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A Group Consensus Measure That Takes into Account the Relative Importance of the Decision-Makers

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Abstract: In group decision making, the knowledge, skills, and experience of the decision-makers may not be at the same level. Hence, the need arises to take into account not only the opinion, but also the relative importance of the opinion of each decision-maker. These relative importance values can be treated as weights. In a group decision making situation, it is not only the weighted aggregate output that matters, but also the weighted measure of the group consensus. Noting that weighted group consensus measures have not yet been intensely studied, in this study, based on well-known requirements for non-weighted consensus measures, we define six reasonable requirements for the weighted case. Then, we propose a function family and prove that it satisfies the above requirements for a weighted consensus measure. Hence, the proposed measure can be used in group decision making situations where the decision-makers have various weight values that reflect the relative importance of their opinions. The proposed weighted consensus measure is based on the fuzziness degree of the decumulative distribution function of the input scores, taking into account the weights. Hence, it may be viewed as a weighted adaptation of the so-called fuzziness measure-based consensus measure. The novel weighted consensus measure is determined by a fuzzy entropy function; i.e., this function may be regarded as a generator of the consensus measure. This property of the proposed weighted consensus measure family makes it very versatile and flexible. The nice properties of the proposed weighted consensus measure family are demonstrated by means of concrete numerical examples.

Keywords: decision support; group decision making; fuzziness measure; weighted consensus measures

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1. Introduction

In a group decision making (GDM) situation, the final decision is usually based on an aggregate score of the inputs provided by the decision-makers. This aggregate score is quite similar to the concept of measure of central tendency in descriptive statistics; i.e., it may be viewed as a mean value. It is well known that just using a measure of central tendency might be misleading in a decision making situation as this measure does not tell us about the spread of the individual inputs. Additionally, what is quantified in descriptive statistics by measures of spread is quantified in GDM by consensus measures. That is, consensus measures are used to quantify the level of agreement among a group of decision-makers.

Consensus measures can be classified based on whether or not they use weights to express the differences among the importance of the decision-makers' opinions. This distinction can also be formulated as there are homogeneous and heterogeneous GDM situations based on [1]. In this article, we focus on the latter as it might be of more practical use for the following reasons:

- Heterogeneous GDM situations cover the homogeneous ones as a special case, i.e., with each decision-maker's weight being equal to the others;
- Heterogeneous GDM tasks frequently occur because, in real-life situations, the differences in knowledge, expertise, experience, position, etc., among the decision-makers are often taken into account (e.g., in a corporate environment, the weight of a decision-maker's opinion is determined by the number or portion of shares he or she holds).

The following example demonstrates the importance of using weighted consensus measures to characterize the level of agreement in heterogeneous GDM situations.

Example 1. *Suppose that a decision making committee in a commercial bank evaluates a credit proposal. The committee consists of four members who all evaluate the proposal with a number from the unit interval, i.e., the input of the ith decision-maker is* $x_i \in [0,1]$, $i = 1,2,\ldots,4$. *Here,* $x_i = 0$ *means that the ith decision-maker is completely against the proposal, while* $x_i = 1$ *means that the ith decision-maker fully supports the proposal. That is, the bigger the value of* x_i, *the more the ith decision-maker supports the proposal.*

Table 1 depicts two situations:

- *In case A, the opinion of each expert has the same importance, i.e., all the input weights are equal (*$w_i^{(A)} = 0.25$, $i = 1,2,3,4$*).*
- *In case B, the first expert has a much higher weight value (*$w_1^{(B)} = 0.5$*) than the other decision-makers. The relatively high value of* $w_1^{(B)}$ *may indicate that the first decision-maker has a higher number of shares, professional expertise, etc., than the other decision-makers.*

In case A, we may conclude that there is a relatively high level of consensus in the group that the proposal shall not be supported, as three members are very much against it ($x_2 = 0.3$, $x_3 = x_4 = 0.2$*) and only one member supports it (*$x_1 = 0.9$*). Unlike case A, in case B, one can intuitively say that there is a big disagreement in the group, as the first group member, who supports the proposal with* $x_1 = 0.9$, *counterbalances the non-supportive opposition (*$x_2 = 0.3$, $x_3 = x_4 = 0.2$*) with their outstanding weight (*$w_1^{(B)} = 0.5$*). That is, in this case, the opinion of the first group member is worth the same as the opinions of the other three taken together. This suggests that the decision weights may be of the utmost importance. In such situations, weighted consensus measures are of great use in characterizing the degree of consensus in a group of decision-makers.*

Table 1. Introductory example.

	i :	1	2	3	4
	x_i	0.9	0.3	0.2	0.2
Case *A*	$w_i^{(A)}$:	0.25	0.25	0.25	0.25
Case *B*	$w_i^{(B)}$:	0.5	0.167	0.167	0.167

Weighted group consensus measures have not yet been intensely studied. With the intention of filling this research gap, our study concentrates on the measurement of consensus degree in situations where the input providers' opinions may have various relative importance values expressed by weights. Weighted consensus measures may be viewed as a generalization of unweighted consensus measures. The practical application of weighted consensus measures is also more general as these measures can handle group decision

making situations where the group member evaluations have different weight values. In a recent paper (see [2]), Dombi and Jónás introduced a measure of consensus that is founded on the fuzziness degree of the decumulative distribution function of the decision inputs. In this article, the authors formulate a weighted version of the proposed consensus measure. Now, we intend to elaborate on this idea. The main motivation of our study is to generalize an existing family of unweighted consensus measures so that they can be applied in situations where the relative importance of the decision inputs needs to be taken into account in the decision making process. In our study, we present very easy to compute instances of the proposed weighted consensus measure family (see, e.g., Equation (32)).

The concept of consensus and its measurement have been intensely studied over the past few decades. Here, without any claim for completeness, we would like to highlight some avenues of related research. Some authors have devised some newly constructed consensus measures. For example, in [3], a measure of consensus with reciprocal preference relations was presented, and in [4], the authors introduced a Mahalanobis distance-based cardinal dissensus measure. Others formulated requirements against consensus measures. For instance, ranking ordinal scales using the consensus measure was discussed in [5], measuring consensus; the concepts, comparisons, and properties were presented in [6]. The authors of [7] focused on measuring the cohesiveness of preferences using an axiomatic analysis, and measuring consensus in a preference–approval context was introduced in [8]. Consensus measures constructed from aggregation functions and fuzzy implications were presented in [9].

Most authors deal with continuous inputs, but there are some papers with dichotomous inputs (see, e.g., [10–13]). Providing approaches for experts to make recommendations for increasing the consensus in the form of identification and direction rules is also a growing area of research (see, e.g., [14–17]). Consensus measures are frequently based on similarity or distance functions. The authors of [18,19] analyzed the effect of distance functions on the consensus reaching process (CRP), for which [20] provided a thorough literature review and a framework for a comparison of their efficiency. Another avenue of research takes into account the quality of relationships among the experts, where 'trust' is an important term in the consensus process, as one can intuitively think. This area is the social network group decision making (SN-GDM), and the interested reader is referred to [21–23]. For the most recent contributions, see, e.g., [2,22,24–26]. During the calculation of a consensus level, the inputs are often aggregated by means of quantifier-guided ordered weighted averaging (OWA) operators (see, e.g, [18–20,27]), originally introduced by Yager in [28]. Though OWA has an associated weighting vector, it is generally used for implementing the notion of fuzzy majority in the aggregation phase. With the use of the so-called weighted ordered weighted averaging (WOWA) operator, not only the concept of fuzzy majority, but also the importance weight of the decision-makers can be incorporated into the calculation [29]. For other papers with consensus measures that take into account the importance levels of the decision-makers, see, e.g., [30–32].

The rest of this paper is structured as follows. In Section 2, we summarize the basic concepts we employ in our study. Here, we also state the requirements that a consensus measure should satisfy, and then, in Section 3, we adapt these requirements to the weighted case and explain the rationale behind the modifications. In Section 4, we propose a fuzziness measure-based weighted consensus measure and prove that it satisfies the above-mentioned requirements. Here, we also demonstrate how a class of quasi-arithmetic means can be utilized to construct fuzziness measure-based weighted consensus measures. In Section 5, we present examples that demonstrate the good properties of the proposed weighted consensus measure. Lastly, we make some concluding remarks and outline possible directions for future research.

2. Preliminaries

Here, we present a brief review of the basic concepts and notations that we make use of in our study.

2.1. Fuzziness Measures

Based on articles [33,34], we make use of the following definition for a measure of fuzziness.

Definition 1 (cf. [2]). *Let X be a nonempty set and let $(X, \mathcal{A}, m)$ be a measure space, where $\mathcal{A}$ is a σ-algebra on X and m is a measure on $(X, \mathcal{A})$ such that $0 < m(X) < \infty$. Furthermore, let $\mathcal{F}(X)$ be the set of all fuzzy membership functions (fuzzy sets) $\mu\colon X \to [0,1]$ that are measurable as real-valued functions. The mapping $d\colon \mathcal{F}(X) \to [0,1]$ is said to be a fuzziness measure on $\mathcal{F}(X)$, if d satisfies the following requirements:*

P_1 *For any $\mu \in \mathcal{F}(X)$, $d(\mu) = 0$ holds if and only if μ is sharp almost everywhere in X. That is, either $\mu(x) = 0$ or $\mu(x) = 1$ holds for almost every $x \in X$.*

P_2 *$d(\mu)$ has a unique maximum value if and only if $\mu(x) = \frac{1}{2}$ holds almost for every $x \in X$.*

P_3 *For any $\mu, \mu^* \in \mathcal{F}(X)$, $d(\mu) \geq d(\mu^*)$ holds whenever $\mu^*(x) \geq \mu(x)$ if $\mu(x) > \frac{1}{2}$, and $\mu^*(x) \leq \mu(x)$ if $\mu(x) < \frac{1}{2}$.*

P_4 *$d(\mu) = d(\overline{\mu})$, where $\overline{\mu}$ is the 'complement' of μ, i.e., the function $\overline{\mu}\colon X \to [0,1]$ is given as $\overline{\mu}(x) = 1 - \mu(x)$, where $x \in X$.*

The authors of [33] proposed the following measure:

$$d(f) = \frac{1}{n} \sum_{i=1}^{n} S(f(x_i)), \tag{1}$$

where x_i is an element of a finite set $X = \{x_1, x_2, \ldots, x_n\}$ and

$$S(x) = -\frac{1}{\ln(2)} (x \ln(x) + (1-x) \ln(1-x)), \quad x \in [0,1] \tag{2}$$

(with the convention $0 \cdot \ln(0) = 0$). It was proved in [33] that the function $d(f)$ given in Equation (1) satisfies the criteria for a fuzziness measure stated in P_1–P_3. The function S in Equation (2) is formally similar to the well-known Shannon entropy. Hence, this function is often referred to as fuzzy entropy. Here, we will use the following, more general, definition for fuzzy entropy.

