

Article

Exploring Optimisation Strategies Under Jump-Diffusion Dynamics

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Abstract: This paper addresses the portfolio optimisation problem within the jump-diffusion stochastic differential equations (SDEs) framework. We begin by recalling a fundamental theoretical result concerning the existence of solutions to the Black–Scholes–Merton partial differential equation (PDE), which serves as the cornerstone for subsequent analysis. Then, we explore a range of financial applications, spanning scenarios characterised by the absence of jumps, the presence of jumps following a log-normal distribution, and jumps following a distribution of greater generality. Additionally, we delve into optimising more complex portfolios composed of multiple risky assets alongside a risk-free asset, shedding new light on optimal allocation strategies in these settings. Our investigation yields novel insights and potentially groundbreaking results, offering fresh perspectives on portfolio management strategies under jump-diffusion dynamics.

Keywords: portfolio optimisation; merton problem; optimal allocation strategies; jump-diffusion stochastic differential equations; Lévy processes; verification theorem

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1. Introduction

Portfolio optimisation has been a cornerstone of financial economics since the seminal works of Markowitz [1] and Merton [2,3]. While Markowitz introduced the mean-variance framework in a discrete-time setting, Merton extended it to a continuous-time formulation, laying the foundation for the modern theory of intertemporal portfolio choice. Known as the Merton problem, this framework models an investor seeking to maximise the expected utility of terminal wealth over a finite investment horizon. Investors can allocate their wealth between a risk-free asset, such as a bond or savings account with deterministic returns, and one or more risky assets, typically stocks, whose prices evolve stochastically.

In the classical formulation, the prices of risky assets follow a geometric Brownian motion (GBM), a model that accounts for continuous market fluctuations but excludes sudden and significant price changes. However, financial markets are often subject to abrupt events, such as crashes or spikes, caused by macroeconomic news, policy decisions, or systemic shocks. These events, termed jumps, cannot be captured by GBM but are well-represented by Lévy processes, which generalise Brownian motion to include discontinuities. Jump diffusion models, combining continuous Brownian motion with discrete jumps, offer a more realistic representation of asset price dynamics.

This paper extends Merton’s classical problem to a setting where risky assets are modelled using stochastic differential equations (SDEs) driven by Lévy processes. These models incorporate drift, diffusion, and jump components, enabling a more detailed description

of market behaviour. Our objective is to determine the optimal investment strategy that maximises the expected utility of terminal wealth, assuming the utility function belongs to the constant relative risk aversion (CRRA) class.

The solution to this optimisation problem is based on stochastic control theory, precisely, the dynamic programming approach and the associated Hamilton–Jacobi–Bellman (HJB) integrodifferential equation. Using a verification theorem, we validate candidate solutions and derive the optimal strategy. To illustrate the theory, we consider three cases: (i) the classic Merton problem without jumps, (ii) the inclusion of jumps in the asset dynamics, and (iii) a multi-asset portfolio with correlated jump components.

The incorporation of jumps into portfolio optimisation has been extensively studied in the literature. Early work by Aase [4] considered optimal consumption and portfolio selection in a jump-diffusion market, modelling asset prices as càdlàg functions. Framstad et al. [5] extended this approach to geometric Lévy processes, offering a more general formulation of jump dynamics. Kallsen [6] employed the duality or martingale approach to characterise optimal portfolio and consumption strategies using the characteristic triplet of the Lévy process. Transaction costs and other market frictions have also been explored in this context. Framstad et al. [5] studied optimal strategies in the presence of proportional transaction costs, showing how these constraints alter portfolio allocation. Benth et al. [7] investigated infinite-horizon investment–consumption problems with jump-diffusion processes, accounting for both transaction costs and Hindy–Huang–Kreps preferences. These studies demonstrate the challenges of balancing jump risks and market frictions in portfolio optimisation. Recent advancements include incorporating stochastic volatility and predictive signals into the optimisation framework. Benth et al. [8] extended the Merton problem to include stochastic volatility, highlighting its impact on portfolio strategies. Bank and Körber [9] introduced a novel approach using Meyer σ -fields to model signals that alert investors to impending jumps, providing a richer decision-making framework. Other important works are [10–14]. In contrast to most existing studies on portfolio optimisation that used deterministic methods, in [15] the authors develop a probabilistic portfolio selection framework that uses real stock data to calculate optimal values, assess risk rankings, and evaluate the robustness of the model through Monte Carlo simulations and Gaussian noise analysis. A more comprehensive review of portfolio optimisation can be found in [16], which includes detailed case studies, comparisons of different models, and an insightful discussion on their strengths and limitations.

The paper’s contributions, while grounded in the existing literature, introduce refinements that enhance the mathematical rigour and applicability of jump-diffusion models in portfolio optimisation. In particular, we extend the classical Merton framework by incorporating stochastic dynamics governed by a complete Lévy process, allowing for both continuous fluctuations and discrete jumps in asset prices, hence improving traditional approaches based on geometric Brownian motion and/or those containing a jump component limited to specific distributions of the jumps themselves. The latter characteristic allows us to capture abrupt market movements, providing a more realistic representation of financial markets. Unlike prior works that often restricted the analysis to specific jump distributions or single-asset cases, this paper generalises the formulation to multi-asset portfolios with correlated jump components, with this last characteristic being a cornerstone in dealing with concrete scenarios, as witnessed by the interest practitioners always give to it.

This work is structured as follows. Section 2 presents the theoretical foundations for analysing jump diffusion processes and optimal control problems. We introduce Lévy processes and their properties, discuss the existence and uniqueness of solutions to stochastic differential equations (SDEs) driven by such processes, and provide a verification theorem that establishes sufficient conditions for optimality in the context of jump-diffusion dy-

namics. In Section 3, we apply these theoretical results to portfolio optimisation problems, demonstrating how the verification theorem can be used to derive and validate optimal strategies in various market scenarios. We begin with the classical Merton problem without jumps, extend the analysis to include jumps in asset prices, and finally consider the case of multiple risky assets with correlated jump dynamics. These examples illustrate the flexibility and power of the framework in addressing diverse financial applications. We conclude in Section 4 with a summary of the key findings, a discussion of the practical implications of the results, and potential avenues for future research. By bridging advanced stochastic control theory and real-world portfolio optimisation, this paper contributes to a deeper understanding of how investors can manage wealth in the presence of jump risks.

2. Stock Price Models and Lévy Processes

2.1. Basic Definitions on Lévy Processes

Lévy processes are a class of stochastic processes that generalise Brownian motion by incorporating both continuous paths and discrete jumps, making them ideal for financial modelling, where assets often experience gradual fluctuations and abrupt changes in value. A stochastic process $\{\eta(t)\}$ is a Lévy process if it is continuous in probability and has stationary and independent increments.

Theorem 1. *Let $t \mapsto \eta(t)$ be a Lévy process. Then, η has a càdlàg version (right continuous with left limits), which is also a Lévy process.*

Because of this result, we will, from now on, assume that the Lévy processes we work with are càdlàg.

