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Ordered Kernels of OBCI-Algebras in the Homomorphism Environment

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Abstract: Yang, Roh and Jun recently introduced kernels of homomorphisms in OBCI-algebras and left an ordered generalization of those kernels as a future work. As its answer, we introduce the concept of ordered kernels of OBCI-algebras in the homomorphism environment. To be more concrete, first of all, the notion of ordered kernels of OBCI-algebras is intruduced. Next, properties of those ordered kernels related to (ordered) subalgebras, (ordered) filters and functional compositions are discussed in homomorphisms of OBCI-algebras.

Keywords: homomorphism; ordered kernel; (ordered) filter; (ordered) subalgebra; functional composition

MSC: 03B05; 03G25; 06F35

1. Introduction

Ordered algebras, algebras with an underlying partial order, are important structures in universal logic (see [1]), because lots of logic systems require at least such order. For instance, the basic substructural logic GL is characterized by residuated lattice-ordered groupoids with unit [2,3]. Note that lattice orders are partial orders with supremum and infimum. In particular, logic classes based on ordered algebras have been recently introduced. Tonoid and partial gaggle logics [4,5], weakly implicative logics [6,7], and implicational logics [7–9] are such classes.

Unlike universal logic, algebraic structures are in general introduced in a strict sense in universal algebra. For example, BCH-algebras [10,11] and B-algebras [12,13], which were introduced as generalizations of BCI-algebras, are all algebras defined by equations, i.e., algebras in the strict sense. Thus, such algebras do not require any underlying partial order. To follow up the above research trend in universal logic, Yang, Roh and Jun [14] introduced very recently an ordered generalization of BCI-algebras, called OBCI-algebras. In addressing various relations between algebras, homomorphism is a significant implementation. Interestingly, they [15] further investigated homomorphisms of OBCI-algebras. More precisely, they introduced kernels of OBCI-algebras and studied the relations of those kernels to (ordered) subalgebras and (ordered) filters in the homomorphism environment.

We note here that while Yang, Roh and Jun [14,15] introduced ordered subalgebras and ordered filters as ordered generalizations of subalgebras and filters, respectively, in the

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homomorphism environment, they did not introduce an ordered generalization of kernels in the same environment. They instead left it as a future work.

The purpose of this work is to introduce such a generalization of kernels and investigate similar relations to (ordered) subalgebras and (ordered) filters in the same environment. To accomplish this goal, first, the notion of ordered kernels of OBCI-algebras is introduced and the relations between kernels and ordered kernels are briefly dealt with. Next, the relations between ordered kernels of OBCI-algebras and (ordered) subalgebras of OBCI-algebras are considered in the homomorphism environment. Analogously, the relations between ordered kernels of OBCI-algebras and (ordered) filters of OBCI-algebras are dealt with in the same environment. Finally, composite functions of ordered kernels of OBCI-algebras are studied in the homomorphism environment.

2. Preliminaries

Definition 1 ([14]). *Suppose that S is a set with a binary relation " $\leq_\iota$ ", a binary operation " $\to$ " and a constant "ι". A structure $\boldsymbol{S} := (S, \to, \iota, \leq_\iota)$ is said to be an OBCI-algebra whenever the following conditions are satisfied in it:*

$$(\forall s, t, u \in S)(\iota \leq_\iota (s \to t) \to ((t \to u) \to (s \to u))), \tag{1}$$
$$(\forall s, t \in S)(\iota \leq_\iota s \to ((s \to t) \to t)), \tag{2}$$
$$(\forall s \in S)(\iota \leq_\iota s \to s), \tag{3}$$
$$(\forall s, t \in S)(\iota \leq_\iota s \to t, \iota \leq_\iota t \to s \ \Rightarrow \ s = t), \tag{4}$$
$$(\forall s, t \in S)(s \leq_\iota t \ \Leftrightarrow \ \iota \leq_\iota s \to t), \tag{5}$$
$$(\forall s, t \in S)(\iota \leq_\iota s, s \leq_\iota t \ \Rightarrow \ \iota \leq_\iota t). \tag{6}$$

Proposition 1 ([14]). *Every OBCI-algebra $\boldsymbol{S} := (S, \to, \iota, \leq_\iota)$ satisfies:*

$$(\forall s \in S)(\iota \to s = s). \tag{7}$$

By $\mathbf{S} := (S, \to, \iota, \leq_\iota)$, we henceforth denote the OBCI-algebra if we do not have to specify it. Note that $(S, \leq_\iota)$ in $\mathbf{S}$ forms a partial order, and so is an ordered set. Note also that every BCI-algebra is an OBCI-algebra but the converse is not necessarily true [14].

