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The Existence of Positive Solutions for a p-Laplacian Tempered Fractional Diffusion Equation Using the Riemann–Stieltjes Integral Boundary Condition

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Abstract: In this paper, we focus on the existence of positive solutions for a class of p-Laplacian tempered fractional diffusion equations involving a lower tempered integral operator and a Riemann–Stieltjes integral boundary condition. By introducing certain new local growth conditions and establishing an a priori estimate for the Green's function, several sufficient conditions on the existence of positive solutions for the equation are derived by using a fixed point theorem. Interesting points are that the tempered fractional diffusion equation contains a lower tempered integral operator and that the boundary condition involves the Riemann–Stieltjes integral, which can be a changing-sign measure.

Keywords: tempered fractional equations; Riemann–Stieltjes integral; fixed point theorem; positive solutions; p-Laplacian operator

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1. Introduction

In this paper, we focus on the existence of positive solutions for a class of p-Laplacian tempered fractional diffusion equations involving a lower tempered integral operator and the Riemann–Stieltjes integral boundary condition:

$$\begin{cases} {}_0^R\mathbb{D}_t^{\alpha,\lambda}\left(\varphi_p(-{}_0^R\mathbb{D}_t^{\beta,\lambda}(u(t)))\right) = f\left(u(t), \mathcal{I}^{\beta}(e^{\lambda t}u(t))\right), \\ u(0) = 0, \quad \mathbb{D}_t^{\beta,\lambda}(u(0)) = 0, \ u(1) = \int_0^1 u(t)dA(t), \end{cases} \tag{1}$$

where $0 < \alpha \leq 1, 1 < \beta \leq 2$, λ is a positive constant; $\varphi_p(s) = |s|^{p-2}s$ is a p-Laplacian operator with $\frac{1}{p} + \frac{1}{q} = 1, p > 1$; A is a function of bounded variation and dA can be a signed measure; $\int_0^1 u(t)dA(t)$ denotes the Riemann–Stieltjes integral; and $f(u, v)$ is a continuous function. ${}_0^R\mathbb{D}_t^{\alpha,\lambda}$ is defined as the tempered fractional-order derivative, which is associated with the Riemann–Liouville fractional derivatives with the following mathematical relationship:

$$\mathbb{D}_t^{\alpha,\lambda}z(t) = e^{-\lambda t}\mathcal{D}_t^{\alpha,\lambda}(e^{\lambda t}z(t)),$$

where $\mathcal{D}_t^{\alpha,\lambda}z(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n}(\mathcal{I}_t^{n-\alpha}z(t))$ denotes the standard Riemann–Liouville fractional derivative, and $\mathcal{I}_t^{n-\alpha}z(t)$ is the Riemann–Liouville fractional integral operator, defined as

$$\mathcal{I}_t^{n-\alpha}z(t) = \int_0^t (t-s)^{\alpha-1}z(s)\mathrm{d}s,$$

From this relation, it is easy to see that when $\lambda = 0$, the tempered fractional-order derivative becomes the Riemann-Liouville fractional derivative, and so the tempered fractional-order derivative is an exponential optimisation of the Riemann-Liouville fractional derivative [1].

Given that a fractional derivative possesses nonlocal characteristics, in recent decades, fractional differential equations have made remarkable progress in both theoretical and applied fields, which not only show a wide range of applications in several disciplines, such as mathematical physics [2], biochemistry [3,4], economics [5], chemical engineering, automatic control, and thermoelasticity [6–13], but also play an important role in the modelling and simulation of complex systems [14–21]. Among these, fractional differential equations with a p-Laplacian operator can describe the turbulent flow in a porous medium [22,23]; see also Leibenson's work [24]. However, the transport of solute with Brownian motion in highly heterogeneous porous media often exhibits long-range dependent anomalous diffusion phenomena, known a semi-heavy tail feature. These phenomena have also occurred in financial time series, Nile river data, fractal analyses, etc. [25]. In order to describe the characteristic of a semi-heavy tail, using a Fourier transform, Sabzikar introduced an exponential factor into the particle jump density [26] and obtained a tempered anomalous diffusion equation. Compared to classical Riemann–Liouville fractional differential equations, a tempered fractional equation has an exponential decay advantage over long time scales and possesses more practical applications in tempered Lévy flight diffusion [27], geophysics [28,29], finance [30], and applied mathematics [31,32]. In particular, introducing the p-Laplacian operator into the tempered fractional equation can make the tempered model more accurately and effectively simulate the turbulent velocity fluctuations in porous media with exponential law behaviour [33].

In recent years, in order to solve various nonlinear problems arising from the field of science, useful mathematical tools and methods, such as spaces and smooth theories [34–38], operator theories [39–42], monotone iterative techniques [43–47], spectral analysis [48,49], the variational method [50–56], and the method of upper and lower solutions [57,58], have been developed to handle these problems. In a recent work [33], by using the method of upper and lower solutions, the existence of positive solutions to the singular tempered fractional-order turbulence model in porous media,

$$\begin{cases} {}_0^R\mathbb{D}_t^{\alpha,\lambda}\left(\varphi_p({}_0^R\mathbb{D}_t^{\beta,\lambda}(u(t)))\right) = f(t,u(t)), \\ u(0) = 0, \quad \mathbb{D}_t^{\beta,\lambda}(u(0)) = 0, u(1) = \int_0^1 e^{-\lambda(1-t)}u(t)\mathrm{d}t, \end{cases} \tag{2}$$

was established, where $0 < \alpha \leq 1, 1 < \beta \leq 2$, λ is a positive constant, and f is decreasing in the second variable. In ref. [59], Ricceri's variational principle was employed to study the existence of weak solutions to the following tempered sub-diffusion fractional equation involving an oscillating term using the Dirichlet boundary condition:

$$\begin{cases} \mathbb{D}_{d^-}^{\alpha,\mu}\left({}^C\mathbb{D}_{c^+}^{\alpha,\mu}z(t)\right) = \lambda p(t)f(z(t)), \ \ t \in (c,d), \\ z(c) = z(d) = 0, \end{cases} \tag{3}$$

where $p \in L^\infty(c,d)$, λ is a parameter, and $\alpha \in (\frac{1}{2},1)$, $\mu > 0$ are constants. When the nonlinearity f fulfils suitable oscillating conditions near the origin or at infinity, the tempered fractional sub-diffusion Equation (3) has infinitely many weak solutions.

By reviewing the existing work, we find that no work has been conducted on p-Laplacian tempered fractional diffusion equations involving the Riemann–Stieltjes integral boundary conditions. Thus, motivated by the previous work, the aim of this paper is to investigate the existence of positive solutions for tempered fractional diffusion equations

involving p-Laplacian operators and a lower tempered integral operator and subject to the Riemann–Stieltjes integral boundary conditions. By introducing certain new local growth conditions and establishing an a priori estimate for the associated Green's function, several sufficient conditions on the existence of positive solutions for the equation are derived by using the fixed point theorem. Different from [33], the tempered fractional diffusion Equation (1) we study not only contains a lower tempered integral operator a and p-Laplacian operator, but also the boundary condition involves the Riemann–Stieltjes integral, which is allowed to be a changing-sign measure. Here, we also point out that the Riemann–Stieltjes integral boundary condition is a class of a more general nonlocal boundary condition, which has more advantages and accuracy than the local condition in describing natural phenomena with memory effects and hereditary features, such as in studying polymers, viscoelasticity, and biomathematics; moreover, it contains the classical integral boundary condition ($A(t) = t$) and the Dirichlet boundary condition ($A(t) = 0$) as special cases. Thus, in the processing method, the Riemann–Stieltjes integral boundary condition is more difficult than the local condition because A is a function of bounded variation and dA is a signed measure.

2. Preliminaries and Lemmas

In this section, we give some preliminaries and lemmas to be used in the rest of this paper.

Lemma 1 ([1]). *Suppose $x(t) \in C[0,1] \cap L^1[0,1]$ and $\alpha > \beta > 0$, and let $m = [\alpha] + 1$. Then,*

(i) $\mathcal{I}_t^{\gamma}\mathcal{D}_t^{\gamma}(x(t)) = x(t) + a_1 t^{\gamma-1} + a_2 t^{\gamma-2} + ... + a_m t^{\gamma-m}$,
where $a_i \in \mathbb{R}, i = 1, 2, 3, ..., m$.