The jump of η at $t \geq 0$ is defined by [17]

$$\Delta\eta(t) = \eta(t) - \eta(t^-). \quad (1)$$

Given a Borel set $U \subset \mathbb{R}$ whose closure does not contain 0, the quantity

$$N(t, U) = \sum_{s \in (0, t]} \mathbb{1}_U(\Delta\eta(s)) \quad (2)$$

counts the number of jumps of size $\Delta\eta(s) \in U$ occurred before or at time t . $N(t, U)$ is called the jump measure of η . The jump measure $N(t, U)$ allows us to define an important object that will be useful later, which is the Lévy measure of η :

$$\nu(U) = \mathbb{E}[N(1, U)]. \quad (3)$$

The Brownian motion has stationary and independent increments; thus, it is a Lévy process. Another simple example of a Lévy process is the Poisson process. The Poisson process π of intensity λ is a Lévy process with values in $\mathbb{N} \cup \{0\}$ such that

$$\mathbb{P}[\pi(t) = n] = \frac{(\lambda t)^n}{n!} e^{-\lambda t}, \quad n = 0, 1, 2, \dots$$

The Compound Poisson process generalises the Poisson process by allowing the jumps to have variable sizes. Formally, a compound Poisson process $t \mapsto Y(t)$ with rate λ and jump size distribution μ_Z can be represented as

$$Y(t) = \sum_{i=1}^{N(t)} Z_i \quad (4)$$

where $t \mapsto N(t)$ is a Poisson process with intensity λ , and Z_i are i.i.d. random variables drawn from distribution μ_Z , representing the sizes of the jumps. The Lévy measure ν of Y is

$$\begin{aligned}\nu(U) &= \mathbb{E}[N(1, U)] = \mathbb{E}\left[\sum_{s \in (0,1]} \mathbb{1}_U(\Delta Y(s))\right] \\ &= \mathbb{E}[(\text{number of jumps}) \times \mathbb{1}_U(\text{jump})] = \mathbb{E}[\pi(1)\mathbb{1}_U(Z)] = \lambda\mu_Z(U).\end{aligned}$$

This shows that a Lévy process can be represented by a compound Poisson process if and only if its Lévy measure is finite.

This extension enables the model to capture a wider range of jump behaviours in asset prices, such as larger shocks or heterogeneous jump distributions. In a financial context, the compound Poisson process can be used to model asset price jumps, where each jump could represent events like earnings announcements or macroeconomic shocks that cause abrupt price movements.

The Lévy–Khintchine representation characterises the infinitesimal generator (or characteristic function) of a Lévy process. It provides a detailed mathematical description of the process regarding its continuous and jump components. Given a Lévy process η with Lévy measure ν , then $\int_{\mathbb{R}} \min(1, z^2)\nu(dz) < \infty$ and

$$\mathbb{E}[e^{iu\eta(t)}] = e^{t\psi(u)}, \quad u \in \mathbb{R}, \quad (5)$$

where

$$\psi(u) = i\mu u - \frac{1}{2}\sigma^2 u^2 + \int_{|z| < R} (e^{iuz} - 1 - iuz)\nu(dz) + \int_{|z| \geq R} (e^{iuz} - 1)\nu(dz). \quad (6)$$

Conversely, given constants μ, σ , and a measure ν on the Borel sets s.t. $\int_{\mathbb{R}} \min(1, z^2)\nu(dz) < \infty$, there exists a Lévy process ν (unique in law) such that (5)–(6) hold. In other words, the Lévy process is uniquely determined by the “Lévy–Khintchine triplet” (μ, σ, ν) . This suggests that a Lévy process has three independent components: a linear drift, a Brownian motion, and a Lévy jump process.

More formally, any Lévy process admits the Itô–Lévy decomposition

$$\eta(t) = \mu t + \sigma B(t) + \int_{|z| < R} z \tilde{N}(t, dz) + \int_{|z| \geq R} z N(t, dz), \quad (7)$$

for some constants $\mu \in \mathbb{R}, \sigma \in \mathbb{R}, R \in [0, \infty]$. Here,

$$\tilde{N}(dt, dz) = N(dt, dz) - \nu(dz)dt \quad (8)$$

is called compensated Poisson random measure of η , and B is a Brownian motion independent of $\tilde{N}(dt, dz)$. For each Borel set A , the process $M(t) = \tilde{N}(t, A)$ is a martingale. We can always choose $R = 1$. If $\mathbb{E}[|\eta(t)|] < \infty$ for all $t \geq 0$, then

$$\int_{|z| \geq 1} |z|\nu(dz) < \infty$$

and, hence, we may choose $R = \infty$ and write [17]

$$\eta(t) = \mu t + \sigma B(t) + \int_{\mathbb{R}} z \tilde{N}(t, dz).$$

Finally, one of the most fundamental Lévy processes is the geometric Brownian motion (GBM).

Example 1 (The Geometric Lévy process). Consider the stochastic differential equation

$$dX(t) = X(t^-) \left[\mu dt + \sigma dB(t) + \int_{\mathbb{R}} \gamma(t, z) \bar{N}(dt, dz) \right],$$

where μ, σ are constant and $\gamma(t, z) \geq -1$. To find the solution $X(t)$ of this equation, we rewrite it as follows:

$$\frac{dX(t)}{X(t^-)} = \mu dt + \sigma dB(t) + \int_{\mathbb{R}} \gamma(t, z) \bar{N}(dt, dz).$$

Now define

$$Y(t) = \ln X(t),$$

then, by Itô's formula,

$$dY(t) = \left(\mu - \frac{1}{2} \sigma^2 \right) dt + \beta dB(t) + \int_{|z| < R} [\ln(1 + \gamma(t, z)) - \gamma(t, z)] \nu(dz) dt + \int_{\mathbb{R}} \ln(1 + \gamma(t, z)) \bar{N}(dt, dz).$$

Hence,

$$Y(t) = Y(0) + \left(\mu - \frac{1}{2} \sigma^2 \right) t + \beta B(t) + \int_0^t \int_{|z| < R} [\ln(1 + \gamma(s, z)) - \gamma(s, z)] \nu(dz) ds + \int_0^t \int_{\mathbb{R}} \ln(1 + \gamma(s, z)) \bar{N}(ds, dz),$$

and this gives the solution

$$X(t) = X(0) \exp \left\{ \left(\mu - \frac{1}{2} \sigma^2 \right) t + \beta B(t) + \int_0^t \int_{|z| < R} [\ln(1 + \gamma(s, z)) - \gamma(s, z)] \nu(dz) ds + \int_0^t \int_{\mathbb{R}} \ln(1 + \gamma(s, z)) \bar{N}(ds, dz) \right\}$$

2.2. Lévy SDEs

The geometric Lévy process is an example of a Lévy diffusion, i.e., the solution of an SDE driven by Lévy processes.

