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On Spectral Graph Determination

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Abstract: The study of spectral graph determination is a fascinating area of research in spectral graph theory and algebraic combinatorics. This field focuses on examining the spectral characterization of various classes of graphs, developing methods to construct or distinguish cospectral nonisomorphic graphs, and analyzing the conditions under which a graph's spectrum uniquely determines its structure. This paper presents an overview of both classical and recent advancements in these topics, along with newly obtained proofs of some existing results, which offer additional insights.

Keywords: spectral graph theory; spectral graph determination; cospectral nonisomorphic graphs; Haemers' conjecture; Turán graphs; graph operations

MSC: 05C50; 05C75; 05C76

1. Introduction

Spectral graph theory lies at the intersection of combinatorics and matrix theory and explores the structural and combinatorial properties of graphs through the analysis of the eigenvalues and eigenvectors of matrices associated with these graphs [1–5]. Spectral properties of graphs offer powerful insights into a variety of useful graph characteristics, enabling the determination or estimation of features, such as the independence number, clique number, chromatic number, and the Shannon capacity of graphs, which are notoriously NP-hard to compute.

A particularly intriguing topic in spectral graph theory is the study of cospectral graphs, i.e., graphs that share identical multisets of eigenvalues with respect to one or more matrix representations. While isomorphic graphs are always cospectral, nonisomorphic graphs may also share spectra, leading to the study of nonisomorphic cospectral (NICS) graphs. This phenomenon raises profound questions about the extent to which a graph's spectrum encodes its structural properties. Conversely, graphs determined by their spectrum (DS graphs) are uniquely identifiable, up to isomorphism, by their eigenvalues. In other words, a graph is DS if and only if no other nonisomorphic graph shares the same spectrum.

The problem of spectral graph determination and the characterization of DS graphs dates back to the pioneering 1956 paper by Günthard and Primas [6], which explored the interplay between graph theory and chemistry. This paper posed the question of whether graphs can be uniquely determined by their spectra with respect to their adjacency matrix $\mathbf{A}$.

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While every graph can be determined by its adjacency matrix, which enables the determination of every graph by its eigenvalues and a basis of corresponding eigenvectors, the characterization of graphs for which eigenvalues alone suffice for identification forms a fertile area of research in spectral graph theory. This research holds both theoretical interest and practical implications.

Subsequent studies have broadened the scope of this question to include determination by the spectra of other significant matrices, such as the Laplacian matrix (**L**), signless Laplacian matrix (**Q**), and normalized Laplacian matrix ($\mathcal{L}$), among many other matrices associated with graphs. The study of cospectral and DS graphs with respect to these matrices has become a cornerstone of spectral graph theory. This line of research has far-reaching applications in diverse fields, including chemistry and molecular structure analysis, physics and quantum computing, network communication theory, machine learning, and data science.

One of the most prominent conjectures in this area is Haemers' conjecture [7,8], which posits that most graphs are determined by the spectrum of their adjacency matrices (**A**-DS). Despite the difficulty of proving this open conjecture, some theoretical and experimental progress on the theme of this conjecture has been recently presented in [9,10], while graphs or graph families that are not DS also continue to be discovered. Haemers' conjecture has spurred significant interest in classifying DS graphs and understanding the factors that influence spectral determination, particularly among special families of graphs such as regular graphs, strongly regular graphs, trees, graphs of pyramids, as well as the construction of NICS graphs by a variety of graph operations. Studies in these directions of research have been covered in the seminal works by Schwenk [11] and by van Dam and Haemers [12,13], as well as in more recent studies (in part by the authors) such as [9,14–45], and references therein. Specific contributions of these papers to the problem of the spectral determination of graphs are addressed in the continuation of this article.

This paper surveys both classical and recent results on spectral graph determination and also presents newly obtained proofs of some existing results, which offer additional insights.

The paper emphasizes the significance of adjacency spectra (**A**-spectra), and it provides conditions for **A**-cospectrality, **A**-NICS, and **A**-DS graphs, offering examples that support or refute Haemers' conjecture. We furthermore address the cospectrality of graphs with respect to the Laplacian, signless Laplacian, and normalized Laplacian matrices. For regular graphs, cospectrality with respect to any one of these matrices (or the adjacency matrix) implies cospectrality with respect to all the others, which enables a unified framework for studying DS and NICS graphs across different matrix representations. However, for irregular graphs, cospectrality with respect to one matrix does not necessarily imply cospectrality with respect to another. This distinction underscores the complexity of analyzing spectral properties in irregular graphs, where the interplay among different matrix representations becomes more intricate and often necessitates distinct techniques for characterization and comparison.

The structure of the paper is as follows: Section 2 provides preliminary material in matrix theory, graph theory, and graph-associated matrices. Section 3 focuses on graphs determined by their spectra (with respect to one or multiple matrices). Section 4 examines special families of graphs and their determination by adjacency spectra. Section 5 analyzes unitary and binary graph operations, emphasizing their impact on spectral determination and construction of NICS graphs. Finally, Section 6 concludes the paper with open questions and an outlook on spectral graph determination, highlighting areas for further research.

2. Preliminaries

The present section provides preliminary material and notation in matrix theory, graph theory, and graph-associated matrices, which are used for the presentation of this paper.

2.1. Matrix Theory Preliminaries

The following standard notation in matrix theory is used in this paper:

- $\mathbb{R}^{n\times m}$ denotes the set of all $n \times m$ matrices with real entries,
- $\mathbb{R}^n \triangleq \mathbb{R}^{n\times 1}$ denotes the set of all n-dimensional column vectors with real entries,
- $\mathbf{I}_n \in \mathbb{R}^{n\times n}$ denotes the $n \times n$ identity matrix,
- $\mathbf{0}_{k,m} \in \mathbb{R}^{k\times m}$ denotes the $k \times m$ all-zero matrix,
- $\mathbf{J}_{k,m} \in \mathbb{R}^{k\times m}$ denotes the $k \times m$ all-ones matrix,
- $\mathbf{1}_n \triangleq \mathbf{J}_{n,1} \in \mathbb{R}^n$ denotes the n-dimensional column vector of ones.

Throughout this paper, we deal with real matrices. The concepts of Schur complement and interlacing of eigenvalues are useful in papers on spectral graph determination and cospectral graphs, and are also addressed in this paper.

Definition 1. *Let* $\mathbf{M}$ *be a block matrix*

$$\mathbf{M} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}, \tag{1}$$

where the block $\mathbf{D}$ *is invertible. The Schur complement of D in M is*

$$\mathbf{M}/\mathbf{D} = \mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C}. \tag{2}$$

Schur proved the following remarkable theorem:

Theorem 1 (Theorem on the Schur complement [46])**.** *If D is invertible, then*

$$\det \mathbf{M} = \det(\mathbf{M}/\mathbf{D}) \det \mathbf{D}. \tag{3}$$

Theorem 2 (Cauchy Interlacing Theorem [47])**.** *Let* $\lambda_1 \geqslant \ldots \geqslant \lambda_n$ *be the eigenvalues of a symmetric matrix* $\mathbf{M}$ *and let* $\mu_1 \geqslant \ldots \geqslant \mu_m$ *be the eigenvalues of a* principal $m \times m$ submatrix of $\mathbf{M}$ *(i.e., a submatrix that is obtained by deleting the same set of rows and columns from M). Then* $\lambda_i \geqslant \mu_i \geqslant \lambda_{n-m+i}$ *for* $i = 1, \ldots, m$.

Definition 2 (Completely Positive Matrices)**.** *A matrix* $\mathbf{A} \in \mathbb{R}^{n\times n}$ *is called* completely positive *if there exists a matrix* $\mathbf{B} \in \mathbb{R}^{n\times m}$ *whose all entries are nonnegative such that* $\mathbf{A} = \mathbf{B}\mathbf{B}^{\mathrm{T}}$.

A completely positive matrix is, therefore, symmetric, and all its entries are nonnegative. The interested reader is referred to the textbook [48] on completely positive matrices, also addressing their connections to graph theory.

Definition 3 (Positive Semidefinite Matrices)**.** *A matrix* $\mathbf{A} \in \mathbb{R}^{n\times n}$ *is called* positive semidefinite *if* $\mathbf{A}$ *is symmetric, and the inequality* $\underline{x}^{\mathrm{T}}\,\mathbf{A}\,\underline{x} \geqslant 0$ *holds for every column vector* $\underline{x} \in \mathbb{R}^n$.

Proposition 1. *A symmetric matrix is positive semidefinite if and only if one of the following conditions hold:*

1. *All its eigenvalues are nonnegative (real) numbers.*
2. *There exists a matrix* $\mathbf{B} \in \mathbb{R}^{n\times m}$ *such that* $\mathbf{A} = \mathbf{B}\mathbf{B}^{\mathrm{T}}$.

The next result readily follows.

Corollary 1. *A completely positive matrix is positive semidefinite.*

Remark 1. *Regarding Corollary 1, it is natural to ask whether, under certain conditions, a positive semidefinite matrix whose all entries are nonnegative is also completely positive. By [48] (Theorem 3.35), this holds for all square matrices of order $n \leqslant 4$. Moreover, Ref. [48] (Example 3.45) also presents an explicit example of a matrix of order 5 that is positive semidefinite with all nonnegative entries but is not completely positive.*

2.2. Graph Theory Preliminaries

A graph $\mathsf{G} = (\mathsf{V}(\mathsf{G}), \mathsf{E}(\mathsf{G}))$ forms a pair where $\mathsf{V}(\mathsf{G})$ is a set of vertices and $\mathsf{E}(\mathsf{G}) \subseteq \mathsf{V}(\mathsf{G}) \times \mathsf{V}(\mathsf{G})$ is a set of edges.

In this paper, all the graphs are assumed to be as follows:

- *Finite*—$\left|\mathsf{V}(\mathsf{G})\right| < \infty$;
- *Simple*—G has no parallel edges and no self loops;
- *Undirected*—the edges in G are undirected.

We use the following terminology:

- The *degree*, $d(v)$, of a vertex $v \in \mathsf{V}(\mathsf{G})$ is the number of vertices in G that are adjacent to v.
- A *walk* in a graph G is a sequence of vertices in G, where every two consecutive vertices in the sequence are adjacent in G.
- A *path* in a graph is a walk with no repeated vertices.
- A *cycle* C is a closed walk, obtained by adding an edge to a path in G.
- The *length of a path or a cycle* is equal to its number of edges. A *triangle* is a cycle of length 3.
- A *connected graph* is a graph in which every pair of distinct vertices is connected by a path.
- The *distance* between two vertices in a connected graph is the length of a shortest path that connects them.
- The *diameter* of a connected graph is the maximum distance between any two vertices in the graph, and the diameter of a disconnected graph is set to be infinity.
- The *connected component* of a vertex $v \in \mathsf{V}(\mathsf{G})$ is the subgraph whose vertex set $\mathcal{U} \subseteq \mathsf{V}(\mathsf{G})$ consists of all the vertices that are connected to v by any path (including the vertex v itself), and its edge set consists of all the edges in $\mathsf{E}(\mathsf{G})$ whose two endpoints are contained in the vertex set $\mathcal{U}$.
- A *tree* is a connected graph that has no cycles (i.e., it is a connected and *acyclic* graph).
- A *spanning tree* of a connected graph G is a tree with the vertex set $\mathsf{V}(\mathsf{G})$ and some of the edges of G.
- A graph is *regular* if all its vertices have the same degree.
- A *d-regular* graph is a regular graph whose all vertices have degree d.
- A *bipartite graph* is a graph G whose vertex set is a disjoint union of two subsets such that no two vertices in the same subset are adjacent.
- A *complete bipartite graph* is a bipartite graph where every vertex in each of the two partite sets is adjacent to all the vertices in the other partite set.

Definition 4 (Complement of a graph)**.** *The complement of a graph G, denoted by $\overline{\mathsf{G}}$, is a graph whose vertex set is $\mathsf{V}(\mathsf{G})$, and its edge set is the complement set $\overline{\mathsf{E}(\mathsf{G})}$. Every vertex in $\mathsf{V}(\mathsf{G})$ is nonadjacent to itself in G and $\overline{\mathsf{G}}$, so $\{i, j\} \in \mathsf{E}(\overline{\mathsf{G}})$ if and only if $\{i, j\} \notin \mathsf{E}(\mathsf{G})$ with $i \neq j$.*

Definition 5 (Disjoint union of graphs)**.** *Let $\mathsf{G}_1, \ldots, \mathsf{G}_k$ be graphs. If the vertex sets in these graphs are not pairwise disjoint, let $\mathsf{G}'_2, \ldots, \mathsf{G}'_k$ be isomorphic copies of $\mathsf{G}_2, \ldots, \mathsf{G}_k$, respectively, such*

that none of the graphs $\mathsf{G}_1, \mathsf{G}_2', \ldots \mathsf{G}_k'$ *have a vertex in common. The disjoint union of these graphs, denoted by* $\mathsf{G} = \mathsf{G}_1 \dot\cup \ldots \dot\cup \mathsf{G}_k$*, is a graph whose vertex and edge sets are equal to the disjoint unions of the vertex and edge sets of* $\mathsf{G}_1, \mathsf{G}_2', \ldots, \mathsf{G}_k'$ *(*G *is defined up to an isomorphism).*

Definition 6. *Let* $k \in \mathbb{N}$ *and let* G *be a graph. Define* $k\mathsf{G} = \mathsf{G} \dot\cup \mathsf{G} \dot\cup \ldots \dot\cup \mathsf{G}$ *to be the disjoint union of* k *copies of* G*.*

Definition 7 (Join of graphs)**.** *Let* G *and* H *be two graphs with disjoint vertex sets. The join of* G *and* H *is defined to be their disjoint union together with all the edges that connect the vertices in* G *with the vertices in* H*. It is denoted by* $\mathsf{G} \vee \mathsf{H}$*.*

Definition 8 (Induced subgraphs)**.** *Let* $\mathsf{G} = (\mathsf{V}, \mathsf{E})$ *be a graph, and let* $\mathcal{U} \subseteq \mathsf{V}$*. The* subgraph of G induced by $\mathcal{U}$ *is the graph obtained by the vertices in* $\mathcal{U}$ *and the edges in* G *that has both ends on* $\mathcal{U}$*. We say that* H *is an* induced subgraph of G*, if it is induced by some* $\mathcal{U} \subseteq \mathsf{V}$*.*

Definition 9 (Strongly regular graphs)**.** *A regular graph* G *that is neither complete nor empty is called a* strongly regular *graph with parameters* (n, d, λ, μ)*, where* λ *and* μ *are nonnegative integers, if the following conditions hold:*

1. G *is a* d*-regular graph on* n *vertices.*
2. *Every two adjacent vertices in* G *have exactly* λ *common neighbors.*
3. *Every two distinct and nonadjacent vertices in* G *have exactly* μ *common neighbors.*

The family of strongly regular graphs with these four specified parameters is denoted by $\mathsf{srg}(n, d, \lambda, \mu)$*. It is important to note that a family of the form* $\mathsf{srg}(n, d, \lambda, \mu)$ *may contain multiple nonisomorphic strongly regular graphs. Throughout this work, we refer to a strongly regular graph as* $\mathsf{srg}(n, d, \lambda, \mu)$ *if it belongs to this family.*

Proposition 2 (Feasible parameter vectors of strongly regular graphs)**.** *The four parameters of a strongly regular graph* $\mathsf{srg}(n, d, \lambda, \mu)$ *satisfy the equality*

$$(n - d - 1)\mu = d(d - \lambda - 1). \tag{4}$$

Remark 2. *Equality (4) provides a necessary, but not sufficient, condition for the existence of a strongly regular graph* $\mathsf{srg}(n, d, \lambda, \mu)$*. For example, as shown in [49], no* $(76, 21, 2, 7)$ *strongly regular graph exists, even though the condition* $(n - d - 1)\mu = 378 = d(d - \lambda - 1)$ *is satisfied in this case.*

Notation 1 (Classes of graphs)**.**

- K_n *is the complete graph on* n *vertices.*
- P_n *is the path graph on* n *vertices.*
- $\mathsf{K}_{\ell,r}$ *is the complete bipartite graph whose degrees of partite sets are* ℓ *and* r *(with possible equality between* ℓ *and* r*).*
- S_n *is the star graph on* n *vertices* $\mathsf{S}_n = \mathsf{K}_{1,n-1}$*.*

Definition 10 (Integer-valued functions of a graph)**.**

- *Let* $k \in \mathbb{N}$*. A proper* k*-coloring of a graph* G *is a function* $c\colon \mathsf{V}(\mathsf{G}) \to \{1, 2, ..., k\}$*, where* $c(v) \neq c(u)$ *for every* $\{u, v\} \in \mathsf{E}(\mathsf{G})$*. The chromatic number of* G*, denoted by* $\chi(\mathsf{G})$*, is the smallest* k *for which there exists a proper* k*-coloring of* G*.*
- *A clique in a graph* G *is a subset of vertices* $U \subseteq \mathsf{V}(\mathsf{G})$ *where the subgraph induced by* U *is a complete graph. The clique number of* G*, denoted by* $\omega(\mathsf{G})$*, is the largest size of a clique in* G*; i.e., it is the largest order of an induced complete subgraph in* G*.*

- *An independent set in a graph* G *is a subset of vertices* $U \subseteq \mathsf{V}(\mathsf{G})$*, where* $\{u, v\} \notin \mathsf{E}(\mathsf{G})$ *for every* $u, v \in U$*. The independence number of* G*, denoted by* $\alpha(\mathsf{G})$*, is the largest size of an independent set in* G*.*

Definition 11 (Orthogonal and orthonormal representations of a graph)**.** *Let* G *be a finite, simple, and undirected graph, and let* $d \in \mathbb{N}$*.*

- *An orthogonal representation of the graph* G *in the* d*-dimensional Euclidean space* $\mathbb{R}^d$ *assigns to each vertex* $i \in \mathsf{V}(\mathsf{G})$ *a nonzero vector* $\mathbf{u}_i \in \mathbb{R}^d$ *such that* $\mathbf{u}_i^{\mathrm{T}} \mathbf{u}_j = 0$ *for every* $\{i, j\} \notin \mathsf{E}(\mathsf{G})$ *with* $i \neq j$*. In other words, for every two distinct and nonadjacent vertices in the graph, their assigned nonzero vectors should be orthogonal in* $\mathbb{R}^d$*.*
- *An orthonormal representation of* G *is additionally represented by unit vectors, i.e.,* $\|\mathbf{u}_i\| = 1$ *for all* $i \in \mathsf{V}(\mathsf{G})$*.*
- *In an orthogonal (orthonormal) representation of* G*, every two nonadjacent vertices in* G *are mapped (by definition) into orthogonal (orthonormal) vectors, but adjacent vertices may not necessarily be mapped into nonorthogonal vectors. If* $\mathbf{u}_i^{\mathrm{T}} \mathbf{u}_j \neq 0$ *for all* $\{i, j\} \in \mathsf{E}(\mathsf{G})$*, then such a representation of* G *is called faithful.*

Definition 12 (Lovász ϑ-function [50])**.** *Let* G *be a finite, simple, and undirected graph. Then, the Lovász* ϑ*-function of* G *is defined as*

$$\vartheta(\mathsf{G}) \triangleq \min_{\mathbf{c}, \{\mathbf{u}_i\}} \max_{i \in \mathsf{V}(\mathsf{G})} \frac{1}{\left(\mathbf{c}^{\mathrm{T}} \mathbf{u}_i\right)^2}, \tag{5}$$

where the minimum on the right-hand side of (5) *is taken over all unit vectors* $\mathbf{c}$ *and all orthonormal representations* $\{\mathbf{u}_i : i \in \mathsf{V}(\mathsf{G})\}$ *of* G*. In* (5)*, it suffices to consider orthonormal representations in a space of dimension at most* $n = |\mathsf{V}(\mathsf{G})|$*.*

The Lovász ϑ-function of a graph G can be calculated by solving (numerically) a convex optimization problem. Let $\mathbf{A} = (A_{i,j})$ be the $n \times n$ adjacency matrix of G with $n \triangleq |\mathsf{V}(\mathsf{G})|$. The Lovász ϑ-function $\vartheta(\mathsf{G})$ can be expressed as the solution of the following semidefinite programming (SDP) problem:

$$\boxed{\begin{array}{l} \text{maximize} \quad \mathrm{Tr}(\mathbf{B}\,\mathbf{J}_n) \\ \text{subject to} \\ \quad \begin{cases} \mathbf{B} \succeq 0, \\ \mathrm{Tr}(\mathbf{B}) = 1, \\ A_{i,j} = 1 \;\Rightarrow\; B_{i,j} = 0, \quad i, j \in [n]. \end{cases} \end{array}} \tag{6}$$

There exist efficient convex optimization algorithms (e.g., interior-point methods) to compute $\vartheta(\mathsf{G})$, for every graph G, with a precision of r decimal digits, and a computational complexity that is polynomial in n and r. The reader is referred to Section 2.5 of [41] for an account of the various interesting properties of the Lovász ϑ-function. Among these properties, the sandwich theorem states that for every graph G,

$$\alpha(\mathsf{G}) \leqslant \vartheta(\mathsf{G}) \leqslant \chi(\overline{\mathsf{G}}), \tag{7}$$

$$\omega(\mathsf{G}) \leqslant \vartheta(\overline{\mathsf{G}}) \leqslant \chi(\mathsf{G}). \tag{8}$$

The usefulness of (7) and (8) lies in the fact that while the independence, clique, and chromatic numbers of a graph are NP-hard to compute, the Lovász ϑ-function can be

efficiently computed as a bound in these inequalities by solving the convex optimization problem in (6).