Definition 2 ([14,15]). *A non-empty subset Θ of S is said to be*

- *a subalgebra of $\boldsymbol{S} := (S, \to, \iota, \leq_\iota)$ if Θ satisfies:*

$$(\forall s, t \in S)(s, t \in \Theta \ \Rightarrow \ s \to t \in \Theta). \tag{8}$$

- *an ordered subalgebra (This definition is called "OBCI-subalgebra" in [14]. Since it can be applied to any other ordered algebras as well as OBCI-algebras, we call it "ordered subalgebra" following [15].) of $\boldsymbol{S} := (S, \to, \iota, \leq_\iota)$ if Θ satisfies:*

$$(\forall s, t \in S)(s, t \in \Theta, \iota \leq_\iota s, \iota \leq_\iota t \ \Rightarrow \ s \to t \in \Theta). \tag{9}$$

Note that subalgebras in OBCI-algebras are ordered subalgebras but the converse is not necessarily true [14].

Proposition 2 ([14]). *Let Θ be an ordered subalgebra of S satisfying:*

$$(\forall s \in S)(s \in \Theta \ \Rightarrow \ \iota \leq_\iota s). \tag{10}$$

Then, Θ is a subalgebra of S.

Definition 3 ([14,15]). *A subset* Γ *of* S *is said to be*

- *a filter of* $\mathbf{S} := (S, \to, \iota, \leq_\iota)$ *if* Γ *satisfies:*

$$\iota \in \Gamma, \tag{11}$$
$$(\forall s, t \in S)(s \to t \in \Gamma, s \in \Gamma \ \Rightarrow \ t \in \Gamma). \tag{12}$$

- *an ordered filter (Like OBCI-subalgebra, it is called "OBCI-filter" in [14]. We call this "ordered filter" following [15].) of* $\mathbf{S} := (S, \to, \iota, \leq_\iota)$ *if* Γ *satisfies* (11) *and*

$$(\forall s, t \in S)(s \in \Gamma, \iota \leq_\iota s \to t \ \Rightarrow \ t \in \Gamma). \tag{13}$$

Proposition 3 ([14]).

(i) *If a filter* Γ *of* $\mathbf{S} := (S, \to, \iota, \leq_\iota)$ *satisfies*

$$(\forall s \in S)(\iota \leq_\iota s \ \Rightarrow \ s \in \Gamma), \tag{14}$$

then Γ *is an ordered filter of* $\mathbf{S}$.

(ii) *If an ordered filter* Γ *of* $\mathbf{S} := (S, \to, \iota, \leq_\iota)$ *satisfies* (10)*, then* Γ *is a filter of* $\mathbf{S}$.

Note that filters and ordered filters of $\mathbf{S} := (S, \to, \iota, \leq_\iota)$ are independent of each other [14].

Definition 4 ([14,15]). *An (ordered) filter* Γ *of* $\mathbf{S} := (S, \to, \iota, \leq_\iota)$ *is said to be closed if* Γ *is a subalgebra of* $\mathbf{S}$*; ordered closed (O-closed for brevity) if* Γ *is an ordered subalgebra of* $\mathbf{S}$.

Definition 5 ([15]). *Let* $\mathbf{S} := (S, \to_S, \iota_S, \leq_S)$ *and* $\mathbf{T} := (T, \to_T, \iota_T, \leq_T)$ *be OBCI-algebras. A map* $\varsigma : S \to T$ *is said to be a homomorphism if* ς *satisfies:*

$$(\forall s, t \in S)(\varsigma(s \to_S t) = \varsigma(s) \to_T \varsigma(t)). \tag{15}$$

Proposition 4 ([15]). *Let* $\mathbf{S} := (S, \to_S, \iota_S, \leq_S)$ *and* $\mathbf{T} := (T, \to_T, \iota_T, \leq_T)$ *be OBCI-algebras. If* $\varsigma : S \to T$ *is a homomorphism, then*

$$\iota_T \leq_T \varsigma(\iota_S) \to_T \varsigma(\iota_S), \tag{16}$$
$$\iota_T \leq_T \varsigma(\iota_S). \tag{17}$$