Theorem 2 (Existence and uniqueness of solutions of Lévy SDEs). Consider the following Lévy SDE in $\mathbb{R}^n$: $X^{t,x}(t) = x \in \mathbb{R}^n$ and, for $s \geq t$,

$$dX^{t,x}(s) = \mu(s, X^{t,x}(s)) ds + \sigma(s, X^{t,x}(s)) dB(s) + \int_{\mathbb{R}^n} \gamma(s, X^{t,x}(s^-), z) \tilde{N}(ds, dz), \quad (9)$$

where $\mu: [0, +\infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, $\sigma: [0, +\infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$ and $\gamma: [0, +\infty) \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times \ell}$ satisfy the following conditions:

1. (At most linear growth) There exists a constant $C_1 < \infty$ such that

$$|\mu(t, x)|^2 + \|\sigma(t, x)\|^2 + \int_{\mathbb{R}} \sum_{k=1}^{\ell} |\gamma_k|^2 \nu_k(dz_k) \leq C_1(1 + |x|^2)$$

for all $x \in \mathbb{R}^n$.

2. (Lipschitz continuity) There exists a constant $C_2 < \infty$ such that

$$\begin{aligned} & |\mu(t, x) - \mu(t, y)|^2 + \|\sigma(t, x) - \sigma(t, y)\|^2 \\ & + \int_{\mathbb{R}} \sum_{k=1}^{\ell} |\gamma_k(t, x, z_k) - \gamma_k(t, y, z_k)|^2 \nu_k(dz_k) \leq C_2|x - y|^2 \end{aligned}$$

for all $x, y \in \mathbb{R}^n$.

Then, there exists a unique càdlàg adapted solution $X^{t,x}(s)$ such that

$$\mathbb{E}[|X^{t,x}(s)|^2] < \infty \quad \forall s \geq t.$$

Solutions of Lévy SDEs in the time-homogeneous case, i.e., when $\mu(t, x) = \mu(x)$, $\sigma(t, x) = \sigma(x)$, and $\gamma(t, x, z) = \gamma(x, z)$, are called jump diffusions (or Lévy diffusions).

Theorem 3. A jump diffusion is a strong Markov process.

Theorem 4. Suppose $\phi \in C_0^2([0, +\infty) \times \mathbb{R}^n)$. Then, $\mathcal{L}\phi(t, x)$ exists and is given by

$$\begin{aligned} \mathcal{L}\phi(t, x) = & \frac{\partial \phi}{\partial t}(t, x) + \sum_{i=1}^n \mu_i(x) \frac{\partial \phi}{\partial x_i}(t, x) + \frac{1}{2} \sum_{i,j=1}^n (\sigma \sigma^\top)_{ij}(x) \frac{\partial^2 \phi}{\partial x_i \partial x_j}(t, x) \\ & + \int_{\mathbb{R}} \sum_{k=1}^{\ell} [\phi(t, x + \gamma_k(x, z)) - \phi(t, x) - \nabla_x \phi(t, x) \cdot \gamma_k(x, z)] \nu_k(dz_k). \quad (10) \end{aligned}$$

From now on, we define $\mathcal{L}\phi(t, x)$ by the expression (10) for all ϕ such that the partial derivatives of ϕ and the integrals in (10) exist at x .

2.3. Optimal Control Theory

In this section, we will discuss key components of optimal control theory, focusing on the Hamilton–Jacobi–Bellman (HJB) equation and its modifications in a jump-diffusion setting, as well as the verification theorem, which provides sufficient conditions for optimality.

Let $\alpha: [0, +\infty) \rightarrow A$, $A \subset \mathbb{R}^d$ be a càdlàg and adapted process, called control. Let us consider a controlled jump diffusion in $\mathbb{R}^n$

$$\begin{aligned} dX^{t,x,\alpha}(s) = & \mu(X^{t,x,\alpha}(s), \alpha(s))ds + \sigma(X^{t,x,\alpha}(s), \alpha(s))dB(s) \\ & + \int_{\mathbb{R}^\ell} \gamma(z, \alpha(s^-))\tilde{N}(ds, dz), \quad X^{t,x,\alpha}(t) = x \in \mathbb{R}^n, \quad (11) \end{aligned}$$

where

$$\mu: \mathbb{R}^n \times A \rightarrow \mathbb{R}^n, \quad \sigma: \mathbb{R}^n \times A \rightarrow \mathbb{R}^{n \times m}, \quad \gamma: \mathbb{R}^\ell \times A \rightarrow \mathbb{R}^{n \times \ell}.$$

Let $0 \leq T \leq \infty$ be a time horizon and assume at most linear growth and Lipschitz continuity of μ , σ and γ , uniformly in a . As we know, these conditions ensure, for all α and any initial condition $(t, x) \in [0, T] \times \mathbb{R}$, the existence and uniqueness of a strong solution for SDE starting from x at t . Let us denote this solution by $\{X^{t,x,\alpha}(s)\}_{s \in [t, T]}$.

Let $f: [0, T] \times \mathbb{R}^n \times A \rightarrow \mathbb{R}$ be a measurable function and for $(t, x) \in [0, T] \times \mathbb{R}^n$ let $\mathcal{A}(t, x)$, denote the set of controls $\alpha \in \mathcal{A}$ such that

$$\mathbb{E} \left[\int_t^T |f(s, X^{t,x,\alpha}(s), \alpha(s))| ds \right] < \infty. \quad (12)$$

Given a measurable function $g: \mathbb{R}^n \rightarrow \mathbb{R}$, let $\mathcal{J}: [0, T] \times \mathbb{R}^n \times \mathcal{A}$ be the gain functional, defined as

$$\mathcal{J}(t, x, \alpha) = \mathbb{E} \left[\int_t^T f(s, X^{t,x,\alpha}(s), \alpha(s)) ds + g(X^{t,x,\alpha}(T)) \right] \quad (13)$$

The associated value function $v: [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$ is defined by

$$v(t, x) = \sup_{\alpha \in \mathcal{A}(t, x)} \mathcal{J}(t, x, \alpha). \quad (14)$$

The stochastic control problem is to find, for any given initial condition $(t, x) \in [0, T] \times \mathbb{R}^n$, the value function $v(t, x)$ and an optimal control $\hat{\alpha} \in \mathcal{A}(t, x)$ such that

$$v(t, x) = \mathcal{J}(t, x, \hat{\alpha}), \quad (15)$$

with terminal condition $v(T, x) = g(x)$.

Under mild conditions (see, e.g., Theorem 11.2.3 in [17]), it suffices to consider Markov controls, i.e., controls $\alpha(t)$ of the form

$$\alpha(t) = \alpha_0(t, X(t^-)), \quad (16)$$

for some function $\alpha_0: [0, T] \times \mathbb{R}^n \rightarrow A$. Therefore, from now on, we will only consider Markov controls, and we will, with a slight abuse of notation, write $\alpha(t) = \alpha(t, X(t^-))$.

Note that if $\alpha = \alpha(t, x)$ is a Markov control, then $\{X^{t,x,\alpha}(s)\}_{s \in [t, T]}$ is a Lévy diffusion with generator

$$\begin{aligned} \mathcal{L}^\alpha \varphi(t, x) &= D_x \varphi(t, x)^\top \mu(x, \alpha) + \frac{1}{2} \text{Tr} \left(\sigma(x, \alpha)^\top D_{xx} \varphi(t, x) \sigma(x, \alpha) \right) \\ &+ \int_{\mathbb{R}^\ell} \left[\varphi(t, x + \gamma(z, \alpha)) - \varphi(t, x) - D_x \varphi(t, x)^\top \gamma(z, \alpha) \right] \nu(dz). \end{aligned} \quad (17)$$

We now formulate a verification theorem for the optimal control problem (15), analogous to the classical Hamilton–Jacobi–Bellman (HJB) for (continuous) Itô diffusions.

