G') - \lambda(G') + 1$, since $\{v_1, v_2, w\}$ turns out to be a metric basis.

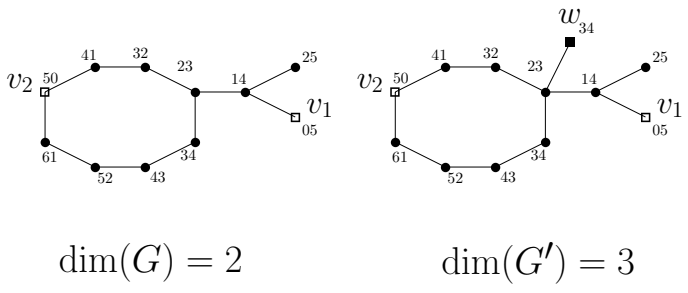


Figure 5. (**Left**): Even unicyclic graph G with $\ell(G) = 2$, $\lambda(G) = 1$, $\rho(G) = 1$. (**Right**): Even unicyclic graph G' with $\ell(G) = 3$, $\lambda(G) = 2$, $\rho(G) = 1$.

As an immediate consequence of the previous results, the following one is obtained.

Corollary 2. *Let G be an even unicyclic graph, where C_g is its unique cycle, with ℓ leaves, λ exterior major vertices, ρ branch-active vertices and c_2 vertices in the cycle with degree two. If $g \geq 3$, then $\dim(G) = \ell - \lambda + \max\{2 - \rho, 0\}$ whenever any the following conditions hold:*

1. $\rho = 0, g \geq 8$ *and* $c_2 \geq 2$.
2. $\rho = 0$ *and* $g = 4$.
3. $\rho = 0, g = 6$ *and* $c_2 \neq 0$.
4. $\rho \geq 3$ *with three branching-active vertices forming a geodesic triple.*

5. Strong Metric Dimension

The strong metric dimension is the most atypical one among the metric dimensions. To start with, it does not rely on coordinates to distinguish the vertices of the graph but on a certain notion of betweenness in shortest paths. Let $G = (V, E)$ be a graph, then a vertex $v \in V$ is said to *strongly resolve* two vertices $x, y \in V(G)$ if there exists either a $v - x$ geodesic that contains y, or a $v - y$ geodesic that contains x.

Definition 3. A set of vertices S of G is called *strong locating* if every pair of distinct vertices $x, y \in V(G)$ is strong resolved by a vertex $v \in S$.

In other words, S is a strong locating set if, for every distinct vertex $x, y \in V(G)$, there is a vertex $v \in S$, such that either $d_G(v, x) = d_G(v, y) + d_G(y, x)$ or $d_G(v, y) = d_G(v, x) + d_G(x, y)$.

A strong locating set of minimum cardinality is called a *strong metric basis* of G. The *strong metric dimension* of G, denoted by $\mathrm{sdim(G)}$, is the cardinality of a strong metric basis. This parameter was formally introduced by Sébo and Tannier in [5]. To know more about this parameter, see mainly [5,14,35,36] and also [17,19,37–41].

The next result is mentioned in many papers, and mainly for this reason, we include here the proof for the sake of completeness.

Theorem 6. *Let S be a proper subset of vertices of a graph $G = (V, E)$. Then, the following statements are equivalent:*

1. *S is a strong locating set.*
2. *G is uniquely determined by the matrix of distances between the vertices of S and the vertices of V.*

Proof. Assume that S is a strong locating set of G. Take $u, v \in V \setminus S$. Let $w \in S$ such that $d(w, v) = d(w, u) + d(u, v)$. We distinguish the following cases:

Case (a): If $d(w, v) \geq d(w, u) + 2$, then $d(u, v) \geq 2$, i.e., $uv \notin E$.

Case (b): If $d(w, v) = d(w, u) + 1$, then $d(u, v) = 1$, i.e., $uv \in E$.

Conversely, suppose that S is not a strong locating set of G. Take $u, v \in V \setminus S$ such that for every vertex $w \in S$, $d(w, v) < d(w, u) + d(u, v)$ and $d(w, u) < d(w, v) + d(v, u)$.

Take $w \in S$ and $z \in V \setminus S$. If ρ is a $w - z$ geodesic, then either u or v does not belong to ρ, since otherwise either $d(w, v) = d(w, u) + d(u, v)$ or $d(w, u) = d(w, v) + d(v, u)$. Hence, both if $uv \in E$ and $uv \notin E$, the matrix of distances between the vertices of S and the vertices of V is the same. □

As it was mentioned above, strong locating sets do not rely on coordinates so it would be desirable to find a characterization of some of the vertices that belong to a strong locating set. The next pair of results gives us precisely that characterization.

Lemma 8. *Let S be a strong locating set of a graph G. If $u, v \in V(G)$ are* MMD*, then $\{u, v\} \cap S \neq \emptyset$.*

Proof. In contrast, assume $u, v \notin S$. Let $w \in S$ be such that the pair u, v is strongly resolved by w. Suppose that u lies in a $w - v$ geodesic. Thus, there exists a neighbor of u in this path that is farther away from v than u, contradicting the fact that u, v are *MMD*. □

Theorem 7. *Consider the cycle C_g. Then,* $\mathrm{sdim}(\mathrm{C_g}) = \lceil \frac{g}{2} \rceil$. *Moreover, a $\lceil \frac{g}{2} \rceil$-set S is a strong metric basis of C_g if and only if the set $V(G) \setminus S$ contains no antipodal pair.*

Proof. Assume that $V(C_g) = [g]$ and take the set $W = [\lceil \frac{g}{2} \rceil]$. Notice that for every pair of vertices i, j such that $\lceil \frac{g}{2} \rceil < i < j \leq g$, $d(i, 1) = d(i, j) + d(j, 1)$, which means that there is a (unique) $(i - 1)$ geodesic containing vertex j. Hence, W is a strong locating set of C_g of cardinality $\lceil \frac{g}{2} \rceil$.

Let S be a strong locating set of C_g. According to Lemma 8, the set $V(C_g) \setminus S$ contains no antipodal pair, which means that the cardinality of $V(G) \setminus S$ is at most $\lfloor \frac{g}{2} \rfloor$. Hence, $\mathrm{sdim}(\mathrm{C_g}) = \mathrm{g} - \lfloor \frac{\mathrm{g}}{2} \rfloor = \lceil \frac{\mathrm{g}}{2} \rceil$.

Conversely, let S be a $\lceil \frac{g}{2} \rceil$-set such that $V(G)| \setminus S$ contains no antipodal pair. Let $i, j \in V(C_g) \setminus S$ such that $d(i, j) = h < \lceil \frac{g}{2} \rceil$. W.l.o.g., we may assume that $i = 1$ and $j = h + 1$. If $\lceil \frac{g}{2} \rceil = k$, take the vertex $k + 1$, which is an antipodal vertex of 1, and check that there is a (unique) $1 - (k + 1)$ geodesic containing vertex h. □

Lemma 9. *Let U be a k-set of non-consecutive vertices of the odd cycle C_{2k+1}. Then, it contains at least a pair of antipodal vertices.*

Proof. Suppose that U contains no antipodal pairs. Let $\{U_i\}_{i=1}^{\alpha}$ be the maximal disjoint partition of U such that, for every $i \in [\alpha]$, U_i is a β_i-set consisting of β_i consecutive vertices. If W_i denotes the set of antipodal vertices of U_i, then $\{W_i\}_{i=1}^{\alpha}$ is a disjoint partition contained in $V(C_g) \setminus U$ such that, for every $i \in [\alpha]$, W_i is a $(\beta_i + 1)$-set consisting of $\beta_i + 1$ consecutive vertices. Then, $\sum_{i=1}^{\alpha} \beta_i = k$, $|\cup_{i=1}^{\alpha} W_i| \leq k + 1$, $|\cup_{i=1}^{\alpha} W_i| = \sum_{i=1}^{\alpha} (\beta_i + 1) = \sum_{i=1}^{\alpha} \beta_i + \alpha = k + \alpha$, and thus $\alpha = 1$, i.e., U is a k-set of consecutive vertices. □

As a straight consequence of both the previous theorem and lemma, the following result holds.