Proposition 5. *Let* $\mathbf{S} := (S, \to_S, \iota_S, \leq_S)$ *and* $\mathbf{T} := (T, \to_T, \iota_T, \leq_T)$ *be OBCI-algebras, and* $\varsigma : S \to T$ *be a homomorphism. Then, the following assertion holds.*

$$(\forall s \in S)(\iota_T \leq_T \varsigma(s) \ \Leftrightarrow \ \varsigma(\iota_S) \leq_T \varsigma(s)). \tag{18}$$

Proof. Let $\iota_T \leq_T \varsigma(s)$ for $s \in S$. By (7), one has $\iota_T \leq_T \varsigma(\iota_S \to_S s)$, and so $\iota_T \leq_T \varsigma(\iota_S) \to_T \varsigma(s)$ by (15). Thus, one obtains $\varsigma(\iota_S) \leq_T \varsigma(s)$ by (5). Similarly, the other direction can be proved. □

Definition 6 ([15]). *Let* $\mathbf{S} := (S, \to_S, \iota_S, \leq_S)$ *and* $\mathbf{T} := (T, \to_T, \iota_T, \leq_T)$ *be OBCI-algebras. Given a map* $\varsigma : S \to T$*, a subset, denoted by* $\mathrm{kr}(\varsigma)$*, of* S *is said to be the kernel of* ς *if it satisfies* (11) *(As a mistake, this condition was dropped in [15].) and:*

$$(\forall s \in S)(s \in \mathrm{kr}(\varsigma) \ \Leftrightarrow \ \iota_T \leq_T \varsigma(s)). \tag{19}$$

Remark 1. *The kernel* $\mathrm{kr}(\varsigma)$ *of* ς *in Definition 6 is unique. Otherwise, we may assume that there are two kernels* $\mathrm{kr}(\varsigma)_1$ *and* $\mathrm{kr}(\varsigma)_2$ *such that* $\mathrm{kr}(\varsigma)_1 \neq \mathrm{kr}(\varsigma)_2$. *However, by (19), one has that for all* $s \in S$, $s \in \mathrm{kr}(\varsigma)_1$ *if and only if (iff)* $\iota_T \leq_T \varsigma(s)$ *iff* $s \in \mathrm{kr}(\varsigma)_2$. *Hence, one obtains* $\mathrm{kr}(\varsigma)_1 = \mathrm{kr}(\varsigma)_2$, *a contradiction.*

If we take the following in place of (19):

$$(\forall s \in S)(s \in \mathrm{kr}(\varsigma) \ \Leftrightarrow \ \iota_T = \varsigma(s)), \tag{20}$$

we can provide the definition of the kernel in BCI-algebras, more generally the definition of the kernel in abstract algebra in the strict sense above. Note that every kernel in BCI-algebras is a kernel in OBCI-algebras but its converse does not hold. The homomorphic images of the elements of kernels in BCI-algebras are unique in the sense that the value of related homomorphisms is identity (see (20)), whereas the homomorphic images of the elements of kernels in OBCI-algebras need not be because the kernels in OBCI-algebras are defined by inequations (see (19)).

Theorem 1 ([14]). *Let* $\boldsymbol{S} := (S, \to_S, \iota_S, \leq_S)$ *and* $\boldsymbol{T} := (T, \to_T, \iota_T, \leq_T)$ *be OBCI-algebras, and* $\varsigma : S \to T$ *be a homomorphism.*

(i) *If the kernel* $\mathrm{kr}(\varsigma)$ *of* ς *is closed, then it satisfies:*

$$(\forall s \in S)(\varsigma(\iota_S) \leq_T \varsigma(s) \ \Rightarrow \ s \to_S \iota_S \in \mathrm{kr}(\varsigma)). \tag{21}$$

(ii) *If the kernel* $\mathrm{kr}(\varsigma)$ *of* ς *is O-closed and has (10), then it satisfies (21).*

By the expression '(*O*-)closed', we refer ambiguously to both 'closed' and '*O*-closed' together whenever they need not be distinguished.