Definition 2 (cf. [2]). *A function $F\colon [0,1] \to [0,1]$ is said to be fuzzy entropy if F satisfies the following requirements:*

(a) *F is continuous function on $[0,1]$.*

(b) *$F(0) = 0$ and $F(1) = 0$.*

(c) *F is a strictly increasing function on $\left(0, \frac{1}{2}\right)$ and F is a strictly decreasing function on $\left(\frac{1}{2}, 1\right)$.*

(d) *$F(x)$ has a unique maximum at $x = \frac{1}{2}$, and $F\left(\frac{1}{2}\right) = 1$.*

(e) *$F(x) = F(1-x)$ holds for any $x \in [0,1]$.*

In the sense of Definition 2, the function

$$F(x) = 4\,x\,(1-x), \quad x \in [0,1], \tag{3}$$

is an example of fuzzy entropy. This function was proposed in [35–37]. We should add that there are infinitely many functions that satisfy the requirements given in Definition 2.

We utilize the following theorem based on [34,35].

Theorem 1 (cf. [34,35]). *Let X be a nonempty set and let $(X, \mathcal{A}, m)$ be a measure space, where $\mathcal{A}$ is a σ-algebra on X and m is a measure on $(X, \mathcal{A})$ such that $0 < m(X) < \infty$. Suppose that $\mu\colon X \to [0,1]$ is a fuzzy membership function (fuzzy set), and $F\colon [0,1] \to [0,1]$ is a fuzzy entropy satisfying the criteria given in Definition 2. Then, the function*

$$d_F(\mu) = \frac{1}{m(X)} \int_X F(\mu(x)) \mathrm{d}m(x) \tag{4}$$

satisfies the criteria P_1–P_4 for a fuzziness measure given by Definition 1.

We will differentiate the concepts of fuzziness measure and fuzzy entropy. A fuzziness measure is used for measuring how much a fuzzy membership function differs from the characteristic functions, while fuzzy entropy is used for generating a fuzziness measure.

2.2. Strong Negations

Later in this paper, we utilize the following definition for a strong negation (see, e.g., Definition 1.2 in [38], or Definition 11.3 in [39]).

Definition 3 (cf. [38,39]). *A strictly decreasing continuous function $N\colon [0,1] \to [0,1]$ is a strong negation if $N(0) = 1$, $N(1) = 0$, and N is an involution, i.e., $N(N(x)) = x$ holds for any $x \in [0,1]$.*

Trillas, in 1979, presented the following representation theorem of strong negations (see [40]).

Theorem 2 (Trillas representation). *The function $n\colon [0,1] \to [0,1]$ is a strong negation if and only if, for any $x \in [0,1]$,*

$$n(x) = g^{-1}(1 - g(x)), \tag{5}$$

where $g\colon [0,1] \to [0,1]$ is an automorphism of $[0,1]$.

Another representation theorem of strong negations was given by Dombi in 2011 (see [41]).

Theorem 3 (Dombi representation). *The function $\eta_A\colon [0,1] \to [0,1]$ is a strong negation if and only if, for any $x \in [0,1]$,*

$$\eta_A(x) = f^{-1}\left(\frac{A}{f(x)}\right), \tag{6}$$

where $f\colon [0,1] \to [0,\infty]$ is an additive generator function of some strict triangular norm or strict triangular conorm, $A \in \mathbb{R}$ and $A > 0$.

For more details on triangular norms and conorms, see [39]. It is worth noting that, in [42], Dombi and Jónás presented a necessary and sufficient condition for the equivalence of the above two representations of strong negations. Also note that, here, we use only the strictly decreasing monotonicity and the involutivity properties of strong negation operators.

2.3. Consensus Measures

In our study, we suppose that there is a group of n entities ($n \in \mathbb{N}$, $n \geq 2$), and every member of this group expresses its opinion on the unit interval $[0,1]$. This means that we have the x_i individual input value (evaluation) of the ith group member, where

$i \in \{1, 2, \ldots, n\}$. Hence, the inputs of a group are represented by an n-dimensional vector $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in [0,1]^n$.

A consensus measure is a function that expresses how much the individual inputs of the group members agree with one another. In the seminal paper of Beliakov et al. (see [9]), a consensus measure is defined as a mapping $C\colon [0,1]^n \to [0,1]$ which satisfies the following two requirements: (1) for any $a \in [0,1]$, $C(a, a, \ldots, a) = 1$ (unanimity property of C); (2) for the special case of two inputs, it holds that $C(0,1) = C(1,0) = 0$ (called the minimum consensus for $n = 2$). In [9], there were five reasonable additional criteria set for a consensus measure: symmetry, maximum dissension, reciprocity, replication invariance, and monotonicity with respect to the majority. Using this approach, the authors of [2] employed the following definition for a group consensus measure.

Definition 4 (cf. [2,9]). *A function $C\colon [0,1]^n \to [0,1]$ is said to be a group consensus measure if it satisfies the following requirements:*

(C1) (Unanimity) For any $a \in [0,1]$, $C(a, a, \ldots, a) = 1$.

(C2) (Minimum consensus for $n = 2$) For the special case of two inputs, it holds that $C(0,1) = C(1,0) = 0$.

(C3) (Symmetry) For any permutation $\rho\colon \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ and input vector $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in [0,1]^n$, it holds that $C(x_1, x_2, \ldots, x_n) = C\left(x_{\rho(1)}, x_{\rho(2)}, \ldots, x_{\rho(n)}\right)$.

(C4) (Maximum dissension) For $n = 2k$, if k of the inputs are equal to zero and k of the inputs are equal to 1, then $C(0, 0, \ldots, 1, 1) = 0$ for all permutations of the input vector.

(C5) (Reciprocity) For any input vector $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in [0,1]^n$, it holds that $C(x_1, x_2, \ldots, x_n) = C(N_s(x_1), N_s(x_2), \ldots, N_s(x_n))$, where $N_s(a) = 1 - a$ is the standard fuzzy negation and $a \in [0,1]$.

(C6) (Replication invariance) For any input vector $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in [0,1]^n$, replicating the inputs does not alter the degree of consensus, i.e., $C(\mathbf{x}) = C(\mathbf{x}, \mathbf{x}) = \ldots = C(\mathbf{x}, \mathbf{x} \ldots, \mathbf{x})$.

(C7) (Monotonicity with respect to the majority) For $n = 2k$, let half of the inputs be equal and be denoted by $\mathbf{a} = (a, a, \ldots, a)$, where $\mathbf{a} \in [0,1]^k$. Furthermore, let $\mathbf{x} = (x_1, x_2, \ldots, x_k)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_k)$ be two input vectors, where $\mathbf{x}, \mathbf{y} \in [0,1]^k$. If $|a - x_j| \leq |a - y_j|$ for all $j = 1, 2, \ldots, k$; then, $C(\mathbf{a}, x_1, x_2, \ldots, x_k) \geq C(\mathbf{a}, y_1, y_2, \ldots, y_k)$ holds for any permutation of the inputs.

Remark 1. *We should add that the (C2) requirement is a special case of the (C4) requirement. That is, with the choice $n = 2$, criterion (C4) is the same as criterion (C2).*

In [2], Dombi and Jónás proved that the following function $C_F\colon [0,1]^n \to [0,1]$ satisfies the requirements (C1)–(C7) for a consensus measure given in Definition 4:

$$C_F(\mathbf{x}) = 1 - \sum_{i=1}^{n} \left(x_{\pi(i)} - x_{\pi(i-1)}\right) F\left(\frac{n-i+1}{n}\right), \tag{7}$$

where $\pi\colon \{0, 1, 2, \ldots, n+1\} \to \{0, 1, 2, \ldots, n+1\}$ is a permutation such that

$$0 = x_{\pi(0)} \leq x_{\pi(1)} \leq x_{\pi(2)} \leq \ldots \leq x_{\pi(n)} \leq x_{\pi(n+1)} = 1$$

and $F\colon [0,1] \to [0,1]$ is a mapping satisfying the requirements for a fuzzy entropy given in Definition 2.

Here, function C_F is called the fuzziness measure-based consensus measure induced by the fuzzy entropy F. Since C_F is induced by F, the fuzzy entropy F may be regarded as a generator function of the consensus measure C_F.

The notation $(\mathbf{x}, \mathbf{y})$ will be used for the concatenation of the vectors $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$. That is,

$$(\mathbf{x}, \mathbf{y}) = (x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n),$$

where $n \in \mathbb{N}$, $n \geq 2$. Furthermore, the notation $(\mathbf{x}, \mathbf{x}, \ldots, \mathbf{x})$ will stand for the vector $(x_1, x_2, \ldots, x_n, x_1, x_2, \ldots, x_n, \ldots, x_1, x_2, \ldots, x_n)$.

3. Weighted Consensus Measures

In the following definition, we modify the requirements for a consensus measure given in Definition 4 so that they apply to the weighted case. Then, in Remark 2, we explain the rationale behind our modifications.

Definition 5. *Let $n \in \mathbb{N}$, $n \geq 2$ and let $\mathbf{w} = (w_1, w_2, \ldots, w_n) \in [0,1]^n$ be a weight vector such that $w_i > 0$ for $i = 1, 2, \ldots, n$ and $\sum_{i=1}^{n} w_i = 1$. We say that a function $C_{\mathbf{w}} \colon [0,1]^n \to [0,1]$ is a weighted consensus measure with respect to the weight vector $\mathbf{w}$, if $C_{\mathbf{w}}$ satisfies the following requirements:*

(cw1) (Unanimity) For any $a \in [0,1]$, $C_{\mathbf{w}}(a, a, \ldots, a) = 1$.

(cw2) (Symmetry) For any permutation $\rho \colon \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ and input vector $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in [0,1]^n$, it holds that $C_{\mathbf{w}}(x_1, x_2, \ldots, x_n) = C_{\mathbf{w}'}\left(x_{\rho(1)}, x_{\rho(2)}, \ldots, x_{\rho(n)}\right)$, where $\mathbf{w}' = (w_{\rho(1)}, w_{\rho(2)}, \cdots, w_{\rho(n)})$.