(ii) $\mathcal{I}_t^{\alpha}\mathcal{I}_t^{\beta}(x(t)) = \mathcal{I}_t^{\alpha+\beta}x(t), \mathcal{D}_t^{\beta}\mathcal{I}_t^{\alpha}(x(t)) = \mathcal{I}_t^{\alpha-\beta}x(t), \mathcal{D}_t^{\beta}\mathcal{I}_t^{\beta}(x(t)) = x(t)$.

Lemma 2 ([2]). *Let $g(t)$ be a positive continuous function in $[0,1]$; then, the following tempered fractional-order equation*

$$\begin{cases} -{}_0^R\mathbb{D}_t^{\beta,\lambda}(u(t)) = g(t), \\ u(0) = u(1) = 0, \end{cases} \tag{4}$$

has a unique positive solution

$$u(t) = \int_0^1 H(t,s)g(s)ds. \tag{5}$$

where

$$H(t,s) = \begin{cases} \dfrac{t^{\beta-1}(1-s)^{\beta-1} - (t-s)^{\beta-1}}{\Gamma(\beta)} e^{-\lambda t}e^{\lambda s}, & 0 \le s \le t \le 1, \\ \dfrac{t^{\beta-1}(1-s)^{\beta-1}}{\Gamma(\beta)} e^{-\lambda t}e^{\lambda s}, & 0 \le t \le s \le 1 \end{cases} \tag{6}$$

is the Green's function of Equation (4).

According to Lemma 1 and simple computation, the unique solution to the problem

$$\begin{cases} -{}_0^R\mathbb{D}_t^{\beta,\lambda}(u(t)) = 0, \ 0 < t < 1, \\ u(0) = 0, u(1) = 1, \end{cases}$$

is $e^{\lambda(1-t)}t^{\beta-1}$. Let

$$\mathcal{A} = \int_0^1 e^{\lambda(1-t)}t^{\beta-1}dA(t), \ \mathcal{H}_A(s) = \int_0^1 H(t,s)dA(t).$$

It follows from [60] that Green's function for the following fractional equation with the Riemann–Stieltjes integral boundary condition

$$\begin{cases} -{}_0^R\mathbb{D}_t^{\beta,\lambda}(u(t)) = g(t), \\ u(0) = 0,\ u(1) = \int_0^1 u(t)dA(t), \end{cases} \tag{7}$$

is

$$G(t,s) = \frac{e^{\lambda(1-t)}t^{\beta-1}}{1-\mathcal{A}}\mathcal{H}_A(s) + H(t,s). \tag{8}$$

In order to preserve the nonnegativity of Green's function, we list a basic assumption used in this paper.

(**A0**) A is a function of bounded variation satisfying $\mathcal{H}_A(s) \geq 0$ for $s \in [0,1]$ and

$$0 < \int_0^1 e^{\lambda(1-t)}t^{\beta-1}dA(t) < 1.$$

Lemma 3. *Suppose* (**A0**) *holds; then, Green's function defined by* (8) *satisfies the following properties:*

(1) $G(t,s) \geq 0, forall\ t,s \in [0,1]$;

(2) $e^{-\lambda t}t^{\beta-1}(1-t)\vartheta(s) \leq G(t,s) \leq \vartheta(s),\ t,s \in [0,1]$,

where

$$\vartheta(s) = \frac{e^{\lambda}}{1-\mathcal{A}}\mathcal{H}_A(s) + \frac{(1-s)^{\beta-1}se^{\lambda s}}{\Gamma(\beta)}.$$

Proof. (1) It follows from (**A0**) that $0 < \mathcal{A} < 1$ and $\mathcal{H}_A(s) \geq 0, s \in [0,1]$; on the other hand, it is obvious that $H(t,s) > 0, t,s \in (0,1)$. Thus, condition (1) holds.

For (2), according to Lemma 2 and the definition of $H(t,s)$, we have

$$\begin{aligned} G(t,s) &= \frac{e^{\lambda(1-t)}t^{\beta-1}}{1-\mathcal{A}}\mathcal{H}_A(s) + H(t,s) \geq \frac{e^{\lambda(1-t)}t^{\beta-1}(1-t)}{1-\mathcal{A}}\mathcal{H}_A(s) + \frac{t^{\beta-1}(1-t)e^{-\lambda t}(1-s)^{\beta-1}se^{\lambda s}}{\Gamma(\beta)} \\ &= \left(\frac{e^{\lambda}}{1-\mathcal{A}}\mathcal{H}_A(s) + \frac{(1-s)^{\beta-1}se^{\lambda s}}{\Gamma(\beta)}\right)e^{-\lambda t}t^{\beta-1}(1-t) = e^{-\lambda t}t^{\beta-1}(1-t)\vartheta(s). \end{aligned}$$

On the other hand, since $1 < \beta \leq 2$, we have

$$G(t,s) = \frac{e^{\lambda(1-t)}t^{\beta-1}}{1-\mathcal{A}}\mathcal{H}_A(s) + H(t,s) \leq \frac{e^{\lambda}}{1-\mathcal{A}}\mathcal{H}_A(s) + \frac{(\beta-1)(1-s)^{\beta-1}se^{\lambda s}}{\Gamma(\beta)} \leq \vartheta(s),\ t,s \in [0,1].$$

□

Next, from Lemmas 1 and 2, we have the following lemma.

Lemma 4. *Let $g(t)$ be a positive continuous function in $[0,1]$; then, the following tempered fractional equation*

$$\begin{cases} {}_0^R\mathbb{D}_t^{\alpha,\lambda}\left(\varphi_p(-{}_0^R\mathbb{D}_t^{\beta,\lambda}(u(t)))\right) = g(t), \\ u(0) = 0, \quad \mathbb{D}_t^{\beta,\lambda}(u(0)) = 0,\ u(1) = \int_0^1 u(t)dA(t), \end{cases} \tag{9}$$

has a unique positive solution

$$u(t) = \int_0^1 G(t,s)\left(\int_0^s \frac{(s-\tau)^{\alpha-1}e^{-\lambda s}e^{\lambda\tau}g(\tau)}{\Gamma(\alpha)}\mathrm{d}\tau\right)^{q-1}\mathrm{d}s. \tag{10}$$

Proof. Let

$$v = -\mathbb{D}_t^{\beta,\lambda} u(t), x = \varphi_p(v),$$

and then from Lemma 1 and the definition of the tempered fractional-order derivative, the solution of the initial value problem

$$\begin{cases} {}_0^R\mathbb{D}_t^{\alpha,\lambda} x(t) = g(t), \\ x(0) = 0, \end{cases} \tag{11}$$

can be expressed by:

$$e^{\lambda t} x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(s) e^{\lambda s} \mathrm{d}s - c_1 t^{\alpha-1}.$$

Since $x(0) = 0, 0 < \alpha \leq 1$, so $c_1 = 0$,

$$x(t) = \int_0^t \frac{(t-s)^{\alpha-1} e^{-\lambda t}}{\Gamma(\alpha)} g(s) e^{\lambda s} \mathrm{d}s. \tag{12}$$

Consequently,

$$x(t) = \varphi_p\Big(-{}_0^R\mathbb{D}_t^{\beta,\lambda}(u(t))\Big) = \int_0^t \frac{(t-s)^{\alpha-1} e^{-\lambda t}}{\Gamma(\alpha)} g(s) e^{\lambda s} \mathrm{d}s,$$

i.e.,

$$-{}_0^R\mathbb{D}_t^{\beta,\lambda}(u(t)) = \varphi_p^{-1}\left(\int_0^t \frac{(t-s)^{\alpha-1} e^{-\lambda t}}{\Gamma(\alpha)} g(s) e^{\lambda s} \mathrm{d}s\right).$$

Since $g(s) \geq 0, s \in [0,1]$, we have

$$\varphi_p^{-1}\left(\int_0^t \frac{(t-s)^{\alpha-1} e^{-\lambda t}}{\Gamma(\alpha)} g(s) e^{\lambda s} \mathrm{d}s\right) = \left(\int_0^t \frac{(t-s)^{\alpha-1} e^{-\lambda t}}{\Gamma(\alpha)} g(s) e^{\lambda s} \mathrm{d}s\right)^{q-1}.$$

Thus, according to (7), Equation (9) has a unique solution

$$u(t) = \int_0^1 G(t,s)\left(\int_0^s \frac{(s-\tau)^{\alpha-1} e^{-\lambda s}}{\Gamma(\alpha)} g(\tau) e^{\lambda \tau} \mathrm{d}\tau\right)^{q-1} \mathrm{d}s. \tag{13}$$

□

Choose $E = C([0,1], \mathbb{R})$ with the norm

$$\|u(t)\| = \max_{t \in [0,1]} |u(t)|,$$

as our work space. Define a cone in E

$$P = \Big\{u(t) \in E : u(t) \geq t^{\beta-1}(1-t) e^{-\lambda t} \|u\|\Big\},$$

and an operator T in E

$$Tu(t) = \int_0^1 G(t,s)\left(\int_0^s \frac{(s-\tau)^{\alpha-1} e^{-\lambda s} e^{\lambda \tau}}{\Gamma(\alpha)} f(u(\tau), \mathcal{I}^{\beta}(e^{\lambda \tau} u(\tau))) \mathrm{d}\tau\right)^{q-1} \mathrm{d}s, \tag{14}$$

and then the fixed point of the operator T in E is a solution of the tempered fractional-order Equation (1).