Theorem 5. Let φ be a function in $C^{1,2}([0, T] \times \mathbb{R}^n)$.

- (a) Suppose that $\mathcal{L}^a \varphi(t, x) + f(t, x, a) \leq 0$ for all $t \in [0, T]$, $x \in \mathbb{R}^n$, $a \in A$, and $\varphi(T, x) = g(x)$, for all $x \in \mathbb{R}^n$. Then, $\varphi(t, x) \geq v(t, x)$ on $[0, T] \times \mathbb{R}^n$.
- (b) Suppose, further, that for each $t \in (0, T)$, $x \in \mathbb{R}^n$, there exists a measurable function $\hat{\alpha}(t, x)$ valued in A such that
 - (i) $\mathcal{L}^{\hat{\alpha}} \varphi(t, x) + f(t, x, \hat{\alpha}(t)) = 0$
 - (ii) the SDE

$$\begin{aligned} dX^{t,x,\hat{\alpha}}(s) &= \mu(X^{t,x,\hat{\alpha}}(s), \hat{\alpha}(s)) ds + \sigma(X^{t,x,\hat{\alpha}}(s), \hat{\alpha}(s)) dB(s) \\ &+ \int_{\mathbb{R}^\ell} \gamma(z, \hat{\alpha}(s), z) N(ds, dz), \end{aligned} \quad (18)$$

admits a unique solution, denoted by $\hat{X}^{t,x,\hat{\alpha}}$, given an initial condition $X^{t,x,\hat{\alpha}}(t) = x$, and

(iii) the process $\{\hat{\alpha}(s, X^{t,x,\alpha}(s))\}_{s \in [t,T]}$ lies in $\mathcal{A}(t, x)$.

Then, $\hat{\alpha}$ is an optimal Markovian control and

$$\varphi = v \quad \text{on } [0, T] \times \mathbb{R}^n. \quad (19)$$

Proof. (i) By hypothesis, $\varphi \in C^{1,2}([0, T] \times \mathbb{R}^n)$. Then, applying Itô's formula, we have

$$\begin{aligned} \varphi(u, X^{t,x,\alpha}(u)) &= \varphi(t, x) \\ &+ \int_t^u D_x \varphi(s, X^{t,x,\alpha}(s))^\top \mu(X^{t,x,\alpha}(s), \alpha(s)) ds \\ &+ \int_t^u \frac{1}{2} \text{Tr} \left(\sigma(X^{t,x,\alpha}(s), \alpha(s))^\top D_{xx} \varphi(s, X^{t,x,\alpha}(s)) \sigma(X^{t,x,\alpha}(s), \alpha(s)) \right) ds \\ &+ \int_{\mathbb{R}^\ell} \left[\varphi(s, X^{t,x,\alpha}(s) + \gamma(z, \alpha(s))) - \varphi(s, X^{t,x,\alpha}(s)) \right] \tilde{N}(ds, dz) \\ &+ \int_{\mathbb{R}^\ell} \left[\varphi(s, X^{t,x,\alpha}(s) + \gamma(z, \alpha(s))) - \varphi(s, X^{t,x,\alpha}(s)) \right. \\ &\quad \left. - D_x \varphi(s, X^{t,x,\alpha}(s))^\top \gamma(z, \alpha(s)) \right] \nu(dz) ds \end{aligned}$$

for all $(t, x) \in [0, T] \times \mathbb{R}$, $\alpha \in \mathcal{A}(t, x)$ and $u \in [t, T]$ fixed. Taking the expectation, we obtain

$$\begin{aligned} \mathbb{E}[\varphi(u, X^{t,x,\alpha}(u))] &= \varphi(t, x) + \mathbb{E} \left[\int_t^u D_x \varphi(s, X^{t,x,\alpha}(s))^\top \mu(X^{t,x,\alpha}(s), \alpha(s)) ds \right. \\ &\quad \left. + \int_t^u \frac{1}{2} \text{Tr} \left(\sigma(X^{t,x,\alpha}(s), \alpha(s))^\top D_{xx} \varphi(s, X^{t,x,\alpha}(s)) \sigma(X^{t,x,\alpha}(s), \alpha(s)) \right) ds \right. \\ &\quad \left. - D_x \varphi(s, X^{t,x,\alpha}(s))^\top \gamma(z, \alpha(s)) \right] \nu(dz) ds \quad (20) \end{aligned}$$

By hypothesis (a), we can affirm that $\forall \alpha \in \mathcal{A}(t, x)$,

$$\mathbb{E}[\varphi(u, X^{t,x,\alpha}(u))] \leq \varphi(t, x) - \mathbb{E} \left[\int_t^u f(s, X^{t,x,\alpha}(s), \alpha(s)) ds \right]. \quad (21)$$

Now, sending $u \rightarrow T$, we have $\forall \alpha \in \mathcal{A}(t, x)$

$$\mathbb{E}[g(X^{t,x,\alpha}(T))] = \mathbb{E}[\varphi(T, X^{t,x,\alpha}(T))] \leq \varphi(t, x) - \mathbb{E} \left[\int_t^T f(s, X^{t,x,\alpha}(s), \alpha(s)) ds \right], \quad (22)$$

that is,

$$\mathcal{J}(t, x, \alpha) \leq \varphi(t, x), \quad \forall \alpha \in \mathcal{A}(t, x). \quad (23)$$

Taking the supremum over $\alpha \in \mathcal{A}(t, x)$, we obtain

$$v(t, x) \leq \varphi(t, x) \quad \forall (t, x) \in [0, T] \times \mathbb{R}^n. \quad (24)$$

(ii) Now, apply the above argument to $\alpha(s) = \hat{\alpha}(s, X^{t,x,\alpha}(s))$, where $\hat{\alpha}$ is as in (b). Then, we obtain equality in (23) and hence,

$$\varphi(t, x) = \mathcal{J}(t, x, \hat{\alpha}) \leq v(t, x) \quad (t, x) \in [0, T] \times \mathbb{R}. \quad (25)$$

Combining (24) and (25), we obtain (19).

□

The abstract theoretical framework of optimal control theory for jump-diffusion processes developed in this section underlies a broad spectrum of applications in finance, with particular emphasis on portfolio optimisation. Within this framework, the controlled process $X^{t,x,\alpha}$ denotes the investor's wealth, whose evolution is governed by stochastic dynamics subject to the chosen portfolio allocation $\alpha(t)$, which functions as the control variable. The value function $v(t, X)$ represents the maximal expected utility of terminal wealth attainable from time t , encapsulating the trade-off between risk and return. The verification theorem delineates the conditions under which a candidate value function $v(t, X)$ and its associated control $\alpha(t)$ resolve the optimisation problem, thereby providing a robust connection between theoretical constructs and practical implementation, enabling the derivation of explicit solutions and validation of their optimality.