(cw3) (Maximum dissension) If

(a) for $\{i_1, i_2, \ldots, i_k\} \subset \{1, 2, \ldots, n\}$, $\sum_{j=1}^{k} w_{i_j} = \frac{1}{2}$ and

(b) for all $j \in \{i_1, i_2, \ldots, i_k\}$, $x_j = 0$ and

(c) for all $j \in \{1, 2, \ldots, n\} \setminus \{i_1, i_2, \ldots, i_k\}$, $x_j = 1$,

then $C_{\mathbf{w}}(x_1, x_2, \ldots, x_n) = 0$.

(cw4) (Reciprocity) For any input vector $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in [0,1]^n$, it holds that $C_{\mathbf{w}}(x_1, x_2, \ldots, x_n) = C_{\mathbf{w}}(N_s(x_1), N_s(x_2), \ldots, N_s(x_n))$, where $N_s(a) = 1 - a$ is the standard fuzzy negation and $a \in [0,1]$.

(cw5) (Replication invariance) For any input vector $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in [0,1]^n$, replicating the inputs along with re-normalizing the weights to 1 does not alter the degree of weighted consensus; i.e., for any $k \in \mathbb{N}$, $k \geq 2$,

$$C_{\mathbf{w}}(\mathbf{x}) = C_{\mathbf{w}^{(k)}}(\mathbf{x}^{(k)}),$$

where

$$\mathbf{x}^{(k)} = (\underbrace{\mathbf{x}, \mathbf{x}, \ldots, \mathbf{x}}_{k\text{-times}}), \quad \mathbf{w}^{(k)} = (\underbrace{\mathbf{w}^*, \mathbf{w}^*, \ldots, \mathbf{w}^*}_{k\text{-times}})$$

and

$$\mathbf{w}^* = \left(\frac{w_1}{k}, \frac{w_2}{k}, \ldots, \frac{w_n}{k}\right).$$

(cw6) (Monotonicity with respect to the majority) Let $a \in [0,1]$, and let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ be two input vectors, where $\mathbf{x}, \mathbf{y} \in [0,1]^n$. If

(a) for $\{i_1, i_2, \ldots, i_k\} \subset \{1, 2, \ldots, n\}$, $\sum_{j=1}^{k} w_{i_j} \geq \frac{1}{2}$ and

(b) for all $j \in \{i_1, i_2, \ldots, i_k\}$, $x_j = y_j = a$ and

(c) for all $j \in \{1, 2, \ldots, n\} \setminus \{i_1, i_2, \ldots, i_k\}$, $|a - x_j| \leq |a - y_j|$,

then $C_{\mathbf{w}}(x_1, x_2, \ldots, x_n) \geq C_{\mathbf{w}}(y_1, y_2, \ldots, y_n)$.

Remark 2. *Here, we explain the modifications made in Definition 5 compared to Definition 4. These modifications mainly concern the requirements that contain the number of decision-makers.*

(cw1) (Unanimity) The original requirement (C1) states that if all the decision-makers have the same opinion, then the level of consensus should be maximal. This requirement does not need to be modified as the same opinion shared by all the group members results in a maximal consensus, independent of the actual weight values.

(cw2) (Symmetry) This requirement is related to the order of inputs. The original criterion of symmetry (i.e., (C3)) states that if the inputs are permuted, the level of consensus should not change. We can adapt it to the weighted case by demanding the same permutation of the weights and inputs.

(cw3) (Maximum dissension) The original criterion concerns the maximal possible dispersion of the inputs and the lowest possible level of consensus in this case. The introductory example in Table 1 suggests that, in the weighted case, it is not the number of decision-makers but their total weight that counts. Consequently, the original (C4) criterion (a special case of which is (C2)) should be modified such that not half of the decision-makers but some of them representing half of the total weights shall produce zeros as inputs, and all the others shall produce ones as inputs to obtain the lowest possible level of consensus.

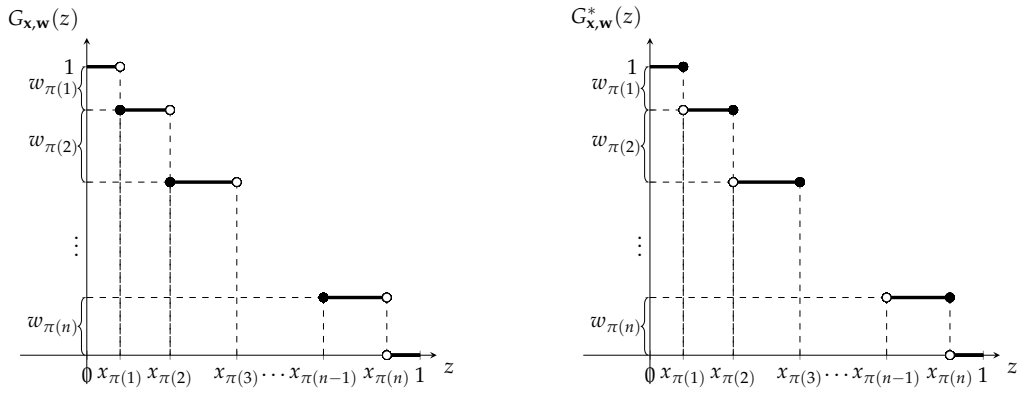
(cw4) (Reciprocity) The (C5) criterion is about reversing each decision-maker's opinion (the level of consensus is independent of which end of the evaluation scale the inputs come from [9]). As this criterion is independent of the number of decision-makers, it need not be modified.

(cw5) (Replication invariance) The original criterion (C6) in [43] states that replicating the inputs should not alter the degree of consensus. In other words, if the proportion of each evaluation remains the same, the level of consensus should not change. In the weighted case, the constant proportion of each input can be guaranteed by re-normalizing the weights.

(cw6) (Monotonicity with respect to the majority) The original requirement (C7) states that if the majority of the decision-makers have the same input value and the minority is approaching, in absolute terms in its inputs, to the constant input of the majority, then the level of consensus should not decrease. We can adapt this criterion to the weighted case using the fact that in the weighted case, it is not the number of decision-makers, but their total weight that really counts. Consequently, the modified criterion states that if each input of the weight minority is approaching the constant input of weight majority, the level of consensus should not decrease. At this point, we refer to the introductory example given in Table 1. In case A, the third and fourth decision-makers form a weight-majority as $w_3 + w_4 = 0.5$. At the same time, their inputs are identical ($x_3 = x_4 = 0.2 = a$). Hence, in this case, if expert 1 or 2 modifies their evaluation such that it becomes closer to $a = 0.2$ than it is now, then the level of consensus should not decrease. Similarly, in case B, decision-maker 1 constitutes the weight majority, and so if any other decision-maker is closer to $x_1 = 0.9 = a$ with their evaluation, then the level of consensus should not decrease.

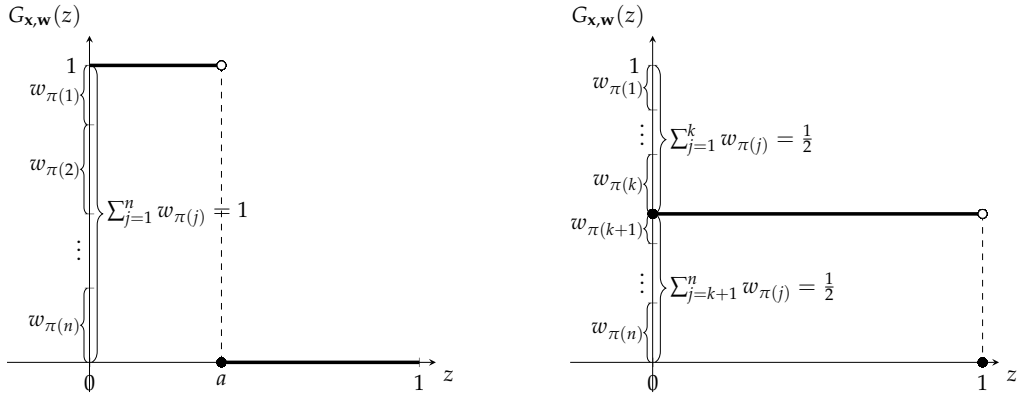
4. Weighted Consensus Measures Based on Fuzziness Measure

Now, we introduce an auxiliary function that we call the decumulative distribution function (DDF for short) of an n-dimensional vector of evaluations with respect to the weight of each input. This function may be viewed as a generalization of the decumulative distribution function of the inputs presented by Dombi and Jónás in [2]. The DDF of an input vector with respect to the input weights is an auxiliary function. This is a step function, which has the advantage that its shape tells us how much the inputs are scattered (see Figures 1 and 2), given the weights. Here, we demonstrate that a fuzziness measure of this function can be used to construct a family of weighted consensus measures, and the members of this measure family satisfy all the criteria set in Definition 5.



(**a**) (**b**)

Figure 1. The decumulative distribution functions $G_{\mathbf{x},\mathbf{w}}(z)$ and $G^*_{\mathbf{x},\mathbf{w}}(z)$ of the input vector $\mathbf{x}$ with respect to the weight vector $\mathbf{w}$. (**a**) The $G^*_{\mathbf{x},\mathbf{w}}(z)$ function; (**b**) the $G_{\mathbf{x},\mathbf{w}}(z)$ function.



(**a**) (**b**)

Figure 2. Two terminal cases of $G_{\mathbf{x},\mathbf{w}}(z)$. (**a**) Maximal consensus; (**b**) maximal dissension.

4.1. The Decumulative Distribution Function of an Input Vector with Respect to the Input Weights

We will define the decumulative distribution function of an input vector with respect to a weight vector as follows.

Definition 6. *Let $n \in \mathbb{N}$, $n \geq 2$, let $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in [0,1]^n$ be an input vector, and let $\mathbf{w} = (w_1, w_2, \ldots, w_n) \in [0,1]^n$ be a weight vector such that $w_i > 0$ for $i = 1, 2, \ldots, n$ and $\sum_{i=1}^{n} w_i = 1$. Furthermore, let $\pi\colon \{0, 1, 2, \ldots, n+1\} \to \{0, 1, 2, \ldots, n+1\}$ be a permutation such that*

$$0 = x_{\pi(0)} \leq x_{\pi(1)} \leq x_{\pi(2)} \leq \ldots \leq x_{\pi(n)} \leq x_{\pi(n+1)} = 1. \tag{8}$$

A function $G_{\mathbf{x},\mathbf{w}}\colon [0,1] \to [0,1]$, which is given by

$$G_{\mathbf{x},\mathbf{w}}(z) = \sum_{\substack{i: z < x_{\pi(i)} \\ i \in \{1,\ldots,n\}}} w_{\pi(i)}, \tag{9}$$

is said to be the decumulative distribution function (DDF) of the input vector $\mathbf{x}$ with respect to the weight vector $\mathbf{w}$.