Next, we give the following assumptions, which will be used in the rest of this paper.

(**A1**) $f \in C([0,+\infty)\times[0,+\infty),[0,+\infty))$

(**A2**) There exists a positive constant n such that, for any $0 < u+v \le (1+\frac{e^{\lambda}}{\Gamma(\beta+1)})n$,

$$f(u,v) < Nn^{\frac{1}{q-1}},$$

where

$$N = \left(\frac{\Gamma^{q-1}(\alpha+1)}{\frac{e^{\lambda}\|\mathcal{H}_A\|}{1-\mathcal{A}} + \frac{e^{\lambda}}{\Gamma(\beta)}}\right)^{\frac{1}{q-1}}.$$

(**A3**) There is a constant $\xi > 0$ such that for any

$$\left[\frac{e^{-\lambda}\Gamma(\beta+1)}{4^{2\beta-1}\Gamma(2\beta+1)} + \frac{e^{-\frac{3}{4}\lambda}}{4^{\beta}}\right]\xi \le u+v \le \left(1+\frac{e^{\lambda}}{\Gamma(\beta+1)}\right)\xi,$$

$$f(u,v) \ge L\xi^{\frac{1}{q-1}},$$

where

$$L = \left(\Gamma(\beta)\Gamma^{q-1}(\alpha+1)2^{3\beta+2\alpha(q-1)}e^{\frac{\lambda(3q-2)}{4}}\right)^{\frac{1}{q-1}}.$$

Lemma 5. *Assume that* (**A0**) *and* (**A1**) *hold; then, the operator* $T : P \to P$ *is completely continuous.*

Proof. From (**A1**), it is clear that T is continuous in $[0,1]$, and for any $u \in P$, there exists a constant $Q > 0$ such that $\|u\| \le Q$, and thus, we have

$$0 < \mathcal{I}^{\beta}(e^{\lambda t}u(t)) = \int_0^t \frac{(t-s)^{\beta-1}e^{\lambda s}u(s)}{\Gamma(\beta)}\mathrm{d}s \le \frac{e^{\lambda}}{\Gamma(\beta)}Q. \tag{15}$$

Denote

$$N^* = \max_{(u,v)\in[0,Q]\times[0,\frac{e^{\lambda}}{\Gamma(\beta)}Q]} f(u,v),$$

and then for any $u \in P$, it follows from (14), (15) and Lemma 3 that

$$\begin{aligned}
\|Tu\| &= \max_{t\in[0,1]}\int_0^1 G(t,s)\left(\int_0^s \frac{(s-\tau)^{\alpha-1}e^{-\lambda s}e^{\lambda\tau}}{\Gamma(\alpha)}f(u(\tau),\mathcal{I}^{\beta}(e^{\lambda\tau}u(\tau)))\mathrm{d}\tau\right)^{q-1}\mathrm{d}s \\
&\le \int_0^1 \vartheta(s)\left(\int_0^s \frac{(s-\tau)^{\alpha-1}e^{-\lambda s}e^{\lambda\tau}}{\Gamma(\alpha)}f(u(\tau),\mathcal{I}^{\beta}(e^{\lambda\tau}u(\tau)))\mathrm{d}\tau\right)^{q-1}\mathrm{d}s \\
&\le N^{*q-1}\int_0^1 \vartheta(s)\left(\int_0^s \frac{(s-\tau)^{\alpha-1}e^{-\lambda s}e^{\lambda\tau}}{\Gamma(\alpha)}\mathrm{d}\tau\right)^{q-1}\mathrm{d}s \\
&\le N^{*q-1}\int_0^1 \left(\frac{e^{\lambda}}{1-\mathcal{A}}\mathcal{H}_A(s) + \frac{(1-s)^{\beta-1}se^{\lambda s}}{\Gamma(\beta)}\right)\left(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}\mathrm{d}\tau\right)^{q-1}ds \\
&\le N^{*q-1}\int_0^1 \left(\frac{e^{\lambda}}{1-\mathcal{A}}\mathcal{H}_A(s) + \frac{(1-s)^{\beta-1}se^{\lambda s}}{\Gamma(\beta)}\right)\left(\frac{s^{\alpha}}{\Gamma(\alpha+1)}\right)^{q-1}ds \\
&\le \frac{N^{*q-1}}{\Gamma^{q-1}(\alpha+1)}\left(\frac{e^{\lambda}||\mathcal{H}_A||}{1-\mathcal{A}} + \frac{e^{\lambda}}{\Gamma(\beta)}\right) < +\infty,
\end{aligned} \tag{16}$$

which implies that $T : P \to E$ is well defined. Moreover, according to Lemmas 3 and (16), we have

$$\begin{aligned} Tu(t) &\geq \int_0^1 e^{-\lambda t} t^{\beta-1}(1-t)\vartheta(s)\left(\int_0^s \frac{(s-\tau)^{\alpha-1}e^{-\lambda s}e^{\lambda\tau}}{\Gamma(\alpha)} f(u(\tau), \mathcal{I}^{\beta}(e^{\lambda\tau}u(\tau)))\mathrm{d}\tau\right)^{q-1}\mathrm{d}s \\ &\geq t^{\beta-1}(1-t)e^{-\lambda t}\|Tu\|. \end{aligned} \tag{17}$$

This indicates that $T(P) \subset P$. Finally, according to the Ascoli–Arzela theorem, $T(P) \subset P$ is completely continuous. □

The proof of our results relies on the following fixed point theorem.

Lemma 6 ([61]). *Suppose P is a cone of real Banach space E, and the bounded open subsets Ω_1, Ω_2 of E satisfy $\theta \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2$. Let $A : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ be a completely continuous operator such that either*

(1) $\|Ax\| \leq \|x\|, x \in P \cap \partial\Omega_1 \ \|Ax\| \geq \|x\|, x \in P \cap \partial\Omega_2$, *or*
(2) $\|Ax\| \geq \|x\|, x \in P \cap \partial\Omega_1 \ \|Ax\| \leq \|x\|, x \in P \cap \partial\Omega_2$.

Then, A has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. The Main Results

Theorem 1. *Assume that* (**A0**), (**A1**), *and* (**A2**) *hold; moreover,*

$$\lim_{u+v\to 0} \frac{f(u,v)}{(u+v)^{\frac{1}{q-1}}} = +\infty. \tag{18}$$

Then, the tempered fractional Equation (1) *has at least one positive solution u, and two constants $A_1, B_1 > 0$ exist such that the solution has the following asymptotic estimate:*

$$A_1 t^{\beta-1}(1-t)e^{-\lambda t} \leq u(t) \leq B_1.$$

Proof. Firstly, from Lemma 5, we know that the operator $T : P \to P$ is a completely continuous operator. Next, it follows from (18) that a sufficiently small constant m and a sufficiently large constant M exist satisfying

$$0 < m < n, \ \ M \geq \Gamma(\alpha+1)\left[2^{5\beta+2\alpha(q-1)+1}e^{\left(\frac{3}{2}+q\right)\lambda}\right]^{\frac{1}{q-1}},$$

such that for any $0 < u+v \leq (1+\frac{e^{\lambda}}{\Gamma(\beta+1)})m$, we have

$$f(u,v) \geq M(u+v)^{\frac{1}{q-1}}. \tag{19}$$

Now, take $\Omega_m = \{u \in E : ||u|| < m\}$, and $\partial\Omega_m = \{u \in E : ||u|| = m\}$; then, for any $u \in P \cap \partial\Omega_m$, one has

$$\mathcal{I}^{\beta}(e^{\lambda\tau}u(\tau)) = \int_0^t \frac{(t-\tau)^{\beta-1}}{\Gamma(\beta)} e^{\lambda\tau}u(\tau)d\tau \leq \frac{e^{\lambda}mt^{\beta}}{\Gamma(\beta+1)} \leq \frac{e^{\lambda}m}{\Gamma(\beta+1)},$$

and then

$$u(\tau) + \mathcal{I}^{\beta}(e^{\lambda\tau}u(\tau)) \leq \left(1+\frac{e^{\lambda}}{\Gamma(\beta+1)}\right)m. \tag{20}$$