In the subsequent section, this framework is applied to portfolio optimisation scenarios, demonstrating its utility across various market situations, both with and without asset price jumps and involving multiple assets. These examples illustrate the translation of abstract theory into effective strategies aimed at maximising terminal wealth under conditions of uncertainty.

3. Financial Applications

In the previous section, we established the theoretical foundations for analysing stochastic systems driven by Lévy processes, including key properties, the dynamics of stochastic differential equations, and a verification theorem for solving optimal control problems. Building upon this framework, we now turn to its application in the context of portfolio optimisation, specifically addressing Merton's portfolio problem under increasingly complex market conditions. We derive and validate optimal investment strategies for three distinct cases: a portfolio composed of a risk-free asset and a risky asset with no jumps, a portfolio with a risky asset exhibiting jump-diffusion dynamics, and a more intricate setting with multiple risky assets driven by pure jump processes. This progression illustrates the versatility and power of the mathematical framework, bridging abstract theory with practical financial applications.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ be a filtered probability space satisfying the usual hypotheses. We consider an investor who divides her wealth between one risk-free asset (bank account or bond) paying a fixed interest rate $r > 0$ and one or more risky assets (stocks).

3.1. Without Jumps

Let us first consider an application without a jump component. In particular, let $S_0 = (S_0(t))_{t \in [0, T]}$ and $S_1 = (S_1(t))_{t \in [0, T]}$ denote the amount of money the investor has in the risk-free asset and risky asset, respectively. The processes evolve according to

$$\begin{cases} dS_0(t) = rS_0(t)dt, \\ dS_1(t) = S_1(t)(\mu(t)dt + \sigma(t)dB(t)), \\ S_0(0) = 1, S_1(0) = s, \end{cases}$$

where $B(t)$ is a standard Brownian motion, $\mu(t)$ is the expected return of the asset at time t , and $\sigma(t)$ is the volatility of the asset at time t . We assume that $\mu(t)$ and $\sigma(t)$ are both stochastic processes adapted to a filtration $\mathcal{F}_t$.

Let $\pi = (\pi_0(t), \pi_1(t))_{t \in [0, T]}$ denote a small investor portfolio where $\pi_0(t)$ and $\pi_1(t)$ represent the proportion of wealth invested in the risk-free asset and in the risky asset, respectively. Therefore, for every $t \in [0, T]$, they satisfy the relation

$$\pi_0(t) + \pi_1(t) = 1.$$

The value at time t of such portfolio, or wealth, is

$$X(t) = \pi_0(t)S_0(t) + \pi_1(t)S_1(t),$$

and represents the total amount of money invested in the market.

The portfolio is assumed to be self-financing, that is,

$$dX(t) = \pi_0(t)dS_0(t) + \pi_1(t)dS_1(t).$$

The investor's wealth process is given by

$$dX(t) = X(t)rdt + \pi_1(t)S_1(t)(\mu(t) - r)dt + \pi_1(t)S_1(t)\sigma(t)dB(t).$$

Let $\alpha(t)$ denote the fraction of the total wealth invested in stocks at time t . Then,

$$\alpha(t) = \frac{\pi_1(t)S_1(t)}{X(t)} \iff \pi_1(t)S_1(t) = \alpha(t)X(t).$$

Plugging into the SDE, one obtains

$$dX^{t,x,\alpha}(s) = X^{t,x,\alpha}(s) \left[(\alpha(s)(\mu(s) - r) + r)ds + \alpha(s)\sigma(s)dB(s) \right], \quad (26)$$

where $(X^{t,x,\alpha}(s))_{s \in [t,T]}$ denotes the solution of (26) starting from x at time $s = t$.

The investor's objective is to maximise the expected utility over $[0, T]$. The functional to be optimised is

$$\mathcal{J}(t, x, \alpha) = \mathbb{E}[U(X^{t,x,\alpha}(T))],$$

where U is the investor's utility function. We introduce the following assumptions on the utility function:

- (U1) $U(x)$ is a continuous, nondecreasing, and concave function on $[0, \infty)$ with $U(0) = 0$.
- (U2) There exists $\gamma \in (0, 1)$ and constant $K > 0$ such that $U(x) \leq K(1 + x)^\gamma$ for all $x \in [0, \infty)$.

The Merton portfolio optimisation problem is to find a function v and an optimal control $\hat{\alpha}$ such that

$$v(t, x) = \sup_{\alpha \in \mathcal{A}} \mathcal{J}(t, x, \alpha) = \mathcal{J}(t, x, \hat{\alpha}), \quad (t, x) \in [0, T] \times [0, +\infty).$$

We know that v solves the Hamilton–Jacobi–Bellman (HJB) equation:

$$\begin{cases} \frac{\partial v}{\partial t}(t, x) + H\left(t, x, \frac{\partial v}{\partial x}(t, x), \frac{\partial^2 v}{\partial x^2}(t, x)\right) = 0, & (t, x) \in [0, T] \times [0, +\infty), \\ v(T, x) = U(x), & x \in [0, +\infty), \end{cases}$$

where, for $H(t, x, p, q) \in [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$,

$$H(t, x, p, q) = \sup_{a \in A} \left\{ x[a(\mu - r) + r]p + \frac{1}{2}a^2x^2\sigma^2q \right\}$$

is the Hamiltonian of the problem.

Let $\psi(a) := pxr + px(\mu - r)a + \frac{1}{2}x^2\sigma^2qa^2$. Assume $\mu > r$. Then, if $q \geq 0$, $\sup_a \psi(a) = H(t, x, p, q) = +\infty$ or $\sup_a \psi(a) = H(t, x, p, q) = 0$; whereas, if $q < 0$, we have

$$\psi'(a) = px(\mu - r) + q\sigma^2x^2a.$$

Let $\hat{a}$ such that $\psi'(\hat{a}) = 0$. So,

$$\hat{a} = -\frac{\mu - r}{\sigma^2} \frac{p}{qx}.$$

Then,

$$H(t, x, p, q) = prx - \frac{1}{2} \frac{p^2}{q} \left(\frac{\mu - r}{\sigma} \right)^2.$$

Let us consider a power utility function (strictly concave) $U(x) = \frac{x^\gamma}{\gamma}$, $x > 0$, risk $\gamma < 1$. We will look for a function φ of the form

$$\varphi(t, x) = \omega(t)U(x) = \omega(t) \frac{x^\gamma}{\gamma}.$$

Taking derivatives, we have

$$\frac{\partial \varphi}{\partial t}(t, x) = \omega'(t) \frac{x^\gamma}{\gamma}, \quad \frac{\partial \varphi}{\partial x}(t, x) = \omega(t)x^{\gamma-1}, \quad \frac{\partial^2 \varphi}{\partial x^2}(t, x) = \omega(t)(\gamma - 1)x^{\gamma-2}, \quad (27)$$

and

$$U(x) = \varphi(T, x) = \omega(T)U(x) \implies \omega(T) = 1.$$

Then, we plug (27) into the (HJB) equation, and isolating $\omega'(t)$, we obtain

$$\omega'(t) = \left[-r\gamma + \frac{1}{2} \frac{\gamma}{\gamma - 1} \left(\frac{\mu(t) - r}{\sigma(t)} \right)^2 \right] \omega(t). \quad (28)$$