Corollary 3. *Every strong metric basis of the odd cycle C_{2k+1} consists of $k + 1$ consecutive vertices.*

Theorem 8 ([5])**.** *Let T be a tree with ℓ leaves. Then, any set consisting of all the leaves except one is a minimum strong locating set of T. Hence,* $\mathrm{sdim}(\mathrm{T}) = \ell - 1$.

Proof. According to Lemma 8, for any pair of vertices mutually maximally distant, at least one of them must belong to a given strong locating code S. In a tree, the vertices that are mutually maximally distant are the leaves, so S must contain all the leaves except one.

Conversely, let S be a set containing all the leaves of T except one. It is clear that the shortest path between any pair of vertices u, v can be extended until both of its extremes are leaves. Hence, at least one of them is in S and strongly resolves the pair u, v. □

To compute the strong metric dimension, it is necessary, in most cases, to use the strong resolving graph introduced by Oellermann and Peters-Fransen in [36]. The *strong resolving graph* G'_{SR} of a connected graph G is the graph with vertex set $V(G_{SR}) = V(G)$ where two vertices are adjacent if they are mutually maximally distant in G. Another definition, given in [14], reduces the vertex set to be $V(G_{SR}) = \partial(G)$. Thus, the strong resolving graph G_{SR} defined in [14] is the subgraph of the strong resolving graph G'_{SR} defined in [36] obtained

from the latter by deleting its isolated vertices (see Figure 6). Although we will stick formally with the second definition, this does not affect the proof of any of the results given.

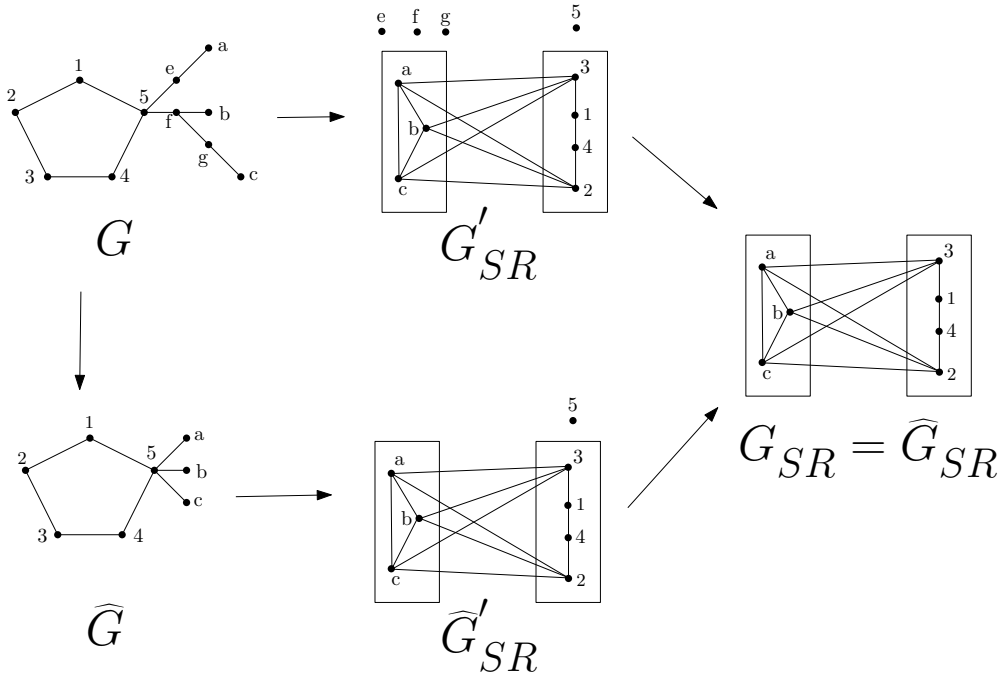


Figure 6. A unicyclic graph G having vertices 1 to 5 in the cycle, a, b and c as leaves, and e, f, g as inner vertices in a tree; its closed necklace $\widehat{G}$ and its strong resolving graph G_{SR}.

Theorem 9 ([14,36]). *Let G be a connected graph. Then,*

$$\mathrm{sdim(G)} = |\partial(\mathrm{G})| - \alpha(\mathrm{G_{SR}}).$$

Observe that $[P_n]_{SR} = K_2$, $[C_n]_{SR} = C_n$ and $[T_\ell]_{SR} = K_\ell$, where T_ℓ denotes any tree with ℓ leaves. Notice also that $\alpha(K_r) = 1$ and $\alpha(C_n) = \lfloor \frac{g}{2} \rfloor$. Hence, as a direct consequence of the previous theorem, it is derived that $\mathrm{sdim(P_n)} = 1$, $\mathrm{sdim(C_n)} = \lceil \frac{n}{2} \rceil$ and $\mathrm{sdim(T_\ell)} = \ell - 1$.

The so-called *closed necklace* $\widehat{G}$ associated to a unicyclic graph G of girth g, is the unicyclic graph where, for every vertex $i \in [g]$, the branching tree of i is replaced by the star K_{1,ℓ_i}, where ℓ_i is the number of leaves of T_i, other than i. It is a routine exercise to check, first, that $\partial(\widehat{G}) = \partial(G)$, and second, that $\widehat{G}_{SR} = G_{SR}$ (see Figure 6 for an example).

Let $G = (V, E)$ be a closed necklace of order n, girth g with ℓ leaves. Clearly, if C_g is this cycle and $c_2(C_g) = h$, then $|\partial(G)| = \ell + h$. Moreover, the vertex set $V(G_{SR}) = \partial(G)$ of its strong resolving graph G_{SR}, can be partitioned into two sets U and W, in such a way that $G_{SR}[U] = K_\ell$, and $G_{SR}[W]$ is a graph of order h and maximum degree at most 2.