Thus, for any $u \in P \cap \partial\Omega_m$ and for any $\tau \in [\frac{1}{4}, \frac{3}{4}]$, it follows from (19) and (20) that

$$\begin{aligned} f(u(\tau),\mathcal{I}^{\beta}(e^{\lambda\tau}u(\tau))) &\geq M(u(\tau)+\mathcal{I}^{\beta}(e^{\lambda\tau}u(\tau)))^{\frac{1}{q-1}} \\ &\geq Mu^{\frac{1}{q-1}}(\tau) \\ &\geq M\left[\tau^{\beta-1}(1-\tau)e^{-\lambda t}\|u\|\right]^{\frac{1}{q-1}} \\ &\geq M\left[\frac{1}{4^{\beta}e^{\lambda}}\|u\|\right]^{\frac{1}{q-1}}. \end{aligned} \tag{21}$$

Consequently, for any $u \in P \cap \partial\Omega_m$, we have

$$\begin{aligned} \|Tu\| \geq Tu\left(\frac{1}{2}\right) &= \int_0^1 G\left(\frac{1}{2},s\right)\left(\int_0^s \frac{(s-\tau)^{\alpha-1}e^{-\lambda s}e^{\lambda\tau}}{\Gamma(\alpha)}f(u(\tau),\mathcal{I}^{\beta}(e^{\lambda\tau}u(\tau)))\mathrm{d}\tau\right)^{q-1}\mathrm{d}s \\ &\geq \left(\frac{1}{2}\right)^{\beta}e^{-\frac{\lambda}{2}}\int_0^1 \vartheta(s)e^{-\lambda(q-1)s}\left(\int_0^s \frac{(s-\tau)^{\alpha-1}e^{\lambda\tau}f(u(\tau),\mathcal{I}^{\beta}(e^{\lambda\tau}u(\tau)))}{\Gamma(\alpha)}\mathrm{d}\tau\right)^{q-1}\mathrm{d}s \\ &\geq 2^{-\beta}e^{-\frac{\lambda}{2}}\int_{\frac{1}{4}}^{\frac{3}{4}} \vartheta(s)e^{-\lambda(q-1)s}\left(\int_0^s \frac{(s-\tau)^{\alpha-1}e^{\lambda\tau}f(u(\tau),\mathcal{I}^{\beta}(e^{\lambda\tau}u(\tau)))}{\Gamma(\alpha)}\mathrm{d}\tau\right)^{q-1}\mathrm{d}s \\ &\geq 2^{-\beta}e^{-\frac{\lambda}{2}}\int_{\frac{1}{4}}^{\frac{3}{4}} \vartheta(s)e^{-\lambda(q-1)s}\left(\int_0^s \frac{(s-\tau)^{\alpha-1}e^{\lambda\tau}}{\Gamma(\alpha)}M\left[\frac{1}{4^{\beta}e^{\lambda}}\|u\|\right]^{\frac{1}{q-1}}\mathrm{d}\tau\right)^{q-1}\mathrm{d}s \\ &\geq \frac{M^{q-1}}{2^{3\beta}e^{\frac{3}{2}\lambda}}\int_{\frac{1}{4}}^{\frac{3}{4}} \vartheta(s)e^{-\lambda(q-1)s}\left(\int_0^s \frac{(s-\tau)^{\alpha-1}e^{\lambda\tau}}{\Gamma(\alpha)}\mathrm{d}\tau\right)^{q-1}\mathrm{d}s\|u\| \\ &\geq \frac{M^{q-1}}{2^{3\beta}e^{\frac{3}{2}\lambda}}\int_{\frac{1}{4}}^{\frac{3}{4}} \vartheta(s)e^{-\lambda(q-1)s}\left(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}\mathrm{d}\tau\right)^{q-1}\mathrm{d}s||u|| \\ &\geq \frac{M^{q-1}}{2^{3\beta}e^{\frac{3}{2}\lambda}}\int_{\frac{1}{4}}^{\frac{3}{4}} \frac{(1-s)^{\beta-1}se^{\lambda s}}{\Gamma(\beta)}e^{-\lambda(q-1)s}\left(\frac{s^{\alpha}}{\Gamma(\alpha+1)}\right)^{q-1}\mathrm{d}s\|u\| \\ &\geq \frac{M^{q-1}}{\Gamma^{q-1}(\alpha+1)2^{5\beta+2\alpha(q-1)+1}e^{\left(\frac{3}{2}+q\right)\lambda}}\|u\| \\ &\geq \|u\|, \end{aligned} \tag{22}$$

which implies that for any $u \in P \cap \partial\Omega_m$, we have $||Tu|| \geq ||u||$.

Next, let $\Omega_n = \{u \in E : ||u|| < n\}$, and $\partial\Omega_n = \{u \in E : ||u|| = n\}$. Similarly to (20), for any $u \in P \cap \partial\Omega_n$, we have

$$u(\tau)+\mathcal{I}^{\beta}(e^{\lambda\tau}u(\tau)) \leq \left(1+\frac{e^{\lambda}}{\Gamma(\beta+1)}\right)n. \tag{23}$$

Thus, for any $u \in P \cap \partial\Omega_n$, it follows from (23) and (**A2**) that

$$\begin{aligned} Tu(t) &= \max\int_0^1 G(t,s)\left(\int_0^s \frac{(s-\tau)^{\alpha-1}e^{-\lambda s}e^{\lambda\tau}}{\Gamma(\alpha)}f(u(\tau),\mathcal{I}^{\beta}(e^{\lambda\tau}u(\tau)))\mathrm{d}\tau\right)^{q-1}\mathrm{d}s \\ &\leq \int_0^1 \vartheta(s)\left(\int_0^s \frac{(s-\tau)^{\alpha-1}e^{-\lambda s}e^{\lambda\tau}}{\Gamma(\alpha)}f(u(\tau),\mathcal{I}^{\beta}(e^{\lambda\tau}u(\tau)))\mathrm{d}\tau\right)^{q-1}\mathrm{d}s \\ &\leq N^{q-1}n\int_0^1 \vartheta(s)\left(\int_0^s \frac{(s-\tau)^{\alpha-1}e^{-\lambda s}e^{\lambda\tau}}{\Gamma(\alpha)}\mathrm{d}\tau\right)^{q-1}\mathrm{d}s \\ &\leq N^{q-1}n\int_0^1 \vartheta(s)\left(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}\mathrm{d}\tau\right)^{q-1}\mathrm{d}s \\ &\leq N^{q-1}n\int_0^1 \vartheta(s)\left(\frac{s^{\alpha}}{\Gamma(\alpha+1)}\right)^{q-1}\mathrm{d}s \\ &\leq \frac{N^{q-1}n}{\Gamma^{q-1}(\beta-1)}\left(\frac{e^{\lambda}}{1-\mathcal{A}}\|\mathcal{H}_A\|+\frac{e^{\lambda}}{\Gamma(\beta)}\right) \\ &= n = \|u\|. \end{aligned} \tag{24}$$

Therefore, for any $u \in P \cap \partial\Omega_n$, we have $||Tu|| \leq ||u||$.