Theorem 2 ([14]). *Let* $\boldsymbol{S} := (S, \to_S, \iota_S, \leq_S)$ *and* $\boldsymbol{T} := (T, \to_T, \iota_T, \leq_T)$ *be OBCI-algebras, and* $\varsigma : S \to T$ *be a homomorphism satisfying*

$$\iota_T = \varsigma(\iota_S). \tag{22}$$

If $\mathrm{kr}(\varsigma)$ *has (21), then it is (O-)closed.*

Let $\mathcal{K}(\mathcal{OBCI})$ and $\mathcal{K}(\mathcal{OBCI})_{(21)}^{(22)}$ be the class of kernels and the class of kernels satisfying (21) with homomorphisms satisfying (22), respectively, in OBCI-algebras. Also, let $\mathcal{SA}(\mathcal{OBCI})_K$, $\mathcal{OSA}(\mathcal{OBCI})_K$, and $\mathcal{OSA}(\mathcal{OBCI})_K$ + (10) be the class of subalgebras, the class of ordered subalgebras and the class of ordered subalgebras satisfying (10), respectively, in the same domain as kernels in the OBCI-algebras. Note that the subalgebras and ordered subalgebras need not be kernels. Then, since subalgebras are ordered subalgebras, we can summarize the relations between the classes as follows.

$$(S1)\ \mathcal{K}(\mathcal{OBCI})_{(21)}^{(22)}, \mathcal{OSA}(\mathcal{OBCI})_K + (10) \subsetneq \mathcal{SA}(\mathcal{OBCI})_K \subsetneq \mathcal{OSA}(\mathcal{OBCI})_K.$$

Theorem 3 ([14]). *Let* $\varsigma : S \to T$ *be a homomorphism from an OBCI-algebra* $\boldsymbol{S} := (S, \to_S, \iota_S, \leq_S)$ *to an OBCI-algebra* $\boldsymbol{T} := (T, \to_T, \iota_T, \leq_T)$. *Then,* $\mathrm{kr}(\varsigma)$ *is a filter of* $\boldsymbol{S}$.

Theorem 4 ([14]). *Let* $\varsigma : S \to T$ *be a homomorphism from an OBCI-algebra* $\boldsymbol{S} := (S, \to_S, \iota_S, \leq_S)$ *to an OBCI-algebra* $\boldsymbol{T} := (T, \to_T, \iota_T, \leq_T)$ *and satisfy the assertion*

$$(\forall s \in S)(\iota_S \leq_S s \ \Rightarrow \ \iota_T \leq_T \varsigma(s)). \tag{23}$$

Then, $\mathrm{kr}(\varsigma)$ *is an ordered filter of* $\boldsymbol{S}$.

Let $\mathcal{F}(\mathcal{OBCI})_K$, $\mathcal{OF}(\mathcal{OBCI})_K$ and $\mathcal{OF}(\mathcal{OBCI})_K$ + (23) be the class of filters, the class of ordered filters and the class of ordered filters with homomorphisms satisfying (23), respectively, in the same domain as kernels in OBCI-algebras. Note that the filter and ordered filter are independent of each other [14]. Then, we can summarize the relations between the classes as follows.

$$(S2) \qquad \mathcal{K}(\mathcal{OBCI}) \subsetneq \mathcal{F}(\mathcal{OBCI})_K.$$

$$(S3) \qquad \mathcal{K}(\mathcal{OBCI}) + (23) \subsetneq \mathcal{OF}(\mathcal{OBCI})_K.$$

Theorem 5 ([15]). *Let* $\varsigma : S \to T$ *be injective (We add "injective" to the Theorem in [15] since* ς^{-1} *works when* ς *is injective, and similarly for the other theorems.) and a homomorphism from an OBCI-algebra* $\boldsymbol{S} := (S, \to_S, \iota_S, \leq_S)$ *to an OBCI-algebra* $\boldsymbol{T} := (T, \to_T, \iota_T, \leq_T)$ *satisfying* (22).

(i) *Let* Δ *also be a filter of* $\boldsymbol{T}$. *Then,* $\varsigma^{-1}(\Delta)$ *is a filter of* $\boldsymbol{S}$.
(ii) *Let* ς *be surjective and* Γ *be a filter of* $\boldsymbol{S}$. *Then,* $\varsigma(\Gamma)$ *is a filter of* $\boldsymbol{T}$.