2.3. Matrices Associated with a Graph

2.3.1. Four Matrices Associated with a Graph

Let $\mathsf{G} = (\mathsf{V}, \mathsf{E})$ be a graph with vertices $\{v_1, \ldots, v_n\}$. There are several matrices associated with G. In this survey, we consider four of them, all are symmetric matrices in $\mathbb{R}^{n\times n}$: the *adjacency matrix* ($\mathbf{A}$), *Laplacian matrix* (LM), *signless Laplacian matrix* ($\mathbf{Q}$), and the *normialized Laplacian matrix* ($\mathcal{L}$).

1. The adjacency matrix of a graph G, denoted by $\mathbf{A} = \mathbf{A}(\mathsf{G})$, has the binary-valued entries

$$(\mathbf{A}(\mathsf{G}))_{i,j} = \begin{cases} 1 & \text{if } \{v_i, v_j\} \in \mathsf{E}(\mathsf{G}), \\ 0 & \text{if } \{v_i, v_j\} \notin \mathsf{E}(\mathsf{G}). \end{cases} \tag{9}$$

2. The Laplacian matrix of a graph G, denoted by $\mathbf{L} = \mathbf{L}(\mathsf{G})$, is given by

$$\mathbf{L}(\mathsf{G}) = \mathbf{D}(\mathsf{G}) - \mathbf{A}(\mathsf{G}), \tag{10}$$

where

$$\mathbf{D}(\mathsf{G}) = \operatorname{diag}\big(d(v_1), d(v_2), \ldots, d(v_n)\big) \tag{11}$$

is the *diagonal matrix* whose entries in the principal diagonal are the degrees of the n vertices of G.

3. The signless Laplacian matrix of a graph G, denoted by $\mathbf{Q} = \mathbf{Q}(\mathsf{G})$, is given by

$$\mathbf{Q}(\mathsf{G}) = \mathbf{D}(\mathsf{G}) + \mathbf{A}(\mathsf{G}). \tag{12}$$

4. The normalized Laplacian matrix of a graph G, denoted by $\mathcal{L}(\mathsf{G})$, is given by

$$\mathcal{L}(\mathsf{G}) = \mathbf{D}^{-\frac{1}{2}}(\mathsf{G})\ \mathbf{L}(\mathsf{G})\ \mathbf{D}^{-\frac{1}{2}}(\mathsf{G}), \tag{13}$$

where

$$\mathbf{D}^{-\frac{1}{2}}(\mathsf{G}) = \operatorname{diag}\big(d^{-\frac{1}{2}}(v_1), d^{-\frac{1}{2}}(v_2), \ldots, d^{-\frac{1}{2}}(v_n)\big), \tag{14}$$

with the convention that if $v \in \mathsf{V}(\mathsf{G})$ is an isolated vertex in G (i.e., $d(v) = 0$), then $d^{-\frac{1}{2}}(v) = 0$. The entries of $\mathcal{L} = (\mathcal{L}_{i,j})$ are given by

$$\mathcal{L}_{i,j} = \begin{cases} 1, & \text{if } i = j \text{ and } d(v_i) \neq 0, \\ -\dfrac{1}{\sqrt{d(v_i)\, d(v_j)}}, & \text{if } i \neq j \text{ and } \{v_i, v_j\} \in \mathsf{E}(\mathsf{G}), \\ 0, & \text{otherwise.} \end{cases} \tag{15}$$

In the continuation of this section, we also occasionally refer to two other matrices that are associated with undirected graphs.

Definition 13. *Let* G *be a graph with n vertices and m edges. The* incidence matrix *of* G*, denoted by* $\mathbf{B} = \mathbf{B}(\mathsf{G})$*, is an* $n \times m$ *matrix with binary entries, and it is defined as follows:*

$$B_{i,j} = \begin{cases} 1 & \text{if vertex } v_i \in \mathsf{V}(\mathsf{G}) \text{ is incident to edge } e_j \in \mathsf{E}(\mathsf{G}), \\ 0 & \text{if vertex } v_i \in \mathsf{V}(\mathsf{G}) \text{ is not incident to edge } e_j \in \mathsf{E}(\mathsf{G}). \end{cases} \tag{16}$$

For an undirected graph, each edge e_j *connects two vertices* v_i *and* v_k*, and the corresponding column in* $\mathbf{B}$ *has exactly two* 1*'s, one for each vertex.*

Definition 14. *Let* G *be a graph with n vertices and m edges. An oriented incidence matrix of* G*, denoted by* $\mathbf{N} = \mathbf{N}(\mathsf{G})$*, is an* $n \times m$ *matrix with ternary entries from* $\{-1, 0, 1\}$*. This is defined as follows. One first selects an arbitrary orientation to each edge in* G*. Then, define*

$$N_{i,j} = \begin{cases} -1 & \text{if vertex } v_i \in \mathsf{V}(\mathsf{G}) \text{ is the tail (starting vertex) of edge } e_j \in \mathsf{E}(\mathsf{G}), \\ +1 & \text{if vertex } v_i \in \mathsf{V}(\mathsf{G}) \text{ is the head (ending vertex) of edge } e_j \in \mathsf{E}(\mathsf{G}), \\ 0 & \text{if vertex } v_i \in \mathsf{V}(\mathsf{G}) \text{ is not incident to edge } e_j \in \mathsf{E}(\mathsf{G}). \end{cases} \tag{17}$$

Consequently, each column of $\mathbf{N}$ *contains exactly one entry equal to 1 and one entry equal to* -1*, representing the head and tail of the corresponding oriented edge in the graph, respectively, with all other entries in the column being zeros.*

For $X \in \{A, L, Q, \mathcal{L}\}$, the *X-spectrum* of a graph G, $\sigma_X(G)$, is the multiset of the eigenvalues of $X(G)$. We denote the elements of the multiset of eigenvalues of $\{\mathbf{A}, \mathbf{L}, \mathbf{Q}, \mathcal{L}\}$, respectively, by

$$\lambda_1(\mathsf{G}) \geqslant \lambda_2(\mathsf{G}) \geqslant \ldots \geqslant \lambda_n(\mathsf{G}), \tag{18}$$

$$\mu_1(\mathsf{G}) \leqslant \mu_2(\mathsf{G}) \leqslant \ldots \leqslant \mu_n(\mathsf{G}), \tag{19}$$

$$\nu_1(\mathsf{G}) \geqslant \nu_2(\mathsf{G}) \geqslant \ldots \geqslant \nu_n(\mathsf{G}), \tag{20}$$

$$\delta_1(\mathsf{G}) \leqslant \delta_2(\mathsf{G}) \leqslant \ldots \leqslant \delta_n(\mathsf{G}). \tag{21}$$

Example 1. *Consider the complete bipartite graph* $\mathsf{G} = \mathsf{K}_{2,3}$ *with the adjacency matrix*

$$\mathbf{A}(\mathsf{G}) = \begin{pmatrix} \mathbf{0}_{2,2} & \mathbf{J}_{2,3} \\ \mathbf{J}_{3,2} & \mathbf{0}_{3,3} \end{pmatrix}.$$

It can be verified that the spectra of G *can be given as follows:*

1. *The* $\mathbf{A}$*-spectrum of* G *is*

$$\sigma_{\mathbf{A}}(\mathsf{G}) = \left\{-\sqrt{6}, [0]^3, \sqrt{6}\right\}, \tag{22}$$

   with the notation that $[\lambda]^m$ *means that* λ *is an eigenvalue with multiplicity m.*
2. *The* $\mathbf{L}$*-spectrum of* G *is*

$$\sigma_{\mathbf{L}}(\mathsf{G}) = \left\{0, [2]^2, 3, 5\right\}. \tag{23}$$

3. *The* $\mathbf{Q}$*-spectrum of* G *is*

$$\sigma_{\mathbf{Q}}(\mathsf{G}) = \left\{0, [2]^2, 3, 5\right\}. \tag{24}$$

4. *The $\mathcal{L}$-spectrum of G is*

$$\sigma_{\mathcal{L}}(\mathsf{G}) = \left\{0, [1]^3, 2\right\}. \tag{25}$$

Remark 3. *If H is an induced subgraph of a graph G, then $\mathbf{A}(\mathsf{H})$ is a principal submatrix of $A(\mathsf{G})$. However, since the degrees of the remaining vertices are affected by the removal of vertices when forming the induced subgraph H from the graph G, this property does not hold for the other three associated matrices discussed in this paper (namely, the Laplacian, signless Laplacian, and normalized Laplacian matrices).*

Definition 15. *Let G be a graph, and let $\overline{\mathsf{G}}$ be the complement graph of G. Define the following matrices:*

1. $\overline{\mathbf{A}}(\mathsf{G}) = \mathbf{A}(\overline{\mathsf{G}})$.
2. $\overline{\mathbf{L}}(\mathsf{G}) = \mathbf{L}(\overline{\mathsf{G}})$.
3. $\overline{\mathbf{Q}}(\mathsf{G}) = \mathbf{Q}(\overline{\mathsf{G}})$.
4. $\overline{\mathcal{L}}(\mathsf{G}) = \mathcal{L}(\overline{\mathsf{G}})$.

Definition 16. *Let $\mathcal{X} \subseteq \{\mathbf{A}, \mathbf{L}, \mathbf{Q}, \mathcal{L}, \overline{\mathbf{A}}, \overline{\mathbf{L}}, \overline{\mathbf{Q}}, \overline{\mathcal{L}}\}$. The $\mathcal{X}$-spectrum of a graph G is a list with $\sigma_X(\mathsf{G})$ for every $X \in \mathcal{X}$.*

Observe that if $\mathcal{X} = \{X\}$ is a singleton, then the $\mathcal{X}$ spectrum is equal to the X-spectrum. We now describe some important applications of the four matrices.

2.3.2. Properties of the Adjacency Matrix

Theorem 3 (Number of walks of a given length between two fixed vertices)**.** *Let $\mathsf{G} = (\mathsf{V}, \mathsf{E})$ be a graph with a vertex set $\mathsf{V} = \mathsf{V}(\mathsf{G}) = \{v_1, \ldots, v_n\}$, and let $\mathbf{A} = \mathbf{A}(\mathsf{G})$ be the adjacency matrix of G. Then, the number of walks of length ℓ, with the fixed endpoints v_i and v_j, is equal to $(\mathbf{A}^\ell)_{i,j}$.*

Corollary 2 (Number of closed walks of a given length)**.** *Let $\mathsf{G} = (\mathsf{V}, \mathsf{E})$ be a simple undirected graph on n vertices with an adjacency matrix $\mathbf{A} = \mathbf{A}(\mathsf{G})$, and let its spectrum (with respect to $\mathbf{A}$) be given by $\{\lambda_j\}_{j=1}^n$. Then, for all $\ell \in \mathbb{N}$, the number of closed walks of length ℓ in G is equal to $\sum_{j=1}^n \lambda_j^\ell$.*

Corollary 3 (Number of edges and triangles in a graph)**.** *Let G be a simple undirected graph with $n = |\mathsf{V}(\mathsf{G})|$ vertices, $e = |\mathsf{E}(\mathsf{G})|$ edges, and t triangles. Let $\mathbf{A} = \mathbf{A}(\mathsf{G})$ be the adjacency matrix of G, and let $\{\lambda_j\}_{j=1}^n$ be its adjacency spectrum. Then,*

$$\sum_{j=1}^{n} \lambda_j = \operatorname{tr}(\mathbf{A}) = 0, \tag{26}$$

$$\sum_{j=1}^{n} \lambda_j^2 = \operatorname{tr}(\mathbf{A}^2) = 2e, \tag{27}$$

$$\sum_{j=1}^{n} \lambda_j^3 = \operatorname{tr}(\mathbf{A}^3) = 6t. \tag{28}$$

For a d-regular graph, the largest eigenvalue of its adjacency matrix is equal to d. Consequently, by Equation (27), for d-regular graphs, $\sum_j \lambda_j^2 = 2e = nd = n\lambda_1$. Interestingly, this turns out to be a necessary and sufficient condition for the regularity of a graph, which means that the adjacency spectrum enables us to identify whether a graph is regular.

Theorem 4 ([4] (Corollary 3.2.2)). *A graph* G *on* n *vertices is regular if and only if*

$$\sum_{i=1}^{n} \lambda_i^2 = n\lambda_1, \tag{29}$$

where λ_1 *is the largest eigenvalue of the adjacency matrix of* G.

Theorem 5 (The eigenvalues of strongly regular graphs). *The following spectral properties are satisfied by the family of strongly regular graphs:*

1. *A strongly regular graph has three distinct eigenvalues.*
2. *Let* G *be a connected strongly regular graph, and let its parameters be* $\mathsf{srg}(n, d, \lambda, \mu)$. *Then, the largest eigenvalue of its adjacency matrix is* $\lambda_1(\mathsf{G}) = d$ *with multiplicity 1, and the other two distinct eigenvalues of its adjacency matrix are given by*

$$p_{1,2} = \tfrac{1}{2}\left(\lambda - \mu \pm \sqrt{(\lambda-\mu)^2 + 4(d-\mu)}\right), \tag{30}$$

   with the respective multiplicities

$$m_{1,2} = \tfrac{1}{2}\left(n - 1 \mp \frac{2d + (n-1)(\lambda-\mu)}{\sqrt{(\lambda-\mu)^2 + 4(d-\mu)}}\right). \tag{31}$$

3. *A connected regular graph with exactly three distinct eigenvalues is strongly regular.*
4. *Strongly regular graphs for which* $2d + (n-1)(\lambda-\mu) \neq 0$ *have integral eigenvalues, and the multiplicities of* $p_{1,2}$ *are distinct.*
5. *A connected regular graph is strongly regular if and only if it has three distinct eigenvalues, where the largest eigenvalue is of multiplicity 1.*

The following result follows readily from Theorem 5.

Corollary 4. *Strongly regular graphs with identical parameters* (n, d, λ, μ) *are cospectral.*

Remark 4. *Strongly regular graphs having identical parameters* (n, d, λ, μ) *are cospectral but may not be isomorphic. For instance, Chang graphs form a set of three nonisomorphic strongly regular graphs with identical parameters* $\mathsf{srg}(28, 12, 6, 4)$ *[51] (Section 10.11). Consequently, the three Chang graphs are strongly regular NICS graphs.*

An important class of strongly regular graphs, for which $2d + (n-1)(\lambda-\mu) = 0$, is given by the family of conference graphs.

Definition 17 (Conference graphs). *A conference graph on* n *vertices is a strongly regular graph with the parameters* $\mathsf{srg}(n, \frac{1}{2}(n-1), \frac{1}{4}(n-5), \frac{1}{4}(n-1))$, *where* n *must satisfy* $n = 4k+1$ *with* $k \in \mathbb{N}$.

If G is a conference graph on n vertices, then so is its complement $\overline{\mathsf{G}}$; it is, however, not necessarily self-complementary. By Theorem 5, the distinct eigenvalues of the adjacency matrix of G are given by $\frac{1}{2}(n-1)$, $\frac{1}{2}(\sqrt{n}-1)$, and $-\frac{1}{2}(\sqrt{n}+1)$ with multiplicities 1, $\frac{1}{2}(n-1)$, and $\frac{1}{2}(n-1)$, respectively. In contrast to Item 4 of Theorem 5, the eigenvalues $\pm\frac{1}{2}(\sqrt{n}+1)$ are not necessarily integers. For instance, the cycle graph C_5, which is a conference graph, has an adjacency spectrum $\{2, [\frac{1}{2}(\sqrt{5}-1)]^{(2)}, [-\frac{1}{2}(\sqrt{5}+1)]^{(2)}\}$. Thus, apart from the largest eigenvalue, the other eigenvalues are irrational numbers.

2.3.3. Properties of the Laplacian Matrix

Theorem 6. *Let* G *be a finite, simple, and undirected graph, and let* $\mathbf{L}$ *be the Laplacian matrix of* G*. Then,*

1. *The Laplacian matrix* $\mathbf{L} = \mathbf{N}\mathbf{N}^{\mathrm{T}}$ *is positive semidefinite, where* $\mathbf{N}$ *is the oriented incidence matrix of* G *(see Definition 14 and [4] (p. 185)).*
2. *The smallest eigenvalue of* $\mathbf{L}$ *is zero, with a multiplicity equal to the number of components in* G *(see [4] (Theorem 7.1.2)).*
3. *The size of the graph,* $\left|\mathsf{E}(\mathsf{G})\right|$*, equals one-half of the sum of the eigenvalues of* $\mathbf{L}$*, counted with multiplicities (see [4] (Equation (7.4))).*

The following celebrated theorem provides an operational meaning of the $\mathbf{L}$-spectrum of graphs in counting their number of spanning subgraphs.

Theorem 7 (Kirchhoff's Matrix Tree Theorem [52])**.** *The number of spanning trees in a connected and simple graph* G *on* n *vertices is determined by the* $n-1$ *nonzero eigenvalues of the Laplacian matrix, and it is equal to* $\frac{1}{n}\prod_{\ell=2}^{n}\mu_{\ell}(\mathsf{G})$*.*

Corollary 5 (Cayley's Formula [53])**.** *The number of spanning trees of* K_n *is* n^{n-2}*.*

Proof. The $\mathbf{L}$-spectrum of K_n is given by $\{0,[n]^{n-1}\}$, and the result readily follows from Theorem 7. □

2.3.4. Properties of the Signless Laplacian Matrix

Theorem 8. *Let* G *be a finite, simple, and undirected graph, and let* $\mathbf{Q}$ *be the signless Laplacian matrix of* G*. Then,*

1. *The matrix* $\mathbf{Q}$ *is positive semidefinite. Moreover, it is a completely positive matrix, expressed as* $\mathbf{Q} = \mathbf{B}\mathbf{B}^{\mathrm{T}}$*, where* $\mathbf{B}$ *is the incidence matrix of* G *(see Definition 13 and [4] (Section 2.4)).*
2. *If* G *is a connected graph, then it is bipartite if and only if the least eigenvalue of* $\mathbf{Q}$ *is equal to zero. In this case,* 0 *is a simple* $\mathbf{Q}$*-eigenvalue (see [4] (Theorem 7.8.1)).*
3. *The multiplicity of 0 as an eigenvalue of* $\mathbf{Q}$ *is equal to the number of bipartite components in* G *(see [4] (Corollary 7.8.2)).*
4. *The size of the graph* $\left|E(\mathsf{G})\right|$ *is equal to one-half the sum of the eigenvalues of* $\mathbf{Q}$*, counted with multiplicities (see [4] (Corollary 7.8.9)).*

The interested reader is referred to [54] for bounds on the $\mathbf{Q}$-spread (i.e., the difference between the largest and smallest eigenvalues of the signless Laplacian matrix), expressed as a function of the number of vertices in the graph. In regard to Item 2 of Theorem 8, the interested reader is referred to [55] for a lower bound on the least eigenvalue of signless Laplacian matrix for connected non-bipartite graphs, and to [56] for a lower bound on the least eigenvalue of signless Laplacian matrix for a general simple graph with a fixed number of vertices and edges.