From Lemma 6, T has a fixed point u on $P \cap (\overline{\Omega}_n \setminus \Omega_m)$ satisfying $m \leq ||u|| \leq n$, and therefore the tempered fractional diffusion Equation (1) has a positive solution u satisfying

$$A_1 t^{\beta-1}(1-t)e^{-\lambda t} \leq u(t) \leq B_1,$$

where $A_1 = m, B_1 = n$. □

Theorem 2. *Assume that* (**A0**) ,(**A1**), *and* (**A3**) *hold, and*

$$\lim_{u+v\to 0} \frac{f(u,v)}{(u+v)^{\frac{1}{q-1}}} = 0. \tag{25}$$

Then, the tempered fractional Equation (1) *has at least one positive solution* u, *and two constants* $A_2, B_2 > 0$ *exist such that the solution has the following asymptotic estimate:*

$$A_2 t^{\beta-1}(1-t)e^{-\lambda t} \leq u(t) \leq B_2.$$

Proof. Firstly, it follows from Lemma 5 that the operator $T : P \to P$ is a completely continuous operator. Next, select a sufficiently small positive constant

$$\varepsilon < \Gamma(\alpha+1)\left[\left(1+\frac{e^\lambda}{\Gamma(\beta+1)}\right)\left(\frac{e^\lambda}{1-\mathcal{A}}\|\mathcal{H}_A\| + \frac{e^\lambda}{\Gamma(\beta)}\right)\right]^{\frac{1}{1-q}},$$

and according to (25), a constant $K > 0$ exists such that for any $0 < u+v \leq (1+\frac{e^\lambda}{\Gamma(\beta+1)})K$, we have

$$f(u,v) \leq \varepsilon(u+v)^{\frac{1}{q-1}}. \tag{26}$$

Now, let $\Omega_K = \{u \in E : ||u|| < K\}$, and $\partial\Omega_K = \{u \in E : ||u|| = K\}$; then, for any $u \in P \cap \partial\Omega_K$, as with (20), one also has

$$u(s) + \mathcal{I}^\beta(e^{\lambda s}u(s)) \leq \left(1+\frac{e^\lambda}{\Gamma(\beta+1)}\right)K. \tag{27}$$

Thus, for any $u \in P \cap \partial\Omega_K$, it follows from (26) and (27) that

$$\begin{aligned}
Tu(t) &= \int_0^1 G(t,s)\left(\int_0^s \frac{(s-\tau)^{\alpha-1}e^{-\lambda s}e^{\lambda\tau}}{\Gamma(\alpha)} f(u(\tau), \mathcal{I}^\beta(e^{\lambda\tau}u(\tau)))\mathrm{d}\tau\right)^{q-1}\mathrm{d}s \\
&\leq \int_0^1 \vartheta(s)\left(\int_0^s \frac{(s-\tau)^{\alpha-1}e^{-\lambda s}e^{\lambda\tau}}{\Gamma(\alpha)} f(u(\tau), \mathcal{I}^\beta(e^{\lambda\tau}u(\tau)))\mathrm{d}\tau\right)^{q-1}\mathrm{d}s \\
&\leq \int_0^1 \vartheta(s)\left(\int_0^s \frac{(s-\tau)^{\alpha-1}e^{-\lambda s}e^{\lambda\tau}}{\Gamma(\alpha)} \varepsilon\Big(u(\tau) + \mathcal{I}^\beta(e^{\lambda t}u(\tau))\Big)^{\frac{1}{q-1}}\mathrm{d}\tau\right)^{q-1}\mathrm{d}s \\
&\leq \varepsilon^{q-1}K\left(1+\frac{e^\lambda}{\Gamma(\beta+1)}\right)\int_0^1 \vartheta(s)\left(\int_0^s \frac{(s-\tau)^{\alpha-1}e^{-\lambda s}e^{\lambda\tau}}{\Gamma(\alpha)}\mathrm{d}\tau\right)^{q-1}\mathrm{d}s \\
&\leq \varepsilon^{q-1}K\left(1+\frac{e^\lambda}{\Gamma(\beta+1)}\right)\int_0^1 \vartheta(s)\left(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}\mathrm{d}\tau\right)^{q-1}\mathrm{d}s \\
&\leq \varepsilon^{q-1}K\left(1+\frac{e^\lambda}{\Gamma(\beta+1)}\right)\int_0^1 \vartheta(s)\left(\frac{s^\alpha}{\Gamma(\alpha+1)}\right)^{q-1}\mathrm{d}s \\
&\leq \varepsilon^{q-1}K\left(1+\frac{e^\lambda}{\Gamma(\beta+1)}\right)\left(\frac{e^\lambda}{1-\mathcal{A}}\|\mathcal{H}_A\| + \frac{e^\lambda}{\Gamma(\beta)}\right)\left(\frac{1}{\Gamma(\alpha+1)}\right)^{q-1} \\
&\leq K = \|u\|.
\end{aligned} \tag{28}$$

So, we have $||Tu|| \leq ||u||$, $u \in P \cap \partial\Omega_K$.

Next, let $\Omega_\xi = \{u \in E : ||u|| < \xi\}$, and $\partial\Omega_\xi = \{u \in E : ||u|| = \xi\}$; then, for any $u \in P \cap \partial\Omega_\xi$ and for any $\tau \in [\frac{1}{4}, \frac{3}{4}]$, we have

$$u(\tau) \geq (1-\tau)\tau^{\beta-1}e^{-\lambda\tau}\|u\| \geq \left(\frac{1}{4}\right)^{\beta} e^{-\frac{3}{4}\lambda}\xi, \tag{29}$$

and

$$\begin{aligned}
\mathcal{I}^{\beta}(e^{\lambda\tau}u(\tau)) &= \int_0^{\tau} \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)} e^{\lambda s}u(s)ds \\
&\geq \frac{1}{\Gamma(\beta)} \int_0^{\tau} (\tau-s)^{\beta-1}s^{\beta-1}(1-s)e^{-\lambda s}ds\|u\| \\
&\geq \frac{\xi e^{-\lambda}}{\Gamma(\beta)} \int_0^{\tau} (\tau-s)^{\beta-1}s^{\beta-1}(1-s)ds \\
&\geq \frac{\xi e^{-\lambda}\tau^{2\beta-1}}{\Gamma(\beta)} \int_0^{1} (1-s)^{\beta-1}s^{\beta-1}(1-\tau s)ds \\
&\geq \frac{\xi e^{-\lambda}\tau^{2\beta-1}}{\Gamma(\beta)} \int_0^{1} (1-s)^{\beta}s^{\beta-1}ds \\
&= \frac{e^{-\lambda}\tau^{2\beta-1}\Gamma(\beta+1)}{\Gamma(2\beta+1)}\xi \\
&\geq \frac{\left(\frac{1}{4}\right)^{2\beta-1}e^{-\lambda}\Gamma(\beta+1)}{\Gamma(2\beta+1)}\xi.
\end{aligned} \tag{30}$$

Thus, (20), (29), and (30) yield

$$\left[\frac{\left(\frac{1}{4}\right)^{2\beta-1}e^{-\lambda}\Gamma(\beta+1)}{\Gamma(2\beta+1)} + \left(\frac{1}{4}\right)^{\beta}e^{-\frac{3}{4}\lambda}\right]\xi \leq u(\tau) + \mathcal{I}^{\beta}(e^{\lambda\tau}u(\tau)) \leq \left(1 + \frac{e^{\lambda}}{\Gamma(\beta+1)}\right)\xi. \tag{31}$$

Consequently, according to (31) and (**A3**), for any $u \in P \cap \partial\Omega_\xi$, $t \in [\frac{1}{4}, \frac{3}{4}]$, one has