Remark 1. *If g is odd, then G_{SR} is a Hamiltonian graph and $G_{SR}[W]$ is a disjoint union of paths (see Figure 7). Hence, $\left\lceil \frac{h}{2} \right\rceil \leq \alpha(G_{SR}) \leq h + 1$.*

Remark 2. *If g is even, then $G_{SR}[W] = rK_2 + sK_1$, where r is the number of antipodal pairs of $C_2(C_g)$, t is the number of antipodal pairs of $C_3(C_g)$ and $2s = g - 2r - 2t$ (see Figure 8). Hence, $h = 2r + s$ and $\left\lceil \frac{h}{2} \right\rceil = r + \left\lceil \frac{s}{2} \right\rceil \leq r + s \leq \alpha(G_{SR}) \leq r + s + 1$.*

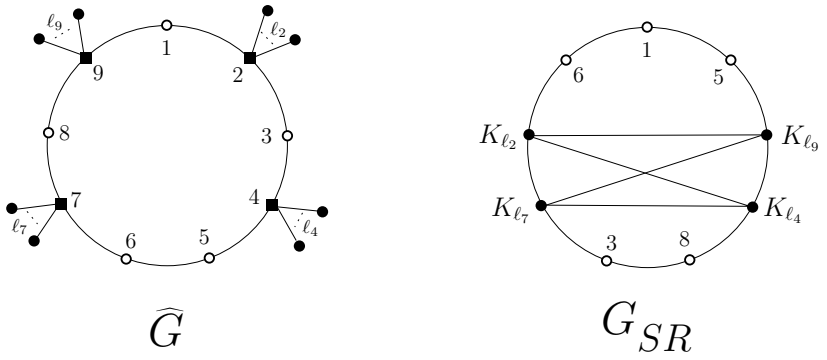


Figure 7. Closed necklace $\widehat{G}$ of a unicyclic graph G of girth 9 such that $\ell = \ell_2 + \ell_4 + \ell_7 + \ell_9$, $c_2(C_g) = 5$ and $G_{SR}[W] = P_3 + P_2$ and its strong resolving graph G_{SR}.

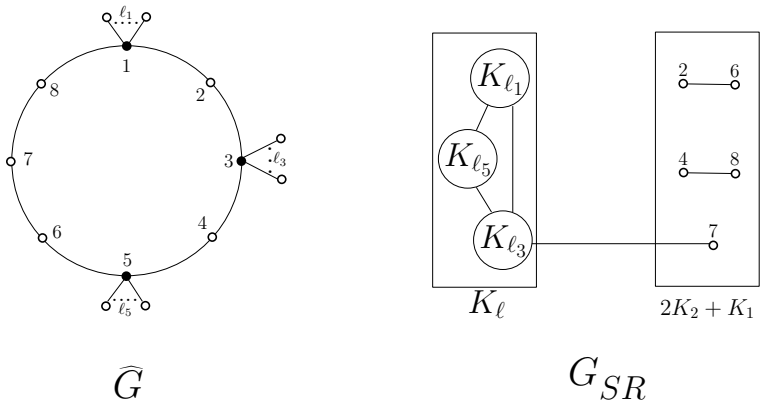


Figure 8. Closed necklace $\widehat{G}$ of a unicyclic graph G of girth 8 with $\ell = \ell_1 + \ell_3 + \ell_5$ leaves, $r = 2$, $s = 1$ and $t = 1$ and its strong resolving graph G_{SR}.

The next result, proved by Kuziak in [35], provides the range of the value of the strong metric dimension for unicyclic graphs.

Theorem 10 ([35])**.** *Let G be a proper unicyclic graph of girth g, with C_g as its unique cycle and ℓ the number of leaves. If $c_2(C_g) = h$, then*

$$\max\left\{\left\lceil \frac{g}{2} \right\rceil, \ell - 1\right\} \leq \mathrm{sdim(G)} \leq \ell + \left\lfloor \frac{\mathrm{h}}{2} \right\rfloor.$$

Now, we address the problem not only of proving that the bounds displayed in Theorem 10 are tight but also of characterizing the families of graphs achieving each of them.

Proposition 3. *Let G be a proper unicyclic graph of girth g, where C_g is its unique cycle and ℓ the number of leaves. If $c_2(C_g) = h$, then* $\mathrm{sdim(G)} = \ell - 1$ *if and only if either $g = 3$ and $h = 0$ or $g \geq 4$ and any of the following conditions hold:*

1. $0 \leq h \leq 1$.
2. *$g = 2k$ is even, $2 \leq h \leq k - 1$ and $C_2(C_g)$ contains no pair of antipodal vertices.*
3. *$g = 2k + 1$ is odd, $2 \leq h \leq k - 1$, $C_2(C_g)$ contains no pair of antipodal vertices and $C_3(C_g)$ contains at least one antipodal triple.*

Proof. According to Theorem 9, $\mathrm{sdim(G)} = |\partial(\mathrm{G})| - \alpha(\mathrm{G_{SR}}) = \ell + \mathrm{h} - \alpha(\mathrm{G_{SR}})$. Hence, $\mathrm{sdim(G)} = \ell - 1$ if and only if $\alpha(G_{SR}) = h + 1$.

If $g = 3$, then, in all cases, $\alpha(G_{SR}) = 1$. Hence, $\mathrm{sdim(G)} = \ell + \mathrm{h} - 1 = \ell - 1$ if and only if $h = 0$.

Suppose that $g \geq 4$. We distinguish the following cases:

Case (a): If $h = 0$, then $G_{SR} \cong K_\ell$. Thus, $\alpha(G_{SR}) = 1 = h + 1$.

Case (b): If $h = 1$, then G_{SR} is a graph of order $\ell + 1$ containing the clique K_ℓ and minimum degree either 1 or 2, depending of the parity of g. Thus, $\alpha(G_{SR}) = 2 = h + 1$.

Case (c): Suppose that $h \geq 2$ and $g = 2k$ is even. According to Remark 2, $h = 2r + s$ and $r + s \leq \alpha(G_{SR}) \leq r + s + 1$. Hence, $r = 0$, i.e., $C_2(C_g)$ contains no pair of antipodal vertices, if and only if both $h = s$ and $s \leq \alpha(G_{SR}) \leq s + 1$. Moreover, $\alpha(G_{SR}) = s$ if and only if $h = k$, which means that $\alpha(G_{SR}) = s + 1 = h + 1$ if and only if $r = 0$ and $h \leq k - 1$.

Case (d): Suppose that $h \geq 2$ and $g = 2k + 1$ is odd. According to Remark 1, $\left\lceil \frac{h}{2} \right\rceil \leq \alpha(G_{SR}) \leq h + 1$. In addition, $\alpha(G_{SR}) = h + 1$ if and only if both $[W]G_{SR} = hK_1$ and $C_3(C_g)$ contains at least one antipodal triple, which also includes the fact that $h \leq k - 1$. □

Proposition 4. *Let G be a proper unicyclic graph of girth g, with C_g its unique cycle and ℓ itd leaves. If $c_2(C_g) = h$, then* $\mathrm{sdim(G)} = \ell + \left\lfloor \frac{\mathrm{h}}{2} \right\rfloor$ *if and only if any of the following conditions hold:*

1. *g is even and $h = g - 1$.*
2. *g is odd and $g - 2 \leq h \leq g - 1$.*

Proof. According to Theorem 5 and Remarks 1 and 2, $\mathrm{sdim(G)} = \ell + \left\lfloor \frac{\mathrm{h}}{2} \right\rfloor$ if and only if $\alpha(G_{SR}) = \left\lceil \frac{h}{2} \right\rceil$. We distinguish the following cases:

Case (a): Suppose that g is even. According to Remark 2, $h = 2r + s$ and $r + s \leq \alpha(G_{SR}) \leq r + s + 1$. Assume that $h = g - 1$. Then, $2r + s = 2r + 2s + 2t - 1$, which means that $t = 0$, $s = 1$ and $h = 2r + 1$. So, $\alpha(G_{SR}) = r + 1 = \left\lceil \frac{h}{2} \right\rceil$.