2.3.5. Properties of the Normalized Laplacian Matrix

The normalized Laplacian matrix of a graph, defined in (13), exhibits several interesting spectral properties, which are introduced below.

Theorem 9 ([4,57])**.** *Let* G *be a finite, simple, and undirected graph, and let* $\mathcal{L}$ *be the normalized Laplacian matrix of* G*. Then,*

1. *The eigenvalues of* $\mathcal{L}$ *lie in the interval* $[0,2]$ *(see [4] (Section 7.7)).*

2. *The number of components in* G *is equal to the multiplicity of 0 as an eigenvalue of* $\mathcal{L}$ *(see [4] (Theorem 7.7.3)).*
3. *The largest eigenvalue of* $\mathcal{L}$ *is equal to 2 if and only if the graph has a bipartite component (see [4] (Theorem 7.7.2(v))). Furthermore, the number of the bipartite components of* G *is equal to the multiplicity of 2 as an eigenvalue of* $\mathcal{L}$*.*
4. *The sum of its eigenvalues (including multiplicities) is less than or equal to the graph order* (n)*, with equality if and only if the graph has no isolated vertices (see [4] (Theorem 7.7.2(i))).*

2.3.6. More on the Spectral Properties of the Four Associated Matrices

The following theorem considers equivalent spectral properties of bipartite graphs.

Theorem 10. *Let* G *be a graph. The following are equivalent:*

1. G *is a bipartite graph.*
2. G *does not have cycles of odd length.*
3. *The* $\mathbf{A}$*-spectrum of* G *is symmetric around zero, and for every eigenvalue* λ *of* $\mathbf{A}(G)$*, the eigenvalue* $-\lambda$ *is of the same multiplicity [4] (Theorem 3.2.3).*
4. *The* $\mathbf{L}$*-spectrum and* $\mathbf{Q}$*-spectrum are identical (see [4] (Proposition 7.8.4)).*
5. *The* $\mathcal{L}$*-spectrum has the same multiplicity of* 0*'s and* 2*'s as eigenvalues (see [4] (Corollary 7.7.4)).*

Remark 5. *Item 3 of Theorem 10 can be strengthened if* G *is a connected graph. In that case,* G *is bipartite if and only if* $\lambda_1 = -\lambda_n$ *(see [4] (Theorem 3.2.4)).*

Table 1, borrowed from [58], lists properties of a graph that can or cannot be determined by the X-spectrum for $X \in \{\mathbf{A}, \mathbf{L}, \mathbf{Q}, \mathcal{L}\}$. From the $\mathbf{A}$-spectrum of a graph G, one can determine the number of edges and the number of triangles in G (by Equations (27) and (28), respectively), and whether the graph is bipartite or not (by Item 3 of Theorem 10). However, the $\mathbf{A}$ spectrum does not indicate the number of components (see Example 2). From the $\mathbf{L}$-spectrum of a graph G, one can determine the number of edges (by Item 3 of Theorem 6), the number of spanning trees (by Theorem 7), the number of components of G (by Item 2 of Theorem 6), but not the number of its triangles, and whether the graph G is bipartite. From the $\mathbf{Q}$-spectrum, one can determine whether the graph is bipartite, the number of bipartite components, and the number of edges (respectively, by Items 3 and 4 of Theorem 8), but not the number of components of the graph or whether the graph is bipartite (see Remark 6). From the $\mathcal{L}$-spectrum, one can determine the number of components and the number of bipartite components in G (by Theorem 9), and whether the graph is bipartite (by Items 1 and 5 of Theorem 10). The number of closed walks in G is determined by the $\mathbf{A}$-spectrum (by Corollary 2) but not by the spectra with respect to the other three matrices.

Table 1. Some properties of a finite, simple, and undirected graph that one can or cannot determine by the X-spectrum for $X \in \{\mathbf{A}, \mathbf{L}, \mathbf{Q}, \mathcal{L}\}$.

Matrix	# Edges	Bipartite	# Components	# Bipartite Components	# of Closed Walks
$\mathbf{A}$	Yes	Yes	No	No	Yes
$\mathbf{L}$	Yes	No	Yes	No	No
$\mathbf{Q}$	Yes	No	No	Yes	No
$\mathcal{L}$	No	Yes	Yes	Yes	No

Remark 6. *By Item 2 of Theorem 8, a connected graph is bipartite if and only if the least eigenvalue of its signless Laplacian matrix is equal to zero. If the graph is disconnected and it has a bipartite*

component and a non-bipartite component, then the least eigenvalue of its signless Laplacian matrix is equal to zero, although the graph is not bipartite. According to Table 1, the **Q***-spectrum alone does not determine whether a graph is bipartite. This is due to the fact that the* **Q***-spectrum does not provide information about the number of components in the graph or whether the graph is connected. It is worth noting that while neither the* **L***-spectrum nor the* **Q***-spectrum independently determines whether a graph is bipartite, the combination of these spectra does. Specifically, by Item 4 of Theorem 10, the combined knowledge of both spectra enables us to establish this property.*

3. Graphs Determined by Their Spectra

The spectral determination of graphs has long been a central topic in spectral graph theory. A major open question in this area is as follows: "Which graphs are determined by their spectrum (DS)?" This section begins our survey of both classical and recent results on spectral graph determination. We explore the spectral characterization of various graph classes, methods for constructing or distinguishing cospectral nonisomorphic graphs, and conditions under which a graph's spectrum uniquely determines its structure. Additionally, we present newly obtained proofs of existing results, offering further insights into this field.

Definition 18. *Let* G, H *be two graphs. A mapping* $\phi\colon\ \mathsf{V}(\mathsf{G}) \to \mathsf{V}(\mathsf{H})$ *is a graph isomorphism if*

$$\{u, v\} \in \mathsf{E}(\mathsf{G}) \iff \{\phi(u), \phi(v)\} \in \mathsf{E}(\mathsf{H}). \tag{32}$$

If there is an isomorphism between G *and* H*, we say that these graphs are isomorphic.*

Definition 19. *A permutation matrix is a* $\{0, 1\}$*–matrix in which each row and each column contains exactly one entry equal to* 1*.*

Remark 7. *In terms of the adjacency matrix of a graph,* G *and* H *are cospectral graphs if* $\mathbf{A}(\mathsf{G})$ *and* $\mathbf{A}(\mathsf{H})$ *are similar matrices, and* G *and* H *are isomorphic if the similarity of their adjacency matrices is through a permutation matrix* $\mathbf{P}$*, i.e.,*

$$A(\mathsf{G}) = \mathbf{P}\ \mathbf{A}(\mathsf{H})\, \mathbf{P}^{-1}. \tag{33}$$

3.1. Graphs Determined by Their Adjacency Spectrum (DS Graphs)

Theorem 11 ([12])**.** *All of the graphs with fewer than five vertices are DS.*

Example 2. *The star graph* S_5 *and a graph formed by the disjoint union of a length-4 cycle and an isolated vertex,* $\mathsf{C}_4 \dot\cup \mathsf{K}_1$*, have the same* $\mathbf{A}$*-spectrum* $\{-2, [0]^3, 2\}$*. They are, however, not isomorphic since* S_5 *is connected and* $\mathsf{C}_4 \dot\cup \mathsf{K}_1$ *is disconnected (see Figure 1).*

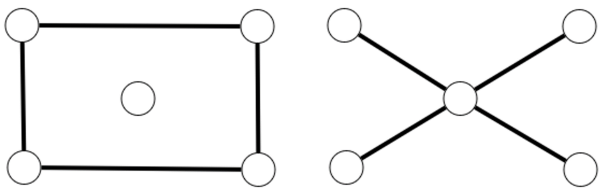


Figure 1. The graphs $\mathsf{S}_4 = \mathsf{K}_{1,4}$ and $\mathsf{C}_4 \dot\cup \mathsf{K}_1$ (i.e., a union of a 4-length cycle and an isolated vertex) are cospectral and nonisomorphic graphs ($\mathbf{A}$-NICS graphs) on five vertices. These two graphs, therefore, cannot be determined by their adjacency matrix.

It can be verified computationally that all the connected nonisomorphic graphs on five vertices can be distinguished by their $\mathbf{A}$*-spectrum (see [4] (Appendix A1)).*

Theorem 12 ([12]). *All the regular graphs with fewer than ten vertices are DS (and, as will be clarified later, also $\mathcal{X}$-DS for every $\mathcal{X} \subseteq \{\mathbf{A}, \mathbf{L}, \mathbf{Q}\}$).*

Example 3 ([12]). *The two regular graphs in Figure 2 are $\{\mathbf{A}, \mathbf{L}, \mathbf{Q}, \mathcal{L}\}$-NICS.*

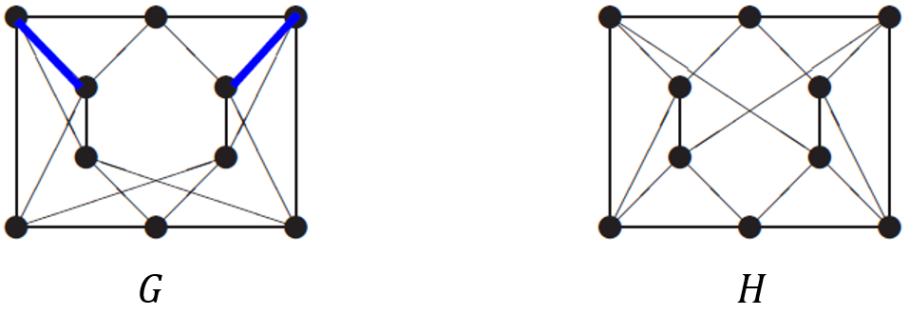


Figure 2. $\{\mathbf{A}, \mathbf{L}, \mathbf{Q}, \mathcal{L}\}$-NICS regular graphs on 10 vertices. These cospectral graphs are nonisomorphic because each of the blue edges in G belongs to three triangles, whereas no such an edge exists in H.

The regular graphs G and H in Figure 2 can be verified to be cospectral with the common characteristic polynomial

$$P(x) = x^{10} - 20x^8 - 16x^7 + 110x^6 + 136x^5 - 180x^4 - 320x^3 + 9x^2 + 200x + 80.$$

These graphs are also nonisomorphic because each of the two blue edges in G belongs to three triangles, whereas no such an edge exists in H. Furthermore, it is shown in Example 4.18 of [41] that each pair of the regular NICS graphs on 10 vertices, denoted by $\{\mathsf{G}, \mathsf{H}\}$ and $\{\overline{\mathsf{G}}, \overline{\mathsf{H}}\}$, exhibits distinct values of the Lovász ϑ-functions, whereas the graphs G, $\overline{\mathsf{G}}$, H, and $\overline{\mathsf{H}}$ share identical independence numbers (3), clique numbers (3), and chromatic numbers (4). Furthermore, based on these two pairs of graphs, it is constructively shown in Theorem 4.19 of [41] that for every even integer $n \geqslant 14$, there exist connected, irregular, cospectral, and nonisomorphic graphs on n vertices, being jointly cospectral with respect to their adjacency, Laplacian, signless Laplacian, and normalized Laplacian matrices, while also sharing identical independence, clique, and chromatic numbers, but being distinguished by their Lovász ϑ-functions.

Remark 8. *In continuation to Example 3, it is worth noting that closed-form expressions for the Lovász ϑ-functions of regular graphs, which are edge-transitive or strongly regular, were derived in [50] (Theorem 9) and [59] (Proposition 1), respectively. In particular, it follows from [59] (Proposition 1) that strongly regular graphs with identical four parameters (n, d, λ, μ) are cospectral and they have identical Lovász ϑ-numbers, although they need not be necessarily isomorphic. For such an explicit counterexample, the reader is referred to [59] (Remark 3).*

We next introduce friendship graphs to address their possible determination by their spectra with respect to several associated matrices.

Definition 20. *Let $p \in \mathbb{N}$. The friendship graph F_p, also known as the windmill graph, is a graph with $2p + 1$ vertices, consisting of a single vertex (the central vertex) that is adjacent to all the other $2p$ vertices. Furthermore, every pair of these $2p$ vertices shares exactly one common neighbor, namely the central vertex (see Figure 3). This graph has $3p$ edges and p triangles.*

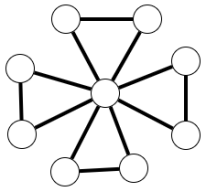


Figure 3. The friendship (windmill) graph F_4 has 9 vertices, 12 edges, and 4 triangles.

The term friendship graph in Definition 20 originates from the *Friendship Theorem* [60]. This theorem states that if G is a finite graph where any two vertices share exactly one common neighbor, then there exists a vertex that is adjacent to all other vertices. In this context, the adjacency of vertices in the graph can be interpreted socially as a representation of friendship between the individuals represented by the vertices (assuming friendship is a mutual relationship). For a nice exposition of the proof of the Friendship Theorem, the interested reader is referred to Chapter 44 of [61].

Theorem 13. *The following graphs are DS:*

1. *All graphs with fewer than five vertices, and also all regular graphs with fewer than 10 vertices [12] (recall Theorems 11 and 12).*
2. *The graphs* K_n, C_n, P_n, $\mathsf{K}_{m,m}$, *and* $\overline{\mathsf{K}_\mathsf{n}}$ *[12].*
3. *The complement of the path graph* $\overline{\mathsf{P}_\mathsf{n}}$ *[62].*
4. *The disjoint union of k path graph with no isolated vertices, the disjoint union of k complete graphs with no isolated vertices, and the disjoint union of k cycles (i.e., every 2-regular graph) [12].*
5. *The complement graph of a DS regular graph [4].*
6. *Every* $(n-3)$*-regular graph on n vertices [4].*
7. *The friendship graph* F_p *for* $p \neq 16$ *[63].*
8. *Sandglass graphs, which are obtained by appending a triangle to each of the pendant (i.e., degree-1) vertices of a path [64].*
9. *If* H *is a subgraph of a graph* G*, and* $\mathsf{G}\backslash\mathsf{H}$ *denotes the graph obtained from* G *by deleting the edges of* H*, then, in addition, the following graphs are DS [21]:*
   - $\mathsf{K}_n \setminus (\ell\,\mathsf{K}_2)$ *and* $\mathsf{K}_n \setminus \mathsf{K}_m$*, where* $m \leqslant n-2$*;*
   - $\mathsf{K}_n \setminus \mathsf{K}_{\ell,m}$*;*
   - $\mathsf{K}_n \setminus \mathsf{H}$*, where* H *has at most four edges.*

3.2. Graphs Determined by Their Spectra with Respect to Various Matrices (X-DS Graphs)

In this section, we consider graphs that are determined by the spectra of various associated matrices beyond the adjacency matrix spectrum.

Definition 21. *Let* G, H *be two graphs and let* $\mathcal{X} \subseteq \{\mathbf{A}, \mathbf{L}, \mathbf{Q}, \mathcal{L}, \overline{\mathbf{A}}, \overline{\mathbf{L}}, \overline{\mathbf{Q}}, \overline{\mathcal{L}}\}$.

1. G *and* H *are said to be* $\mathcal{X}$*-cospectral if they have the same X-spectrum, i.e.,* $\sigma_X(\mathsf{G}) = \sigma_X(\mathsf{H})$.
2. *Nonisomorphic graphs* G *and* H *that are* $\mathcal{X}$*-cospectral are said to be* $\mathcal{X}$*-NICS, where* NICS *is an abbreviation of nonisomorphic and cospectral.*
3. *A graph* G *is said to be determined by its* $\mathcal{X}$*-spectrum (*$\mathcal{X}$*-DS) if every graph that is* $\mathcal{X}$*-cospectral to* G *is also isomorphic to* G.

Notation 2. *For a singleton* $\mathcal{X} = \{X\}$*, we abbreviate* $\{X\}$*-cospectral,* $\{X\}$*-DS, and* $\{X\}$*-NICS by X-cospectral, X-DS, and X-NICS, respectively. For the adjacency matrix, we will abbreviate* $\mathbf{A}$*-DS by DS.*

Remark 9. *Let $\mathcal{X} \subseteq \mathcal{Y} \subseteq \{\mathbf{A}, \mathbf{L}, \mathbf{Q}, \mathcal{L}, \overline{\mathbf{A}}, \overline{\mathbf{L}}, \overline{\mathbf{Q}}, \overline{\mathcal{L}}\}$. The following holds by definition:*

- *If two graph G, H are $\mathcal{Y}$-cospectral, then they are $\mathcal{X}$-cospectral.*
- *If a graph G is $\mathcal{X}$-DS, then it is $\mathcal{Y}$-DS.*

Definition 22. *Let G be a graph. The generalized spectrum of G is the $\{\mathbf{A}, \overline{\mathbf{A}}\}$-spectrum of G.*

The following result on the cospectrality of regular graphs can be readily verified.

Proposition 3. *Let G and H be regular graphs that are $\mathcal{X}$-cospectral for* some *$\mathcal{X} \subseteq \{\mathbf{A}, \mathbf{L}, \mathbf{Q}, \mathcal{L}\}$. Then, G and H are $\mathcal{Y}$-cospectral for* every *$\mathcal{Y} \subseteq \{\mathbf{A}, \overline{\mathbf{A}}, \mathbf{L}, \overline{\mathbf{L}}, \mathbf{Q}, \overline{\mathbf{Q}}, \mathcal{L}, \overline{\mathcal{L}}\}$. In particular, the cospectrality of regular graphs (and their complements) stays unaffected by the chosen matrix among $\{\mathbf{A}, \mathbf{L}, \mathbf{Q}, \mathcal{L}\}$.*

Definition 23. *A graph G is said to be determined by its generalized spectrum (DGS) if it is uniquely determined by its generalized spectrum. In other words, a graph G is DGS if and only if every graph H with the same $\{\mathbf{A}, \overline{\mathbf{A}}\}$-spectrum as G is necessarily isomorphic to G.*

If a graph is not DS, it may still be DGS, as additional spectral information is available. Conversely, every DS graph is also DGS. For further insights into DGS graphs, including various characterizations, conjectures, and studies, we refer the reader to [65–67].

The continuation of this section characterizes graphs that are X-DS, where $X \in \{\mathbf{L}, \mathbf{Q}, \mathcal{L}\}$, with pointers to various studies. We first consider regular DS graphs.