Remark 3. *We will also make use of the following form of the DDF for the input vector $\mathbf{x}$ with respect to the weight vector $\mathbf{w}$:*

$$G^*_{\mathbf{x},\mathbf{w}}(z) = \sum_{\substack{i: z \leq x_{\pi(i)} \\ i \in \{1,\ldots,n\}}} w_{\pi(i)}, \tag{10}$$

where $z \in [0,1]$.

Figure 1 shows example plots of the $G_{\mathbf{x},\mathbf{w}}(z)$ and $G^*_{\mathbf{x},\mathbf{w}}(z)$ functions.

Now, we present some key properties of the functions $G_{\mathbf{x},\mathbf{w}}$ and $G^*_{\mathbf{x},\mathbf{w}}$.

Theorem 4. *Let* $n \in \mathbb{N}$, $n \geq 2$, *let* $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in [0,1]^n$ *be an input vector, and let* $\mathbf{w} = (w_1, w_2, \ldots, w_n) \in [0,1]^n$ *be a weight vector such that* $w_i > 0$ *for* $i = 1, 2, \ldots, n$ *and* $\sum_{i=1}^{n} w_i = 1$. *Furthermore, let* $G_{\mathbf{x},\mathbf{w}}$ *and* $G^*_{\mathbf{x},\mathbf{w}}$ *be the DDFs of* $\mathbf{x}$ *with respect to the weight vector* $\mathbf{w}$ *given by Definition 6 and Equation (10), respectively. Then, the following statements hold:*

(a) $G_{\mathbf{x},\mathbf{w}}$ *and* $G^*_{\mathbf{x},\mathbf{w}}$ *are non-increasing functions on* $(0,1)$.

(b) $G_{\mathbf{x},\mathbf{w}}$ *is right-continuous on* $(0,1)$, *and* $G^*_{\mathbf{x},\mathbf{w}}$ *is left-continuous on* $(0,1)$.

(c) *If* $x_{\pi(1)} = 0$, *then* $G^*_{\mathbf{x},\mathbf{w}}(0) = 1$, *and if* $x_{\pi(n)} = 1$, *then* $G_{\mathbf{x},\mathbf{w}}(1) = 0$, *where* π *is a permutation on* $\{0, 1, 2, \ldots, n+1\}$ *satisfying Equation (8).*

(d) *For any strong negation* $N\colon [0,1] \to [0,1]$, *if the negated input vector* $\overline{\mathbf{x}}$ *is given by* $\overline{\mathbf{x}} = (N(x_1), N(x_2), \ldots, N(x_n))$, *then, for any* $z \in (0,1)$, *it holds that*

$$G_{\overline{\mathbf{x}},\mathbf{w}}(z) = 1 - G^*_{\mathbf{x},\mathbf{w}}(N(z)). \tag{11}$$

(e) *For any* $z \in \left(x_{\pi(i-1)}, x_{\pi(i)}\right)$,

$$G_{\mathbf{x},\mathbf{w}}(z) = \sum_{j=i}^{n} w_{\pi(j)}, \tag{12}$$

where $i \in \{1, 2, \ldots, n\}$ *and* π *is a permutation on* $\{0, 1, 2, \ldots, n+1\}$ *satisfying Equation (8).*

(f) *The areas under the curves of* $G_{\mathbf{x},\mathbf{w}}$ *and* $G^*_{\mathbf{x},\mathbf{w}}$ *on the interval* $[0,1]$ *are both equal to the weighted arithmetic mean of the inputs. That is,*

$$\int_0^1 G_{\mathbf{x},\mathbf{w}}(z)\mathrm{d}z = \int_0^1 G^*_{\mathbf{x},\mathbf{w}}(z)\mathrm{d}z = \sum_{i=1}^{n} w_i x_i.$$

Proof. With the definitions of $G_{\mathbf{x},\mathbf{w}}$ and $G^*_{\mathbf{x},\mathbf{w}}$, we immediately know that properties (a), (b), and (c) hold.

Using the involutive and strictly decreasing properties of the strong negation N, we can write

$$\begin{aligned} G_{\overline{\mathbf{x}},\mathbf{w}}(z) &= \sum_{\substack{i:z<N\left(x_{\pi(i)}\right)\\ i\in\{1,\ldots,n\}}} w_{\pi(i)} = \sum_{\substack{i:N(z)>x_{\pi(i)}\\ i\in\{1,\ldots,n\}}} w_{\pi(i)} = 1 - \sum_{\substack{i:N(z)\leq x_{\pi(i)}\\ i\in\{1,\ldots,n\}}} w_{\pi(i)} \\ &= 1 - G^*_{\mathbf{x},\mathbf{w}}(N(z)), \end{aligned}$$

which means that property (d) holds.

Based on the definition of $G_{\mathbf{x},\mathbf{w}}$, we also see that for any $z \in \left(x_{\pi(i-1)}, x_{\pi(i)}\right)$,

$$G_{\mathbf{x},\mathbf{w}}(z) = G_{\mathbf{x},\mathbf{w}}\left(x_{\pi(i-1)}\right) = \sum_{j=i}^{n} w_{\pi(j)},$$

where $i \in \{1, 2, \ldots, n\}$. Hence, property (e) holds, too.

Noting the construction of $G_{\mathbf{x},\mathbf{w}}$, we can see that the area under the curve of function $G_{\mathbf{x},\mathbf{w}}$ on $[0,1]$ is

$$\int_0^1 G_{\mathbf{x},\mathbf{w}}(z)\mathrm{d}z = \sum_{i=1}^{n} w_{\pi(i)} x_{\pi(i)} = \sum_{i=1}^{n} w_i x_i,$$

where π is a permutation on $\{0, 1, 2, \ldots, n+1\}$ satisfying Equation (8). As the functions $G_{\mathbf{x},\mathbf{w}}$ and $G^*_{\mathbf{x},\mathbf{w}}$ are identical almost everywhere on $[0, 1]$, the areas under their curves are equal. That is, we find that

$$\int_0^1 G_{\mathbf{x},\mathbf{w}}(z)\mathrm{d}z = \int_0^1 G^*_{\mathbf{x},\mathbf{w}}(z)\mathrm{d}z = \sum_{i=1}^{n} w_i x_i.$$

□

4.2. The Fuzziness Degree of the Decumulative Distribution Function Viewed as a Degree of Dissension

Figure 2 shows example plots of two terminal cases of the DDF $G_{\mathbf{x},\mathbf{w}}(z)$ for the input vector $\mathbf{x}$ and weight vector $\mathbf{w}$.

Figure 2a corresponds to the case where $x_{\pi(i)} = a$, $a \in [0, 1]$, $i = 1, 2, \ldots, n$; i.e., all the individual input values are identical. Recall that π is a permutation on $\{0, 1, 2, \ldots, n+1\}$ satisfying Equation (8). In this case, the group consensus is maximal regardless of the weights of the inputs, and so the value of the weighted group consensus measure should be 1. At the same time, the corresponding DDF with respect to the weights is sharp everywhere on $[0, 1]$ (i.e., either $G_{\mathbf{x},\mathbf{w}}(z) = 0$ or $G_{\mathbf{x},\mathbf{w}}(z) = 1$). That is, the fuzziness measure of this function is zero.

In Figure 2b, there are k inputs ($k \geq 1$) with zero value, and the sum of the weights corresponding to these inputs is $\frac{1}{2}$. Additionally, there are $n - k$ inputs with a value of one, and the sum of the weights for these inputs is also $\frac{1}{2}$. This means that with respect to the weights, the dissension in the group is maximal, and hence, the value of the weighted group consensus measure in this case is expected to be 0. Notice that the corresponding DDF has a value of $\frac{1}{2}$ almost everywhere on $[0, 1]$, and so its fuzziness measure is one.

Based on the above findings, we may conclude the following:

- When the consensus among the decision-makers is maximal, and hence, the value of the weighted group consensus measure is expected to be 1, the fuzziness measure value of the corresponding DDF with respect to the weights is 0.
- When the dissension (which takes into account the input weights) among the group members is maximal, and hence, the value of the group consensus measure should be 0, the fuzziness measure value of the corresponding DDF with respect to the weights is 1.

Based on the above line of thought, we can conclude that the fuzziness measure of the DDF of the input scores taking into account the corresponding weights may be regarded as a characterizing degree of the weighted dissension. Hence, subtracting the fuzziness measure value of the DDF from one results in a value that can be viewed as the degree of a weighted consensus measure. Considering sharpness as the opposite of fuzziness, we can interpret the sharpness of the DDF of an input vector with regard to the input weights as a weighted consensus metric for that input vector. Building on this idea, we can formulate the following theorem.

Theorem 5. *Let $n \in \mathbb{N}$, $n \geq 2$, let $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in [0, 1]^n$ be an input vector, and let $\mathbf{w} = (w_1, w_2, \ldots, w_n) \in [0, 1]^n$ be a weight vector such that $w_i > 0$ for $i = 1, 2, \ldots, n$ and $\sum_{i=1}^{n} w_i = 1$. Suppose that $G_{\mathbf{x},\mathbf{w}} \colon [0, 1] \to [0, 1]$ is the DDF of the input vector $\mathbf{x}$ with regard to the weight vector $\mathbf{w}$ given in Definition 6. Furthermore, let $F \colon [0, 1] \to [0, 1]$ be a mapping that satisfies the criteria for fuzzy entropy given in Definition 2. Then, the function $C_{F,\mathbf{w}} \colon [0, 1]^n \to [0, 1]$, which is given by*

$$C_{F,\mathbf{w}}(\mathbf{x}) = 1 - \int_0^1 F(G_{\mathbf{x},\mathbf{w}}(z))\mathrm{d}z, \tag{13}$$

satisfies requirements (cw1)–(cw6) for the weighted consensus measure stated in Definition 5.