$$\begin{aligned}
\|Tu\| \geq Tu\left(\frac{1}{2}\right) &= \int_0^1 G\left(\frac{1}{2}, s\right)\left(\int_0^s \frac{(s-\tau)^{\alpha-1}e^{-\lambda s}e^{\lambda\tau}}{\Gamma(\alpha)} f(u(\tau), \mathcal{I}^{\beta}(e^{\lambda\tau}u(\tau)))\mathrm{d}\tau\right)^{q-1}\mathrm{d}s \\
&\geq \left(\frac{1}{2}\right)^{\beta} e^{-\frac{\lambda}{2}} \int_0^1 \vartheta(s)e^{-\lambda(q-1)s}\left(\int_0^s \frac{(s-\tau)^{\alpha-1}e^{\lambda\tau}f(u(\tau), \mathcal{I}^{\beta}(e^{\lambda\tau}u(\tau)))}{\Gamma(\alpha)}\mathrm{d}\tau\right)^{q-1}\mathrm{d}s \\
&\geq 2^{-\beta}e^{-\frac{\lambda}{2}} \int_{\frac{1}{4}}^{\frac{3}{4}} \vartheta(s)e^{-\lambda(q-1)s}\left(\int_0^s \frac{(s-\tau)^{\alpha-1}e^{\lambda\tau}}{\Gamma(\alpha)}\mathrm{d}\tau\right)^{q-1}\mathrm{d}sL^{q-1}\xi \\
&\geq L^{q-1}2^{-\beta}e^{-\frac{\lambda}{2}}\xi \int_{\frac{1}{4}}^{\frac{3}{4}} \vartheta(s)e^{-\lambda(q-1)s}\left(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}\mathrm{d}\tau\right)^{q-1}\mathrm{d}s \\
&\geq L^{q-1}2^{-\beta}e^{-\frac{\lambda}{2}}\xi \int_{\frac{1}{4}}^{\frac{3}{4}} \frac{(1-s)^{\beta-1}se^{\lambda s}}{\Gamma(\beta)} e^{-\lambda(q-1)s}\left(\frac{s^{\alpha}}{\Gamma(\alpha+1)}\right)^{q-1}\mathrm{d}s \\
&\geq L^{q-1}\xi\left(\frac{2^{-3\beta-2\alpha(q-1)}e^{\frac{\lambda(2-3q)}{4}}}{\Gamma(\beta)\Gamma^{q-1}(\alpha+1)}\right) \\
&= \xi = \|u\|,
\end{aligned} \tag{32}$$

which implies that $||Tu|| \geq ||u||$, $u \in P \cap \partial\Omega_\xi$.

It follows from (28), (32) and Lemma 6 that T has a fixed point u on $P \cap (\overline{\Omega}_\xi \setminus \Omega_K)$ such that $K \leq ||u|| \leq \xi$, and therefore the tempered fractional Equation (1) has at least one positive solution u satisfying

$$A_2 t^{\beta-1}(1-t)e^{-\lambda t} \leq u(t) \leq B_2,$$

where $A_2 = K, B_2 = \xi$. □

4. Examples

In this section, we give some examples to illustrate the validity of our results.

Example 1. *Let* $\alpha = \frac{1}{2}, \beta = \frac{3}{2}, \lambda = 2$, *and* $p = \frac{5}{4}$, *and* $A(t)$ *is a bounded variational function*

$$A(t) = \begin{cases} 0, & 0 \leq t < \frac{1}{2}, \\ \frac{1}{2}t - \frac{1}{8}, & \frac{1}{2} \leq t < \frac{3}{4}, \\ \frac{1}{4}t + \frac{1}{8}, & \frac{3}{4} \leq t \leq 1. \end{cases}$$

Consider the existence of positive solutions for the following p-Laplacian tempered fractional diffusion equations involving a lower tempered integral operator and the Riemann–Stieltjes integral boundary condition

$$\begin{cases} {}_0^R\mathbb{D}_t^{\frac{1}{2},2}\left(\varphi_{\frac{5}{4}}(-{}_0^R\mathbb{D}_t^{\frac{3}{2},2}(u(t)))\right) = \frac{\left(u(t)+\mathcal{I}^{\frac{3}{2}}(e^{2t}u(t))\right)^{\frac{1}{8}}}{5}, \\ u(0) = 0, \quad \mathbb{D}_t^{\frac{3}{2},2}(u(0)) = 0, \quad u(1) = \int_0^1 u(t)dA(t). \end{cases} \tag{33}$$

Then, the tempered fractional Equation (33) *has at least one positive solution* u, *and two constants* $A_1, B_1 > 0$ *exist such that*

$$A_1 t^{\frac{1}{2}}(1-t)e^{-2t} \leq u(t) \leq B_1.$$

Proof. Since

$$A(t) = \begin{cases} 0, & 0 \leq t < \frac{1}{2}, \\ \frac{1}{2}t - \frac{1}{8}, & \frac{1}{2} \leq t < \frac{3}{4}, \\ \frac{1}{4}t + \frac{1}{8}, & \frac{3}{4} \leq t \leq 1, \end{cases}$$

we have

$$0 < \mathcal{A} = \int_0^1 e^{2(1-t)} t^{\frac{1}{2}} dA(t) \approx 0.3891 < 1,$$

$$0 < \mathcal{H}_A(s) = \int_0^1 H(t,s)dA(t) \leq \int_0^1 \frac{(\beta-1)e^2}{\Gamma(\beta)} dA(t) \leq 0.5973,$$

and

$$||\mathcal{H}_A|| \approx 0.5973.$$

So, (**A0**) holds.

Here,

$$f(u,v) = \frac{(u+v)^{\frac{1}{8}}}{5}.$$

Clearly, (**A1**) holds, and

$$\lim_{u+v\to 0} \frac{f(u,v)}{(u+v)^{\frac{1}{4}}} = +\infty.$$

Now, we take $n = 2$; then,

$$0 < u + v \leq (1 + \frac{e^{\lambda}}{\Gamma(\beta+1)})n = 2(1 + \frac{e^2}{\Gamma(\frac{5}{2})}) \approx 13.1172,$$

and

$$N = \left(\frac{\Gamma^4(\frac{3}{2})}{\frac{e^2\|\mathcal{H}_A(s)\|}{1-\mathcal{A}} + \frac{e^2}{\Gamma(\frac{3}{2})}} \right)^{\frac{1}{4}} \approx 0.4478,$$

so for any $0 \leq u + v \leq 13.1172$, we have

$$f(u,v) = \frac{(u+v)^{\frac{1}{8}}}{5} \leq 0.2759 \leq Nn^{\frac{1}{q-1}} = 0.5325$$

which holds.

This implies that (**A2**) holds. Therefore, according to Theorem 1, the tempered fractional Equation (33) has at least one positive solution $u(t)$, and two constants $A_1, B_1 > 0$ exist such that

$$A_1 t^{\frac{1}{2}}(1-t)e^{-2t} \leq u(t) \leq B_1.$$

□

Example 2. *Let* $\alpha = \frac{1}{2}, \beta = \frac{3}{2}, \lambda = 2, p = \frac{5}{4}$, *and* $A(t)$ *is a bounded variational function*

$$A(t) = \begin{cases} 0, & 0 \leq t < \frac{1}{2}, \\ \frac{1}{2}t - \frac{1}{8}, & \frac{1}{2} \leq t < \frac{3}{4}, \\ \frac{1}{4}t + \frac{1}{8}, & \frac{3}{4} \leq t \leq 1, \end{cases}$$

Consider the following p-Laplacian tempered fractional diffusion equations involving a lower tempered integral operator and a Riemann–Stieltjes integral boundary condition

$$\begin{cases} {}_0^R\mathbb{D}_t^{\frac{1}{2},2}\left(\varphi_{\frac{5}{4}}(-{}_0^R\mathbb{D}_t^{\frac{3}{2},2}(u(t)))\right) = 7200\left(u(t) + \mathcal{I}^{\frac{3}{2}}(e^{2t}u(t))\right)^2, \\ u(0) = 0, \quad \mathbb{D}_t^{\frac{3}{2},2}(u(0)) = 0, \quad u(1) = \int_0^1 u(t)dAt. \end{cases} \tag{34}$$

Then, the tempered fractional Equation (34) *has at least one positive solution* u, *and two constants* $A_2, B_2 > 0$ *exist such that*

$$A_2 t^{\beta-1}(1-t)e^{-\lambda t} \leq u(t) \leq B_2.$$

Proof. Since

$$A(t) = \begin{cases} 0, & 0 \leq t < \frac{1}{2}, \\ \frac{1}{2}t - \frac{1}{8}, & \frac{1}{2} \leq t < \frac{3}{4}, \\ \frac{1}{4}t + \frac{1}{8}, & \frac{3}{4} \leq t \leq 1, \end{cases}$$

one has

$$0 < \mathcal{A} = \int_0^1 e^{2(1-t)}t^{\frac{1}{2}}dA(t) \approx 0.3891 < 1,$$

and

$$0 < \mathcal{H}_A(s) = \int_0^1 H(t,s)dA(t) \leq \int_0^1 \frac{(\beta-1)e^2}{\Gamma(\beta)}dA(t) \leq 0.5973,$$

then (**A0**) holds.