Theorem 14 ([12] (Proposition 3))**.** *For regular graphs, the properties of being DS, $\mathbf{L}$-DS, and $\mathbf{Q}$-DS are equivalent.*

Remark 10. *To avoid any potential confusion, it is important to emphasize that in statements such as Theorem 14, the only available information is the spectrum of the graph. There is no indication or prior knowledge that the spectrum corresponds to a regular graph. In such cases, the regularity of the graph is not part of the revealed information and, therefore, cannot be used to determine the graph. This recurring approach, which states that G is to be a graph that satisfies certain properties (e.g., regularity, strong regularity, etc.) and then examines whether the graph can be determined from its spectrum, appears throughout this paper. It should be understood that the only available information is the spectrum of the graph, and no additional properties of the graph beyond its spectrum are disclosed.*

Remark 11. *The crux of the proof of Theorem 14 is that there are no two NICS graphs, with respect to either $\mathbf{A}$, $\mathbf{L}$, or $\mathbf{Q}$, where one graph is regular, and the other is irregular (see [12] (Proposition 2.2)). This, however, does not extend to NICS graphs with respect to the normalized Laplacian matrix $\mathcal{L}$, and regular DS graphs are not necessarily $\mathcal{L}$-DS. For instance, the cycle C_4 and the bipartite complete graph $\mathsf{K}_{1,3}$ (i.e., S_3) share the same $\mathcal{L}$-spectrum, which is given by $\{0, 1^2, 2\}$, but these graphs are nonisomorphic (as C_4 is regular, in contrast to $\mathsf{K}_{1,3}$). It therefore follows that the 2-regular graph C_4 is* not *$\mathcal{L}$-DS, although it is DS (see Item 2 of Theorem 13). More generally, it is conjectured in [19] that C_n is $\mathcal{L}$-DS if and only if $n > 4$ and $4 \mid n$.*

Theorem 15. *The following graphs are $\mathbf{L}$-DS:*

1. *$\mathsf{P}_n, \mathsf{C}_n, \mathsf{K}_n, \mathsf{K}_{m,m}$ and their complements [12].*
2. *The disjoint union of k paths, $\mathsf{P}_{n_1} \dot\cup \mathsf{P}_{n_2} \dot\cup \ldots \dot\cup \mathsf{P}_{n_k}$ each having at least one edge [12].*
3. *The complete bipartite graph $\mathsf{K}_{m,n}$ with $m, n \geqslant 2$ and $\frac{5}{3}n < m$ [68].*
4. *The star graphs S_n with $n \neq 3$ [36,39].*
5. *Trees with a single vertex having a degree greater than 2 (referred to as starlike trees) [36,39].*

6. *The friendship graph* F_p *[36].*
7. *The path-friendship graphs, where a friendship graph and a starlike tree are joined by merging their vertices of degree greater than 2 [38].*
8. *The wheel graph* $\mathsf{W}_{n+1} \triangleq \mathsf{K}_1 \vee \mathsf{C}_n$ *for* $n \neq 7$ *(otherwise, if* $n = 7$*, then it is not* **L***-DS) [43].*
9. *The join of a clique and an independent set on n vertices,* $\mathsf{K}_{n-m} \vee \overline{\mathsf{K_m}}$*, where* $m \in [n-1]$ *[69].*
10. *Sandglass graphs (see also Item 8 in Theorem 13) [64].*
11. *The join graph* $\mathsf{G} \vee \mathsf{K}_m$*, for every* $m \in \mathbb{N}$*, where* G *is a disconnected graph [45].*
12. *The join graph* $\mathsf{G} \vee \mathsf{K}_m$*, for every* $m \in \mathbb{N}$*, where* G *is an* **L***-DS connected graph on n vertices and m edges with* $m \leqslant 2n-6$*,* $\overline{\mathsf{G}}$ *is a connected graph, and either one of the following conditions holds [45]:*
    - $\mathsf{G} \vee \mathsf{K}_1$ *is* **L***-DS;*
    - *the maximum degree of* G *is smaller than* $\frac{1}{2}(n-2)$*.*
13. *Specifically, the join graph* $\mathsf{G} \vee \mathsf{K}_m$*, for every* $m \in \mathbb{N}$*, where* G *is an* **L***-DS tree on* $n \geqslant 5$ *vertices (since, the equality* $m = n-1$ *holds for a tree on n vertices and m edges) [45].*

Remark 12. *In general, a disjoint union of complete graphs is not determined by its Laplacian spectrum.*

Theorem 16. *The following graphs are* **Q***-DS:*

1. *The disjoint union of k paths,* $\mathsf{P}_{n_1} \dot\cup \mathsf{P}_{n_2} \dot\cup \ldots \dot\cup \mathsf{P}_{n_k}$ *each having at least one edge [12].*
2. *The star graphs* S_n *with* $n \geqslant 3$ *[16,40].*
3. *Trees with a single vertex having a degree greater than 2 [16,40].*
4. *The friendship graph* F_k *[70].*
5. *The lollipop graphs, where a lollipop graph, denoted by* $\mathrm{H}_{n,p}$ *where* $n, p \in \mathbb{N}$ *and* $p < n$*, is obtained by appending a cycle* C_p *to a pendant vertex of a path* P_{n-p} *[26,44].*
6. $\mathsf{G} \vee \mathsf{K}_1$ *where* G *is a either a 1-regular graph, an* $(n-2)$*-regular graph of order n or a 2-regular graph with at least 11 vertices [15].*
7. *If* $n \geqslant 21$ *and* $0 \leqslant q \leqslant n-1$*, then* $\mathsf{K}_1 \vee (\mathsf{P}_q \dot\cup (n-q-1)\,\mathsf{K}_1)$ *[42].*
8. *If* $n \geqslant 21$ *and* $3 \leqslant q \leqslant n-1$*, then* $\mathsf{K}_1 \vee (\mathsf{C}_q \dot\cup (n-q-1)\,\mathsf{K}_1)$ *is* **Q***-DS if and only if* $q \neq 3$ *[42].*
9. *The join of a clique and an independent set on n vertices,* $\mathsf{K}_{n-m} \vee \overline{\mathsf{K_m}}$*, where* $m \in [n-1]$ *and* $m \neq 3$ *[69].*

Since the regular graphs K_n, $\overline{\mathsf{K_n}}$, $\mathsf{K}_{m,m}$ and C_n are DS, they are also $\mathcal{X}$-DS for every $\mathcal{X} \subseteq \{\mathbf{A}, \mathbf{L}, \mathbf{Q}\}$ (see Theorem 14). This, however, does not apply to regular $\mathcal{L}$-DS graphs (see Remark 11), which are next addressed.

Theorem 17. *The following graphs are* $\mathcal{L}$*-DS:*

- K_n*, for every* $n \in \mathbb{N}$ *[20].*
- *The friendship graph* F_k*, for* $k \geqslant 2$ *[14] (Corollary 1).*
- *More generally,* $\mathsf{F}_{p,q} = \mathsf{K}_1 \vee (p\,\mathsf{K}_q)$ *if* $q \geqslant 2$*, or* $q = 1$ *and* $p \geqslant 2$ *[14] (Theorem 1).*

4. Special Families of Graphs

This section introduces special families of structured graphs, and it states the conditions for their unique determination by their spectra.

4.1. Stars and Graphs of Pyramids

Definition 24. *For every* $k, n \in \mathbb{N}$ *with* $k < n$*, define the graph* $T_{n,k} = \mathsf{K}_k \vee \overline{\mathsf{K}_{n-k}}$*. For* $k = 1$*, the graph* $T_{n,k}$ *represents the* star graph S_n*. For* $k = 2$*, it represents a graph comprising* $n-2$

triangles sharing a common edge, which is referred to as a crown. *For n, k satisfying $1 < k < n$, the graphs $T_{n,k}$ are referred to as* graphs of pyramids *[31].*

Theorem 18 ([31]). *The graphs of pyramids are DS for every $1 < k < n$.*

Theorem 19 ([31]). *The star graph S_n is DS if and only if $n-1$ is prime.*

To prove these theorems, a closed-form expression for the spectrum of $T_{n,k}$ is derived in [31], which also presents a generalized result. Subsequently, using Theorem 3, the number of edges and triangles in any graph cospectral with $T_{n,k}$ are calculated. Finally, Schur's theorem (Theorem 1) and Cauchy's interlacing theorem (Theorem 2) are applied in [31] to prove Theorems 18 and 19.

4.2. Complete Bipartite Graphs

By Theorem 19, the star graph $\mathsf{S}_n = \mathsf{K}_{1,n-1}$ is DS if and only if $n-1$ is prime. By Theorem 13, the regular complete bipartite graph $\mathsf{K}_{m,m}$ is DS for every $m \in \mathbb{N}$. Here, we generalize these results and provide a characterization for the DS property of $\mathsf{K}_{p,q}$ for every $p, q \in \mathbb{N}$.

Theorem 20 ([12]). *The spectrum of the complete bipartite graph $\mathsf{K}_{p,q}$ is $\{-\sqrt{pq}, [0]^{p+q-2}, \sqrt{pq}\}$.*

This theorem can be proved by Theorem 1. An alternative simple proof is next presented.

Proof. The adjacency matrix of $\mathsf{K}_{p,q}$ is given by

$$\mathbf{A}(\mathsf{K}_{p,q}) = \begin{pmatrix} \mathbf{0}_{p,p} & \mathbf{J}_{p,q} \\ \mathbf{J}_{q,p} & \mathbf{0}_{q,q} \end{pmatrix} \in \mathbb{R}^{(p+q)\times(p+q)} \tag{34}$$

The rank of $\mathbf{A}(\mathsf{K}_{p,q})$ is equal to 2, so the multiplicity of 0 as an eigenvalue is $p+q-2$. By Corollary 3, the two remaining eigenvalues are given by $\pm\lambda$ for some $\lambda \in \mathbb{R}$ since the eigenvalues sum to zero. Furthermore,

$$2\lambda^2 = \sum_{i=1}^{p+q} \lambda_i^2 = 2\,|\,\mathsf{E}(\mathsf{K}_{p,q})| = 2pq, \tag{35}$$

so $\lambda = \sqrt{pq}$. □

For $p, q \in \mathbb{N}$, the arithmetic and geometric means of p, q are, respectively, given by $\mathrm{AM}(p,q) = \frac{1}{2}(p+q)$ and $\mathrm{GM}(p,q) = \sqrt{pq}$. The AM-GM inequality states that for every $p, q \in \mathbb{N}$, we have $\mathrm{GM}(p,q) \leqslant \mathrm{AM}(p,q)$ with equality if and only if $p = q$.

Definition 25. *Let $p, q \in \mathbb{N}$. The two-element multiset $\{p, q\}$ is said to be an* AM-minimizer *if it attains the minimum arithmetic mean for their given geometric mean, i.e.,*

$$AM(p,q) = \min\Big\{AM(a,b) :\ a,b \in \mathbb{N},\ GM(a,b) = GM(p,q)\Big\} \tag{36}$$

$$= \min\Big\{\tfrac{1}{2}(a+b) :\ a,b \in \mathbb{N},\ ab = pq\Big\}. \tag{37}$$

Example 4. *The following are AM-minimizers:*

- *$\{k, k\}$ for every $k \in \mathbb{N}$. By the AM-GM inequality, it is the only case where $GM(p,q) = AM(p,q)$.*

- $\{p, q\}$ *where* p, q *are prime numbers. In this case, the following family of multisets*

$$\Big\{\{a, b\} : a, b \in \mathbb{N},\ GM(a, b) = GM(p, q)\Big\} \tag{38}$$

only contains the two multisets $\{p, q\}, \{pq, 1\}$, *and* $p + q \leqslant pq < pq + 1$ *since* $p, q \geqslant 2$.
- $\{1, q\}$ *where* q *is a prime number.*

Theorem 21. *The following holds for every* $p, q \in \mathbb{N}$*:*

1. *Let* G *be a graph that is cospectral with* $\mathsf{K}_{p,q}$*. Then, up to isomorphism,* $G = \mathsf{K}_{a,b} \cup \mathsf{H}$ *(i.e.,* G *is a disjoint union of the two graphs* $\mathsf{K}_{a,b}$ *and* H*), where* H *is an empty graph and* $a, b \in \mathbb{N}$ *satisfy* $GM(a, b) = GM(p, q)$.
2. *The complete bipartite graph* $\mathsf{K}_{p,q}$ *is DS if and only if* $\{p, q\}$ *is an AM-minimizer.*

Remark 13. *Item 2 of Theorem 21 is equivalent to Corollary 3.1 of [37], for which an alternative proof is presented here.*

Proof. (Proof of Theorem 21):

1. Let G be a graph cospectral with $\mathsf{K}_{p,q}$. The number of edges in G equals the number of edges in $\mathsf{K}_{p,q}$, which is pq. As $\mathsf{K}_{p,q}$ is bipartite, so is G. Since $\mathbf{A}(\mathsf{G})$ is of rank 2, and $\mathbf{A}(\mathsf{P}_3)$ has rank 3, it follows from the Cauchy's Interlacing Theorem (Theorem 2) that P_3 is not an induced subgraph of G.
   It is claimed that G has a single nonempty connected component. Suppose to the contrary that G has (at least) two nonempty connected components $\mathsf{H}_1, \mathsf{H}_2$. For $i \in \{1, 2\}$, since H_i is a nonempty graph, $\mathbf{A}(\mathsf{H}_i)$ has at least one eigenvalue $\lambda \neq 0$. Since G is a simple graph, the sum of the eigenvalues of $\mathbf{A}(\mathsf{H}_i)$ is $\mathrm{Tr}(\mathbf{A}(\mathsf{H}_i)) = 0$, so H_i has at least one positive eigenvalue. Thus, the induced subgraph $\mathsf{H}_1 \cup \mathsf{H}_2$ has at least two positive eigenvalues while G has only one positive eigenvalue, which contradicts Cauchy's Interlacing Theorem.
   Hence, G can be decomposed as $\mathsf{G} = \mathsf{K}_{a,b} \cup \mathsf{H}$ where H is an empty graph. Since G and $\mathsf{K}_{p,q}$ have the same number of edges, $pq = ab$, so $\mathrm{GM}(p, q) = \mathrm{GM}(a, b)$.
2. First, we will show that if $\{p, q\}$ is not an AM-minimizer, then the graph $\mathsf{K}_{p,q}$ is not $\mathbf{A}$-DS. This is performed by finding a nonisomorphic graph to $\mathsf{K}_{p,q}$ that is $\mathbf{A}$-cospectral with it. By assumption, since $\{p, q\}$ is not an AM-minimizer, there exist $a, b \in \mathbb{N}$ satisfying $\mathrm{GM}(a, b) = \mathrm{GM}(p, q)$ and $a + b < p + q$. Define the graph $\mathsf{G} = \mathsf{K}_{a,b} \vee \overline{\mathsf{K}_r}$ where $r = p + q - a - b$. Observe that $r \in \mathbb{N}$. The $\mathbf{A}$-spectrum of both of these graphs is given by

$$\sigma_{\mathbf{A}}(\mathsf{G}) = \sigma_{\mathbf{A}}(\mathsf{K}_{p,q}) = \big\{-\sqrt{pq}, [0]^{pq-2}, \sqrt{pq}\big\}, \tag{39}$$

   so these two graphs are nonisomorphic and cospectral, which means that G is not $\mathbf{A}$-DS.
   We next prove that if $\{p, q\}$ is an AM-minimizer, then $\mathsf{K}_{p,q}$ is $\mathbf{A}$-DS. Let G be a graph that is cospectral with $\mathsf{K}_{p,q}$. From the first part of this theorem, $\mathsf{G} = \mathsf{K}_{a,b} \cup \mathsf{H}$ where $\mathrm{GM}(a, b) = \mathrm{GM}(p, q)$ and H is an empty graph. Consequently, it follows that $\mathrm{AM}(a, b) = \frac{1}{2}(a + b) \leqslant \frac{1}{2}(p + q) = \mathrm{AM}(p, q)$. Since $\{p, q\}$ is assumed to be an AM-minimizer, it follows that $\mathrm{AM}(a, b) \geqslant \mathrm{AM}(p, q)$, and thus equality holds. Both equalities $\mathrm{GM}(a, b) = \mathrm{GM}(p, q)$ and $\mathrm{AM}(a, b) = \mathrm{AM}(p, q)$ can be satisfied simultaneously if and only if $\{a, b\} = \{p, q\}$, so $r = p + q - a - b = 0$ and $\mathsf{G} = \mathsf{K}_{p,q}$.

□

Corollary 6. *Almost all of the complete bipartite graphs are not DS. More specifically, for every $n \in \mathbb{N}$, there exists a single complete bipartite graph on n vertices that is DS.*

Proof. Let $n \in \mathbb{N}$. By the *fundamental theorem of arithmetic*, there is a unique decomposition $n = \prod_{i=1}^{k} p_i$ where $k \in \mathbb{N}$ and $\{p_i\}$ are prime numbers for every $1 \leqslant i \leqslant k$. Consider the family of multisets

$$\mathcal{D} = \left\{ \{a, b\} : a, b \in \mathbb{N}, \mathrm{GM}(a, b) = \sqrt{n} \right\}. \tag{40}$$

This family has 2^k members since every prime factor p_i of n should be in the prime decomposition of a or b. Since the minimization of $\mathrm{AM}(a, b)$ under the equality constraint $\mathrm{GM}(a, b) = \sqrt{n}$ forms a convex optimization problem, only one of the multisets in the family $\mathcal{D}$ is an AM-minimizer. Thus, if $n = \prod_{i=1}^{k} p_i$, then the number of complete bipartite graphs of n vertices is $O(2^k)$, and (by Item 2 of Theorem 21) only one of them is DS. □

4.3. Turán Graphs

The Turán graphs are a significant and well-studied class of graphs in extremal graph theory, and they form an important family of multipartite complete graphs. Turán graphs are particularly known for their role in Turán's theorem, which provides a solution to the problem of finding the maximum number of edges in a graph that does not contain a complete subgraph of a given order [71]. Before delving into formal definitions, it is noted that the distinction of the Turán graphs as multipartite complete graphs is that they are as balanced as possible, ensuring their vertex sets are divided into parts of nearly equal size.

Definition 26. *Let $n_1, \ldots, n_k$ be natural numbers. Define the complete k-partite graph*

$$\mathsf{K}_{n_1,\ldots,n_k} = \bigvee_{i=1}^{k} \overline{\mathsf{K}_{n_i}}. \tag{41}$$

A graph is multipartite if it is k-partite for some $k \geqslant 2$.

Definition 27. *Let $2 \leqslant k \leqslant n$. The* Turán graph *$T(n, k)$ (not to be confused with the graph of pyramids $T_{n,k}$) is formed by partitioning a set of n vertices into k subsets, with sizes as equal as possible, and then every two vertices are adjacent in that graph if and only if they belong to different subsets. It is, therefore, expressed as the complete k-partite graph $K_{n_1,\ldots,n_k}$, where $|n_i - n_j| \leqslant 1$ for all $i, j \in [k]$ with $i \neq j$. Let q and s be the quotient and remainder, respectively, of dividing n by k (i.e., $n = qk + s$, $s \in \{0, 1, \ldots, k-1\}$), and let $n_1 \leqslant \ldots \leqslant n_k$. Then,*

$$n_i = \begin{cases} q, & 1 \leqslant i \leqslant k - s, \\ q + 1, & k - s + 1 \leqslant i \leqslant k. \end{cases} \tag{42}$$

By construction, the graph $T(n, k)$ has a clique of order k (any subset of vertices with a single representative from each of the k subsets is a clique of order k), but it cannot have a clique of order $k + 1$ (since vertices from the same subset are nonadjacent). Note also that, by (42), the Turán graph $T(n, k)$ is a q-regular graph if and only if n is divisible by k, and then $q = \frac{n}{k}$.

Definition 28. *Let $q, k \in \mathbb{N}$. Define the regular complete multipartite graph, $\mathrm{K}_q^k := \bigvee_{i=1}^{k} \overline{\mathsf{K}_q}$, to be the k-partite graph with q vertices in each part. Observe that $\mathrm{K}_q^k = T(kq, k)$.*

Let G be a simple graph on n vertices that does not contain a clique of order greater than a fixed number $k \in \mathbb{N}$. Turán investigated a fundamental problem in extremal graph theory of determining the maximum number of edges that G can have [71].

Theorem 22 (Turán's Graph Theorem)**.** *Let G be a graph on n vertices with a clique of order at most k for some $k \in \mathbb{N}$. Then,*

$$|\mathsf{E}(\mathsf{G})| \leqslant |\mathsf{E}(T(n,k))| \tag{43}$$

$$= \left(1 - \frac{1}{k}\right)\frac{n^2 - s^2}{2} + \binom{s}{2}, \quad s \triangleq n - k\left\lfloor \frac{n}{k} \right\rfloor. \tag{44}$$

For a nice exposition of five different proofs of Turán's Graph Theorem, the interested reader is referred to Chapter 41 of [61].