Theorem 6 ([15]). *Let* $\varsigma : S \to T$ *be injective and a homomorphism from an OBCI-algebra* $\boldsymbol{S} := (S, \to_S, \iota_S, \leq_S)$ *to an OBCI-algebra* $\boldsymbol{T} := (T, \to_T, \iota_T, \leq_T)$ *satisfying* (22).

(i) *Let* ς *satisfy* (23) *and* Δ *be an ordered filter of* $\boldsymbol{T}$. *Then,* $\varsigma^{-1}(\Delta)$ *is an ordered filter of* $\boldsymbol{S}$.
(ii) *Let* ς *be surjective and satisfy the assertion*

$$(\forall s \in S)(\iota_T \leq_T \varsigma(s) \;\Rightarrow\; \iota_S \leq_S s), \tag{24}$$

and Γ *be an ordered filter of* $\boldsymbol{S}$. *Then,* $\varsigma(\Gamma)$ *is an ordered filter of* $\boldsymbol{T}$.

Theorem 7 ([15]). *Let* $\varsigma : S \to T$ *be injective and a homomorphism from an OBCI-algebra* $\boldsymbol{S} := (S, \to_S, \iota_S, \leq_S)$ *to an OBCI-algebra* $\boldsymbol{T} := (T, \to_T, \iota_T, \leq_T)$. *Let* ς *also be surjective and satisfy* (22). *Take the following two sets:*

$$\mathcal{J} := \{\Gamma \mid \Gamma \text{ is a filter of } S \text{ containing } \mathrm{kr}(\varsigma)\},$$
$$\mathcal{K} := \{\Delta \mid \Delta \text{ is a filter of } T\},$$

One can construct a bijective function

$$\xi : \mathcal{J} \to \mathcal{K},\ \Gamma \mapsto \xi(\Gamma) \tag{25}$$

such that $\xi^{-1}(\Delta) = \varsigma^{-1}(\Delta)$.

Theorem 8 ([15]). *Let* $\varsigma : S \to T$ *be injective and a homomorphism from an OBCI-algebra* $\boldsymbol{S} := (S, \to_S, \iota_S, \leq_S)$ *to an OBCI-algebra* $\boldsymbol{T} := (T, \to_T, \iota_T, \leq_T)$. *Let* ς *also be surjective and satisfy* (22). *Take the following two sets:*

$$\mathcal{J} := \{\Gamma \mid \Gamma \text{ is an ordered filter of } S \text{ containing } \mathrm{kr}(\varsigma) \text{ and satisfying (10)}\},$$
$$\mathcal{K} := \{\Delta \mid \Delta \text{ is an ordered filter of } T\},$$

One can construct a bijective function ξ *satisfying* (25) *as in Theorem 7.*

3. Homomorphisms and Ordered Kernels

3.1. Ordered Kernels

Definition 7. *Let ς be a map from an OBCI-algebra $\mathbf{S} := (S, \to_S, \iota_S, \leq_S)$ to an OBCI-algebra $\mathbf{T} := (T, \to_T, \iota_T, \leq_T)$. A subset Θ of S is said to be an ordered kernel of ς if it satisfies (11) and:*

$$(\forall s, t \in S)(s \in \Theta, \iota_T \leq_T \varsigma(s) \to_T \varsigma(t) \ \Rightarrow \ t \in \Theta). \tag{26}$$

We first show that in OBCI-algebras, kernels are ordered kernels.

Theorem 9. *Let $\mathbf{S} := (S, \to_S, \iota_S, \leq_S)$ and $\mathbf{T} := (T, \to_T, \iota_T, \leq_T)$ be OBCI-algebras and $\varsigma : S \to T$ be a map. If Θ is the kernel of $\mathbf{S}$, then it is an ordered kernel of $\mathbf{S}$.*

Proof. Let Θ be the kernel of $\mathbf{S}$. We further assume that $s \in \Theta$ and $\iota_T \leq_T \varsigma(s) \to_T \varsigma(t)$. We need to prove that $t \in \Theta$. Using (19), we have $\iota_T \leq_T \varsigma(s)$. Then, by (5) and (6), we further obtain $\iota_T \leq_T \varsigma(t)$. Hence, $t \in \Theta$ by (19). Therefore, Θ forms an ordered kernel of $\mathbf{S}$. □

Note that the kernel $\mathrm{kr}(\varsigma)$ is unique (see Remark 1). However, the ordered kernels of ς do not need to be, as the example below illustrates.