Conversely, suppose that $\alpha(G_{SR}) = \left\lceil \frac{h}{2} \right\rceil = r + \left\lceil \frac{s}{2} \right\rceil$. Thus, $r + s \leq r + \left\lceil \frac{s}{2} \right\rceil \leq r + s + 1$, i.e., $s \leq \left\lceil \frac{s}{2} \right\rceil \leq s + 1$, which means that $s = \left\lceil \frac{s}{2} \right\rceil$ since $\left\lceil \frac{s}{2} \right\rceil < s + 1$, for every positive integer s. After noticing that $t = 0$ and $g = 2r + 2s + 2t$, we distinguish the following cases:

Case (a.1): If $s = 0$, then $G = C_g$, which is a contradiction.

Case (a.2): If $s = 1$, then $h = 2r + 1$ and $g = 2r + 2$, i.e., $h = g - 1$.

Case (b): Suppose that g is odd. If $g - 2 \leq h \leq g - 1$, then according to Remark 1, in all possible cases, $\alpha(G_{SR}) = \left\lceil \frac{h}{2} \right\rceil$.

Conversely, suppose both that $\alpha(G_{SR}) = \left\lceil \frac{h}{2} \right\rceil$ and $h \leq g - 3$. If $G[W] = \sum_{i=1}^{\rho} P_{h_i}$, then

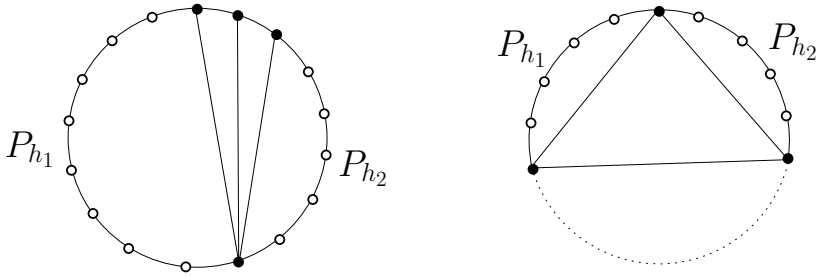
$$\sum_{i=1}^{\rho} \alpha(P_{h_i}) \leq \alpha(G_{SR}) \leq \sum_{i=1}^{\rho} \alpha(P_{h_i}) + 1.$$

Hence, $\alpha(G_{SR}) = \left\lceil \frac{h}{2} \right\rceil$ if and only if $\alpha(G_{SR}) = \sum_{i=1}^{\rho} \alpha(P_{h_i}) = \left\lceil \frac{h}{2} \right\rceil = \alpha(P_h)$. After noticing that $\alpha(P_h) = \sum_{i=1}^{\rho} \alpha(P_{h_i})$ if and only if there is at most an index $j \in [\rho]$ such that P_j is an odd path, we distinguish the following cases:

Case (c): There is a unique index $j \in [\rho]$ such that P_j is an odd path. We distinguish the following further cases:

Case (c.1): If $\rho = 2$, then we can suppose w.l.o.g. that P_{h_1} is the even path and that between P_{h_1} and P_{h_2} there are in G_{SR} at least two vertices (see Figure 9, left). Certainly, it is possible to select a maximum independent set in P_1 not containing its last vertex. Hence, $\alpha(G_{SR}) = \sum_{i=1}^{\rho} \alpha(P_{h_i}) + 1 > \left\lceil \frac{h}{2} \right\rceil$, which is a contradiction.

Case (c.2): Assume that $\rho \geq 3$. Take P_{h_1} and P_{h_2} and suppose w.l.o.g. that they are both even paths and consecutive in G_{SR} (see Figure 9, right). Clearly, it is possible to select a maximum independent set in P_{h_1} (resp., in P_{h_2}) not containing its last vertex (resp., its first vertex). Hence, $\alpha(G_{SR}) = \sum_{i=1}^{\rho} \alpha(P_{h_i}) + 1 > \left\lceil \frac{h}{2} \right\rceil$, which is a contradiction.



$g = 17,\ \rho = 2,\ h = 13$ $\qquad$ $g > 11,\ \rho \geq 3,\ h \geq 9$

Figure 9. Strong locating set of a unicyclic graph of odd girth g and $h \leq g - 3$. (**Left**): $\rho = 2$. (**Right**): $\rho \geq 3$.

Case (d): For every index $j \in [\rho]$, P_{h_1} is an even path. Take P_{h_1} and P_{h_2} and suppose w.l.o.g. that they are consecutive in G_{SR} (see Figure 9, right). Clearly, it is possible to select a maximum independent set in P_{h_1} (resp., in P_{h_2}) not containing its last vertex (resp., its first vertex). Hence, $\alpha(G_{SR}) = \sum_{i=1}^{\rho} \alpha(P_{h_i}) + 1 > \left\lceil \frac{h}{2} \right\rceil$, which is a contradiction. □

As for the problem of characterizing the family of proper unicyclic graphs G of girth g with at most $\lceil \frac{g}{2} \rceil$ leaves such that $\mathrm{sdim}(G) = \lceil \frac{g}{2} \rceil$, it was solved when g is even in [35], while the odd case still remains open.

Proposition 5. *Let G be an even proper unicyclic graph of girth g, with C_g its unique cycle and ℓ its leaves. Then,* $\mathrm{sdim}(G) = \frac{g}{2}$ *if and only if the following conditions hold:*

1. $\rho(G) = 0$.
2. *There is at most one pair of antipodal vertices in C_g, each one of degree at least 3.*

Next, we present a result that clearly shows how easy is to compute the strong metric dimension of an even unicyclic graph.

Theorem 11. *Let G be a unicyclic graph of even girth $g = 2k$, with C_g representing its unique cycle and ℓ its leaves. If $C_2(C_g)$ contains r antipodal pairs and $C_3(C_g)$ contains t antipodal pairs, then*

$$\mathrm{sdim}(G) = \begin{cases} \ell + r - 1 & \text{if } t \geq 1, \\ \ell + r & \text{if } t = 0. \end{cases}$$

Proof. Let s such that $2s = g - 2r - 2t$. According to Theorem 9,

$$\mathrm{sdim}(G) = |\partial(G)| - \alpha(G_{SR}) = \ell + c_2(C_g) - \alpha(G_{SR}) = \ell + 2r + s - \alpha(G_{SR}).$$

As shown in Figure 8 (when $g = 8$, $r = 2$, $s = 1$ and $t = 1$), it is easy to check that $\alpha(G_{SR}) = r + s$ (resp., $\alpha(G_{SR} = r + s + 1)$ if $t = 0$ (resp., $t \geq 1$).

Hence, $\mathrm{sdim}(G) = \ell + 2r + s - r - s = \ell + r$ (resp., $\mathrm{sdim}(G) = \ell + 2r + s - r - s - 1 = \ell + r - 1$) if $t = 0$ (resp., $t \geq 1$). □

We finalize this section by showing that the problem of computing the strong metric dimension of an odd unicyclic graph is certainly quite more complicated than in the even case.

Proposition 6. *Let G be a unicyclic graph of odd girth* $5 \leq g = 2k+1$ *with* ℓ *leaves. If* C_g *is its cycle and* $2 \leq c_2(C_g) = h \leq 2k-2$, *then*

$$\max\{\ell - 1, \ell + h - k\} \leq \mathrm{sdim(G)} \leq \ell + \left\lfloor \frac{\mathrm{h}}{2} \right\rfloor - 1.$$

Moreover, both bounds are tight.