Corollary 7. *Let $k \in \mathbb{N}$, and let G be a graph on n vertices where $\omega(\mathsf{G}) \leqslant k$ and $|\mathsf{E}(\mathsf{G})| = |\mathsf{E}(T(n,k))|$. Let G_1 be a graph obtained by adding an arbitrary edge to G. Then, $\omega(\mathsf{G}_1) > k$.*

4.3.1. The Spectrum of the Turán Graph

Theorem 23 ([72])**.** *Let $k \in \mathbb{N}$, and let $n_1 \leqslant n_2 \leqslant \ldots \leqslant n_k$ be natural numbers. Let $\mathsf{G} = \mathsf{K}_{n_1,n_2,\ldots,n_k}$ be a complete multipartite graph on $n = n_1 + \ldots n_k$ vertices. Then,*

- *G has one positive eigenvalue, i.e., $\lambda_1(\mathsf{G}) > 0$ and $\lambda_2(\mathsf{G}) \leqslant 0$.*
- *G has 0 as an eigenvalue with multiplicity $n - k$.*
- *G has $k - 1$ negative eigenvalues, and*

$$n_1 \leqslant -\lambda_{n-k+2}(\mathsf{G}) \leqslant n_2 \leqslant -\lambda_{n-k+3}(\mathsf{G}) \leqslant n_3 \leqslant \ldots \leqslant n_{k-1} \leqslant -\lambda_n(\mathsf{G}) \leqslant n_k. \tag{45}$$

Corollary 8. *The spectrum of the regular complete k-partite graph $\mathsf{K}_{q,\ldots,q} \triangleq \mathsf{K}_q^k$ is given by*

$$\sigma_{\mathbf{A}}(\mathsf{K}_q^k) = \Big\{[-q]^{k-1}, [0]^{(q-1)k}, q(k-1)\Big\}. \tag{46}$$

Proof. This readily follows from Theorem 23 by setting $n_1 = \ldots = n_k = q$. □

Lemma 1 ([73])**.** *Let G_i be r_i-regular graphs on n_i vertices for $i \in \{1,2\}$, with the adjacency spectrum $\sigma_{\mathbf{A}}(\mathsf{G}_1) = (r_1 = \mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant \mu_n)$ and $\sigma_A(\mathsf{G}_2) = (r_2 = \nu_1 \geqslant \nu_2 \geqslant \ldots \geqslant \nu_n)$. The $\mathbf{A}$-spectrum of $\mathsf{G}_1 \vee \mathsf{G}_2$ is given by*

$$\sigma_{\mathbf{A}}(\mathsf{G}_1 \vee \mathsf{G}_2) = \{\mu_i\}_{i=2}^{n_1} \cup \{\nu_i\}_{i=2}^{n_2} \cup \left\{\frac{r_1 + r_2 \pm \sqrt{(r_1 - r_2)^2 + 4n_1n_2}}{2}\right\}. \tag{47}$$

Theorem 24. *Let $q, s \in \mathbb{N}$ such that $n = kq + s$ and $0 \leqslant s \leqslant k - 1$. The following holds with respect to the $\mathbf{A}$-spectrum of $T(n,k)$:*

1. *If $1 \leqslant s \leqslant k - 1$, then the $\mathbf{A}$-spectrum of the irregular Turán graph $T(n,k)$ is given by*

$$\sigma_{\mathbf{A}}(T(n,k)) = \Big\{[-q-1]^{s-1}, [-q]^{k-s-1}, [0]^{n-k}\Big\} \cup \left\{\frac{1}{2}\left[n - 2q - 1 \pm \sqrt{\Big(n - 2(q+1)s + 1\Big)^2 + 4q(q+1)s(k-s)}\right]\right\}. \tag{48}$$

2. *If $s = 0$, then $q = \frac{n}{k}$, and the* **A***-spectrum of the regular Turán graph $T(n,k)$ is given by*

$$\sigma_{\mathbf{A}}(T(n,k)) = \Big\{[-q]^{k-1}, [0]^{n-k}, (k-1)q\Big\}. \tag{49}$$

Proof. Let $1 \leqslant s \leqslant k-1$. We next derive the **A**-spectrum of an irregular Turán graph $T(n,k)$ in Item 1 of this theorem (i.e., its spectrum if n is not divisible by k since $s \neq 0$). By Corollary 8, the spectra of the regular graphs K_q^{k-s} and K_{q+1}^{s} is

$$\sigma_{\mathbf{A}}(\mathsf{K}_q^{k-s}) = \Big\{[-q]^{k-s-1}, [0]^{(q-1)(k-s)}, q(k-s-1)\Big\}, \tag{50}$$

$$\sigma_{\mathbf{A}}(\mathsf{K}_{q+1}^{s}) = \Big\{[-q-1]^{s-1}, [0]^{qs}, (q+1)(s-1)\Big\}. \tag{51}$$

The $(k-s)$-partite graph K_q^{k-s} is r_1-regular with $r_1 = q(k-s-1)$, the s-partite graph K_{q+1}^{s} is r_2-regular with $r_2 = (q+1)(s-1)$, and by Definition 27, we have $T(n,k) = \mathsf{K}_q^{k-s} \vee \mathsf{K}_{q+1}^{s}$. Hence, by Lemma 1, the adjacency spectrum of $T(n,k)$ is given by

$$\begin{aligned} \sigma_{\mathbf{A}}(T(n,k)) &= \sigma_{\mathbf{A}}(\mathsf{K}_q^{k-s} \vee \mathsf{K}_{q+1}^{s}) \\ &= \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3, \end{aligned} \tag{52}$$

where

$$\mathcal{S}_1 = \Big\{[-q]^{k-s-1}, [0]^{(q-1)(k-s)}\Big\}, \tag{53}$$

$$\mathcal{S}_2 = \Big\{[-q-1]^{s-1}, [0]^{qs}\Big\}, \tag{54}$$

$$\begin{aligned} \mathcal{S}_3 &= \left\{\frac{r_1 + r_2 \pm \sqrt{(r_1 - r_2)^2 + 4n_1n_2}}{2}\right\} \\ &= \left\{\tfrac{1}{2}\left[n - 2q - 1 \pm \sqrt{\Big(n - 2(q+1)s + 1\Big)^2 + 4q(q+1)s(k-s)}\right]\right\}, \end{aligned} \tag{55}$$

where the last equality holds since, by the equality $n = kq + s$ and the above expressions of r_1 and r_2, it can be readily verified that $r_1 + r_2 = n - 2q - 1$ and $r_1 - r_2 = n - 2(q+1)s + 1$. Finally, combining (52)–(55) gives the **A**-spectrum in (48) of an irregular Turán graph $T(n,k)$.

We next prove Item 2 of this theorem, referring to a regular Turán graph $T(n,k)$ (i.e., $k|n$ or, equivalently, $s = 0$). In that case, we have $T(n,k) = \mathsf{K}_q^k$ where $q = \frac{n}{k}$, so the **A**-spectrum in (49) holds by Corollary 8. □

Remark 14. *In light of Theorem 24, if $k \geqslant 2$, then the number of negative eigenvalues (including multiplicities) of the adjacency matrix of the Turán graph $T(n,k)$ is $k-1$ if the graph is regular (i.e., if $k|n$), and it is $k-2$ otherwise (i.e., if the graph is irregular). If $k = 1$, which corresponds to an empty graph (having no edges), then all eigenvalues are zeros (having no negative eigenvalues). Furthermore, the adjacency matrix of $T(n,k)$ always has a single positive eigenvalue, which is of multiplicity 1 irrespectively of the values of n and k. We rely on these properties later in this paper (see Section 4.3.2).*

Example 5. *By Theorem 24, let us calculate the* **A***-spectrum of the Turán graph $T(17,7)$, and verify it numerically with the SageMath software [74]. Having $n = 17$ and $k = 7$ gives $q = 2$ and $s = 3$, which by Theorem 24 implies that*

$$\sigma_{\mathbf{A}}\big(T(17,7)\big) = \Big\{[-3]^2, [-2]^3, [0]^{10}, 6(1+\sqrt{2}), 6(1-\sqrt{2})\Big\}. \tag{56}$$

That has been numerically verified by programming in the SageMath software.

4.3.2. Turán Graphs Are DS

The main result of this subsection establishes that all Turán graphs are determined by their **A**-spectrum. This result is equivalent to Theorem 3.3 in [37], while it also presents an alternative proof that offers additional insights.

Theorem 25. *The Turán graph $T(n,k)$ is* **A***-DS.*

In order to prove Theorem 25, we first introduce an auxiliary result from [75], followed by several other lemmata.

Theorem 26 ([75] (Theorem 1))**.** *Let* G *be a graph. Then, the following statements are equivalent:*

- G *has exactly one positive eigenvalue.*
- $\mathsf{G} = \mathsf{H} \cup \overline{\mathsf{K_m}}$ *for some* m*, where* H *is a nonempty complete multipartite graph. In other words, the non-isolated vertices of* G *form a complete multipartite graph.*

Proof of Theorem 25. Let G be a graph that is **A**-cospectral with $T(n,k)$. Denote $n = qk + s$ for $s, q \in \mathbb{N}$ such that $0 \leqslant s < k$.

Lemma 2. *The graph* G *does not have a clique of order* $k+1$*.*

Proof. Suppose to the contrary that the graph G has a clique of order $k+1$, which means that K_{k+1} is an induced subgraph of G. The complete graph K_{k+1} has k negative eigenvalues (-1 with a multiplicity of k). On the other hand, G has at most $k-1$ negative eigenvalues, $n-k$ zero eigenvalues, and exactly one positive eigenvalue; indeed, this follows from Theorem 24 (see Remark 14), and since G and $T(n,k)$ are **A**-cospectral graphs. Hence, by Cauchy's Interlacing Theorem, every induced subgraph of G on $k+1$ vertices has at most $k-1$ negative eigenvalues (i.e., those eigenvalues interlaced between the negative and zero eigenvalues of G that are placed at distance $k+1$ apart in a sorted list of the eigenvalues of G in decreasing order). This contradicts our assumption of the existence of a clique of $k+1$ vertices because of the k negative eigenvalues of K_{k+1}. □

Lemma 3. *The graph* G *is a complete multipartite graph.*

Proof. Since G has exactly one positive eigenvalue, which is of multiplicity one, we obtain from Theorem 26 that $\mathsf{G} = \mathsf{H} \cup \overline{\mathsf{K}_\ell}$ for some $\ell \in \mathbb{N}$, where H is a nonempty multipartite graph. We next show that $\ell = 0$. Suppose to the contrary that $\ell \geqslant 1$, and let v be an isolated vertex of $\overline{\mathsf{K}_\ell}$. Since H is a nonempty graph, there exists a vertex $u \in \mathsf{V}(\mathsf{H})$. Let G_1 be the graph obtained from G by adding the single edge $\{v,u\}$. By Lemma 2, G does not have a clique of order $k+1$. Hence, G_1 does not have a clique of order $k+1$ either, contradicting Corollary 7. Hence, $\mathsf{G} = \mathsf{H}$. □

Lemma 4. *The graph* G *is a complete* k*-partite graph.*

Proof. By Lemma 3, G is a complete multipartite graph. Let r be the number of partite subsets in the vertex set $\mathsf{V}(\mathsf{G})$. We show that $r = k$, which then gives that G is a complete k-partite graph. By Lemma 2, G does not have a clique of order $k+1$. Hence, $r \leqslant k$. Suppose to the contrary that $r < k$. Since G is a complete r-partite graph, the largest order of a clique in G is r. Let G_1 be a graph obtained from G by adding an edge between two vertices within the same partite subset. The graph G_1 becomes an $(r+1)$-partite graph. Consequently, the maximum order of a clique in G_1 is at most $r+1 \leqslant k$. The graph G_1 has exactly one

more edge than G. Since G is $\mathbf{A}$-cospectral to $T(n,k)$, it has the same number of edges as in $T(n,k)$. Hence, G_1 contains more edges than $T(n,k)$, while also lacking a clique of order $k+1$. This contradicts Corollary 7, so we conclude that $r = k$. □

Let $n_1, n_2, \ldots, n_k \in \mathbb{N}$ be the number of vertices in each partite subset of the complete k-partite graph G, i.e., $\mathsf{G} = \mathsf{K}_{n_1,n_2,\ldots,n_k}$. Then, the next two lemmata subsequently hold.

Lemma 5. *For all $i \in [k]$, $n_i \leqslant q+1$.*

Proof. Suppose to the contrary that there exists a partite subset in the complete k-partite graph G with more than $q+1$ vertices. Let $\mathcal{P}_1$ be such a partite subset, and suppose without loss of generality that $n_1 = |\mathcal{P}_1| \geqslant q+1$. By the pigeonhole principle, there exists a partite subset of G with at most q vertices (since $\sum_{i\in[k]} n_i = n = kq+s$, where $0 \leqslant s \leqslant k-1$). Let $\mathcal{P}_2$ be such a partite subset of G, and suppose without loss of generality that $n_2 = |\mathcal{P}_2| \leqslant q$. Let G_1 be the graph obtained from G by removing a vertex $v \in \mathcal{P}_1$, adding a new vertex u to $\mathcal{P}_2$, and connecting u to all the vertices outside its partite subset. The new graph G_1 is also k-partite, so it does not contain a clique of order greater than k. Furthermore, by construction, G_1 has more edges than G, so

$$\left|\mathsf{E}(\mathsf{G}_1)\right| > \left|\mathsf{E}(\mathsf{G})\right| = \left|\mathsf{E}(T(n,k))\right|. \tag{57}$$

Hence, G_1 is a graph with no clique of order greater than k, and it has more edges than $T(n,k)$. That contradicts Theorem 22, so $\{n_i\}_{i=1}^k$ cannot include any element that is larger than $q+1$. □

Lemma 6. *For all $i \in [k]$, $n_i \geqslant q$.*

Proof. The proof of this lemma is analogous to the proof of Lemma 5. Suppose to the contrary that there exists a partite subset of G with less than q vertices. Let $\mathcal{P}_1$ be such a partite subset, so $p_1 \triangleq |\mathcal{P}_1| < q$. By the pigeonhole principle, there exists a partition with at least $q+1$ vertices. Let $\mathcal{P}_2$ be such a partite subset set, and let its number of vertices be denoted by $p_2 \triangleq |\mathcal{P}_2| \geqslant q+1$. Let G_1 be the graph obtained by removing a vertex $v \in \mathcal{P}_2$, adding a new vertex u to $\mathcal{P}_1$, and connecting the vertex u to all the vertices outside its partite subset. G_1 is k-partite, so it does not contain a clique of order greater than k, and G_1 has more edges than G so (57) holds. Hence, G_1 is a graph with no clique of order greater than k, and it has more edges than $T(n,k)$. That contradicts Theorem 22, so $\{n_i\}_{i=1}^k$ cannot include any element that is smaller than q. □

By Lemmas 5 and 6, we conclude that $n_i \in \{q, q+1\}$ for every $1 \leqslant i \leqslant k-1$. Let α be the number of partite subsets of q vertices and β be the number of partite subsets of $q+1$ vertices. Since G has n vertices, where $\sum n_i = n$, it follows that $q\alpha + (q+1)\beta = n$. Moreover, G is k-partite, so it follows that $\alpha + \beta = k$. This gives the linear system of equations

$$\begin{pmatrix} q & q+1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} n \\ k \end{pmatrix}, \tag{58}$$

which has the single solution

$$\alpha = k - s, \quad \beta = n - qk = s. \tag{59}$$

Hence, $\mathsf{G} = T(n,k)$, which completes the proof of Theorem 25. □

Remark 15. *The proof of Theorem 25 is an alternative proof of Theorem 3.3 in [37]. While both proofs rely on Theorem 26, which is Theorem 1 of [75], our proof relies on the adjacency spectral characterization in Theorem 24, noteworthy in its own right, and further builds upon a sequence of results presented in Lemmata 2–6. On the other hand, the proof of Theorem 3.3 in [37] relies on Theorem 26, but then deviates substantially from our proof (see Lemmata 2.4 and 2.5 in [37] and Theorem 3.1 in [37] and serves to prove Theorem 25).*

4.4. Line Graphs

Among the various studied transformations on graphs, the line graphs of graphs are one of the most studied transformations [76]. We first introduce their definition, and then address the spectral graph determination properties.

Definition 29. *The* line graph *of a graph* G*, denoted by* $\ell(\mathsf{G})$*, is a graph whose vertices are the edges in* G*, and two vertices are adjacent in* $\ell(\mathsf{G})$ *if the corresponding edges are incident in* G*.*

A notable spectral property of line graphs is that all the eigenvalues of their adjacency matrix are greater than or equal to -2 (see, e.g., [76] (Theorem 4.6)). For the determination of all graphs whose spectrum is bounded from below by -2, the interested reader is referred to [76] (Section 4.5).

The following theorem characterizes some families of line graphs that are DS.

Theorem 27. *The following line graphs are DS:*

1. *The line graph of the complete graph* K_k*, where* $k \geqslant 4$ *and* $k \neq 8$ *(see [4] (Theorem 4.1.7));*
2. *The line graph of the complete bipartite graph* $\mathsf{K}_{k,k}$*, where* $k \geqslant 2$ *and* $k \neq 4$ *(see [4] (Theorem 4.1.8));*
3. *The line graph* $\ell(\overline{\mathsf{C}_6})$ *(see [4] (Proposition 4.1.5));*
4. *The line graph of the complete bipartite graph* $\mathsf{K}_{m,n}$*, where* $m+n \geqslant 19$ *and* $\{m,n\} \neq \{2s^2+s, 2s^2-s\}$ *with* $s \in \mathbb{N}$ *(see [4] (Proposition 4.1.18)).*

Remark 16. *In regard to Item 1 of Theorem 27, the line graphs of complete graphs are referred to as triangular graphs. These are strongly regular graphs with the parameters* $\mathsf{srg}(\frac{1}{2}k(k-1), 2(k-2), k-2, 4)$*, where* $k \geqslant 4$*. For* $k = 8$*, the corresponding triangular graph is cospectral and nonisomorphic to the three Chang graphs (see Remark 4), which are strongly regular graphs* $\mathsf{srg}(28, 12, 6, 4)$*.*

We next prove the following result in regard to the Petersen graph, which appears in [4] (Problem 4.3) and [51] (Section 10.3) (without a proof).

Corollary 9. *The Petersen graph is DS.*

Proof. The Petersen graph is known to be isomorphic to the complement of the line graph of the complete graph K_5 (i.e., it is isomorphic to $\overline{\ell(\mathsf{K}_5)}$). By Item 1 of Theorem 27, the line graph $\ell(\mathsf{K}_5)$ is DS. It is also a 6-regular graph (as the line graph of a d-regular graph is $(2d-2)$-regular and K_5 is a 4-regular graph). Consequently, by Item 5 of Theorem 13, the complement of $\ell(\mathsf{K}_5)$ is also DS. □

The following definition and theorem provide further elaboration on Item 2 of Theorem 27.

Definition 30 ([51] (Section 1.1.8))**.** *The Hamming graph* $\mathrm{H}(2,q)$*, where* $q \geqslant 2$*, has the vertex set* $[q] \times [q]$*, and any two vertices are adjacent if and only if they differ in one coordinate (i.e., their Hamming distance is equal to 1). These are also referred to* lattice graphs*, and denoted by* $\mathrm{L}_2(q)$*.*

The Lattice graph $\mathrm{L}_2(q)$*, where* $q \geqslant 2$*, is also the line graph of the complete bipartite graph* $\mathsf{K}_{q,q}$*, and it is a strongly regular graph with parameters* $\mathsf{srg}(q^2, 2(q-1), q-2, 2)$.

Theorem 28 ([77])**.** *The lattice graph* $\mathrm{L}_2(q)$ *is a strongly regular DS graph for all* $q \neq 4$*. For* $q = 4$*, the graph* $\mathrm{L}_2(4)$ *is not DS since it is cospectral and nonisomorphic to the Shrikhande graph, which are nonisomorphic strongly regular graphs with the common parameters* $\mathsf{srg}(16, 6, 4, 2)$.

The following result provides an interesting connection between **A** and **Q**-cospectralities of graphs.