Example 1. *Suppose that $S := \{\iota_S, s, t, u\}$ and $T := \{\iota_T, s, t, u\}$ are sets with binary operations " $\to_S$ " and " $\to_T$ " provided by Tables 1 and 2, respectively.*

Table 1. Table for "$\to_S$".

$\to_S$	ι_S	s	t	u
ι_S	ι_S	s	t	u
s	u	ι_S	s	t
t	t	u	ι_S	s
u	s	t	u	ι_S

Table 2. Table for "$\to_T$".

$\to_T$	ι_T	s	t	u
ι_T	ι_T	s	t	u
s	ι_T	ι_T	t	u
t	ι_T	s	ι_T	u
u	u	u	u	ι_T

Let $\leq_S := \{(\iota_S, \iota_S), (s, s), (t, t), (u, u)\}$ *and*

$$\leq_T := \{(\iota_T, \iota_T), (s, s), (t, t), (u, u), (s, \iota_T), (t, \iota_T)\}.$$

Then, $\mathbf{S} := (S, \to_S, \iota_S, \leq_S)$ and $\mathbf{T} := (T, \to_T, \iota_T, \leq_T)$ are certainly OBCI-algebras. A map ς from S to T is defined as follows:

$$\varsigma : S \to T, \ k \mapsto \begin{cases} \iota_T & \text{if } k = \iota_S, \\ t & \text{if } k = s, \\ u & \text{if } k = t, \\ s & \text{if } k = u. \end{cases} \tag{27}$$

Then, ς is not a homomorphism since $\varsigma(s \to_S t) = \varsigma(s) = t \neq u = \varsigma(s) \to_T \varsigma(t)$. One can easily show that $\mathrm{kr}(\varsigma) = \{\iota_S\}$ and the sets $\{\iota_S\}$, $\{\iota_S, s\}$, $\{\iota_S, t\}$, $\{\iota_S, u\}$, $\{\iota_S, s, t\}$, $\{\iota_S, s, u\}$ and $\{\iota_S, t, u\}$ are ordered kernels of ς.

Note that ς is not a homomorphism in Example 1. The following are examples with homomorphisms.

Example 2. *Let Table 3 provide a set* $S = \{1, \iota, \sigma, 0\}$ *with the binary operation* " $\to$ ".

Table 3. Table for "$\to$".

$\to$	**1**	ι	σ	**0**
1	1	0	0	0
ι	1	ι	σ	0
σ	1	σ	ι	0
0	1	1	1	1

If $\leq_\iota := \{(0,0),(\iota,\iota),(\sigma,\sigma),(1,1),(0,\iota),(0,\sigma),(\iota,1),(\sigma,1)\}$, *then* $\boldsymbol{S} := (S, \to, \iota, \leq_\iota)$ *is an OBCI-algebra (see [14]). Consider the identity map* $\varsigma : S \to S$. *Clearly,* ς *is a homomorphism, and* $\mathrm{kr}(\varsigma) = \{1, \iota\}$ *and the sets* $\{1, \iota\}$, $\{1, \iota, \sigma\}$ *and* S *are ordered kernels of* ς.

Example 3. *Let Table 4 provide a set* $S = \{0, 1, \frac{3}{4}, \frac{1}{2}, \frac{1}{4}\}$ *with the binary operation* " $\to$ ", *and* $\leq_\iota$ *be the natural order in* S.

Table 4. Table for "$\to$".

$\to$	**1**	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	**0**
1	1	0	0	0	0
$\frac{3}{4}$	1	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	0
$\frac{1}{2}$	1	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{1}{2}$	0
$\frac{1}{4}$	1	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	0
0	1	1	1	1	1

Then, $\boldsymbol{S} := (S, \to, \iota, \leq_\iota)$, *where* $\iota = \frac{3}{4}$, *is an OBCI-algebra (see [14]). Let* $\varsigma : S \to S$ *be an automorphism as the identity map. Then,* ς *is a homomorphism,* $\mathrm{kr}(\varsigma)$ *is* $\{1, \frac{3}{4}\}$, *and the sets* $\mathrm{kr}(\varsigma)$, $\{1, \frac{3}{4}, \frac{1}{2}\}$, $\{1, \frac{3}{4}, \frac{1}{2}, \frac{1}{4}\}$ *and* S *are ordered kernels of* ς.