Here,

$$f(u,v) = 7200(u+v)^2.$$

Clearly, $(\mathbf{A1})$ holds, and

$$\lim_{u+v\to 0} \frac{f(u,v)}{(u+v)^{\frac{1}{4}}} = 0.$$

Next, we choose $\xi = 2$, and then we have

$$0.05955 \approx \left[\frac{e^{-2}\Gamma(\frac{5}{2})}{4^2\Gamma(4)} + \frac{e^{-\frac{3}{2}}}{4^{\frac{3}{2}}}\right] \times 2 \leq u+v \leq \left(1 + \frac{e^2}{\Gamma(\frac{5}{2})}\right) \times 2 \approx 13.1172,$$

and

$$L = \left(\Gamma^5(\frac{3}{2}) \times 2^{\frac{17}{2}} \times e^{\frac{13}{2}}\right)^{\frac{1}{4}} \approx 19.0472.$$

Thus, for any $0.05955 \leq u+v \leq 13.1172$, one arrives at

$$f(u,v) = 7200(u+v)^2 = 25.5327 \leq L\xi^{\frac{1}{4}} = 19.0480 \times 2^{\frac{1}{4}} = 22.6511,$$

which means that $(\mathbf{A3})$ is satisfied.

Thus, according to Theorem 2, the tempered fractional Equation (34) has at least one positive solution $u(t)$, and two constants $A_2, B_2 > 0$ exist such that

$$A_2 t^{\frac{1}{2}}(1-t)e^{-2t} \leq u(t) \leq B_2.$$

□

5. Conclusions

Tempered fractional equations with a p-Laplacian operator can more accurately and effectively describe turbulent velocity fluctuations in porous media with exponential law behaviour. In this paper, we establish new results on the existence of positive solutions for a class of p-Laplacian tempered fractional diffusion equations involving a lower tempered integral operator and the Riemann–Stieltjes integral boundary condition by introducing new local growth conditions and establishing an a priori estimate for Green's function. In particular, the tempered fractional diffusion equation contains a lower tempered integral operator, and the boundary condition involves the Riemann–Stieltjes integral, which can be a changing-sign measure. Because the Riemann–Stieltjes integral boundary condition is nonlocal, it can be used to describe many natural phenomena with memory effects and hereditary features related to polymers, viscoelasticity, and biomathematics and the dynamic behaviour of the velocity fluctuations in porous media with a semi-heavy tail feature. The examples also indicate that the conditions of our results are easily verified.

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References

1. Kilbas, A.; Srivastava, H.; Trujillo, J. *Theory and Applications of Fractional Differential Equations in North-Holland Mathematics Studies*; Elsevier: Amsterdam, The Netherlands, 2006; Volume 204.
2. He, J.; Zhang, X.; Liu, L.; Wu, Y.; Cui, Y. A singular fractional Kelvin-Voigt model involving a nonlinear operator and their convergence properties. *Bound. Value Probl.* **2019**, *2019*, 112. [CrossRef]
3. Zhang, X.; Mao, C.; Liu, L.; Wu, Y, Exact iterative solution for an abstract fractional dynamic system model for bioprocess. *Qual. Theory Dyn. Syst.* **2017**, *16*, 205–222. [CrossRef]
4. Baleanu, D., Jajarmi, A., Mohammadi, H., Rezapour, S. A new study on the mathematical modelling of human liver with Caputo–Fabrizio fractional derivative. *Chaos Solitons Fractals* **2020**, *134*, 109705. [CrossRef]
5. Ren, T.; Li, S.; Zhang, X.; Liu, L. Maximum and minimum solutions for a nonlocal *p*-Laplacian fractional differential system from eco-economical processes. *Bound. Value Probl.* **2017**, *2017*, 118. [CrossRef]
6. Zhang, J.; Song, J.; Chen, H. A priori error estimates for spectral galerkin approximations of integral state-constrained fractional optimal control problems. *Adv. Appl. Math. Mech.* **2023**, *15*, 568–582. [CrossRef]
7. Wang, C.; Hou, X.; Wu, Q.; Dang, P.; Fu, Z. Fractional Fourier series on the torus and applications. *Fractal Fract.* **2024**, *8*, 494. [CrossRef]
8. Gu, Q.; Chen, Y.; Zhou, J.; Huang, J. A fast linearized virtual element method on graded meshes for nonlinear time-fractional diffusion equations. *Numer. Algorithms* **2024**, *2024*, 1–37. [CrossRef]
9. Yang, Y.; Wu, Q.; Jhang, S.; Kang, Q. Approximation theorems associated with multidimensional fractional fouried reansform and applications in Laplace and heat equations. *Fractal Fract*. **2022**, *6*, 625. [CrossRef]
10. Kuang N.; Xie H. Derivative of self-intersection local time for the sub-bifractional Brownian motion. *AIMS Math.* **2022**, *7*, 10286–10302. [CrossRef]
11. Fu, Z.; Lin, Y.; Yang, D.; Yang, S. Fractional Fourier transforms meet Riesz potentials and image processing. *SIAM J. Imaging Sci.* **2024**, *17*, 476–500. [CrossRef]
12. Duong, X.; Lacey, M.; Li, J.; Wick, B.; Wu, Q. Commutators of Cauchy-Szego type integrals for domains in C^n with minimal smoothness. *Indiana Univ. Math. J.* **2021**, *70*, 1505–1541... [CrossRef]
13. Fu, Z.; Lu, S.; Shi, S. Two characterizations of central BMO space via the commutators of Hardy operators. *Forum Math. Gruyter* **2021**, *33*, 505–529. [CrossRef]
14. Shi, S.; Zhang, L.; Wang, G. Fractional non-linear regularity, potential and balayage. *J. Geom. Anal.* **2022**, *32*, 221. [CrossRef]
15. Wu, J.; Wu, Q.; Yang, Y.; Dang, P.; Ren, G. Riemann-Liouville fractional integrals and derivatives on Morrey spaces and applications to a Cauchy-type problem. *J. Appl. Anal. Comput.* **2024**, *14*, 1078–1096. [CrossRef] [PubMed]
16. Wu, Y.; Wang, G. Fractional Adams-Moser-Trudinger type inequality with singular term in Lorentz space and L^p space. *J. Appl. Anal. Comput.* **2024**, *14*, 133–145. [CrossRef]
17. Wang, Y.; Guo, L.; Zi, Y.; Li, J. Solvability of fractional differential system with parameters and singular nonlinear terms. *AIMS Math.* **2024**, *9*, 22435–22453. [CrossRef]
18. Shi, S.; Zhai, Z.; Zhang, L. Characterizations of the viscosity solution of a nonlocal and nonlinear equation induced by the fractional *p*-Laplace and the fractional *p*-convexity. *Adv. Calc. Var*. **2023**, *17*, 195–207. [CrossRef]
19. Xue, Y.; Wei, Y. Ground states of nonlocal fractional Schrödinger equations with potentials well. *Taiwan J. Math.* **2022**, *26*, 1203–1217. [CrossRef]
20. Guo, L.; Wang, Y.; Li, C.; Cai, J.; Zhang, B. Solvability for a higher-order Hadamard fractional differential model with a sign-changing nonlinearity dependent on the parameter ϱ. *J. Appl. Anal. Comput.* **2024**, *14*, 2762–2776. [CrossRef]
21. Hu, Z.; Chen, P.; Zhou, W. Two-grid finite element method for time-fractional nonlinear Schrödinger equation. *J. Comput. Math* **2024**, *42*, 1124–1144. [CrossRef]
22. Zhang, X.; Xu, P.; Wu, Y.; Wiwatanapataphee, B. The uniqueness and iterative properties of solutions for a general Hadamard-type singular fractional turbulent flow model. *Nonlinear Anal. Model. Control* **2022**, *27*, 428–444. [CrossRef]
23. Zhou, B.; Zhang, L.; Addai, E.; Zhang, N. Multiple positive solutions for nonlinear high-order Riemann-Liouville fractional differential equations boundary value problems with *p*-Laplacian operator. *Bound. Value Probl.* **2020**, *26*, 2020. [CrossRef]
24. Leibenson L. General problem of the movement of a compressible fluid in a porous medium. *Izv. Akad. Nauk Kirg. SSSR* **1983**, *9*, 7–10.
25. Beran J. *Statistics for Long-Memory Processes*; Routledge: London, UK, 2017. [CrossRef]
26. Sabzikar, F.; Meerschaert, M.; Chen, J. Tempered fractional calculus. *J. Comput. Phys.* **2015**, *293*, 14–28. [CrossRef] [PubMed]
27. Cartea, Á.; Negrete, D. Fluid limit of the continuous-time random walk with general Lévy jump distribution functions. *Phys. Rev. E* **2007**, *76*, 41–105. [CrossRef]
28. Meerschaert, M.; Zhang, Y.; Baeumer, B. Tempered anomalous diffusion in heterogeneous systems. *Geophys. Res. Lett.* **2008**, *35*, L17403. [CrossRef]