Theorem 29 ([4] (Proposition 7.8.5))**.** *If two graphs are* **Q***-cospectral, then their line graphs are* **A***-cospectral.*

4.5. Nice Graphs

The family referred to as "nice graphs" has been recently introduced in [9].

Definition 31.

- *A graph* G *is sunlike if it is connected and can be obtained from a cycle* C *by adding some vertices and connecting each of them to some vertex in* C*.*
- *Let* $l, k \in \mathbb{N}$*. A sunlike graph* G *is* (l, k)-nice *if it can be obtained by a cycle* C_ℓ *and*
  - *There is a single vertex* $v_1 \in \mathsf{V}(\mathsf{C}_\ell)$ *of degree* 3*.*
  - *There are* k *vertices* $u_1, \ldots, u_k \in \mathsf{V}(\mathsf{C}_\ell)$ *of degree* 4*. Let* $\mathcal{U} = \{u_1, \ldots, u_k\}$*.*
  - *By starting a walk on* C_ℓ *from* v_1 *at some orientation, then after 4 or 6 steps, we get to a vertex* $u_1 \in \mathcal{U}$*. Then, after another* 4 *or* 6 *steps from* u_1 *we get to* $u_2 \in \mathcal{U}$*, and so on, until we get to the vertex* $u_k \in \mathcal{U}$*.*

Theorem 30 ([9])**.** *Let* $l, k \in \mathbb{N}$ *such that* $l \equiv 2 \,(mod\ 4)$*. Let* G *be an* (l, k)*-nice graph. If the order of* G *is a prime number greater than some* $n_0 \in \mathbb{N}$*, then the line graph* $\ell(\mathsf{G})$ *is DS.*

A more general class of graphs is introduced in [9], where it is shown that for every sufficiently large $n \in \mathbb{N}$, the number of nonisomorphic n-vertex DS graphs is at least e^{cn} for some positive constant c (see [9] (Theorem 1.4)). This recent result represents a significant advancement in the study of Haemers' conjecture because the earlier lower bounds on the number of nonisomorphic n-vertex DS graphs were all of the form $e^{c\sqrt{n}}$, for some positive constant c. As noted in [9], the first form of such a lower bound was derived by van Dam and Haemers [13] (Proposition 6), who proved that a graph G is DS if every connected component of G is a complete subgraph, leading to a lower bound that is approximately of the form $e^{c\sqrt{n}}$ with $c = \sqrt{\frac{2}{3}\,\pi}$. Therefore, the transition to a lower bound in [9] that scales exponentially with n, rather than with $\sqrt{n}$, is both remarkable and noteworthy.

4.6. Friendship Graphs and Their Generalization

The next theorem considers whether friendship graphs (see Definition 20) can be uniquely determined by the spectra of four of their associated matrices.

Theorem 31. *The friendship graph* F_p *satisfies the following properties: It is DS if and only if* $p \neq 16$ *(i.e., the friendship graph is DS unless it has 16 triangles) [63],* **L***-DS [78],* **Q***-DS [70], and* $\mathcal{L}$*-DS [14].*

The friendship graph F_p, where $p \in \mathbb{N}$, can be expressed in the form $\mathsf{F}_p = \mathsf{K}_1 \vee (p\,\mathsf{K}_2)$ (see Figure 3). The last observation follows from a property of a generalized friendship graph, which is defined as follows.

Definition 32. *Let* $p, q \in \mathbb{N}$. *The* generalized friendship graph *is given by* $\mathsf{F}_{p,q} = \mathsf{K}_1 \vee (p\,\mathsf{K}_q)$. *Note that* $\mathsf{F}_{p,2} = \mathsf{F}_p$.

The following theorem addresses the conditions under which generalized friendship graphs can be uniquely determined by the spectra of their normalized Laplacian matrix.

Theorem 32. *The generalized friendship graph* $\mathsf{F}_{p,q}$ *is* $\mathcal{L}$*-DS if and only if* $q \geqslant 2$ *or both* $q = 1$ *and* $p = 2$ *[14].*

Corollary 10. *The friendship graph* F_p *is* $\mathcal{L}$*-DS [14].*

4.7. Strongly Regular Graphs

Strongly regular graphs with an identical vector of parameters (n, d, λ, μ) are cospectral but may not be isomorphic (see, Corollary 4, Remark 4, and Theorem 28). For that reason, strongly regular graphs are not necessarily determined by their spectrum. There are, however, infinite families of strongly regular DS graphs:

Theorem 33 ([1] (Proposition 14.5.1))**.** *If* $n \neq 8$, $m \neq 4$, $a \geqslant 2$, *and* $\ell \geqslant 2$, *then the disjoint union of identical complete graphs* $a\,\mathsf{K}_\ell$, *the line graph of a complete graph* $\ell(\mathsf{K}_n)$, *and the line graph of a complete bipartite graph with partite sets of equal size* $\ell(\mathsf{K}_{m,m})$, *as well as their complements, are strongly regular DS graphs.*

We next show that, although connected strongly regular graphs are not generally DS, the property of strong regularity, as well as the four parameters that characterize strongly regular graphs can be determined by the spectrum of their adjacency matrix.

Theorem 34. *Let* G *be a connected strongly regular graph. Then, its strong regularity, the vector of parameters* (n, d, λ, μ), *Lovász* ϑ*-function* $\vartheta(\mathsf{G})$, *number of edges and triangles, girth, and diameter can all be determined by its* $\mathbf{A}$*-spectrum.*

Proof. The order of a graph n is determined by the $\mathbf{A}$-spectrum, which is the number of eigenvalues (including multiplicities). By Theorem 4, the regularity of a graph is determined by its $\mathbf{A}$-spectrum. By Item 5 of Theorem 5, a connected regular graph is strongly regular if and only if it has three distinct eigenvalues. Hence, the strong regularity property of G is determined by its $\mathbf{A}$-spectrum. For such a connected regular, the largest eigenvalue is simple, $\lambda_1 = d$, and the other two distinct eigenvalues of the adjacency matrix of G are given by λ_2 and λ_n with $\lambda_n < \lambda_2$. We next show that the number of common neighbors of any pair of adjacent vertices (λ), and the number of common neighbors of any pair of nonadjacent vertices (μ) in G are, respectively, given by

$$\lambda = \lambda_1 + (1 + \lambda_2)(1 + \lambda_n) - 1, \tag{60}$$

$$\mu = \lambda_1 + \lambda_2\lambda_n. \tag{61}$$

So, these parameters are explicitly expressed in terms of the adjacency spectrum of the strongly regular graph. Indeed, by Theorem 5, the second-largest and least eigenvalues of the adjacency matrix of G are given by

$$\begin{cases} \lambda_2 = \frac{1}{2}\left(\lambda - \mu + \sqrt{(\lambda - \mu)^2 + 4(d - \mu)}\right), \\ \lambda_n = \frac{1}{2}\left(\lambda - \mu - \sqrt{(\lambda - \mu)^2 + 4(d - \mu)}\right), \end{cases} \tag{62}$$

from which it follows that (noting that $d = \lambda_1$)

$$\begin{cases} \lambda_2 + \lambda_n = \lambda - \mu, \\ \lambda_2 \lambda_n = \mu - \lambda_1. \end{cases} \tag{63}$$

This gives (60) and (61) from, respectively, the second equality in (63) and by adding the two equalities in (63).

The Lovász ϑ-function of a strongly regular graph is given by (see [59] (Proposition 1))

$$\vartheta(\mathsf{G}) = -\frac{n\lambda_n}{d - \lambda_n}, \tag{64}$$

so $\vartheta(\mathsf{G})$ is determined by its **A**-spectrum since the strong regularity property of G was first determined. The number of edges of a d-regular graph G is given by $\frac{1}{2}nd$, and the number of triangles of the strongly regular graph G is given by $\frac{1}{6}nd\lambda$, so they are both determined once the four parameters of the strongly regular graphs are revealed. The diameter of a connected strongly regular graph is equal to 2 (note that complete graphs are excluded from the family of strongly regular graphs). Finally, the girth of the strongly regular graph G is determined as follows [79]:

1. If $\lambda > 0$, then the girth of G is equal to 3;
2. If $\lambda = 0$ and $\mu \geqslant 2$, then the girth of G is equal to 4;
3. If $\lambda = 0$ and $\mu = 1$, then the girth of G is equal to 5.

□

Remark 17. *A strongly regular graph is connected if and only if $\mu > 0$.*

By [12] (Proposition 2), no pair of **A**-cospectral graphs exists where one graph is regular and the other is not. The following result extends this observation to strong regularity.

Corollary 11. *There are no two* **A***-cospectral connected graphs where one is strongly regular and the other is not.*

Proof. For a connected strongly regular graph, the strong regularity is determined by the **A**-spectrum. □

Another corollary that follows from Theorem 34 applies to strongly regular DS graphs.

Corollary 12. *Let G be a connected strongly regular graph such that there is no other nonisomorphic strongly regular graph with an identical vector of parameters (n, d, λ, μ). Then, G is a DS graph.*

Corollary 12 naturally raises the following question.

Question 1. *Which connected strongly regular graphs are determined by their vector of parameters (n, d, λ, μ)?*

A partial answer to Question 1 is provided below.

By Corollary 12, for connected strongly regular graphs, there exists an equivalence between their spectral determination (due to their regularity and in light of Theorem 14, which is based on the spectrum of their adjacency, Laplacian, or signless Laplacian matrices) and the uniqueness of these graphs for the given parameter vector (n, d, λ, μ).

The study of the number of nonisomorphic strongly regular graphs corresponding to a given set of parameters has been extensively explored. For example, by Theorem 28, there

is a unique (up to isomorphism) strongly regular graph of the form $\mathsf{srg}(q^2, 2(q-1), q-2, 2)$ for any given $q \geqslant 2$ with $q \neq 4$. Specifically, this implies the uniqueness of $\mathsf{srg}(36, 10, 4, 2)$ (setting $q = 6$). On the other hand, a computer search by McKay and Spence established that there are 32,548 strongly regular graphs of the form $\mathsf{srg}(36, 15, 6, 6)$, so none of them is DS (by Corollary 12).

Further results on the uniqueness or non-uniqueness of strongly regular graphs with a given parameter vector (n, d, λ, μ) can be found in [80] and the references therein. Infinite families of strongly regular DS graphs are presented in Theorem 33. Some known sporadic, strongly regular DS graphs are listed in [12] (Table 2), with an update in [13] (Table 1). The uniqueness of further strongly regular graphs with given parameter vectors, which makes them, therefore, DS graphs, was established, e.g., in [81–85].

A combination of [13] (Table 1) and Corollary 9 implies that, apart from complete graphs on fewer than three vertices and all complete bipartite regular graphs (which are known to be DS, as stated in Item 2 of Theorem 13), also all the seven currently known triangle-free strongly regular graphs (see [86]) are DS. These include the following:

- The Pentagon graph C_5 that is $\mathsf{srg}(5, 2, 0, 1)$ (by Item 2 of Theorem 13, and see [51] (Section 10.1));
- The Petersen graph $\mathsf{srg}(10, 3, 0, 1)$ (by Corollary 9, and see [51] (Section 10.3));
- Clebsch graph $\mathsf{srg}(16, 5, 0, 2)$ (see [51] (Section 10.7));
- Hoffman–Singleton srg $\mathsf{srg}(50, 7, 0, 1)$ (see [51] (Section 10.19));
- Gewirtz graph $\mathsf{srg}(56, 10, 0, 2)$ (see [51] (Section 10.20));
- Mesner (M_{22}) graph $\mathsf{srg}(77, 16, 0, 4)$ (see [51] (Section 10.27) and [81]);
- Higman–Sims graph $\mathsf{srg}(100, 22, 0, 6)$ (see [51] (Section 10.31) and [85]).

An up-to-date list of strongly regular DS graphs—strongly regular graphs that are uniquely determined by their parameter vectors—as well as the number of strongly regular NICS graphs for given parameter vectors, is available on Brouwer's website [87]. An exclamation mark placed to the left of a parameter vector (n, d, λ, μ), when it is without a preceding number, indicates a strongly regular DS graph. In contrast, an exclamation mark preceded by a natural number greater than 1 specifies the number of strongly regular NICS graphs with the corresponding parameter vector. For example, as shown in [87], strongly regular graphs with the parameter vectors $(13, 6, 2, 3)$, $(15, 6, 1, 3)$, $(17, 8, 3, 4)$, and $(21, 10, 3, 6)$, among others, are DS graphs. On the other hand, according to [87], there are 15 strongly regular NICS graphs with the parameter vector $(25, 12, 5, 6)$, 10 strongly regular NICS graphs with the parameter vector $(26, 10, 3, 4)$, and so forth.

To conclude, as strongly regular NICS graphs are not DS, **L**-DS, or **Q**-DS, we were recently informed of ongoing research by Cioaba et al. [88], which investigates the spectral properties of higher-order Laplacian matrices associated with these graphs. This research demonstrates that the spectra of these new matrices can distinguish some of the strongly regular NICS graphs. However, in other cases, strongly regular NICS graphs remain indistinguishable even with the spectra of these higher-order Laplacian matrices.

5. Graph Operations for the Construction of Cospectral Graphs

This section presents such graph operations, focusing on unitary and binary transformations that enable the systematic construction of cospectral graphs. These operations are designed to preserve the spectral properties of the original graph while potentially altering its structure, thereby producing nonisomorphic graphs with identical eigenvalues. By employing these techniques, one can generate diverse examples of cospectral graphs, which offer valuable tools for investigating the limitations of spectral characterization and exploring the boundaries between graphs that are or are not determined by their spectrum,

which then relates the scope of the present section to Section 4, which deals with graphs or graph families that are determined by their spectrum.

5.1. Coalescence

A construction of cospectral trees has been offered in [11], implying that almost all trees are not DS.

Definition 33. *Let* $\mathsf{G}_1 = (V_1, E_1), \mathsf{G}_2 = (V_2, E_2)$ *be two graphs with* n_1, n_2 *vertices, respectively. Let* $v_1 \in V_1$ *and* $v_2 \in V_2$ *be an arbitrary choice of vertices in both graphs. The* coalescence of G_1 and G_2 with respect to v_1 and v_2 *is the graph with* $n_1 + n_2 - 1$ *vertices, obtained by the union of* G_1 *and* G_2 *where* v_1 *and* v_2 *are identified as the same vertex in the united graph.*

Theorem 35. *Let* $\mathsf{G}_1 = (V_1, E_1), \mathsf{G}_2 = (V_2, E_2)$ *be two cospectral graphs, and let* $v_1 \in V_1$ *and* $v_2 \in V_2$ *be an arbitrary choice of vertices in both graphs. Let* H_1 *and* H_2 *be the subgraphs of* G_1 *and* G_2 *that are induced by* $V_1 \backslash \{v_1\}$ *and* $V_2 \backslash \{v_2\}$*, respectively. Let* Γ *be a graph and* $u \in V(\Gamma)$*. If* H_1 *and* H_2 *are cospectral, then the coalescence of* G_1 *and* Γ *with respect to* v_1 *and* u *is cospectral to the coalescence of* G_2 *and* Γ *with respect to* v_2 *and* u*.*

Combinatorial arguments that rely on the coalescence operation on graphs lead to a striking asymptotic result in [11], which states that the fraction of n-vertex trees with cospectral and nonisomorphic mates, which are also trees, approaches one as n tends to infinity. Consequently, the fraction of the n-vertex nonisomorphic trees that are determined by their spectrum (DS) approaches zero as n tends to infinity. In other words, this means that almost all trees are not DS (with respect to their adjacency matrix) [11].

5.2. Seidel Switching

Seidel switching is one of the well-known methods for the construction of cospectral graphs.

Definition 34. *Let* G *be a graph, and let* $\mathcal{U} \subseteq \mathsf{V}(\mathsf{G})$*. Constructing a graph* $\mathsf{G}_\mathcal{U}$ *by preserving all the edges in* G *between vertices within* $\mathcal{U}$*, as well as all edges in* G *between vertices within the complement set* $\mathcal{U}^\mathrm{c} = \mathsf{V}(\mathsf{G}) \backslash \mathcal{U}$*, while modifying adjacency and nonadjacency between any two vertices where one is in* $\mathcal{U}$ *and the other is in* $\mathcal{U}^\mathrm{c}$*, is referred to (up to isomorphism) as* Seidel switching *of* G *with respect to* $\mathcal{U}$*.*

By Definition 34, the Seidel switching of G with respect to $\mathcal{U}$ is equivalent to its Seidel switching with respect to $\mathcal{U}^\mathrm{c}$. Let $\mathbf{A}(\mathsf{G})$ and $\mathbf{A}(\mathsf{G}_\mathcal{U})$ be the adjacency matrices of a graph G and its Seidel switching $\mathsf{G}_\mathcal{U}$, and let $\mathbf{A}_\mathcal{U}$ and $\mathbf{A}_{\mathcal{U}^\mathrm{c}}$ be the matrices of $\mathbf{A}(\mathsf{G})$ that, respectively, refer to the adjacency matrices of the subgraphs of G induced by $\mathcal{U}$ and $\mathcal{U}^\mathrm{c}$. Then, for some $\mathbf{B} \in \{0,1\}^{|\mathcal{U}^\mathrm{c}| \times |\mathcal{U}|}$, we obtain

$$\mathbf{A}(\mathsf{G}) = \begin{pmatrix} \mathbf{A}_\mathcal{U} & \mathbf{B}^\mathrm{T} \\ \mathbf{B} & \mathbf{A}_{\mathcal{U}^\mathrm{c}} \end{pmatrix}, \tag{65}$$

and by Definition 34,

$$\mathbf{A}(\mathsf{G}_\mathcal{U}) = \begin{pmatrix} \mathbf{A}_\mathcal{U} & \overline{\mathbf{B}}^\mathrm{T} \\ \overline{\mathbf{B}} & \mathbf{A}_{\mathcal{U}^\mathrm{c}} \end{pmatrix}, \tag{66}$$

where $\overline{\mathbf{B}}$ is obtained from $\mathbf{B}$ by interchanging zeros and ones. If G is a regular graph, the following is a necessary and sufficient condition for $\mathsf{G}_\mathcal{U}$ to be a regular graph of the same degree of its vertices.

Theorem 36 ([4] (Proposition 1.1.7)). *Let* G *be a d-regular graph on n vertices. Then,* $\mathsf{G}_{\mathcal{U}}$ *is also d-regular if and only if* $\mathcal{U}$ *induces a regular subgraph of degree k, where* $|\mathcal{U}| = n - 2(d-k)$.

The next result shows the relevance of Seidel switching for the construction of regular and cospectral graphs.

Theorem 37 ([4] (Proposition 1.1.8)). *Let* G *be a d-regular graph,* $\mathcal{U} \subseteq \mathsf{V}(\mathsf{G})$, *and let* $\mathsf{G}_{\mathcal{U}}$ *be obtained from* G *by Seidel switching. If* $\mathsf{G}_{\mathcal{U}}$ *is also a d-regular graph, then* G *and* $\mathsf{G}_{\mathcal{U}}$ *are cospectral (and due to their regularity, they are* $\mathcal{X}$*-cospectral for every* $\mathcal{X} \in \{\mathbf{A}, \mathbf{L}, \mathbf{Q}, \mathcal{L}\}$).

Remark 18. *Theorem 37 provides a method for finding cospectral regular graphs. These graphs may be, however, also isomorphic. If the graphs are nonisomorphic, then it gives a pair of NICS graphs.*

Remark 19. *A regular graph* G *on n vertices cannot be switched into another regular graph if n is odd (see [4] (Corollary 4.1.10)), which means that the conditions in Theorem 37 cannot be satisfied for any regular graph of an odd order.*

Remark 20. *Seidel switching determines an equivalence relation on graphs. This follows from the fact that switching with respect to a subset* $\mathcal{U} \in \mathsf{V}(\mathsf{G})$, *and then with respect to a subset* $\mathcal{V} \in \mathsf{V}(\mathsf{G})$, *is the same as switching with respect to* $(\mathcal{V} \backslash \mathcal{U}) \cup (\mathcal{U} \backslash \mathcal{V})$ *(see [4] (p. 18)).*

Example 6. *The Shrikhande graph can be obtained through Seidel switching applied to the line graph* $\ell(\mathsf{K}_{4,4})$ *with respect to four independent vertices of the latter (see [4] (Example 1.2.4)). Both are 6-regular graphs (hence, they are cospectral graphs by Theorem 37). Moreover, the former graph is a strongly regular graph* $\mathsf{srg}(16, 6, 2, 2)$, *whereas the line graph* $\ell(\mathsf{K}_{4,4})$ *is not. Consequently, these are nonisomorphic and cospectral (NICS) 6-regular graphs on 16 vertices.*

5.3. The Godsil and McKay Method

Another construction of cospectral pairs of graphs was offered by Godsil and McKay in [24].