Proof. The inequality $\ell - 1 \leq \mathrm{sdim(G)}$ and the upper bound are a direct consequence of Theorem 5 and Proposition 4, respectively. Hence, to prove the lower bound it is enough to show that if $k-1 \leq h \leq 2k-2$, then $\alpha(G_{SR}) \leq k$, since $\mathrm{sdim(G)} = \ell + \mathrm{h} - \alpha(\mathrm{G_{SR}})$.

According to Remark 1, suppose that $G_{SR}[W]$ is the disjoint union of ρ paths, in such a way that $\rho = \mu + \eta$, there are μ odd paths and η even paths. If $G_{SR}[W] = \sum_{\kappa=1}^{\mu} P_{i_\kappa} + \sum_{\kappa=1}^{\eta} P_{j_\kappa}$, then $\sum_{\kappa=1}^{\mu} \alpha(P_{i_\kappa}) + \sum_{\kappa=1}^{\eta} \alpha(P_{j_\kappa}) \leq \alpha(G_{SR}) \leq \sum_{\kappa=1}^{\mu} \alpha(P_{i_\kappa}) + \sum_{\kappa=1}^{\eta} \alpha(P_{j_\kappa}) + 1$. If for every $\kappa \in [\mu]$ (resp., $\kappa \in [\eta]$), $i_\kappa = 2r_\kappa + 1$ (resp., $j_\kappa = 2s_\kappa$), then

$$\sum_{\kappa=1}^{\mu} r_\kappa + \sum_{\kappa=1}^{\eta} s_\kappa + \mu \leq \alpha(G_{SR}) \leq \sum_{\kappa=1}^{\mu} r_\kappa + \sum_{\kappa=1}^{\eta} s_\kappa + \mu + 1$$

and

$$g \geq \sum_{\kappa=1}^{\mu} 2r_\kappa + \mu + \sum_{\kappa=1}^{\eta} 2s_\kappa + \rho = 2\left[\sum_{\kappa=1}^{\mu} r_\kappa + \sum_{\kappa=1}^{\eta} 2s_\kappa + \mu\right] + \eta.$$

We distinguish the following cases:

Case (1): $\eta \geq 2$. In this case, $k \geq \sum_{\kappa=1}^{\mu} r_\kappa + \sum_{\kappa=1}^{\eta} 2s_\kappa + \mu + 1 \geq \alpha(G_{SR})$, since g is odd.

Case (2): $g - h \geq \rho + 2$. In this case,

$$g \geq h + \rho + 2 = h + \mu + \eta + 2 = 2\left[\sum_{\kappa=1}^{\mu} r_\kappa + \sum_{\kappa=1}^{\eta} s_\kappa + \mu + 1\right] + \eta.$$

Hence, $k \geq \sum_{\kappa=1}^{\mu} r_\kappa + \sum_{\kappa=1}^{\eta} s_\kappa + \mu + 1 \geq \alpha(G_{SR})$, since g is odd.

Case (3): $\rho \leq g - h \leq \rho + 1$ and $0 \leq \eta \leq 1$. In this case, $\alpha(G_{SR}) = \sum_{\kappa=1}^{\mu} r_\kappa + \sum_{\kappa=1}^{\eta} s_\kappa + \mu$.

Hence, $k \geq \sum_{\kappa=1}^{\mu} r_\kappa + \sum_{\kappa=1}^{\eta} s_\kappa + \mu = \alpha(G_{SR})$, since g is odd.

To show the tightness of the upper bound, consider the case $h = g - 3 = 2k - 2$ and $\rho = \eta = 1$, and notice that $\alpha(G_{SR}) = \alpha(P_{2k-2}) + 1 = k$.

Hence, $\mathrm{sdim(G)} = \ell + \mathrm{h} - \alpha(\mathrm{G_{SR}}) = \ell + 2\mathrm{k} - 2 - \mathrm{k} = \ell + \mathrm{k} - 2 = \ell + \left\lfloor \frac{\mathrm{h}}{2} \right\rfloor - 1$.

The tightness of the lower bound when $0 \leq h \leq k-1$ is a direct consequence of Proposition 3.

If $h = k$, consider the case $G_{SR}[W] = kK_1$ and notice that $\alpha(G_{SR}) = k$. If $k+1 \leq h \leq 2k-2$, then consider the case $G_{SR}[W] = (g-h-1)K_1 + P_{2h-2k}$. Then, $\alpha(G_{SR}) = g - h - 1 + \alpha(P_{2h-2k}) = 2k + 1 - h - 1 + h - k = k$. Thus, if $k \leq h \leq 2k-2$, then $\mathrm{sdim(G)} = \ell + \mathrm{h} - \alpha(\mathrm{G_{SR}}) = \ell + \mathrm{h} - \mathrm{k}$. □

6. Dominating Metric Dimension

After the introduction of metric dimension, it seems natural to study when a metric-locating set verifies additional properties, and among all the vertex set properties, one of the most important is domination. Thus, Brigham et al. in [4], and also and independently Henning and Oellermann in [42], defined the following parameter.

Definition 4. A set of vertices of a graph G is called *metric-locating-dominating* (an *MLD-set* for short) if it is both locating and dominating.

An MLD-set of minimum cardinality is called an *MLD-basis* of G. The *dominating metric dimension* of G, denoted by $\mathrm{ddim(G)}$, is the cardinality of a MLD-basis. To know more about this parameter, see mainly [4,42,43] and also [44–46].

As usual, the dominating metric dimension of trees is well known.

Theorem 12 ([42]). *Let T be a tree with $\ell(T)$ leaves and $s(T)$ support vertices. Then,* $\mathrm{ddim(T)} = \gamma\mathrm{(T)} + \ell\mathrm{(T)} - \mathrm{s(T)}$.

Corollary 4 ([42]). *Let T be a tree. Then,* $\mathrm{ddim(T)} = \gamma\mathrm{(T)}$ *if and only if T contains no strong support vertex.*

In particular, $\mathrm{ddim}(\mathrm{P_n}) = \gamma(\mathrm{P_n}) = \lceil \frac{n}{3} \rceil$. The same reasoning as used in trees can be applied in general graphs to obtain a lower bound.

Proposition 7. *Let G be a graph with $\ell(G)$ leaves and $s(G)$ support vertices. Then,* $\mathrm{ddim(G)} \geq \gamma\mathrm{(G)} + \ell\mathrm{(G)} - \mathrm{s(G)}$.

Proof. Consider the set $S(G)$ of strong support vertices of G. Let D be a minimum MLD-set of G, containing as few leaves as possible. This means that, for every vertex $v \in S(G)$, D contains all except one leaf v' adjacent to v, as well as vertex v. So, the set $D' = D - \left(\cup_{v \in S(G)} (L_v - v') \right)$ is a dominating set of G. Hence,

$$\gamma(G) \leq |D'| \leq |D| - \sum_{v \in S(G)} (|L_v| - 1) = \mathrm{ddim(G)} - \ell\mathrm{(G)} + \mathrm{s(G)},$$

and the inequality $\mathrm{ddim(G)} \geq \gamma\mathrm{(G)} + \ell\mathrm{(G)} - \mathrm{s(G)}$ follows. □

As an immediate consequence of this result, the following one holds.