Proof. Here, we demonstrate that $C_{F,\mathbf{w}}(\mathbf{x})$ can be calculated as

$$C_{F,\mathbf{w}}(\mathbf{x}) = 1 - \sum_{i=1}^{n} \Big(x_{\pi(i)} - x_{\pi(i-1)} \Big) F\left(\sum_{j=i}^{n} w_{\pi(j)} \right), \tag{14}$$

where $\pi\colon \{0,1,2,\ldots,n+1\} \to \{0,1,2,\ldots,n+1\}$ is a permutation satisfying Equation (8). With definitions of $C_{F,\mathbf{w}}(\mathbf{x})$ and $G_{\mathbf{x},\mathbf{w}}$, and the property (e) of $G_{\mathbf{x},\mathbf{w}}$ in Theorem 4, we obtain

$$\begin{aligned} C_{F,\mathbf{w}}(\mathbf{x}) &= 1 - \int_0^1 F(G_{\mathbf{x},\mathbf{w}}(z))\mathrm{d}z = 1 - \sum_{i=1}^{n} \left(\int_{x_{\pi(i-1)}}^{x_{\pi(i)}} F(G_{\mathbf{x},\mathbf{w}}(z))\mathrm{d}z \right) \\ &= 1 - \sum_{i=1}^{n} \Big(x_{\pi(i)} - x_{\pi(i-1)} \Big) F\left(\sum_{j=i}^{n} w_{\pi(j)} \right). \end{aligned} \tag{15}$$

Proof of (cw1) (Unanimity). Let $a \in [0,1]$ and let $\mathbf{a} = (a,a,\ldots,a) \in [0,1]^n$. Then, based on Definition 6, we see that

(i) If $a = 0$, then for any $z \in (0,1)$, $G_{\mathbf{a},\mathbf{w}}(z) = 0$.
(ii) If $a = 1$, then for any $z \in (0,1)$, $G_{\mathbf{a},\mathbf{w}}(z) = 1$.
(iii) If $a \in (0,1)$, then

$$G_{\mathbf{a},\mathbf{w}}(z) = \begin{cases} 1, \text{if } z \in (0,a) \\ 0, \text{if } z \in [a,1). \end{cases}$$

This means that $G_{\mathbf{a},\mathbf{w}}(z)$ is either 0 or 1 for any $z \in (0,1)$, and so $F(G_{\mathbf{a},\mathbf{w}}(z)) = 0$ almost everywhere on $[0,1]$. Therefore, we see that

$$C_{F,\mathbf{w}}(\mathbf{a}) = 1 - \int_0^1 F(G_{\mathbf{a},\mathbf{w}}(z))\mathrm{d}z = 1.$$

Proof of (cw2) (Symmetry). Let $\rho\colon \{1,2,\ldots,n\} \to \{1,2,\ldots,n\}$ be a permutation, and let $\mathbf{x}' = (x_{\rho(1)}, x_{\rho(2)}, \cdots, x_{\rho(n)})$ and $\mathbf{w}' = (w_{\rho(1)}, w_{\rho(2)}, \cdots, w_{\rho(n)})$. Noting the definition of $G_{\mathbf{x},\mathbf{w}}$ in Definition 6, we immediately see that $G_{\mathbf{x},\mathbf{w}}(z) = G_{\mathbf{x}',\mathbf{w}'}(z)$ holds for any $z \in [0,1]$. Therefore, based on Equation (13), we find that

$$C_{F,\mathbf{w}}(x_1,x_2,\ldots,x_n) = C_{F,\mathbf{w}'}\Big(x_{\rho(1)}, x_{\rho(2)}, \ldots, x_{\rho(n)} \Big).$$

Proof of (cw3) (Maximum dissension). Here, we wish to show that if

(a) for $\{i_1,i_2,\ldots,i_k\} \subset \{1,2,\ldots,n\}$, $\sum_{j=1}^{k} w_{i_j} = \frac{1}{2}$ and
(b) for all $j \in \{i_1,i_2,\ldots,i_k\}$, $x_j = 0$ and
(c) for all $j \in \{1,2,\ldots,n\} \setminus \{i_1,i_2,\ldots,i_k\}$, $x_j = 1$,

then $C_{F,\mathbf{w}}(x_1,x_2,\ldots,x_n) = 0$. Since $C_{F,\mathbf{w}}$ satisfies the symmetry criterion (cw2), it is sufficient to show that $C_{F,\mathbf{w}'}(\mathbf{x}') = 0$, where $\mathbf{x}'$ and $\mathbf{w}'$ are obtained by applying the same permutation to the elements of $\mathbf{x}$ and $\mathbf{w}$, respectively, such that $(i_1,i_2,\ldots,i_k) = (1,2,\ldots,k)$. That is,

$$\mathbf{x}' = (\underbrace{0,0,\ldots,0}_{k\text{-times}}, \underbrace{1,1,\ldots,1}_{n-k\text{-times}}),$$

$1 \le k < n$, $\sum_{i=1}^{k} w'_i = \frac{1}{2}$ and w'_i is the ith element of $\mathbf{w}'$. Following the construction of $G_{\mathbf{x}',\mathbf{w}'}$, we have

$$x'_{\pi(0)} = x'_{\pi(1)} = \cdots = x'_{\pi(k)} = 0 \text{ and } x'_{\pi(k+1)} = x'_{\pi(k+2)} = \cdots = x'_{\pi(n+1)} = 1,$$

where π is the identity permutation on $\{0, 1, \ldots, n+1\}$. Hence, using Equation (14) and the fact that $\sum_{i=1}^{k} w'_{\pi(i)} = \sum_{i=k+1}^{n} w'_{\pi(i)} = \frac{1}{2}$ and $F\left(\frac{1}{2}\right) = 1$, we find that

$$\begin{aligned} C_{F,\mathbf{w}'}(\mathbf{x}') &= 1 - \sum_{i=1}^{n} \left(x'_{\pi(i)} - x'_{\pi(i-1)}\right) F\left(\sum_{j=i}^{n} w'_{\pi(j)}\right) \\ &= 1 - \sum_{i=1}^{k} \left(x'_{\pi(i)} - x'_{\pi(i-1)}\right) F\left(\sum_{j=i}^{n} w'_{\pi(j)}\right) \\ &\quad - \sum_{i=k+1}^{n} \left(x'_{\pi(i)} - x'_{\pi(i-1)}\right) F\left(\sum_{j=i}^{n} w'_{\pi(j)}\right) \\ &= 1 - \left(x'_{\pi(k+1)} - x'_{\pi(k)}\right) F\left(\sum_{j=k+1}^{n} w'_{\pi(j)}\right) = 1 - F\left(\frac{1}{2}\right) = 0. \end{aligned}$$

Proof of (cw4) (Reciprocity). Let $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in [0,1]^n$ be an arbitrary input vector. We need to prove that

$$C_{F,\mathbf{w}}(x_1, x_2, \ldots, x_n) = C_{F,\mathbf{w}}(N_s(x_1), N_s(x_2), \ldots, N_s(x_n)) \tag{16}$$

holds, where $N_s(a) = 1 - a$ is the standard fuzzy negation and $a \in [0,1]$. Let $\overline{\mathbf{x}}$ denote the vector $(N_s(x_1), N_s(x_2), \ldots, N_s(x_n)) \in [0,1]^n$. Taking into account Equation (13) and the property (d) of the DDF $G_{\mathbf{x},\mathbf{w}}$ given in Theorem 4, we can write

$$\begin{aligned} &C_{F,\mathbf{w}}(N_s(x_1), N_s(x_2), \ldots, N_s(x_n)) = C_{F,\mathbf{w}}(\overline{\mathbf{x}}) \\ &= 1 - \int_0^1 F(G_{\overline{\mathbf{x}},\mathbf{w}}(z))\mathrm{d}z = 1 - \int_0^1 F\big(1 - G^*_{\mathbf{x},\mathbf{w}}(N_s(z))\big)\mathrm{d}z. \end{aligned} \tag{17}$$

As F is a fuzzy entropy in the sense of Definition 2, we have that for any $x \in [0,1]$, $F(x) = F(1-x)$. Noting this property of F and the fact that $N_s(z) = 1 - z$, we can continue Equation (17) as follows:

$$\begin{aligned} 1 - \int_0^1 F\big(1 - G^*_{\mathbf{x},\mathbf{w}}(N_s(z))\big)\mathrm{d}z &= 1 - \int_0^1 F\big(G^*_{\mathbf{x},\mathbf{w}}(N_s(z))\big)\mathrm{d}z \\ &= 1 - \int_0^1 F\big(G^*_{\mathbf{x},\mathbf{w}}(1-z)\big)\mathrm{d}z. \end{aligned} \tag{18}$$

Now, with the substitution $u = 1 - z$, we find that

$$\begin{aligned} 1 - \int_0^1 F\big(G^*_{\mathbf{x},\mathbf{w}}(1-z)\big)\mathrm{d}z &= 1 - \left(-\int_1^0 F\big(G^*_{\mathbf{x},\mathbf{w}}(u)\big)\mathrm{d}u\right) \\ &= 1 - \int_0^1 F\big(G^*_{\mathbf{x},\mathbf{w}}(u)\big)\mathrm{d}u. \end{aligned} \tag{19}$$

According to the definitions of $G_{\mathbf{x},\mathbf{w}}$ and $G^*_{\mathbf{x},\mathbf{w}}$, these two functions are identical almost everywhere on $[0,1]$. Therefore, given the continuity of F, we conclude that

$$1 - \int_0^1 F\big(G^*_{\mathbf{x},\mathbf{w}}(u)\big)\mathrm{d}u = 1 - \int_0^1 F(G_{\mathbf{x},\mathbf{w}}(u))\mathrm{d}u = C_{F,\mathbf{w}}(\mathbf{x}),$$

which, based on Equations (17)–(19), means that Equation (16) holds.

Proof of (cw5) (Replication invariance). Let $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in [0,1]^n$ be an input vector, let $k \in \mathbb{N}$, $k \geq 2$, and let

$$\mathbf{x}^{(k)} = (\underbrace{\mathbf{x}, \mathbf{x}, \ldots, \mathbf{x}}_{k\text{-times}}) \quad \text{and} \quad \mathbf{w}^{(k)} = (\underbrace{\mathbf{w}^*, \mathbf{w}^*, \ldots, \mathbf{w}^*}_{k\text{-times}}),$$

where

$$\mathbf{w}^* = \left(\frac{w_1}{k}, \frac{w_2}{k}, \ldots, \frac{w_n}{k}\right).$$

Based on Definition 6, the DDF of $\mathbf{x}$ with regard to the weight vector $\mathbf{w}$ and the DDF of $\mathbf{x}^{(k)}$ with regard to the weight vector $\mathbf{w}^{(k)}$ coincide; i.e., for any $z \in [0,1]$,

$$G_{\mathbf{x},\mathbf{w}}(z) = G_{\mathbf{w}^{(k)},\mathbf{x}^{(k)}}(z).$$

Therefore, with the definition of $C_{F,\mathbf{w}}$, we immediately see that

$$C_{F,\mathbf{w}}(\mathbf{x}) = C_{F,\mathbf{w}^{(k)}}(\mathbf{x}^{(k)}).$$

Proof of (cw6) (Monotonicity with respect to the majority). Let $a \in [0,1]$, and let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ be two input vectors, where $\mathbf{x}, \mathbf{y} \in [0,1]^n$. Furthermore, let

(a) for $\{i_1, i_2, \ldots, i_k\} \subset \{1, 2, \ldots, n\}$, $\sum_{j=1}^{k} w_{i_j} \geq \frac{1}{2}$ and
(b) for all $j \in \{i_1, i_2, \ldots, i_k\}$, $x_j = y_j = a$ and
(c) for all $j \in \{1, 2, \ldots, n\} \setminus \{i_1, i_2, \ldots, i_k\}$, $|a - x_j| \leq |a - y_j|$.