Theorem 38. *Let* G *be a graph with an adjacency matrix of the form*

$$\mathbf{A}(\mathsf{G}) = \begin{pmatrix} \mathbf{B} & \mathbf{N} \\ \mathbf{N}^{\mathrm{T}} & \mathbf{C} \end{pmatrix} \tag{67}$$

where the sum of each column in $\mathbf{N} \in \{0,1\}^{b \times c}$ *is either* $0, b$ *or* $\frac{b}{2}$. *Let* $\widehat{\mathbf{N}}$ *be the matrix obtained by replacing each column* $\underline{c}$ *in* $\mathbf{N}$ *whose sum of elements is* $\frac{b}{2}$ *with its complement* $\mathbf{1}_n - \underline{c}$. *Then, the modified graph* $\widehat{\mathsf{G}}$ *whose adjacency matrix is given by*

$$\mathbf{A}(\widehat{\mathsf{G}}) = \begin{pmatrix} \mathbf{B} & \widehat{\mathbf{N}} \\ \widehat{\mathbf{N}}^{\mathrm{T}} & \mathbf{C} \end{pmatrix} \tag{68}$$

is cospectral with G.

Two examples of pairs of NICS graphs are presented in Section 1.8.3 of [1].

5.4. Graphs Resulting from the Duplication and Corona Graphs

Definition 35 ([89]). *Let* G *be a graph with a vertex set* $\mathsf{V}(\mathsf{G}) = \{v_1, \ldots, v_n\}$, *and consider a copy of* G *with a vertex set* $\mathsf{G} = \{u_1, \ldots, u_n\}$, *where* u_i *is a duplicate of the vertex* v_i. *For each* $i \in [n]$, *connect the vertex* u_i *to all the neighbors of* v_i *in* G, *and then delete all edges in* G. *Similarly, for each* $i \in [n]$, *connect the vertex* v_i *to all the neighbors of* u_i *in the copied graph, and then delete all*

edges in the copied graph. The resulting graph, which has $2n$ *vertices, is called the* duplication graph *of* G*, and is denoted by* $\mathrm{Du}(\mathsf{G})$ *(see Figure 4).*

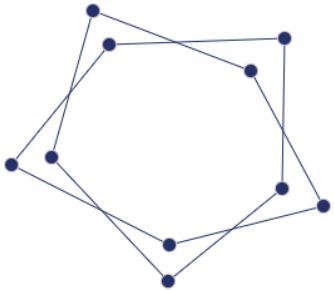


Figure 4. The duplication graph $\mathrm{Du}(\mathsf{C}_5)$ (see Definition 35).

Definition 36 ([90])**.** *Let* G_1 *and* G_2 *be graphs on disjoint vertex sets of* n_1 *and* n_2 *vertices, and with* m_1 *and* m_2 *edges, respectively. The* corona *of* G_1 *and* G_2*, denoted by* $\mathsf{G}_1 \circ \mathsf{G}_2$*, is a graph on* $n_1 + n_1 n_2$ *vertices obtained by taking one copy of* G_1 *and* n_1 *copies of* G_2*, and then connecting, for each* $i \in [n_1]$*, the* i*-th vertex of* G_1 *to each vertex in the* i*-th copy of* G_2 *(see Figure 5).*

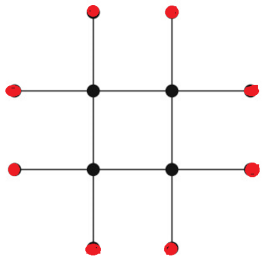


Figure 5. The corona graph $\mathsf{C}_4 \circ (2\,\mathsf{K}_1)$ (see Definition 36). It is composed of one copy of C_4 (referring to the black vertices) and four copies of $2\,\mathsf{K}_1$ (referring to the red vertices).

Definition 37 ([91])**.** *The* edge corona *of* G_1 *and* G_2*, denoted by* $\mathsf{G}_1 \diamondsuit \mathsf{G}_2$*, is defined as the graph obtained by taking one copy of* G_1 *and* $m_1 = \left|\mathsf{E}(\mathsf{G}_1)\right|$ *copies of* G_2*, and then connecting, for each* $j \in [m_1]$*, the two end-vertices of the* j*-th edge of* G_1 *to every vertex in the* j*-th copy of* G_2 *(see Figure 6).*

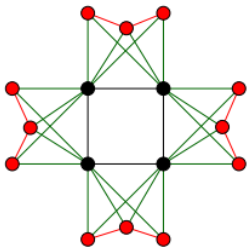


Figure 6. The edge-corona graph $\mathsf{C}_4 \diamondsuit \mathsf{P}_3$ (see Definition 37).

Definition 38. *Let* G_1 *and* G_2 *be graphs with disjoint vertex sets of* n_1 *and* n_2 *vertices, respectively. Let* $Du(\mathsf{G}_1)$ *be the duplication graph of* G_1 *with vertex set* $\mathsf{V}(\mathsf{G}_1) \cup \mathsf{U}(\mathsf{G}_1)$*, where* $\mathsf{V}(\mathsf{G}_1) = \{v_1, \ldots, v_{n_1}\}$ *and with the duplicated vertex set* $\mathsf{U}(\mathsf{G}_1) = \{u_1, \ldots, u_{n_1}\}$ *(see Definition 35). The* duplication corona graph*, denoted by* $\mathsf{G}_1 \boxminus \mathsf{G}_2$*, is the graph obtained from* $\mathrm{Du}(\mathsf{G}_1)$ *and* n_1 *copies of* G_2 *by connecting, for each* $i \in [n_1]$*, the vertex* $v_i \in \mathsf{V}(\mathsf{G}_1)$ *of the graph* $\mathrm{Du}(\mathsf{G}_1)$ *to every vertex in the* i*-th copy of* G_2 *(see Figure 7).*

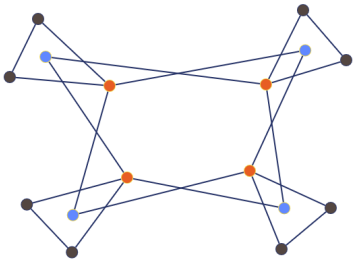


Figure 7. The duplication corona graph $C_4 \boxminus K_2$ (see Definition 38).

Definition 39. *Let* G_1 *and* G_2 *be graphs with disjoint vertex sets of* n_1 *and* n_2 *vertices, respectively. Let* $\mathrm{Du}(G_1)$ *be the duplication graph of* G_1 *with the vertex set* $V(G_1) \cup U(G_1)$*, where* $V(G_1) = \{v_1, \ldots, v_{n_1}\}$ *and the duplicated vertex set* $U(G_1) = \{u_1, \ldots, u_{n_1}\}$ *(see Definition 35). The duplication neighborhood corona, denoted by* $G_1 \boxplus G_2$*, is the graph obtained from* $\mathrm{Du}(G_1)$ *and* n_1 *copies of* G_2 *by connecting the neighbors of the vertex* $v_i \in V(G_1)$ *of* $\mathrm{Du}(G_1)$ *to every vertex in the i-th copy of* G_2 *for* $i \in [n_1]$ *(see Figure 8).*

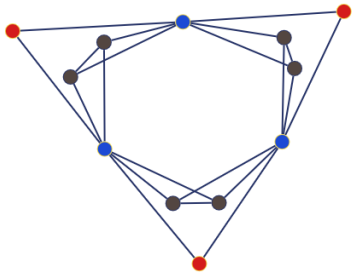


Figure 8. The duplication neighborhood corona $K_3 \boxplus K_2$ (see Definition 39).

Definition 40. *Let* G_1 *and* G_2 *be graphs with disjoint vertex sets of* n_1 *and* n_2 *vertices, respectively. Let* $\mathrm{Du}(G_1)$ *be the duplication graph of* G_1 *with vertex set* $V(G_1) \cup U(G_1)$*, where* $V(G_1) = \{v_1, \ldots, v_{n_1}\}$ *is the vertex set of* G_1 *and* $U(G_1) = \{u_1, \ldots, u_{n_1}\}$ *is the duplicated vertex set. The* duplication edge corona*, denoted by* $G_1 \boxplus G_2$*, is the graph obtained from* $\mathrm{Du}(G_1)$ *and* $|E(G_1)|$ *copies of* G_2 *by connecting each of the two vertices* $v_i, v_j \in V(G_1)$ *of* $\mathrm{Du}(G_1)$ *to every vertex in the k-th copy of* G_2 *whenever* $\{v_i, v_j\} = e_k \in E(G_1)$ *(see Figure 9).*

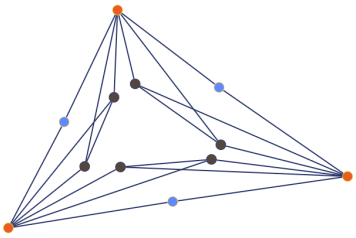


Figure 9. The duplication edge corona $K_3 \boxplus K_2$ (see Definition 40).

Definition 41. *Consider two graphs* G_1 *and* G_2 *with* n_1 *and* n_2 *vertices and, respectively. The closed neighborhood corona of* G_1 *and* G_2*, denoted by* $G_1 \boxtimes G_2$*, is a new graph obtained by creating* n_1 *copies of* G_2*. Each vertex of the* i^{th} *copy of* G_2 *is then connected to the* i^{th} *vertex and neighborhood of the* i^{th} *vertex of* G_1 *(see Figure 10).*

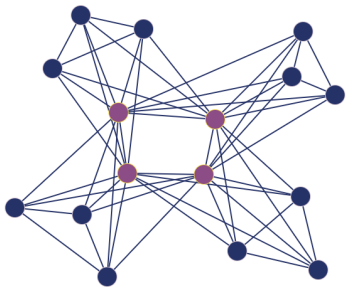


Figure 10. The closed neighborhood corona of the 4-length cycle C_4 and the triangle K_3, denoted by $\mathsf{C}_4 \boxtimes \mathsf{K}_3$ (see Definition 41).

Theorem 39 ([92]). *Let $\mathsf{G}_1, \mathsf{H}_1$ be r_1-regular, cospectral graphs, and let G_2 and H_2 be r_2-regular, cospectral, and nonisomorphic (NICS) graphs. Then, the following holds:*

- *The duplication corona graphs $\mathsf{G}_1 \boxminus \mathsf{G}_2$ and $\mathsf{H}_1 \boxminus \mathsf{H}_2$ are $\{\mathbf{A}, \mathbf{L}, \mathbf{Q}\}$-NICS irregular graphs.*
- *The duplication neighborhood corona $\mathsf{G}_1 \boxplus \mathsf{G}_2$ and $\mathsf{H}_1 \boxplus H_2$ are $\{\mathbf{A}, \mathbf{L}, \mathbf{Q}\}$-NICS irregular graphs.*
- *The duplication edge corona $\mathsf{G}_1 \boxplus \mathsf{G}_2$ and $\mathsf{H}_1 \boxplus \mathsf{H}_2$ are $\{\mathbf{A}, \mathbf{L}, \mathbf{Q}\}$-NICS irregular graphs.*

Question 2. *Are the irregular graphs in Theorem 39 also cospectral with respect to the normalized Laplacian matrix?*

Theorem 40 ([93]). *Let G_1 and G_2 be cospectral regular graphs, and let H be an arbitrary graph. Then, the following holds:*

- *The closed neighborhood corona $\mathsf{G}_1 \boxtimes \mathsf{H}$ and $\mathsf{G}_2 \boxtimes \mathsf{H}$ are $\{\mathbf{A}, \mathbf{L}, \mathbf{Q}\}$-NICS irregular graphs.*
- *The closed neighborhood corona $\mathsf{H} \boxtimes \mathsf{G}_1$ and $\mathsf{H} \boxtimes \mathsf{G}_2$ are $\{\mathbf{A}, \mathbf{L}, \mathbf{Q}\}$-NICS irregular graphs.*

Question 3. *Are the irregular graphs in Theorem 40 also cospectral with respect to the normalized Laplacian matrix?*

5.5. Graph Constructions Based on the Subdivision and Bipartite Incidence Graphs

Definition 42 ([4]). *Let G be a graph. The* subdivision graph *of G, denoted by $\mathsf{S}(\mathsf{G})$, is obtained from G by inserting a new vertex into every edge of G. Subdivision is the process of adding a new vertex along an edge, effectively splitting the edge into two edges connected in series through the new vertex (see Figure 11).*

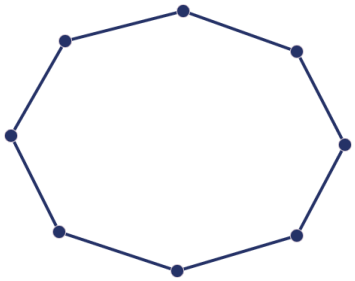


Figure 11. The subdivision graph of a 4-length cycle, denoted by $\mathsf{S}(\mathsf{C}_4)$, which is an 8-length cycle C_8 (see Definition 42).

Definition 43 ([94]). *Let G be a graph. The bipartite incidence graph of G, denoted by $\mathsf{B}(\mathsf{G})$, is a bipartite graph constructed as follows: For each edge $e \in \mathsf{G}$, a new vertex u_e is added to the vertex set of G. The vertex u_e is then made adjacent to both endpoints of e in G (see Figure 12).*

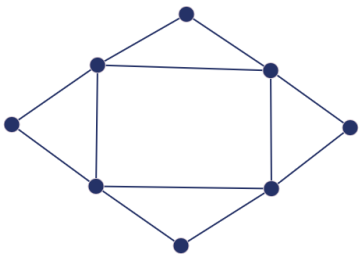


Figure 12. The bipartite incidence graph of a length-4 cycle C_4, denoted by $B(C_4)$ (see Definition 43).

Let G_1 and G_2 be two vertex-disjoint graphs with n_1 and n_2 vertices, and m_1 and m_2 edges, respectively. Four binary operations on these graphs are defined as follows.

Definition 44. *The subdivision-vertex-bipartite-vertex join of graphs* G_1 *and* G_2*, denoted* $S(G_1) \ddot{\vee} B(G_2)$*, is the graph obtained from* $S(G_1)$ *and* $B(G_2)$ *by connecting each vertex of* $V(G_1)$ *(that is the subset of the original vertices in the vertex set of* $S(G_1)$*) with every vertex of* $V(G_2)$ *(that is the subset of the original vertices in the vertex set of* $B(G_2)$*).*

By Definitions 42, 43, and 44, it follows that the graph $S(G_1) \ddot{\vee} B(G_2)$ has $n_1 + n_2 + m_1 + m_2$ vertices and $n_1n_2 + 2m_1 + 3m_2$ edges. Figure 13 displays the graph $S(C_4) \ddot{\vee} B(P_3)$.

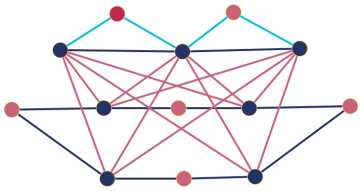


Figure 13. The graph $S(C_4) \ddot{\vee} B(P_3)$ (see Definition 44). The black vertices represent the vertices of the length-4 cycle C_4 and the vertices of the path P_3. The additional vertices in the subdivision graph $S(C_4)$ are the four red vertices located at the bottom of this figure (as shown in Figure 11). Similarly, the additional vertices in the bipartite incidence graph of the path P_3 are the two red vertices located at the top of this figure. The edges of the graph include the following: edges connecting each black vertex of C_4 to each black vertex of P_3, edges between the black and red vertices at the bottom of the figure (that correspond to the subdivision of C_4), and the four (light blue) edges connecting the two top red vertices to the top black vertices of the figure (that correspond to the bipartite incidence graph of P_3).

Definition 45. *The subdivision-edge-bipartite-edge join of* G_1 *and* G_2*, denoted by* $S(G_1) \ddot{\bar{\vee}} B(G_2)$*, is the graph obtained from* $S(G_1)$ *and* $B(G_2)$ *by connecting each vertex of* $V(S(G_1)) \setminus V(G_1)$ *(that is the subset of the added vertices in the vertex set of* $S(G_1)$*) with every vertex of* $V(B(G_2)) \setminus V(G_2)$ *(that is the subset of the added vertices in the vertex set of* $B(G_2)$*).*

By Definitions 42, 43, and 45, it follows that the graph $S(G_1) \ddot{\bar{\vee}} B(G_2)$ has $n_1 + n_2 + m_1 + m_2$ vertices and $m_1m_2 + 2m_1 + 3m_2$ edges. Figure 14 displays the graph $S(C_4) \ddot{\bar{\vee}} B(P_3)$.

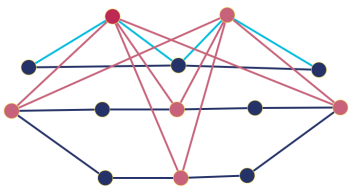


Figure 14. The graph $S(C_4) \ddot{\bar{\vee}} B(P_3)$ (see Definition 45). In comparison to Figure 13, edges connecting each black vertex of C_4 to each black vertex of P_3 are deleted and replaced in this figure by edges connecting each red vertex of C_4 to each red vertex of P_3.

Definition 46. *The subdivision-edge-bipartite-vertex join of* G_1 *and* G_2*, denoted by* $\mathsf{S}(\mathsf{G}_1) \overline{\dot{\vee}}\, \mathsf{B}(\mathsf{G}_2)$*, is the graph obtained from* $\mathsf{S}(\mathsf{G}_1)$ *and* $\mathsf{B}(\mathsf{G}_2)$ *by connecting each vertex of* $\mathsf{V}(\mathsf{S}(\mathsf{G}_1))\setminus \mathsf{V}(\mathsf{G}_1)$ *(that is the subset of the added vertices in the vertex set of* $\mathsf{S}(\mathsf{G}_1)$*) with every vertex of* $\mathsf{V}(\mathsf{G}_2)$ *(that is the subset of the original vertices in the vertex set of* $\mathsf{B}(\mathsf{G}_2)$*).*

By Definitions 42, 43, and 46, it follows that the graph $\mathsf{S}(\mathsf{G}_1) \overline{\dot{\vee}}\, \mathsf{B}(\mathsf{G}_2)$ has $n_1 + n_2 + m_1 + m_2$ vertices and $m_1n_2 + 2m_1 + 3m_2$ edges. Figure 15 displays the graph $\mathsf{S}(\mathsf{C}_4) \overline{\dot{\vee}}\, \mathsf{B}(\mathsf{P}_3)$.

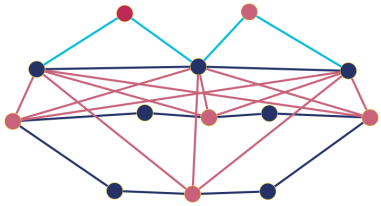


Figure 15. The graph $\mathsf{S}(\mathsf{C}_4) \overline{\dot{\vee}}\, \mathsf{B}(\mathsf{P}_3)$ (see Definition 46). In comparison to Figure 13, edges connecting each black vertex of C_4 to each black vertex of P_3 are deleted and replaced in this figure by edges connecting each red vertex of C_4 to each black vertex of P_3.