Lemma 10. *For every graph G, $\gamma(G) \leq \mathrm{ddim(G)}$. Moreover, if* $\mathrm{ddim(G)} = \gamma\mathrm{(G)}$, *then G has no strong support vertex.*

As a consequence, there exist several situations in which a dominating set is also an MLD-set.

Lemma 11. *Let G be a unicyclic graph of girth g, where C_g is its unique cycle, with no strong support vertex. If D is a dominating set of G, then D is an MLD-set of G whenever any of the following conditions hold:*

1. $g \notin \{3, 4, 6\}$.
2. $g = 3$ *and* $|D \cap V(C_3)| \geq 2$.
3. $g \in \{4, 6\}$ *and* $|D \cap V(C_g)| \geq 3$.

Proof. Let $u, v \in V(G) - D$. Suppose that $N(u) \cap D = N(v) \cap D$. Notice that there is a unique vertex z such that $N(u) \cap D = N(v) \cap D = \{z\}$, since either $g \neq 4$ or $g = 4$ and $|D \cap V(C_4)| \geq 3$. As z is not a strong support vertex, then we can suppose w.l.o.g. that $deg(v) \geq 2$. Take a vertex $w \in N(v) - z$. Notice that $w \notin D$, as $N(v) \cap D = \{z\}$. Let $y \in N(w) \cap D$. Observe that $y \neq z$, as either $g \neq 3$ or $g = 3$ and $|D \cap V(C_3)| \geq 2$. Certainly, $d(v, y) = 2$ and it is also clear that $d(u, y) > 2$, unless $g = 6$ (see Figure 10a). If $g = 6$ and $d(u, y) = 2$, take the vertex $x \in N(u) \cap N(y)$ and notice first that it necessarily belongs to D since $|D \cap V(C_6)| \geq 3$, and second that $d(u, x) = 1$ and $d(v, x) = 3$ (see Figure 10b). □

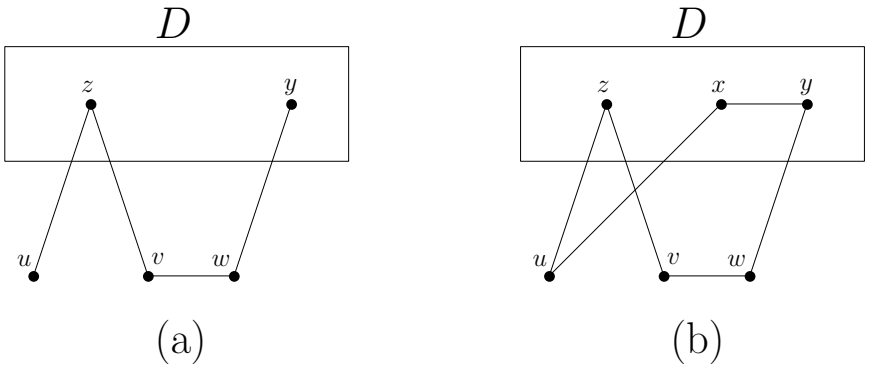


Figure 10. (**a**) D is a dominating set of a unicyclic graph with no strong support vertex, and $d(u, y) > 2$. (**b**) Same case when $d(u, v) = 2$.

From the preceding lemmas, the following result immediately follows.

Corollary 5. *Let G be a unicyclic graph of girth g. If $g \notin \{3, 4, 6\}$, then* $\mathrm{ddim(G)} = \gamma\mathrm{(G)}$ *if and only if G contains no strong support vertex.*

In particular, if $g \notin \{3, 4, 6\}$, then $\mathrm{ddim(C_g)} = \gamma\mathrm{(C_g)} = \lceil \frac{g}{3} \rceil$. The previous results lead us to obtain the same characterization as trees for unicyclic graphs except for the cases $g = 3, 4, 6$.

Theorem 13. *Let G be a unicyclic graph of girth g with $\ell(G)$ leaves and $s(G)$ support vertices. If $g \notin \{3, 4, 6\}$, then* $\mathrm{ddim(G)} = \gamma\mathrm{(G)} + \ell\mathrm{(G)} - \mathrm{s(G)}$.

Proof. According to Proposition 7, $\mathrm{ddim(G)} \geq \gamma\mathrm{(G)} + \ell\mathrm{(G)} - \mathrm{s(G)}$. To prove the other inequality, consider the set $S(G)$ of strong support vertices of G. If $|S(G)|$=0, then $\ell(G) = s(G)$, and according to Corollary 5, the equality holds. Suppose thus that $|S(G)| \geq 1$. For each vertex $v \in S(G)$, take a vertex $v' \in L_v = N(v) \cap \ell(G)$. Consider the unicyclic graph G' obtained from G by deleting, for every vertex $v \in S(G)$, the set $L_v - v'$. Hence, G' has no strong support vertex. Let D' be a minimum dominating set of G' containing all the support vertices of G'. According to Lemma 11, D' is an MLD-set of G'. Thus, $D' \cup \left(\cup_{v \in S(G)} (L_v - v') \right)$ is an MLD-set of G, and so

$$\mathrm{ddim(G)} \leq |\mathrm{D'}| + \sum_{\mathrm{v \in S(G)}} (|\mathrm{L_v}| - 1) = \gamma\mathrm{(G)} + \ell\mathrm{(G)} - \mathrm{s(G)}.$$

□

The cases $g = 3, 4$ and 6 are approached in the following two results.

Lemma 12. *Let G be a unicyclic graph of girth $g = 4$ and C_4 its unique cycle, with no strong support vertex. If D is a minimum dominating set of G and $|D \cap V(C_g)| = 1$, then* $\gamma(G) \leq \mathrm{ddim(G)} \leq \gamma\mathrm{(G)} + 1$.

Proof. Let $u, v \in V(G) - D$. Suppose that $N(u) \cap D = N(v) \cap D$. Notice that there is a unique vertex z such that $N(u) \cap D = N(v) \cap D = \{z\}$, since $|D \cap V(C_4)| = 1$. As z is not a strong support vertex, then we can suppose w.l.o.g. that $deg(v) \geq 2$. Take a vertex $w \in N(v) - z$ and notice that $w \notin D$, as $N(v) \cap D = \{z\}$. Let $y \in N(w) \cap D$. Observe that $y \neq z$, as $g \neq 3$. Certainly, $d(v, y) = 2$. Suppose that $d(u, y) = 2$, as otherwise we are finished. This means that $w \in N(u)$. We distinguish the following cases:

Case (a): If $deg(v) \geq 3$, take $x \in N(v) \setminus \{w, z\}$. Let $\eta \in D \cap N(x)$ and notice that $d(v, \eta) = 2$ and $d(u, \eta) = 4$ (see Figure 11a).