Then, setting

$$k(t) = \left[-r\gamma + \frac{1}{2} \frac{\gamma}{\gamma - 1} \left(\frac{\mu(t) - r}{\sigma(t)} \right)^2 \right]$$

and solving the Cauchy problem

$$\begin{cases} \omega'(t) = k(t)\omega(t) \\ \omega(T) = 1 \end{cases} \quad (29)$$

we obtain

$$\omega(t) = \exp\left(-\int_t^T k(s)ds\right) \implies \varphi(t, x) = \exp\left(-\int_t^T k(s)ds\right) \frac{x^\gamma}{\gamma}.$$

By the verification theorem, the value function is $v(t, x) = \varphi(t, x)$. Finally, the optimal control is

$$\hat{a}(t, x) = \frac{1}{1 - \gamma} \frac{\mu(t) - r}{\sigma^2(t)}.$$

3.2. With Jumps

Suppose that we have a market with two possible investments:

- (i) a risk-free investment (i.e., bond, bank account) with price dynamics

$$dS_0(t) = rS_0(t)dt, \quad S_0(0) = s_0 > 0; \quad (30)$$

(ii) a risky investment (stock) with price dynamics

$$dS_1(t) = S_1(t^-) \left[\mu dt + \sigma dB(t) + \int_{-1}^{\infty} z \tilde{N}(dt, dz) \right], \quad S_1(0) = s_1 > 0; \quad (31)$$

where $r > 0$, $\mu > 0$, and $\sigma \in \mathbb{R}$ are constants. For the sake of simplicity, we assume $\mu(t) = \mu$, $\sigma(t) = \sigma$, and $\gamma(t, z) = z$, with

$$\mu > r \quad \text{and} \quad \int_{-1}^{\infty} |z| d\nu(z) < \infty. \quad (32)$$

The first assumption is very natural since it states that the expected rate of return of the stock is greater than or equal to the risk-free interest rate of the bank account. In contrast, the second assumption ensures that the stock price remains positive.

Let $\pi = (\pi_0(t), \pi_1(t))_{t \in [0, T]}$ denote a small investor portfolio where $\pi_0(t)$ and $\pi_1(t)$ represent the amounts of cash invested in the bond (non-risky asset) and the risky asset, respectively.

The value at time t of such portfolio is denoted by

$$X(t) = \pi_0(t) + \pi_1(t).$$

Assuming that the portfolio is self-financing, that is,

$$dX(t) = \pi_0(t) \frac{dS_0(t)}{S_0(t)} + \pi_1(t) \frac{dS_1(t)}{S_1(t)},$$

the investor's wealth process is given by

$$\begin{aligned} dX(t) &= \pi_0(t) r dt + \pi_1(t) \left[\mu dt + \sigma dB(t) + \int_{-1}^{\infty} \gamma(t, z) \tilde{N}(dt, dz) \right] \\ &= \pi_0(t) r dt + \pi_1(t) \mu dt + \pi_1(t) \sigma dB(t) + \pi_1(t) \int_{-1}^{\infty} \gamma(t, z) \tilde{N}(dt, dz) \pm \pi_1(t) r dt \\ &= (\pi_0(t) + \pi_1(t)) r dt + \pi_1(t) (\mu - r) dt + \pi_1(t) \sigma dB(t) + \pi_1(t) \int_{-1}^{\infty} \gamma(t, z) \tilde{N}(dt, dz) \\ &= X(t) [r + \alpha(t) (\mu - r)] dt + X(t) \alpha(t) \sigma dB(t) + X(t) \alpha(t) \int_{-1}^{\infty} \gamma(t, z) \tilde{N}(dt, dz), \end{aligned}$$

where $\alpha(t)$ denotes the proportion of wealth invested in the stock at time t . Then,

$$\alpha(t) = \frac{\pi_1(t)}{\pi_0(t) + \pi_1(t)} = \frac{\pi_1(t)}{X(t)} \iff \pi_1(t) = \alpha(t) X(t).$$

Given the utility functional

$$J(t, x, \alpha) = \mathbb{E}[U(X^{t, x, \alpha}(T))],$$

the aim is to find

$$v(t, x) = \sup_{\alpha \in \mathcal{A}} J(t, x, \alpha)$$

and $\hat{\alpha}$ that reaches that sup.

The generator $\mathcal{L}^\alpha$ of the controlled process

$$Y(t) = \begin{pmatrix} s+t \\ X(t) \end{pmatrix}, \quad t \geq 0, \quad Y(0) = y = \begin{pmatrix} s \\ x \end{pmatrix}$$

is

$$\begin{aligned} \mathcal{L}^\alpha \varphi(y) &= \frac{\partial \varphi}{\partial s} + x[r + \alpha(\mu - r)] \frac{\partial \varphi}{\partial x} + \frac{1}{2} \sigma^2 x^2 \alpha^2 \frac{\partial^2 \varphi}{\partial x^2} \\ &\quad + \int_{-1}^{\infty} \left[\varphi(t, x + \alpha x \gamma(t, z)) - \varphi(t, x) - \frac{\partial \varphi}{\partial x}(t, x) \alpha x \gamma(t, z) \right] \nu(dz). \end{aligned}$$

If we try

$$v(t, x) = \omega(t) U(x) = \omega(t) \frac{x^\beta}{\beta} \quad (33)$$

we obtain

$$\begin{aligned} \mathcal{L}^\alpha v(t, x) &= \omega'(t) \frac{x^\beta}{\beta} + \omega(t) [r + \alpha(\mu - r)] x^\beta + \omega(t) \frac{1}{2} \sigma^2 \alpha^2 (\beta - 1) x^\beta \\ &\quad + \omega(t) \frac{x^\beta}{\beta} \int_{-1}^{\infty} \left[[1 + \alpha \gamma(t, z)]^\beta - 1 - \beta \alpha \gamma(t, z) \right] \nu(dz). \end{aligned} \quad (34)$$

Let $h(\alpha)$ be the expression on the right-hand side. Then, h is concave in α and the maximum of h is attained at the critical points, i.e., when

$$\begin{aligned} h'(\alpha) &= \omega(t) (\mu - r) x^\beta + \omega(t) \sigma^2 \alpha (\beta - 1) x^\beta \\ &\quad + \omega(t) x^\beta \int_{-1}^{\infty} \left[[1 + \alpha \gamma(t, z)]^{\beta-1} \gamma(t, z) - \gamma(t, z) \right] \nu(dz) = 0 \end{aligned} \quad (35)$$

We obtain that $\alpha = \hat{\alpha}$ should solve the equation

$$\Lambda(\alpha) := \mu - r - \sigma^2 \alpha (1 - \beta) - \int_{-1}^{\infty} \left[1 - [1 + \alpha \gamma(t, z)]^{\beta-1} \right] \gamma(t, z) \nu(dz) = 0. \quad (36)$$

Since $\Lambda(0) = \mu - r > 0$, we see that if

$$\sigma^2 \alpha (1 - \beta) + \int_{-1}^{\infty} \left[1 - [1 + \alpha \gamma(t, z)]^{\beta-1} \right] \gamma(t, z) \nu(dz) \geq \mu - r, \quad (37)$$

then, there exists an optimal $\alpha = \hat{\alpha} \in (0, 1]$.