The above three examples show that every kernel is an ordered kernel but its converse does not hold. Let $\mathcal{K}(\mathcal{BCI})$ and $\mathcal{OK}(\mathcal{OBCI})$ be the class of kernels and the class of ordered kernels, respectively, in OBCI-algebras. Since the kernel of a map ς in BCI-algebras is defined by (20) and BCI-algebras are OBCI-algebras, we can summarize the relations between the classes of kernels in BCI-algebras and kernels and ordered kernels in OBCI-algebras as follows.

$$(S4) \qquad \mathcal{K}(\mathcal{BCI}) \subsetneq \mathcal{K}(\mathcal{OBCI}) \subsetneq \mathcal{OK}(\mathcal{OBCI}).$$

Next, the following theorem provides a condition for ordered kernels to be kernels.

Theorem 10. *Let* $\boldsymbol{S} := (S, \to_S, \iota_S, \leq_S)$ *and* $\boldsymbol{T} := (T, \to_T, \iota_T, \leq_T)$ *be OBCI-algebras and* $\varsigma : S \to T$ *be a homomorphism. If an ordered kernel* Θ *of* $\boldsymbol{S}$ *satisfies*

$$(\forall s \in S)(s \in \Theta \ \Rightarrow \ \varsigma(\iota_S) \leq_T \varsigma(s)), \tag{28}$$

then Θ *is the kernel of* $\boldsymbol{S}$.

Proof. Let Θ be an ordered kernel of $\mathbf{S}$ and satisfy $\varsigma(\iota_S) \leq_T \varsigma(s)$ for all $s \in \Theta$. Let $s \in \Theta$. Then, $\varsigma(\iota_S) \leq_T \varsigma(s)$, and so $\iota_T \leq_T \varsigma(s)$ by (17) and (6). Let $\iota_T \leq_T \varsigma(s)$. Then, $\iota_T \leq_T \varsigma(\iota_S \to_S s)$ by (7), and so

$$\iota_T \leq_T \varsigma(\iota_S \to_S s) = \varsigma(\iota_S) \to_T \varsigma(s)$$

by (15). Since $\iota_S \in \Theta$ by (11), we obtain $s \in \Theta$ by (26). Therefore, Θ forms the kernel of $\mathbf{S}$. □

Corollary 1. *Let $\boldsymbol{S} := (S, \to_S, \iota_S, \leq_S)$ and $\boldsymbol{T} := (T, \to_T, \iota_T, \leq_T)$ be OBCI-algebras and $\varsigma : S \to T$ be a homomorphism. Let $\Theta, \subseteq S$, also satisfy (28). The following are equivalent:*

(1) *Θ is the kernel of $\mathbf{S}$.*
(2) *Θ is the ordered kernel of $\mathbf{S}$.*

Corollary 2. *Let $\boldsymbol{S} := (S, \to_S, \iota_S, \leq_S)$ and $\boldsymbol{T} := (T, \to_T, \iota_T, \leq_T)$ be OBCI-algebras and $\varsigma : S \to T$ be a homomorphism. The following are equivalent:*

(1) *Θ is the kernel of $\mathbf{S}$.*
(2) *Θ is the ordered kernel of $\mathbf{S}$ satisfying (28).*
(3) *Θ is the ordered kernel of $\mathbf{S}$ satisfying*

$$(\forall s \in S)(s \in \Theta \Rightarrow \iota_T \leq_T \varsigma(s)). \tag{29}$$

Proof. (1) ⇒ (2): Let Θ be the kernel of $\mathbf{S}$. We further assume that $s \in \Theta$ and $\iota_T \leq_T \varsigma(s) \to_T \varsigma(t)$. We have $t \in \Theta$ by Theorem 9. Moreover, since $\iota_T \leq_T \varsigma(s)$ by (19), we also obtain $\varsigma(\iota_S) \leq_T \varsigma(s)$ by (18). Hence, the claim (2) holds.

(2) ⇒ (1): This direction follows from Theorem 10.