Case (b): If $deg(u) = deg(v) = 2$, consider the set $D' = D + v$. In order to prove that D' is an MLD-set of G, take a pair of vertices $a, b \in V(G) - D'$. If $N(a) \cap D = N(b) \cap D$, then there is a unique vertex c such that $N(a) \cap D = N(b) \cap D = \{c\}$. As c is not a strong support vertex, then we can suppose w.l.o.g. that $deg(b) \geq 2$. Take a vertex $d \in N(b) - c$ and notice that $d \notin D$, as $N(b) \cap D = \{c\}$. Let $e \in N(d) \cap D$ and check that $e \neq e$. Certainly, $d(b, e) = 2$. Suppose that $d(a, e) = 2$, as otherwise we are finished. This means that $e \in N(c)$ and also that $c = z$, $e = v$, $b = u$ and $d = w$. Hence, $d(b, y) = d(u, y) = 2$ and $d(a, y) = 4$ (see Figure 11b). □

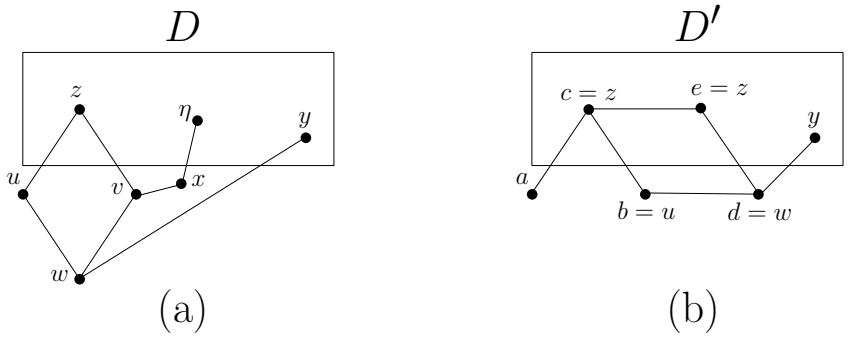


Figure 11. (**a**) D is a dominating set of a unicyclic graph of girth $g = 4$, with no strong support vertex, and where $deg(v) = \geq 3$. (**b**) Same situation where $deg(u) = deg(v) = 2$.

By considering the previous lemma along with items (2) and (3) of Lemma 11, and reasoning in a similar way as in the proof Theorem 13, the following result holds.

Proposition 8. *Let G be a unicyclic graph of girth g with $\ell(T)$ leaves and $s(T)$ support vertices. If $g \in \{3, 4, 6\}$, then $\gamma(G) + \ell(T) - s(T) \leq \mathrm{ddim(G)} \leq \gamma(\mathrm{G}) + \ell(\mathrm{T}) - \mathrm{s(T)} + 1$.*

As for tightness of the above upper bound, in some cases it is required to add an additional vertex to have an MLD-set starting from a minimum dominating set and the set of leaves hanging on the strong support vertices (see Figure 12, for some examples).

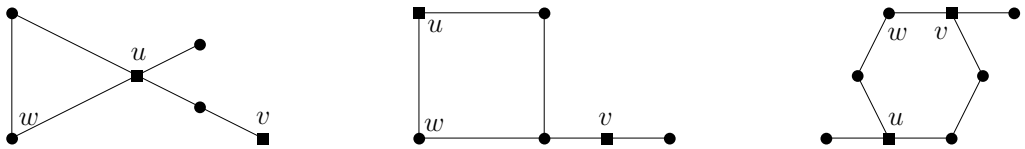


Figure 12. In all cases, $\{u, v\}$ is a dominating set, $\gamma(G) = 2$, $\{u, v, w\}$ is an MLD-basis and $\mathrm{ddim(G)} = 3$.

7. Other Metric Locating Parameters

In this section, we revise the situation of other metric locating parameters regarding unicyclic graphs. For the last two of them, a completed characterization has been obtained by other authors. However, for the three first parameters that characterization is far from being obtained, and here we describe the advances in that matter.

7.1. Fault-Tolerant Metric Dimension

A set of vertices S of a graph G is called 2-*locating* if every pair of distinct vertices $x, y \in V(G)$ is resolved by at least 2 elements of S.

A 2-locating set of minimum cardinality is called a 2-*metric basis* of G. The 2-*metric dimension* of G, denoted by $\dim_2(G)$, is the cardinality of a 2-metric basis.

This parameter was formally introduced by Hernando et al. in [7]. In this work, 2-metric locating sets were called *fault-tolerant sets*, and the 2-metric dimension was named the *fault-tolerant metric dimension* of G. To know more about this parameter, see also [47,48].

It is a routine exercise to check that $\dim_2(P_n) = 2$ and $\dim_2(C_n) = 3$.

Theorem 14 ([7]). *Let T be a tree with ℓ_s strong leaves. Then, $\dim_2(T) = \ell_s$.*

7.2. k-Metric Dimension

Let k, n be a pair of vertices such that $2 \leq k \leq n-1$. A set of vertices S of a graph G of order n is called *k-locating* if every pair of distinct vertices $x, y \in V(G)$ is resolved by at least k elements of S.

A k-locating set of minimum cardinality is called a *k-metric basis* of G. The *k-metric dimension* of G, denoted by $\dim_k(G)$, is the cardinality of a k-metric basis.

This parameter was formally introduced by Estrada-Moreno in [12]. The case $k = 2$ was previously introduced by Hernando et al. in [7] (see previous subsection). To know more about this parameter, see also [9,49–51].

A graph G is called *k-metric dimensional*, if k is the largest integer such that G contains a k-locating set.

Proposition 9 ([12,52]). *Let $k, n \geq 3$ be an integers such that $k+1 \leq n$. Then, the following hold:*

- *P_n is $(n-1)$-dimensional.*
- $\dim_k(P_n) = k+1$.
- *If n is odd (resp., even) , then C_n is $(n-1)$-dimensional (resp., $(n-2)$-dimensional).*
- $$\dim_k(C_n) = \begin{cases} k+2 & \text{if } n \text{ is even and } \frac{n}{2} \leq k \leq n-2, \\ k+1 & \text{otherwise.} \end{cases}.$$

Theorem 15 ([12]). *Let G be a k-metric dimensional graph of order $n \geq 3$. Then, the following hold:*

- $2 \leq k \leq n-1$.
- *$k = 2$ if and only if G has twin vertices.*

Let $\mathcal{M}(T)$ denote the set of strong exterior major vertices of a tree T. For every vertex $w \in \mathcal{M}(T)$, let $Ter(w)$ and $ter(w)$ denote the set and number of terminal vertices of w, respectively.

If $Ter(w) = \{u_1, \ldots, u_h\}$, then $l(w) = \min_{i \in [h]} d(u_i, w)$ and $\zeta(w) = \min_{\substack{i,j \in [h] \\ i \neq j}} [d(u_i, w) + d(w, w_j)]$. Finally, $\zeta(T) = \min_{w \in \mathcal{M}(T)} \zeta(w)$ and

$$I_r(w) = \begin{cases} [ter(w)-1][r - l(w)] + l(w) & \text{if } l(w) \leq \lfloor \frac{r}{2} \rfloor \\ [ter(w)-1]\lceil \frac{r}{2} \rceil + \lfloor \frac{r}{2} \rfloor & \text{otherwise.} \end{cases}.$$

Theorem 16 ([12]). *Let T be a k-metric dimensional tree, other than a path. Then, the following hold:*

- $k = \zeta(T)$.

- *For every* $r \in [\zeta(T)]$, $\dim_r(T) = \sum_{w \in \mathcal{M}(T)} I_r(w)$.

In [52], some partial results for unicyclic graphs are presented. For example, it was given a closed formula for the k-metric dimension of a unicyclic graph G of girth g, whenever $\rho(G) = 0$ and $c_2(C_g) = g - 1$, where C_g is the cycle of G.