We need to show that, under these conditions,

$$C_{F,\mathbf{w}}(x_1, x_2, \ldots, x_n) \geq C_{F,\mathbf{w}}(y_1, y_2, \ldots, y_n) \tag{20}$$

holds.

Suppose that $\mathcal{F}$ is the set of all measurable, real-valued functions $\mu\colon [0,1] \to [0,1]$. Let the function $d_F\colon \mathcal{F} \to [0,1]$ be given by

$$d_F(\mu) = \int_0^1 F(\mu(x))\mathrm{d}x,$$

where $\mu \in \mathcal{F}$. Notice that $d_F(\mu)$ can be written as

$$d_F(\mu) = \frac{1}{m(X)} \int_X F(\mu(x))\mathrm{d}m(x),$$

where $X = [0,1]$, and m is the Lebesgue measure for real-valued intervals (i.e., for $I = [s,t]$, $m(I) = t - s$). Hence, with Theorem 1, we see that d_F is a fuzziness measure on $\mathcal{F}$; i.e., it satisfies criteria P_1–P_4 given in Definition 1.

Now, let $\mathbf{x}$ and $\mathbf{y}$ be two input vectors, and let $G_{\mathbf{x},\mathbf{w}}, G_{\mathbf{y},\mathbf{w}}\colon [0,1] \to [0,1]$ be the DDFs for $\mathbf{x}$ and $\mathbf{y}$, respectively, with regard to the weight vector $\mathbf{w}$. Since $G_{\mathbf{x},\mathbf{w}}, G_{\mathbf{y},\mathbf{w}} \in \mathcal{F}$, we have that

$$d_F(G_{\mathbf{x},\mathbf{w}}) = \int_0^1 F(G_{\mathbf{x},\mathbf{w}}(z))\mathrm{d}z \quad \text{and} \quad d_F\big(G_{\mathbf{y},\mathbf{w}}\big) = \int_0^1 F\big(G_{\mathbf{y},\mathbf{w}}(z)\big)\mathrm{d}z$$

are the fuzziness measures of $G_{\mathbf{x},\mathbf{w}}$ and $G_{\mathbf{y},\mathbf{w}}$ with regard to the fuzzy entropy F. Taking into account the construction of $G_{\mathbf{x},\mathbf{w}}$ and $G_{\mathbf{y},\mathbf{w}}$ and the conditions (a), (b), and (c), we see that, for any $z \in [0,1]$,

$$G_{\mathbf{x},\mathbf{w}}(z) \geq G_{\mathbf{y},\mathbf{w}}(z) \quad \text{if} \quad G_{\mathbf{y},\mathbf{w}}(z) > \frac{1}{2}$$

and

$$G_{\mathbf{x},\mathbf{w}}(z) \leq G_{\mathbf{y},\mathbf{w}}(z) \quad \text{if} \quad G_{\mathbf{y},\mathbf{w}}(z) < \frac{1}{2}.$$

Hence, noting property P_3 in Definition 1, we also have

$$d_F(G_{\mathbf{x},\mathbf{w}}) \leq d_F(G_{\mathbf{y},\mathbf{w}}). \tag{21}$$

Next, using the inequality in Equation (21) and the definition of $C_{F,\mathbf{w}}$, we find that

$$C_{F,\mathbf{w}}(\mathbf{x}) = 1 - d_F(G_{\mathbf{x},\mathbf{w}}) \geq 1 - d_F(G_{\mathbf{y},\mathbf{w}}) = C_{F,\mathbf{w}}(\mathbf{y}).$$

□

Figure 3 demonstrates the (cw6) property of a weighted consensus measure $C_{F,\mathbf{w}}$.

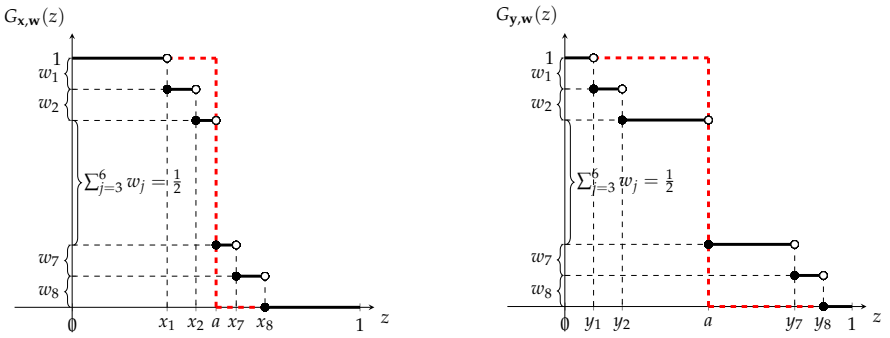


Figure 3. An illustration of the (cw6) property of $C_{F,\mathbf{w}}$. Now, $0 \leq x_1 < x_2 < x_3 = x_4 = x_5 = x_6 = a < x_7 < x_8 \leq 1$, $0 \leq y_1 < y_2 < y_3 = y_4 = y_5 = y_6 = a < y_7 < y_8 \leq 1$, $\sum_{j=3}^{6} w_j = \frac{1}{2}$ and $|a - x_j| \leq |a - y_j|$ for $j \in \{1,2,7,8\}$ imply that $G_{\mathbf{x},\mathbf{w}}(z)$ is sharper than $G_{\mathbf{y},\mathbf{w}}(z)$, and so $C_{F,\mathbf{w}}(x_1, x_2, \ldots, x_8) \geq C_{F,\mathbf{w}}(y_1, y_2, \ldots, y_8)$. The dashed red graph is a plot of the characteristic function $\chi_{[0,a]}(x) = 1$ for any $x \in [0,a]$, and $\chi_{[0,a]}(x) = 0$ for any $x \in (a,1]$.

Based on the above findings, we will call the function $C_{F,\mathbf{w}}$ the fuzziness measure-based weighted consensus measure induced by the fuzzy entropy F. Here, the fuzzy entropy F may be regarded as a generator function of the consensus measure $C_{F,\mathbf{w}}$.

Remark 4. *We should add that in a decision making procedure, the role of the DDF $G_{\mathbf{x},\mathbf{w}}$ ($G^*_{\mathbf{x},\mathbf{w}}$, respectively) is two-fold. First, based on Theorem 4, property (f), the area under the curve of this function is none other than the weighted arithmetic mean of the inputs; i.e.,*

$$\int_0^1 G_{\mathbf{x},\mathbf{w}}(z)\mathrm{d}z = \int_0^1 G^*_{\mathbf{x},\mathbf{w}}(z)\mathrm{d}z = \sum_{i=1}^{n} w_i x_i.$$

Second, Theorem 5 tells us that subtracting the area under the curve of the composition of a fuzzy entropy F and $G_{\mathbf{x},\mathbf{w}}$ ($G^*_{\mathbf{x},\mathbf{w}}$, respectively) from one; i.e.,

$$C_{F,\mathbf{w}}(\mathbf{x}) = 1 - \int_0^1 F(G_{\mathbf{x},\mathbf{w}}(z))\mathrm{d}z,$$

is a weighted group consensus measure in the sense of Definition 5.

Remark 5. *If all the weights are identical, i.e., $w_{\pi(i)} = \frac{1}{n}$ for all $i = 1, 2, \ldots, n$, then the weighted consensus measure induced by the fuzzy entropy F given in Equation (14) is*

$$\begin{aligned} C_{F,\mathbf{w}}(\mathbf{x}) &= 1 - \sum_{i=1}^{n} \left(x_{\pi(i)} - x_{\pi(i-1)} \right) F\left(\sum_{j=i}^{n} w_{\pi(j)} \right) \\ &= 1 - \sum_{i=1}^{n} \left(x_{\pi(i)} - x_{\pi(i-1)} \right) F\left(\frac{n-i+1}{n} \right), \end{aligned}$$

which, noting Equation (7), coincides with the unweighted consensus measure induced by F. That is, the unweighted consensus measure induced by the generator F is a special instance of the weighted consensus measure induced by the same generator.

4.3. Weighted Consensus Measures Induced by Fuzzy Entropies Derived from Quasi-Arithmetic Means

It is well known (see, e.g., Section 5.1 in [39]) that a function $T\colon [0,1]^2 \to [0,1]$ is a strict triangular norm (t-norm for short) if and only if T is continuous, and there exists a continuous and strictly decreasing function $g\colon [0,1] \to [0,\infty]$ with the properties $g(0) = \infty$ and $g(1) = 0$ such that for any $x, y \in [0,1]$,

$$T(x,y) = g^{-1}(g(x) + g(y)).$$

The function g is called an additive generator of the strict t-norm T, which is uniquely determined up to a positive constant multiplier of g. Using Theorem 2 in [35], we see that the function $F_g\colon [0,1] \to [0,1]$, which is given by

$$F_g(x) = 2g^{-1}\left(\frac{1}{2}(g(x) + g(1-x)) \right), \tag{22}$$

satisfies the requirements for a fuzzy entropy function given in Definition 2.

Remark 6. *At this point, it should be mentioned that there is a connection between fuzzy entropies generated by g as in Equation (22) and the quasi-arithmetic mean induced by generator g. Namely,*

$$M_g(x, 1-x) = g^{-1}\left(\frac{1}{2}(g(x) + g(1-x)) \right) \tag{23}$$

is the quasi-arithmetic mean (see Definition 27 in [44]) of x and $1-x$ ($x \in [0,1]$) induced by the generator function g. Noting Equation (22) and Equation (23), we find that

$$F_g(x) = 2M_g(x, 1-x).$$

That is, we can use the quasi-arithmetic mean M_g to generate the fuzzy entropy F_g, while F_g can be used to generate a weighted consensus measure. For other practical applications of quasi-arithmetic means see, e.g., [45–47].