With this choice of $\hat{\alpha}$, we require that $\mathcal{L}^{\hat{\alpha}} v(t, x) = 0$, i.e.

$$\begin{aligned} \omega'(t) \frac{x^\beta}{\beta} + \omega(t) [r + \hat{\alpha}(\mu - r)] x^\beta + \omega(t) \frac{1}{2} \sigma^2 \hat{\alpha}^2 (\beta - 1) x^\beta \\ + \omega(t) \frac{x^\beta}{\beta} \int_{-1}^{\infty} \left[[1 + \hat{\alpha} \gamma(t, z)]^\beta - 1 - \beta \hat{\alpha} \gamma(t, z) \right] \nu(dz) = 0, \end{aligned} \quad (38)$$

which leads to the following ODE in $\omega(t)$:

$$\begin{aligned} \omega'(t) &= - \left(\beta [r + \hat{\alpha}(\mu - r)] + \frac{1}{2} \sigma^2 \hat{\alpha}^2 \beta (\beta - 1) \right. \\ &\quad \left. + \int_{-1}^{\infty} \left[[1 + \hat{\alpha} \gamma(t, z)]^\beta - 1 - \beta \hat{\alpha} \gamma(t, z) \right] \nu(dz) \right) \omega(t). \end{aligned} \quad (39)$$

By the verification theorem, the value function is $v(t, x) = \omega(t) \frac{x^\beta}{\beta}$, where ω is the solution of (39).

3.3. Multiple Assets

Let us now consider a structured scenario, namely, a market with multiple risky assets and a risk-free asset. The available investment opportunities consist of a riskless asset with price $S_0(t)$ and n risky assets with prices $\mathbf{S}(t) = [S_1(t), \dots, S_n(t)]'$. Asset prices follow the dynamics

$$\begin{aligned} dS_0(t) &= S_0(t)rdt, \\ dS_i(t) &= S_i(t^-)[\mu_i(t)dt + \sigma_{ij}(t)dB_j(t) + \Gamma_i(t)dN(t)], \quad i = 1, \dots, n, \end{aligned}$$

with a constant rate of interest $r \geq 0$. $\mathbf{B}(t) = [B_1(t), \dots, B_n(t)]'$ is an n -dimensional standard Brownian motion, and $N(t)$ is a Lévy pure jump process with Lévy measure $\lambda\nu(dz)$, where $\lambda \geq 0$ is a fixed parameter and the measure ν satisfies $\int_{\mathbb{R}} \min(1, |z|)\nu(dz) < \infty$, so the jumps have finite variation. The jump amplitude N is scaled by the scaling factor Γ_i . We assume that the support of ν is such that S_i remains positive, consistent with the limited liability provision. We also assume that the jump process Y and the individual Brownian motions in $\mathbf{B}$ are independent.

For the sake of simplicity, the quantities μ_i , σ_{ij} and Γ_i are assumed to be not time dependent. We write $\boldsymbol{\mu} = [\mu_1, \dots, \mu_n]'$, $\boldsymbol{\Gamma} = [\Gamma_1, \dots, \Gamma_n]'$, and $\Sigma = \sigma\sigma'$, where

$$\sigma = \begin{pmatrix} \sigma_{1,1} & \cdots & \sigma_{1,n} \\ \vdots & \ddots & \vdots \\ \sigma_{n,1} & \cdots & \sigma_{n,n} \end{pmatrix}$$

We assume that Σ is a nonsingular matrix.

Let $\pi_0(t)$ denote the percentage of wealth invested in the riskless asset at time t and $\boldsymbol{\pi}(t) = [\pi_1(t), \dots, \pi_n(t)]'$ denote the vector of portfolio weights in each of the n risky assets, assumed to be adapted predictable càdlàg processes. The portfolio weights satisfy

$$\pi_0(t) + \sum_{i=1}^n \pi_i(t) = 1.$$

The investor's wealth, starting with the initial endowment X_0 , follows the dynamics

$$\begin{aligned} dX(t) &= \pi_0(t)X(t)\frac{dS_0(t)}{S_0(t)} + \sum_{i=1}^n \pi_i(t)X(t)\frac{dS_i(t)}{S_i(t^-)} \\ &= X(t)[r + \boldsymbol{\pi}(t) \cdot (\boldsymbol{\mu} - r\mathbf{1}_n)]dt + X(t)\boldsymbol{\pi}(t) \cdot \sigma d\mathbf{B}(t) + X(t)\boldsymbol{\pi}(t) \cdot \boldsymbol{\Gamma}dN(t), \end{aligned}$$

where $\mathbf{1}_n = (1, \dots, 1)$.

Given the utility functional, the investor's problem at time t is, then, to pick the portfolio weight processes $\{\pi(s)\}_{s \in [t, T]}$ which maximise the expected utility of consumption

$$v(t, x) = \sup_{\boldsymbol{\pi}} J(t, x, \boldsymbol{\pi}),$$

where

$$J(t, x, \boldsymbol{\pi}) = \mathbb{E}[U(X^{t,x,\boldsymbol{\pi}}(T))].$$

The generator $\mathcal{L}^\pi$ of the controlled process is

$$\begin{aligned} \mathcal{L}^\pi \varphi(t, x) &= \frac{\partial \varphi(t, x)}{\partial t} + \frac{\partial \varphi(t, x)}{\partial x} (rx + \boldsymbol{\pi} \cdot (\boldsymbol{\mu} - r\mathbf{1}_n)x) \\ &\quad + \frac{1}{2} \frac{\partial^2 \varphi}{\partial x^2} x^2 \boldsymbol{\pi} \cdot \Sigma \boldsymbol{\pi} + \lambda \int_{-1}^{\infty} [\varphi(t, x + x\boldsymbol{\pi} \cdot \boldsymbol{\Gamma}z) - \varphi(t, x)]\nu(dz). \end{aligned}$$

If we try

$$v(t, x) = \omega(t)U(x) = \omega(t)\frac{x^\beta}{\beta}$$

we obtain

$$\begin{aligned}\mathcal{L}^\pi v(t, x) &= \omega'(t)\frac{x^\beta}{\beta} + \omega(t)x^\beta(r + \pi \cdot (\mu - r\mathbf{1}_n)) \\ &\quad + \frac{1}{2}\omega(t)x^\beta \pi \cdot \Sigma \pi + \lambda \omega(t)\frac{x^\beta}{\beta} \int_{-1}^{\infty} [(1 + \pi \cdot \Gamma z)^\beta - 1] \nu(dz).\end{aligned}$$