7.3. Edge Metric Dimension

Let $G = (V, E)$ be a graph. A vertex $v \in V$ is said to resolve two edges $e, f \in E$ if $d_G(v, e) \neq d_G(v, f)$. A set of vertices S of a graph G is called *edge locating* if every pair of distinct edges $e, f \in E$ is resolved by a vertex of S.

An edge locating set of minimum cardinality is called a *edge metric basis* of G. The *edge metric dimension* of G, denoted by $\mathrm{edim(G)}$, is the cardinality of an edge metric basis. This parameter was formally introduced by Kelenc et al. in [11]. To know more about this parameter, see also [33,34,53–56].

It is a routine exercise to check that $\mathrm{edim}(P_n) = 1$ and $\mathrm{edim}(C_g) = 2$.

Theorem 17 ([11]). *For every tree* T, $\mathrm{edim(T)} = \ell\mathrm{(T)} - \lambda\mathrm{(T)}$.

Theorem 18 ([13]). *Let* G *be a unicyclic graph with* $\ell(G) = \ell$ *leaves,* $\lambda(G) = \lambda$ *exterior major vertices and* $\rho(G) = \rho$ *branch-active vertices. Then,*

$$\ell - \lambda + \max\{2 - \rho, 0\} \leq \mathrm{edim(G)} \leq \ell - \lambda + \max\{2 - \rho, 0\} + 1.$$

Theorem 19 ([57]). *Let* G *be a unicyclic graph. If* $e \in E(G)$ *and* $G - e$ *is connected, then* $\mathrm{edim(G)} \leq \mathrm{edim(G - e)} + 1$.

Theorem 20 ([13]). *If* G *is a unicyclic graph, then* $|\dim(G) - \mathrm{edim(G)}| \leq 1$.

Moreover, a characterization of the sets of unicyclic graphs attaining each of the three possible equalities was also given in [13]. In particular, it was proved that if G is an odd (resp., even) unicyclic graph, then $\dim(G) \leq \mathrm{edim(G)}$ (resp., $\dim(G) \geq \mathrm{edim(G)}$).

7.4. Mixed Metric Dimension

A set of vertices S of a graph G is called *mixed locating* if every pair of distinct elements (vertices or edges) is resolved by a vertex of S.

A mixed locating set of minimum cardinality is called a *mixed metric basis* of G. The *mixed metric dimension* of G, denoted by $\mathrm{mdim(G)}$, is the cardinality of mixed metric basis. This parameter was formally introduced by Kelenc et al. in [10]. To know more about this parameter, see [10,58,59].

It is straightforward to check that, for every integer $n \geq 3$, $\mathrm{mdim}(P_n) = 2$ and $\mathrm{mdim}(C_n) = 3$.

Theorem 21 ([10]). *Let* G *be a graph of order* $n \geq 2$. *Then,* $\mathrm{mdim(G)} = 2$ *if and only if* G *is the path* P_n.

Theorem 22 ([10]). *For any tree* T, $\mathrm{mdim(T)} = \ell$.

Theorem 23 ([59]). *Let* G *be a unicyclic graph whose cycle* C_g *has* t *root vertices. Then,*

$$\mathrm{mdim(G)} = \ell\mathrm{(G)} + \max\{3 - t, 0\} + \epsilon$$

where $\epsilon = 1$ *if* $t \geq 3$ *and there is no geodesic triple of root vertices on* C_g, *while* $\epsilon = 0$ *otherwise.*

7.5. Local Metric Dimension

A set of vertices S of a graph G is called *local locating* if, for every pair of adjacent vertices $x, y \in V(G)$, there is a vertex $v \in S$, such that $d_G(x, v) \neq d_G(y, x)$.

In other words, S is a local locating set of G if, for every pair of adjacent vertices $x, y \in E(G)$, $r(x|S) \neq r(y|S)$.

A local locating set of minimum cardinality is called a *local metric basis* of G. The *local metric dimension* of G, denoted by $\mathrm{ldim(G)}$, is the cardinality of a local metric basis. This parameter was formally introduced by Okamoto et al. in [8]. To know more about this parameter, see also [60–62].

Theorem 24 ([8]). $\mathrm{ldim(G)} = 1$ *if and only if* G *is bipartite.*

Theorem 25 ([63]). *Let* G *be a unicyclic graph, where* C_g *is its unique cycle. Then,*

$$\mathrm{ldim(G)} = \begin{cases} 1 & g \text{ is even} \\ 2 & g \text{ is odd} \end{cases}$$

8. Conclusions and Further Work

The primary purpose of this paper has been to survey the state of art of the main metric locating parameters for the family of pseudotrees, i.e., for paths, cycles, trees and unicyclic graphs.

During the process of writing this survey, the authors noticed that regarding paths, cycles and trees, everything is known about the nine parameters analyzed here (see Table 1). However, complete characterizations have not been obtained for unicyclic graphs in all the metric locations that we have surveyed, so we set out to contribute with a number of new results. As a consequence, we have been able to make significant progress in four of the nine parameters considered in this work.

To be more precise, we have completed the work for the doubly metric dimension and the dominating metric dimension, and we also have contributed with significant new results for the metric dimension and the strong metric dimension.

As for the remaining five parameters, for two of them (the mixed metric dimension and the local metric dimension) the values have already been computed, while for the remaining three parameters (the edge metric dimension, the fault-tolerant metric dimension and the k-metric dimension), there is plenty of work to be performed to obtain both bounds and exact values.

For further details, see Tables 2 and 3, having in mind that G denotes a proper unicyclic graph with girth g, ℓ leaves, ℓ_s strong leaves, s support vertices, λ exterior major vertices, ρ branch-active vertices, c_2 trivial vertices and t root vertices. Moreover, $\hat{\rho} = max\{2 - \rho, 0\}$, $\hat{t} = max\{3 - t, 0\}$ and $\mu(g, \ell) = \max\{\lceil \frac{g}{2} \rceil, \ell - 1\}$.

We conclude with a list of open problems:

1. Starting mainly both from [13] and from Section 4 of this paper, characterizing the family of proper odd unicyclic graphs G such that $\dim(G) = \ell - \lambda + \max\{2 - \rho, 0\}$.
2. Starting mainly both from [13] and from Section 4 of this paper, characterizing the family of proper even unicyclic graphs G such that $\dim(G) = \ell - \lambda + \max\{2 - \rho, 0\}$.
3. Starting mainly both from [35] and from Section 5 of this paper, characterizing the family of proper odd unicyclic graphs G of girth g with at most $\lceil \frac{g}{2} \rceil$ leaves such that $\mathrm{sdim(G)} = \lceil \frac{g}{2} \rceil$.

4. Starting mainly both from [35] and from Section 5 of this paper, either obtaining a result similar to the one displayed in Theorem 11 when g is odd, or at least characterizing the families of graphs achieving the bounds shown in Proposition 6.
5. Starting mainly both from [13] and from Section 4 of this paper, characterizing the family of proper unicyclic graphs G such that $\mathrm{edim}(G) = \ell - \lambda + \max\{2 - \rho, 0\}$, or at least obtaining similar results to those appearing in Sections 4.1 and 4.2.
6. Starting mainly from [7,12,52], obtaining tight bounds, both lower and upper, of the fault-tolerant metric dimension of proper unicyclic graphs.

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