Definition 47. *The subdivision-vertex-bipartite-edge join of* G_1 *and* G_2*, denoted by* $\mathsf{S}(\mathsf{G}_1) \dot{\overline{\vee}}\, \mathsf{B}(\mathsf{G}_2)$*, is the graph obtained from* $\mathsf{S}(\mathsf{G}_1)$ *and* $\mathsf{B}(\mathsf{G}_2)$ *by connecting each vertex of* $\mathsf{V}(\mathsf{G}_1)$ *(that is the subset of the original vertices in the vertex set of* $\mathsf{S}(\mathsf{G}_1)$*) with every vertex of* $\mathsf{V}(\mathsf{B}(\mathsf{G}_2))\setminus \mathsf{V}(\mathsf{G}_2)$ *(that is the subset of the added vertices in the vertex set of* $\mathsf{B}(\mathsf{G}_2)$*).*

By Definitions 42, 43, and 47, it follows that the graph $\mathsf{S}(\mathsf{G}_1) \dot{\overline{\vee}}\, \mathsf{B}(\mathsf{G}_2)$ has $n_1 + n_2 + m_1 + m_2$ vertices and $n_1m_2 + 2m_1 + 3m_2$ edges. Figure 16 displays the graph $\mathsf{S}(\mathsf{C}_4) \dot{\overline{\vee}}\, \mathsf{B}(\mathsf{P}_3)$.

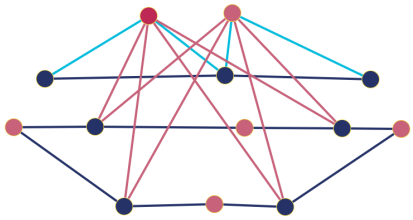


Figure 16. The graph $\mathsf{S}(\mathsf{C}_4) \dot{\overline{\vee}}\, \mathsf{B}(\mathsf{P}_3)$ (see Definition 47). In comparison to Figure 13, edges connecting each black vertex of C_4 to each black vertex of P_3 are deleted and replaced in this figure by edges connecting each black vertex of C_4 to each red vertex of P_3.

We next present the main result of this subsection, which motivates the four binary graph operations introduced in Definitions 44–47.

Theorem 41 ([22])**.** *Let* G_i*,* H_i*, where* $i \in \{1, 2\}$*, be regular graphs, where* G_1 *need not be different from* H_1*. If* G_1 *and* H_1 *are* $\mathbf{A}$*-cospectral, and* G_2 *and* H_2 *are* $\mathbf{A}$*-cospectral, then the following holds:*

- $\mathsf{S}(\mathsf{G}_1) \ddot{\vee}\, \mathsf{B}(\mathsf{G}_2)$ *and* $\mathsf{S}(\mathsf{H}_1) \ddot{\vee}\, \mathsf{B}(\mathsf{H}_2)$ *are irregular* $\{\mathbf{A}, \mathbf{L}, \mathcal{L}\}$*-NICS graphs.*
- $\mathsf{S}(\mathsf{G}_1) \overline{\overline{\vee}}\, \mathsf{B}(\mathsf{G}_2)$ *and* $\mathsf{S}(\mathsf{H}_1) \overline{\overline{\vee}}\, \mathsf{B}(\mathsf{H}_2)$ *are irregular* $\{\mathbf{A}, \mathbf{L}, \mathcal{L}\}$*-NICS graphs.*
- $\mathsf{S}(\mathsf{G}_1) \overline{\dot{\vee}}\, \mathsf{B}(\mathsf{G}_2)$ *and* $\mathsf{S}(\mathsf{H}_1) \overline{\dot{\vee}}\, \mathsf{B}(\mathsf{H}_2)$ *are irregular* $\{\mathbf{A}, \mathbf{L}, \mathcal{L}\}$*-NICS graphs.*
- $\mathsf{S}(\mathsf{G}_1) \dot{\overline{\vee}}\, \mathsf{B}(\mathsf{G}_2)$ *and* $\mathsf{S}(\mathsf{H}_1) \dot{\overline{\vee}}\, \mathsf{B}(\mathsf{H}_2)$ *are irregular* $\{\mathbf{A}, \mathbf{L}, \mathcal{L}\}$*-NICS graphs.*

In light of Theorem 41, the following questions naturally arises.

Question 4. *Are the graphs in Theorem 41 also cospectral with respect to the signless Laplacian matrix (i.e., $\mathbf{Q}$-cospectral)?*

5.6. Connected Irregular NICS Graphs

The (joint) cospectrality of regular graphs with respect to their adjacency, Laplacian, signless Laplacian, and normalized Laplacian matrices can be asserted by verifying their cospectrality with respect to only one of these matrices. In other words, regular graphs are X-cospectral for some $X \in \{\mathbf{A}, \mathbf{L}, \mathbf{Q}, \mathcal{L}\}$ if and only if they are cospectral with respect to all these matrices (see Proposition 3). For irregular graphs, this does not hold in general. Following [17], it is natural to ask the following question:

Question 5. *Are there pairs of irregular graphs that are $\{\mathbf{A}, \mathbf{L}, \mathbf{Q}, \mathcal{L}\}$-NICS, i.e., X-NICS with respect to every $X \in \{\mathbf{A}, \mathbf{L}, \mathbf{Q}, \mathcal{L}\}$?*

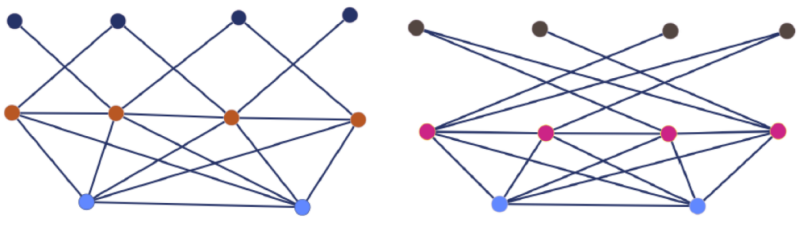
This question remained open until two coauthors of this paper recently proposed a method for constructing pairs of irregular graphs that are X-cospectral with respect to every $X \in \{\mathbf{A}, \mathbf{L}, \mathbf{Q}, \mathcal{L}\}$, providing explicit constructions [25]. Building on that work, another coauthor of this paper demonstrated in [41] that for every even integer $n \geqslant 14$, there exist two connected, irregular $\{\mathbf{A}, \mathbf{L}, \mathbf{Q}, \mathcal{L}\}$-NICS graphs on n vertices with identical independence, clique, and chromatic numbers, yet distinct Lovász ϑ-functions. We now present the preliminary definitions required to outline the relevant results in [25,41], and the construction of such cospectral irregular X-NICS graphs for all $X \in \{\mathbf{A}, \mathbf{L}, \mathbf{Q}, \mathcal{L}\}$.

Definition 48 (Neighbors splitting join of graphs [95])**.** *Let* G *and* H *be graphs with disjoint vertex sets, and let* $\mathsf{V}(\mathsf{G}) = \{v_1, \ldots, v_n\}$*. The* neighbors splitting (NS) join *of* G *and* H *is obtained by adding vertices* $v'_1, \ldots, v'_n$ *to the vertex set of* $\mathsf{G} \vee \mathsf{H}$ *and connecting* v'_i *to* v_j *if and only if* $\{v_i, v_j\} \in \mathsf{E}(\mathsf{G})$*. The NS join of* G *and* H *is denoted by* $\mathsf{G} \veebar \mathsf{H}$*.*

Definition 49 (Nonneighbors splitting join of graphs [25,96])**.** *Let* G *and* H *be graphs with disjoint vertex sets, and let* $\mathsf{V}(\mathsf{G}) = \{v_1, \ldots, v_n\}$*. The nonneighbors splitting (NNS) join of* G *and* H *is obtained by adding vertices* $v'_1, \ldots, v'_n$ *to the vertex set of* $\mathsf{G} \vee \mathsf{H}$ *and connecting* v'_i *to* v_j*, with* $i \neq j$*, if and only if* $\{v_i, v_j\} \notin \mathsf{E}(\mathsf{G})$*. The NNS join of* G *and* H *is denoted by* $\mathsf{G} \barwedge \mathsf{H}$*.*

Remark 21. *In general,* $\mathsf{G} \veebar \mathsf{H} \ncong \mathsf{H} \veebar \mathsf{G}$ *and* $\mathsf{G} \barwedge \mathsf{H} \ncong \mathsf{H} \barwedge \mathsf{G}$ *(unless* $\mathsf{G} \cong \mathsf{H}$*).*

Example 7 (NS and NNS join of graphs [25])**.** *Figure 17 shows the NS and NNS joins of the path graphs* P_4 *and* P_2*, denoted by* $\mathsf{P}_4 \veebar \mathsf{P}_2$ *and* $\mathsf{P}_4 \barwedge \mathsf{P}_2$*, respectively.*



(a) $P_4 \veebar P_2$ (b) $P_4 \underset{=}{\vee} P_2$

Figure 17. The neighbors splitting (NS) and nonneighbors splitting (NNS) joins of the path graphs P_4 and P_2 are depicted, respectively, on the left and right-hand sides of this figure. The NS and NNS joins of graphs are, respectively, denoted by $\mathsf{P}_4 \veebar \mathsf{P}_2$ and $\mathsf{P}_4 \barwedge \mathsf{P}_2$ [25].

Theorem 42 (Irregular $\{\mathbf{A}, \mathbf{L}, \mathbf{Q}, \mathcal{L}\}$-NICS graphs)**.** *Let* G_1 *and* H_1 *be regular and cospectral graphs, and let* G_2 *and* H_2 *be regular, nonisomorphic, and cospectral (NICS) graphs. Then, the following statements hold:*

1. *The NS-join graphs* $\mathsf{G}_1 \veebar \mathsf{G}_2$ *and* $\mathsf{H}_1 \veebar \mathsf{H}_2$ *are irregular* $\{\mathbf{A}, \mathbf{L}, \mathbf{Q}, \mathcal{L}\}$*-NICS graphs.*
2. *The NNS join graphs* $\mathsf{G}_1 \barwedge \mathsf{G}_2$ *and* $\mathsf{H}_1 \barwedge \mathsf{H}_2$ *are irregular* $\{\mathbf{A}, \mathbf{L}, \mathbf{Q}, \mathcal{L}\}$*-NICS graphs.*

The proof of Theorem 42 is provided in [25,96], and it relies heavily on the notion of the Schur complement (see Theorem 1). The interested reader is referred to the recently published paper [25] for further details.

Relying on Theorem 42, the following result is stated and proved in [41].

Theorem 43 (On irregular NICS graphs)**.** *For every even integer* $n \geqslant 14$*, there exist two connected, irregular* $\{\mathbf{A}, \mathbf{L}, \mathbf{Q}, \mathcal{L}\}$*-NICS graphs on* n *vertices with identical independence, clique, and chromatic numbers, yet their Lovász* ϑ*-functions are distinct.*

That result is, in fact, strengthened in Section 4.2 of [41], and the interested reader is referred to that paper for further details. The proof of Theorem 43 is also constructive, providing explicit such graphs.

6. Open Questions and Outlook

We conclude this paper by highlighting some of the most significant open questions in the fascinating area of research related to cospectral nonisomorphic graphs and graphs determined by their spectrum, leaving them as topics for further future study.

6.1. Haemers' Conjecture

Haemers' conjecture [7,8], a prominent topic in spectral graph theory, posits that almost all graphs are uniquely determined by their adjacency spectrum. This conjecture suggests that, for large graphs, the probability of having two nonisomorphic graphs, on a large number of vertices, share the same adjacency spectrum is negligible. The conjecture has inspired extensive research, including studies on specific graph families, cospectrality, and algebraic graph invariants, contributing to deeper insights into the relationship between graph structure and eigenvalues. Haemers' conjecture is stated formally as follows.

Definition 50. *For* $n \in \mathbb{N}$*, let* $I(n)$ *to be the numbers of distinct graphs on* n *vertices, up to isomorphism. For* $\mathcal{X} \subseteq \{\mathbf{A}, \mathbf{L}, \mathbf{Q}, \mathcal{L}, \overline{\mathbf{A}}, \overline{\mathbf{L}}, \overline{\mathbf{Q}}, \overline{\mathcal{L}}\}$*, let* $\alpha_{\mathcal{X}}(n)$ *be the number of* $\mathcal{X}$*-DS graphs on* n *vertices, up to isomorphism, and (for the sake of simplicity of notation) let* $\alpha(n) \triangleq \alpha_{\{A\}}(n)$

Conjecture 1 ([7])**.** *For sufficiently large* n*, almost all of the graphs are DS, i.e.,*

$$\lim_{n\to\infty} \frac{\alpha(n)}{I(n)} = 1. \tag{69}$$

Several results lend support to this conjecture [7,9,10], but a complete proof remains elusive. By [5], the number of graphs with n vertices up to isomorphism is

$$I(n) = \big(1 - \epsilon(n)\big)\, \frac{2^{n(n-1)/2}}{n!}, \tag{70}$$

where $\lim_{n\to\infty} \epsilon(n) = 0$. By Stirling's approximation for $n!$ and straightforward algebra, $I(n)$ can be verified to be given by

$$I(n) = 2^{\frac{n(n-1)}{2}\,\left(1-\epsilon(n)\right)}. \tag{71}$$

It is shown in [9] that the number of DS graphs on n vertices is at least e^{cn} for some constant $c > 0$ (see also the discussion in Section 4.5). In light of Remark 9, Conjecture 1 leads to the following question.

Question 6. *For what minimal subset* $\mathcal{X} \subset \{\mathbf{A}, \mathbf{L}, \mathbf{Q}, \mathcal{L}, \overline{\mathbf{A}}, \overline{\mathbf{L}}, \overline{\mathbf{Q}}, \overline{\mathcal{L}}\}$ *(if any) does the limit*

$$\lim_{n\to\infty} \frac{\alpha_{\mathcal{X}}(n)}{I(n)} = 1 \tag{72}$$

hold?

Some computer results somewhat support Haemers' Conjecture. Approximately 80% of the graphs with 12 vertices are DS [97]. A new class of graphs is defined in [10], which offers an algorithmic method for finding all the $\{\mathbf{A}, \overline{\mathbf{A}}\}$-cospectral mates of a graph G in this class (i.e., graphs that are $\{\mathbf{A}, \overline{\mathbf{A}}\}$-cospectral with G). Using this algorithm, they found that out of $10,000$ graphs with 50 vertices, which were chosen uniformly at random from this class, at least 99.5% of them were $\{\mathbf{A}, \overline{\mathbf{A}}\}$-DS.

6.2. DS Properties of Structured Graphs

This paper explores various structures of graph families determined by their spectrum with respect to one or more of their associated matrices, as well as the related problem of constructing pairs of nonisomorphic graphs that are cospectral with respect to some or all of these matrices. Several questions are interspersed throughout the paper (see Questions 1, 2, 3, 4, and 5), which remain open for further study.

In addition to serving as a survey on spectral graph determination, this paper suggests a new alternative proof to Theorem 3.3 in [37], asserting that all Turán graphs are determined by their adjacency spectrum (see Theorem 25). This proof is based on a derivation of the adjacency spectrum of the family of Turán graphs (see Theorem 24). Since these graphs are generally bi-regular multipartite graphs (i.e., their vertices have only two possible degrees, which in this case are consecutive integers), it does not necessarily imply that Turán graphs are determined by the spectrum of some other associated matrices, such as the Laplacian, signless Laplacian, or normalized Laplacian matrices. Determining whether this holds is an open question.

The distance matrix of a connected graph is the symmetric matrix whose columns and rows are indexed by the graph vertices, and its entries are equal to the pairwise distances between the corresponding vertices. The distance spectrum is the multiset of eigenvalues of the distance matrix, and its characterization has been a subject of fruitful research (see [98,99] for comprehensive surveys on the distance spectra of graphs, and [100,101] on the spectra of graphs with respect to variants of the distance matrix). Nonisomorphic graphs may share an identical adjacency spectrum, as well as an identical distance spectrum, and, therefore, they are graphs that are not determined by either their adjacency or distance spectrum. This holds, e.g., for the Shrikhande graph and the line graph of the complete bipartite graph $\mathsf{K}_{4,4}$, which are nonisomorphic strongly regular graphs with the identical parameters $(16, 6, 2, 2)$; they share an identical $\mathbf{A}$-spectrum given by $\{6, [2]^6, [-2]^6\}$ and an identical distance spectrum given by $\{24, [0]^9, [-4]^6\}$. There exist, however, graphs that are determined by their distance spectrum but are not determined by their adjacency spectrum. In this context, [27] proves that complete multipartite graphs are uniquely determined by their distance spectra (this result confirms a conjecture proposed in [102] and extends the special case established there for complete bipartite graphs). A Turán graph is, in particular, determined by its distance spectrum [27], and by its adjacency spectra (see Theorem 25 here). On the other hand, while complete multipartite graphs are not generally

determined by their adjacency spectra (e.g., for complete bipartite graphs, see Theorem 21), they are necessarily determined by their distance spectra [27]. Another family of graphs that are determined by their distance spectrum but are not determined by their adjacency spectrum, named d-cube graphs, is provided in [29,30]. These graphs have their vertices represented by binary n-length tuples, where any two vertices are adjacent if and only if their corresponding binary n-tuples differ in one coordinate; it is also shown that these graphs have exactly three distinct distance eigenvalues. The question of which multipartite graphs, or graphs in general, are determined by their distance spectra remains open.

Another newly obtained proof presented in this paper refers to a necessary and sufficient condition in [37] for complete bipartite graphs to be **A**-DS (see Theorem 21 and Remark 13 here). These graphs are also bi-regular, and the two possible vertex degrees can differ by more than one. Both of these newly obtained proofs, discussed in Section 4, provide insights into the broader question of which (structured) multipartite graphs are determined by their adjacency spectrum or, more generally, by the spectra of some of their associated matrices.

Even if Haemers' conjecture is eventually proved in its full generality, it remains surprising when a new family of structured graphs is shown to be DS (or X-DS, more generally). This is because, for certain structured graphs, such as strongly regular graphs and trees, their spectra often fail to uniquely determine them [12,13]. This stark contrast between the fact that almost all random graphs of high order are likely to be DS and the existence of interesting structured graphs that are not DS has significant implications.

In addition to the questions posed earlier in this paper, we raise the following additional concrete question:

Question 7. *By Theorem 32, the family of generalized friendship graphs $F_{p,q}$ is $\mathcal{L}$-DS. Are these graphs also DS with respect to their other associated matrices?*

We speculate that the DS property of a graph correlates, to some extent, with the number of symmetries that the graph possesses, and we hypothesize that the size of the automorphism group of a graph can partially indicate whether it is DS.

A justification for this claim is that the automorphism group of a graph reflects its symmetries, which can influence the eigenvalues of its adjacency matrix. Highly symmetric graphs (i.e., those with large automorphism groups) often exhibit eigenvalue multiplicities and patterns that are shared by other nonisomorphic graphs, making such graphs less likely to be DS. Conversely, graphs with trivial automorphism groups are typically less symmetric and may have eigenvalues that uniquely determine their structure, increasing the likelihood that they are DS. As noted in [5], almost all graphs have trivial automorphism groups. This observation aligns with the conjecture that most graphs are DS, as the absence of symmetry reduces the likelihood of two nonisomorphic graphs sharing the same spectrum.

It is noted, however, that the DS property of graphs is not solely dictated by their automorphism groups. Specifically, a graph with a large automorphism group can still be DS if its eigenvalues uniquely encode its structure (see Section 4.7). In contrast to these DS graphs, other graphs with trivial automorphism groups are not guaranteed to be DS; in such cases, the spectrum might not capture enough structural information to uniquely determine the graph. Typically, graphs with a small number of distinct eigenvalues seem to be, in general, the hardest graphs to distinguish by their spectrum. As noted in [103], it seems that most graphs with few eigenvalues (e.g., most of the strongly regular graphs) are not determined by their spectrum, which served as one of the motivations of the work in [103] in studying graphs whose normalized Laplacian has three eigenvalues.

To conclude, the size of the automorphism group of a graph can provide some indication of whether it is DS, but it is not a definitive criterion. While large automorphism

groups often correlate with the graph not being DS due to shared eigenvalues among nonisomorphic graphs, this is not an absolute rule. Therefore, the claim should be understood as a general observation that requires qualification to account for exceptions.

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