29. Zhang, Y.; Meerschaert, M. Gaussian setting time for solute transport in fluvial systems. *Water Resour. Res.* **2011**, *47*, W08601. [CrossRef]
30. Carr, P.; Geman, H.; Madan, D.; Yor, M. The fine structure of asset returns: An empirical investigation. *J. Bus.* **2002**, *75*, 305–333. [CrossRef]
31. Mali, A.; Kucche, K.; Fernandez, A.; Fahad, H. On tempered fractional calculus with respect to functions and the associated fractional differential equations. *Math. Methods Appl. Sci.* **2022**, *45*, 11134–11157. [CrossRef]
32. Wu, J.; Zhang, X.; Liu, L.; Wu, Y.; Cui, Y. Convergence analysis of iterative scheme and error estimation of positive solution for a fractional differential equation. *Math. Model. Anal.* **2018**, *23*, 611–626. [CrossRef]
33. Zhang, X.; Chen, P.; Tian, H.; Wu, Y. Upper and Lower Solution Method for a Singular Tempered Fractional Equation with a p-Laplacian Operator. *Fractal Fract.* **2023**, *7*, 522. [CrossRef]
34. Gong, R.; Vempati, M. N.; Wu, Q.; Xie, P. Boundedness and compactness of Cauchy-type integral commutator on weighted Morrey spaces. *J. Aust. Math. Soc.* **2022**, *113*, 36–56. [CrossRef]
35. Fu, Z.; Hou, X.; Lee, M. Y.; Li, J. A study of one-sided singular integral and function space via reproducing formula. *J. Geom. Anal.* **2023**, *33*, 289. [CrossRef]
36. Wang, W.; Wu, Q.; Wang, W.; Wu, Q. Atomic decomposition theorem for Hardy spaces on products of Siegel upper half spaces and Bi-parameter Hardy spaces. *J. Geom. Anal.* **2023**, *33*, 351. [CrossRef]
37. Dang, P.; Du, J.; Qian, T. Riemann boundary value problems for monogenic functions on the hyperplane. *Adv. Appl. Clifford Algebr.* **2022**, *32*, 29. [CrossRef]
38. Wang, X.; Wang, G. Singular Hardy–Adams inequalities on hyperbolic spaces of any even dimension. *Ann. Pol. Math. Inst. Mat. Pol. Akad. Nauk.* **2022**, *129*, 175–192. [CrossRef]
39. Gu, L.; Liu, Y.; Yang, C. Solvability of some Riemann-Hilbert problems related to dirac operator with gradient potential in $\mathbb{R}^3$. *J. Appl. Anal. Comput.* **2024**, *14*, 976–985. [CrossRef]
40. Chen, W.; Fu, Z.; Grafakos, L.; Wu, Y. Fractional Fourier transforms on L^p and applications. *Appl. Comput. Harmon. Anal.* **2021**, *55*, 71–96. [CrossRef]
41. Gu, L.; Liu, Y.; Lin, R. Some integral representation formulas and Schwarz lemmas related to perturbed Dirac operators. *J. Appl. Anal. Comput.* **2022**, *12*, 2475–2487. [CrossRef]
42. Fu, Z.; Grafakos, L.; Lin, Y.; Wu, Y.; Yang, S. Riesz transform associated with the fractional Fourier transform and applications in image edge detection. *Appl. Comput. Harmon. Anal.* **2023**, *66*, 211–235. [CrossRef]
43. Zhang, X.; Liu, L.; Wu, Y.; Cui, Y. The existence and nonexistence of entire large solutions for a quasilinear Schrodinger elliptic system by dual approach. *J. Math. Anal. Appl.* **2018**, *464*, 1089–1106. [CrossRef]
44. Wu, J.; Zhang, X.; Liu, L.; Wu, Y.; Cui, Y. The convergence analysis and error estimation for unique solution of a p-Laplacian fractional differential equation with singular decreasing nonlinearity. *Bound. Value Probl.* **2018**, *2018*, 82. [CrossRef]
45. Zhang, X.; Chen, P.; Wu, Y.; Wiwatanapataphee, B. A necessary and sufficient condition for the existence of entire large solutions to a Hessian system. *Appl. Math. Lett.* **2023**, *145*, 108745. [CrossRef]
46. Zhang, X.; Jiang, J.; Wu, Y.; Cui, Y. Existence and asymptotic properties of solutions for a nonlinear Schrödinger elliptic equation from geophysical fluid flows. *Appl. Math. Lett.* **2019**, *90*, 229–237. [CrossRef]
47. Zhang, X.; Liu, L.; Wu, Y.; Cui, Y. A sufficient and necessary condition of existence of blow-up radial solutions for a k-Hessian equation with a nonlinear operator. *Nonlinear Anal. Model. Control* **2020**, *25*, 126–143. [CrossRef]
48. Zhou, J.; Li, H.; Zhang, Z. A posteriori error estimates of spectral approximations for second order partial differential equations in spherical geometries. *J. Sci. Comput.* **2022**, *90*, 56. [CrossRef]
49. Zheng, W.; Chen, Y.; Zhou, J. A Legendre spectral method for multidimensional partial Volterra integro-differential equations. *J. Comput. Appl. Math.* **2024**, *436*, 115302. [CrossRef]
50. Jia, Z. Global boundedness of weak solutions for an attraction-repulsion chemotaxis system with p-Laplacian diffusion and nonlinear production. *Discret. Contin. Dyn. Syst.-Ser. B* **2023**, *28*, 4847–4863. [CrossRef]
51. Wu, J.; He, X.; Li, X. Finite-time stabilization of time-varying nonlinear systems based on a novel differential inequality approach. *Appl. Math. Comput.* **2022**, *420*, 126895. [CrossRef]
52. Wu, Y.; Chen, W. On strongly indefinite schrödinger equations with non-periodic potential. *J. Appl. Anal. Comput.* **2023**, *13*, 1–10. [CrossRef] [PubMed]
53. Wang, X.; Wang, Z.; Jia, Z. Global weak solutions for an attraction-repulsion chemotaxis system with p-Laplacian diffusion and logistic source. *Acta Math. Sci.* **2024**, *44*, 909–924. [CrossRef]
54. Gu, L.; Liu, Y. Nonlinear Riemann type problems associated to Hermitian Helmholtz equations. *Complex Var. Elliptic Equ.* **2023**, *68*, 763–775. [CrossRef]
55. Guo, Z.; Jiang, T.; Vasil'ev, V. I.; Wang, G. Complex structure-preserving method for Schrödinger equations in quaternionic quantum mechanics. *Numer. Algorithms* **2024**, *97*, 271–287. [CrossRef]

56. Ai, B.; Jia, Z. The global existence and boundedness of solutions to a Chemotaxis–Haptotaxis model with nonlinear diffusion and signal production. *Mathematics* **2024**, *12*, 2577. [CrossRef]
57. Zhang, X.; Tain, H.; Wu, Y.; Wiwatanapataphee, B. The radial solution for an eigenvalue problem of singular augmented *k*-Hessian equation. *Appl. Math. Lett.* **2022**, *134*, 108330. [CrossRef]
58. Zhang, X.; Xu, P.; Wu, Y. The eigenvalue problem of a singular *k*-Hessian equation. *Appl. Math. Lett.* **2022**, *124*, 107666. [CrossRef]
59. Ledesma, C.; Cuti, H.; Rodríguez, J.; Bonilla, M. Boundary value problem with tempered fractional derivatives and oscillating term. *J. Pseudo-Differ. Oper. Appl.* **2023**, *14*, 62. [CrossRef]
60. Webb, J. Nonlocal conjugate type boundary value problems of higher order. *Nonlinear Anal.* **2009**, *71*, 1933–1940. [CrossRef]
61. Guo, D.J.; Lakshmikantham, V. *Nonlinear Problems in Abstract Cone*; Academic Press Inc.: New York, NY, USA, 1988.

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