Based on Equation (22) and Equation (14), the weighted consensus measure induced by the fuzzy entropy F_g is

$$\begin{aligned} C_{F_g,\mathbf{w}}(\mathbf{x}) &= 1 - \sum_{i=1}^{n} \left(x_{\pi(i)} - x_{\pi(i-1)}\right) F_g\left(\sum_{j=i}^{n} w_{\pi(j)}\right) \\ &= 1 - 2\sum_{i=1}^{n} \left(x_{\pi(i)} - x_{\pi(i-1)}\right) g^{-1}\left(\frac{1}{2}\left(g\left(\sum_{j=i}^{n} w_{\pi(j)}\right)\right.\right. \\ &\quad \left.\left. + g\left(1 - \sum_{j=i}^{n} w_{\pi(j)}\right)\right)\right), \end{aligned} \tag{24}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in [0,1]^n$ is an input vector, $\mathbf{w} = (w_1, w_2, \ldots, w_n) \in [0,1]^n$ is a weight vector such that $w_i > 0$ for $i = 1, 2, \ldots, n$ and $\sum_{i=1}^{n} w_i = 1$, and π is a permutation on $\{0, 1, 2, \ldots, n+1\}$ satisfying Equation (8).

The following theorem provides a necessary and sufficient condition for the identity of two weighted consensus measures induced by fuzzy entropies using Equation (24).

Theorem 6. *Let $n \in \mathbb{N}$, $n \geq 2$, let $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in [0,1]^n$ be an input vector, and let $\mathbf{w} = (w_1, w_2, \ldots, w_n) \in [0,1]^n$ be a weight vector such that $w_i > 0$ for $i = 1, 2, \ldots, n$ and $\sum_{i=1}^{n} w_i = 1$. Suppose that $g_1, g_2 \colon [0,1] \to [0,\infty]$ are additive generators of strict t-norms, and $F_{g_1}, F_{g_2} \colon [0,1] \to [0,1]$ are the fuzzy entropies induced by g_1 and g_2, respectively, according to Equation (22), i.e.,*

$$F_{g_1}(x) = 2g_1^{-1}\left(\frac{1}{2}(g_1(x) + g_1(1-x))\right)$$

and

$$F_{g_2}(x) = 2g_2^{-1}\left(\frac{1}{2}(g_2(x) + g_2(1-x))\right).$$

Furthermore, let $C_{F_{g_1},\mathbf{w}}, C_{F_{g_2},\mathbf{w}} \colon [0,1]^n \to [0,1]$ be the weighted consensus measures with respect to the weight vector $\mathbf{w}$ induced by the fuzzy entropies F_{g_1} and F_{g_2}, respectively, according to Equation (24). That is,

$$C_{F_{g_1},\mathbf{w}}(\mathbf{x}) = 1 - \sum_{i=1}^{n} \left(x_{\pi(i)} - x_{\pi(i-1)}\right) F_{g_1}\left(\sum_{j=i}^{n} w_{\pi(j)}\right) \tag{25}$$

and

$$C_{F_{g_2},\mathbf{w}}(\mathbf{x}) = 1 - \sum_{i=1}^{n} \left(x_{\pi(i)} - x_{\pi(i-1)}\right) F_{g_2}\left(\sum_{j=i}^{n} w_{\pi(j)}\right), \tag{26}$$

where π is a permutation on $\{0, 1, 2, \ldots, n+1\}$ satisfying Equation (8).

Then,

$$C_{F_{g_1},\mathbf{w}}(\mathbf{x}) = C_{F_{g_2},\mathbf{w}}(\mathbf{x}) \tag{27}$$

holds if and only if

$$g_1(x) = a\, g_2(x) \tag{28}$$

for some $a > 0$ constant ($x \in [0,1]$).

Proof. Using the definitions of $C_{F_{g_1},\mathbf{w}}(\mathbf{x})$ and $C_{F_{g_2},\mathbf{w}}(\mathbf{x})$ given in Equations (25) and (26), it is sufficient to show that

$$F_{g_1}(x) = F_{g_2}(x)$$

holds if and only if

$$g_1(x) = a\, g_2(x).$$

Since

$$M_{g_1}(x, 1-x) = \frac{1}{2} F_{g_1}(x) \quad \text{and} \quad M_{g_2}(x, 1-x) = \frac{1}{2} F_{g_2}(x)$$

are the quasi-arithmetic means of x and $1-x$ induced by the generators g_1 and g_2, respectively. Based on [48] (see also p. 226 in [49] and Proposition 7 in [44]), we obtain that

$$F_{g_1}(x) = F_{g_2}(x)$$

holds if and only if

$$g_1(x) = a\, g_2(x) + b$$

for some real a, b constants, $a \neq 0$ ($x \in [0,1]$). Taking into account the fact that g_1 and g_2 are both continuous and strictly decreasing mappings from $[0,1]$ to $[0,\infty]$ with the properties $g_1(0) = g_2(0) = \infty$ and $g_1(1) = g_2(1) = 0$, we find that b necessarily has to have the value of zero and $a > 0$. That is, Equation (27) holds if and only if Equation (28) holds for some constant $a > 0$. □

Theorem 6 tells us that the weighted consensus measure induced by the fuzzy entropy F_g using Equation (24) is uniquely determined up to a positive constant multiplier of the generator function g.

It is well known that the additive generator $g_p\colon [0,1] \to [0,\infty]$, which is given by $g_p(x) = -\ln(x)$, induces the product t-norm. Using Equation (22) with the conventions $\ln(0) = -\infty$ and $\mathrm{e}^{-\infty} = 0$, the fuzzy entropy induced by g_p is

$$F_{g_p}(x) = 2\sqrt{x(1-x)}, \quad x \in [0,1]. \tag{29}$$

Hence, noting Equation (24), the weighted consensus measure induced by the fuzzy entropy F_{g_p} is

$$\begin{aligned} C_{F_{g_p},\mathbf{w}}(\mathbf{x}) &= 1 - \sum_{i=1}^{n} \Big(x_{\pi(i)} - x_{\pi(i-1)}\Big) F_{g_p}\left(\sum_{j=i}^{n} w_{\pi(j)}\right) \\ &= 1 - 2\sum_{i=1}^{n} \Big(x_{\pi(i)} - x_{\pi(i-1)}\Big) \sqrt{\left(\sum_{j=i}^{n} w_{\pi(j)}\right)\left(1 - \sum_{j=i}^{n} w_{\pi(j)}\right)}. \end{aligned} \tag{30}$$

Similarly, with the conventions $\frac{1}{0} = \infty$ and $\frac{1}{\infty} = 0$, the fuzzy entropy induced by the additive generator $g_D\colon [0,1] \to [0,\infty]$, which is given by $g_D(x) = \frac{1-x}{x}$, is

$$F_{g_D}(x) = 4x(1-x), \quad x \in [0,1]. \tag{31}$$

Using Equation (24), the weighted consensus measure induced by the fuzzy entropy F_{g_D} is

$$\begin{aligned} C_{F_{g_D},\mathbf{w}}(\mathbf{x}) &= 1 - \sum_{i=1}^{n} \Big(x_{\pi(i)} - x_{\pi(i-1)}\Big) F_{g_p}\left(\sum_{j=i}^{n} w_{\pi(j)}\right) \\ &= 1 - 4\sum_{i=1}^{n} \Big(x_{\pi(i)} - x_{\pi(i-1)}\Big) \left(\sum_{j=i}^{n} w_{\pi(j)}\right)\left(1 - \sum_{j=i}^{n} w_{\pi(j)}\right). \end{aligned} \tag{32}$$

Note that the function $g_D(x) = \frac{1-x}{x}$ is an additive generator of the Dombi t-norms [36], which is a class of strict t-norms.

5. Numerical Examples

In this section, we demonstrate the properties of the weighted consensus measures $C_{F_{g_D},\mathbf{w}}$ and $C_{F_{g_p},\mathbf{w}}$ given by Equations (30) and (32), respectively. Recall that these two measures are induced by the fuzzy entropies F_{g_p} and F_{g_D} given in Equations (29) and (31), respectively. We utilize nine different input vectors (the components of which are denoted by x_i) and the weight vector $\mathbf{w} = (0.5, 0.167, 0.167, 0.167)$ (the components of which are denoted by w_i), as shown in Table 2.

Table 2. A numerical example.

	i:	1	2	3	4	$C_{F_{g_D},\mathbf{w}}$	$C_{F_{g_p},\mathbf{w}}$	Property
Case	w_i :	0.5	0.167	0.167	0.167			
1	x_i:	0.9	0.3	0.2	0.2	0.311	0.306	$cw4$
2	x_i:	0.1	0.7	0.8	0.8	0.311	0.306	$cw4$
3	x_i:	1	0	0	0	0.000	0.000	$cw3, cw4$
4	x_i:	0	1	1	1	0.000	0.000	$cw3, cw4$
5	x_i:	0.7	0.7	0.7	0.7	1.000	1.000	$cw1$
6	x_i:	0.9	0.6	0.2	0.2	0.344	0.323	$cw6$
7	x_i:	0.9	0.6	0.3	0.3	0.433	0.417	$cw6$
8	x_i:	0.9	0.3	0.3	0.3	0.400	0.400	$cw6$
9	x_i:	0.5	0.3	0.3	0.3	0.800	0.800	$cw6$

From the data in Table 2, we can see the following properties of the weighted consensus measures $C_{F_{g_D},\mathbf{w}}$ and $C_{F_{g_p},\mathbf{w}}$:

- From the comparison of case 1 and 2, as well as that of case 3 and 4, we see that $C_{F_{g_D},\mathbf{w}}$ and $C_{F_{g_p},\mathbf{w}}$ both have the Reciprocity ($cw4$) property.
- Cases 3 and 4 highlight the Maximum dissension ($cw3$) property of these measures. Namely, if the most extreme evaluations (0 and 1) are distributed such that not necessarily half of the decision-makers use zeros as inputs and the other half of them use ones as inputs, but all the inputs corresponding to the half of the weight totals are zeros and all the inputs corresponding to the other half of the weight totals are ones, then the degree of consensus is minimal; i.e., it is equal to zero.
- Noting case 5, we see that if all the decision-makers have the same opinion, i.e., there is unanimous agreement in the group, the consensus is maximal, independent of the weight values assigned to the decision-makers. This means that $C_{F_{g_D},\mathbf{w}}$ and $C_{F_{g_p},\mathbf{w}}$ both satisfy the Unanimity ($cw1$) requirement.
- Comparing case 6 with case 1, we note the Monotonicity with respect to the majority ($cw6$) property of $C_{F_{g_D},\mathbf{w}}$ and $C_{F_{g_p},\mathbf{w}}$. Here, decision-maker 1 is in the weight majority ($w_1 = 0.5$) with the input $x_1 = 0.9 = a$. In case 6, the evaluation of decision-maker 2 is closer to 0.9 than in case 1 ($|0.9 - 0.6| \leq |0.9 - 0.3|$), while all the other inputs remain unchanged. That is, the opinion of the weight minority becomes more 'similar' to that of the weight majority, and it is reflected in the higher values of the $C_{F_{g_D},\mathbf{w}}$ and $C_{F_{g_p},\mathbf{w}}$ measures ($C_{F_{g_D},\mathbf{w}}$ increases from 0.311 to 0.344 and $C_{F_{g_p},\mathbf{w}}$ increases from 0.306 to 0.323).