Let $h(\pi)$ be the expression on the right-hand side. Then, the maximum of h is attained at the critical points, i.e., when

$$\nabla h(\pi) = \omega(t)x^\beta(\mu - r\mathbf{1}_n) + \omega(t)x^\beta(\beta - 1)\Sigma\pi + \lambda\omega(t)x^\beta \int_{-1}^{\infty} ((1 + \pi \cdot \Gamma z)^{\beta-1} \Gamma z \nu(dz)) = 0$$

We obtain that $\pi = \hat{\pi}$ should solve the equation

$$\Lambda(\pi) := \mu - r\mathbf{1}_n + (\beta - 1)\Sigma\pi + \lambda \int_{-1}^{\infty} (1 + \pi \cdot \Gamma z)^{\beta-1} \Gamma z \nu(dz) = 0.$$

With this choice of $\hat{\pi}$, we require that $\mathcal{L}^{\hat{\pi}}v(t, x) = 0$, i.e.

$$\begin{aligned}\mathcal{L}^{\hat{\pi}}v(t, x) &= \omega'(t)\frac{x^\beta}{\beta} + \omega(t)x^\beta(r + \hat{\pi} \cdot (\mu - r\mathbf{1}_n)) \\ &\quad + \frac{1}{2}\omega(t)x^\beta \hat{\pi} \cdot \Sigma \hat{\pi} + \lambda \omega(t)\frac{x^\beta}{\beta} \int_{-1}^{\infty} [(1 + \hat{\pi} \cdot \Gamma z)^\beta - 1] \nu(dz) = 0,\end{aligned}$$

which leads to the following ODE in $\omega(t)$:

$$\omega'(t) = -\beta \left([r + \hat{\pi} \cdot (\mu - r\mathbf{1}_n)] + \frac{1}{2}(\beta - 1)\hat{\pi} \cdot \Sigma \hat{\pi} + \frac{1}{\beta} \int_{-1}^{\infty} [(1 + \hat{\pi} \cdot \Gamma z)^\beta - 1] \nu(dz) \right) \omega(t).$$

4. Conclusions

In this paper, we have advanced the application of optimal control theory to portfolio optimisation problems under various stochastic market conditions, with a particular focus on jump-diffusion models. By integrating the mathematical framework of Lévy processes with optimal control theory, we have provided a comprehensive approach to modelling and solving portfolio optimisation problems in markets characterised by continuous price movements and discrete jumps.

We demonstrated the applicability of our framework in three key scenarios: markets without jumps, markets with jumps in a single asset, and markets with jumps in multiple correlated assets. These analyses highlighted the impact of jump risks on optimal portfolio strategies, showing significant deviations from classical strategies and illustrating the framework's practical relevance for managing real-world financial uncertainty.

While this work provides a strong foundation, we acknowledge that several important extensions were not addressed for the sake of simplicity. These include incorporating transaction costs and market frictions in jump-diffusion models, the development of efficient numerical methods for high-dimensional HJB equations, and including regime-switching dynamics. Exploring these directions represents a promising area for future research and could further enhance our understanding of portfolio optimisation in complex financial environments.

In conclusion, this work bridges the gap between advanced stochastic control theory and practical portfolio management, offering a robust framework for understanding and addressing the complexities of financial markets. By providing tools to optimise wealth under both continuous and jump risks, this study contributes to developing robust strategies for navigating uncertainty and maximising returns in an increasingly volatile financial environment.

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References

1. Markowitz, H. The Utility of Wealth. *J. Political Econ.* **1952**, *60*, 151–158. [\[CrossRef\]](#)
2. Merton, R.C. Lifetime Portfolio Selection under Uncertainty: The Continuous-Time Case. *Rev. Econ. Stat.* **1969**, *51*, 247–257. [\[CrossRef\]](#)
3. Merton, R.C. An Intertemporal Capital Asset Pricing Model. *Econometrica* **1973**, *41*, 867–887. [\[CrossRef\]](#)
4. Aase, K.K. Optimum portfolio diversification in a general continuous-time model. *Stoch. Process. Their Appl.* **1984**, *18*, 81–98. [\[CrossRef\]](#)
5. Framstad, N.C.; Øksendal, B.; Sulem, A. Optimal consumption and portfolio in a jump diffusion market with proportional transaction costs. *J. Math. Econ.* **2001**, *35*, 233–257. [\[CrossRef\]](#)
6. Kallsen, J. Optimal portfolios for exponential Lévy processes. *Math. Methods Oper. Res.* **2000**, *51*, 357–374. [\[CrossRef\]](#)
7. Benth, F.E.; Karlsen, K.H.; Reikvam, K. Optimal portfolio selection with consumption and nonlinear integro-differential equations with gradient constraint: A viscosity solution approach. *Financ. Stochastics* **2001**, *5*, 275–303. [\[CrossRef\]](#)
8. Benth, F.E.; Karlsen, K.H.; Reikvam, K. Merton's portfolio optimization problem in a Black and Scholes market with non-Gaussian stochastic volatility of Ornstein-Uhlenbeck type. *Math. Financ.* **2003**, *13*, 215–244. [\[CrossRef\]](#)
9. Bank, P.; Körber, L. Merton's optimal investment problem with jump signals. *SIAM J. Financ. Math.* **2022**, *13*, 1302–1325. [\[CrossRef\]](#)
10. Emmer, S.; Klüppelberg, C. Optimal portfolios when stock prices follow an exponential Lévy process. *Financ. Stochastics* **2004**, *8*, 17–44. [\[CrossRef\]](#)
11. Aït-Sahalia, Y.; Cacho-Diaz, J.; Hurd, T.R. Portfolio choice with jumps: A closed-form solution. *Ann. Appl. Probab.* **2009**, *19*, 556–584. [\[CrossRef\]](#)
12. Cvitanić, J.; Polimenis, V.; Zapatero, F. Optimal portfolio allocation with higher moments. *Ann. Financ.* **2008**, *4*, 1–28. [\[CrossRef\]](#)
13. Benth, F.E.; Karlsen, K.H.; Reikvam, K. Portfolio optimization in a Lévy market with intertemporal substitution and transaction costs. *Stochastics Stoch. Rep.* **2002**, *74*, 517–569. [\[CrossRef\]](#)
14. Campi, L.; Owen, M.P. Multivariate utility maximization with proportional transaction costs. *Financ. Stochastics* **2011**, *15*, 461–499. [\[CrossRef\]](#)
15. Shadabfar, M.; Cheng, L. Probabilistic approach for optimal portfolio selection using a hybrid Monte Carlo simulation and Markowitz model. *Alex. Eng. J.* **2020**, *59*, 3381–3393. [\[CrossRef\]](#)
16. Kalayci, C.B.; Ertenlice, O.; Akbay, M.A. A comprehensive review of deterministic models and applications for mean-variance portfolio optimization. *Expert Syst. Appl.* **2019**, *125*, 345–368. [\[CrossRef\]](#)
17. Øksendal, B. *Stochastic Differential Equations: An Introduction with Applications*; Google-Books-ID: GizqCAAQBAJ; Springer Science & Business Media: Berlin/Heidelberg, Germany, 2013.

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