  Similarly, based on cases 6 and 7, we find that as the inputs of decision-makers 3 and 4 approach the input of the weight majority ($|0.9 - 0.3| \leq |0.9 - 0.2|$ and $|0.9 - 0.3| \leq |0.9 - 0.2|$), while all the other inputs remain unchanged, the values of both weighted consensus measures increase ($C_{F_{g_D},\mathbf{w}}$ increases from 0.344 to 0.433 and $C_{F_{g_p},\mathbf{w}}$ increases from 0.323 to 0.417).

  Cases 8 and 9 provide us with another demonstration of the Monotonicity with respect to the majority property of $C_{F_{g_D},\mathbf{w}}$ and $C_{F_{g_p},\mathbf{w}}$. Here, decision-makers 2, 3,

and 4 form a weight majority ($w_2 + w_3 + w_4 = 0.5$) with a constant input value $x_2 = x_3 = x_4 = 0.3 = a$. We see that the input of decision-maker 1 is closer to $a = 0.3$ in case 9 than in case 8 ($|0.5 - 0.3| \leq |0.9 - 0.3|$), and this is reflected in the increase in the $C_{F_{g_D},\mathbf{w}}$ and $C_{F_{g_P},\mathbf{w}}$ values (both increase from 0.400 to 0.800).

Looking at Table 2 from the top to the bottom might be viewed as investigating the sensitivity of the weighted consensus with regard to the input vector, while going from left to the right may be viewed as a sensitivity analysis with regard to the fuzzy entropy used for generating the consensus measure. However, it is also interesting how sensitive the level of weighted consensus is to changes in the weight vector $\mathbf{w}$. We shall address this question in the following paragraphs below.

Let $w_1, w_2 \in [0.01, 0.49]$ be the weights of the first two decision-makers. Here, we take all the possible combinations of w_1 and w_2, i.e., $(w_1, w_2) \in [0.01, 0.49] \times [0.01, 0.49]$, and let the w_3, w_4 weights of the remaining decision-makers be equal to each other such that $\sum_{i=1}^{4} w_i = 1$ holds. That is,

$$w_3 = w_4 = (1 - (w_1 + w_2))/2. \tag{33}$$

We now consider the inputs and the fuzzy entropy of $F_{g_D} = 4x(1-x)$ to be fixed and investigate how the level of weighted consensus behaves as the weight vector $\mathbf{w}$ changes.

The colors in Figure 4 indicate, at each (w_1, w_2) point, the level of weighted consensus calculated from a fixed input vector and the weight vector $\mathbf{w} = (w_1, w_2, w_3, w_4)$, where w_3 and w_4 are calculated as shown in Equation (33).

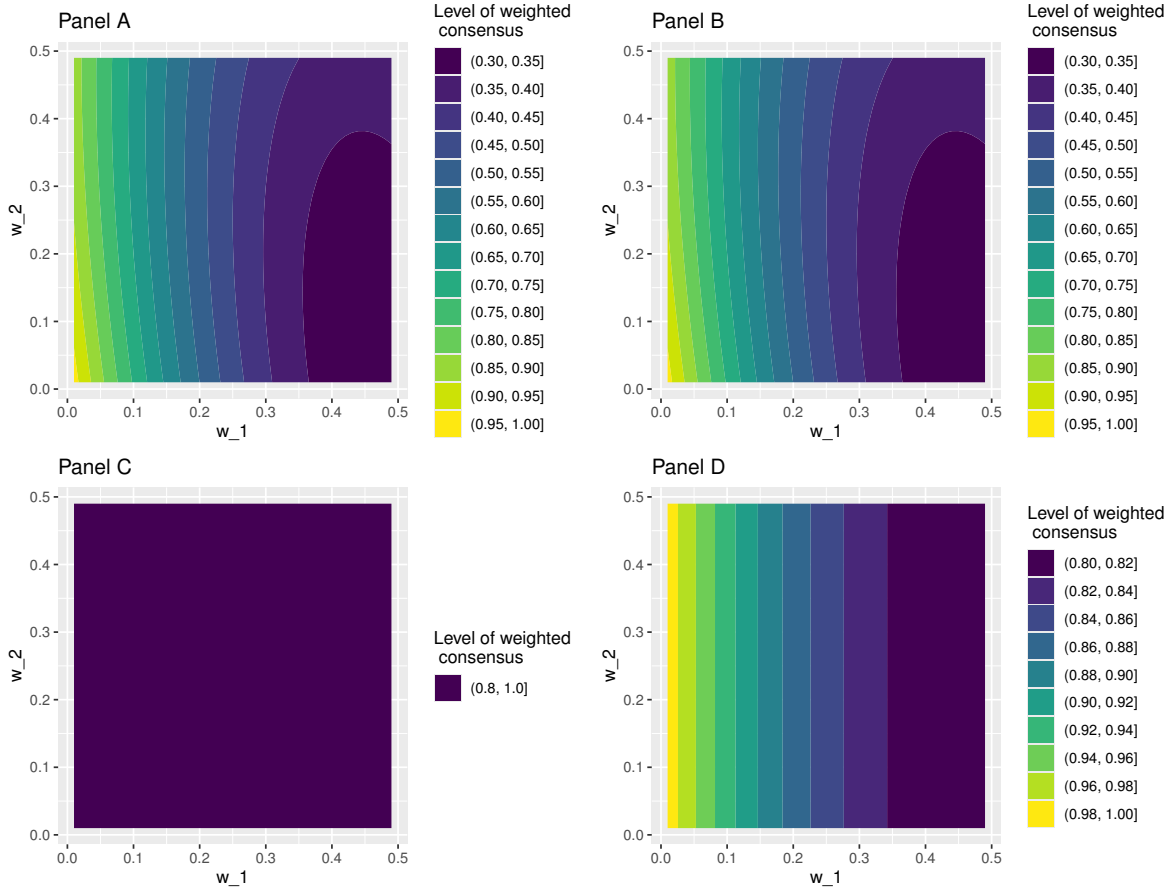


Figure 4. A sensitivity analysis with regard to the weight vector $\mathbf{w}$ with a fixed fuzzy entropy of $F_{g_D} = 4x(1-x)$ and input vectors corresponding to case 1 (Panel A), case 2 (Panel B), case 5 (Panel C), and case 9 (Panel D) of Table 2.

In Panel A of Figure 4, we see that the level of weighted consensus depends on the weight vector **w** to a large extent. Here, the third and fourth decision-makers agree with one another as $x_3 = x_4 = 0.2$, while the first and second decision-makers, especially the first one, have different opinions $(x_1 = 0.9, x_2 = 0.3)$. We also see that if the first decision-maker has a high weight value, then the level of weighted consensus is relatively low. As the opinion of this decision-maker is very different from those of the others, their weight has a big influence on the level of weighted consensus. At the same time, the weight value of the second decision-maker does not really have a high influence, which is reflected in the contour map as moving vertically and does not really alter the color and hence the level of weighted consensus.

The input vector of case 2 in Table 2 can be obtained by applying the standard fuzzy negation to the input vector used in case 1 of Table 2. Hence, we expect the reciprocity property to hold. This is also clearly reflected in Panel B of Figure 4, as this contour map is identical to that of Panel A.

In Panel C of Figure 4, we notice that in the case of unanimity (case 5 in Table 2), the level of weighted consensus is independent of the weight vector **w**.

In Panel D of Figure 4, the input vector of case 9 in Table 2 is taken into account; i.e., $\mathbf{x} = (0.5, 0.3, 0.3, 0.3)$. Here, we see that the level of weighted consensus is high as there is only a little variation in the input values, and the consensus value depends only on w_1, corresponding to the first decision-maker who does not share the opinion of the others. This can be explained as follows. When the first decision-maker's weight value increases, the level of weighted consensus decreases because the opinion of this minority group is assigned a higher weight. At the same time, the level of weighted consensus in this case is independent of w_2 because an increase in the value of this weight results in just a reallocation of the weight values among the decision-makers sharing the same opinion $(x_2 = x_3 = x_4 = 0.3)$.

6. Conclusions and Plans for Future Research

In our study, we argue that in group decision making situations where the decision-makers' inputs are taken into account with various weight values (i.e., with different relative importance), it makes sense to utilize a weighted consensus measure to characterize the degree of group consensus.

Our aim was to generalize the concept of (unweighted) fuzziness measure-based consensus measure, which was introduced in [2]. With this in mind, we adapted the requirements for a consensus measure, which were originally defined by [9], to the weighted case. Next, we introduced a family of weighted measures and proved that the members of this measure family satisfy the requirements for a weighted consensus measure. Since the proposed weighted consensus measure is induced by a fuzzy entropy, this latter one may be regarded as the generator function of the novel measure. This feature of the fuzziness measure-based weighted consensus measure makes it very flexible because users can choose from a variety of functions to generate a weighted consensus measure. We demonstrated that a class of quasi-arithmetic means can be used to generate fuzziness measure-based weighted consensus measures. This means that we have a new application area of this class of quasi-arithmetic means.

As part of our future research, we plan to examine the proposed family of weighted consensus measures from the perspective of the consensus reaching process. In particular, we intend to investigate what recommendations should be made to group members, taking into account their current inputs, to increase the value of the weighted group consensus measure. Thus, we seek to answer the question about which decision-makers should modify their opinions to improve the level of group consensus.

Another interesting topic that we plan to investigate is the robustness of the presented weighted consensus measure when different entropy functions are employed.

In practical applications, there may be situations where the weights are uncertain, disputed, or subjective. Hence, we also plan to study how the presented weighted consensus measure could be enhanced so that it can handle situations where the uncertain weights are given by Z-numbers, ZE-numbers, Trust numbers, fuzzy numbers, triangular numbers, etc.

There are many areas where the value of weighted consensus can serve as an additional input to decision making. For example, reasoning about private and expressed opinions in group decision making systems (see [50]) could be supplemented with a calculated value of weighted consensus. Hence, it is an interesting question of how our theoretical results could be applied in these areas.

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