(2) ⇔ (3): This equivalence follows from (18). □

Let $\mathcal{OK}(\mathcal{OBCI})$ + (28) and $\mathcal{K}(\mathcal{OBCI})_=$ denote the class of ordered kernels satisfying (28) and the class of kernels defined by (20) instead of (19), respectively, in OBCI-algebras. Then, as a specification of $(S4)$, we can summarize the relations between the related classes as follows.

$$(S5)\ \mathcal{K}(\mathcal{BCI}) = \mathcal{K}(\mathcal{OBCI})_= \subsetneq \mathcal{K}(\mathcal{OBCI}) = \mathcal{OK}(\mathcal{OBCI}) + (28) \subsetneq \mathcal{OK}(\mathcal{OBCI}).$$

3.2. (Ordered) Subalgebras with Ordered Kernels

Here, we deal with relations between (ordered) subalgebras and ordered kernels in homomorphisms in contrast to the relations between (ordered) subalgebras and kernels in homomorphisms introduced in [15]. The following example first shows that ordered kernels are not necessarily (ordered) subalgebras.

Example 4. *The ordered kernel $\Theta = \{1, \frac{3}{4}, \frac{1}{2}\}$ in Example 3 is not an (ordered) subalgebra of $\boldsymbol{S} := (S, \to, \iota, \leq_\iota)$ since $\iota = \frac{3}{4} \leq_\iota 1$ and $\iota = \frac{3}{4} \leq_\iota \frac{3}{4}$, but $1 \to \frac{3}{4} = 0 \notin \Theta$.*

Definition 8. *Suppose that $\boldsymbol{S} := (S, \to_S, \iota_S, \leq_S)$ and $\boldsymbol{T} := (T, \to_T, \iota_T, \leq_T)$ are OBCI-algebras, and $\varsigma : S \to T$ is a homomorphism. We call an ordered kernel of ς closed in case it is a subalgebra of $\mathbf{S}$, and O-closed in case it is an ordered subalgebra of $\mathbf{S}$.*

The following is an example of (O-)closed ordered kernels.

Example 5. *Let Table 5 provide a set $S = \{\iota, s, t, u, v\}$ with the binary operation " $\to$ ".*

Table 5. Table for "$\to$".

$\to$	ι	s	t	u	v
ι	ι	s	t	u	v
s	ι	ι	t	u	u
t	ι	s	ι	u	v
u	u	v	u	ι	s
v	u	u	u	ι	ι

Let $\leq_\iota := \{(\iota,\iota),(s,s),(t,t),(u,u),(v,v),(s,\iota),(t,\iota),(v,u)\}$. *Then,* $\boldsymbol{S} := (S, \to, \iota, \leq_\iota)$ *is an OBCI-algebra. Let* $\varsigma : S \to S$ *be the identity map. Clearly, the sets* $\mathrm{kr}(\varsigma) = \{\iota\}$, $\Theta_1 := \{\iota, s\}$ *and* $\Theta_2 := \{\iota, t, u\}$ *are O-closed ordered kernels of* ς. *In particular,* $\mathrm{kr}(\varsigma)$ *is the closed kernel of* ς.

Ordered kernels of ς also have the same results as Theorem 1. We will indirectly verify it as a corollary of the propositions below.

Proposition 6. *Let* $\boldsymbol{S} := (S, \to_S, \iota_S, \leq_S)$ *and* $\boldsymbol{T} := (T, \to_T, \iota_T, \leq_T)$ *be OBCI-algebras, and* $\varsigma : S \to T$ *be a homomorphism.*

(i) *If an ordered kernel* Θ *of* ς *is closed, then it satisfies:*

$$(\forall s \in S)(\iota_T \leq_T \varsigma(s) \ \Rightarrow \ s \to_S \iota_S \in \Theta). \tag{30}$$

(ii) *If an ordered kernel* Θ *of* ς *is O-closed and has* (10)*, then it satisfies* (30).

Proof. (i) Suppose that Θ is a closed ordered kernel of ς and $\iota_T \leq_T \varsigma(s)$ for $s \in S$. We have $\iota_T \leq_T \varsigma(\iota_S \to_S s)$ by (7) and so

$$\iota_T \leq_T \varsigma(\iota_S \to_S s) = \varsigma(\iota_S) \to_T \varsigma(s)$$

by (15). Then, since $\iota_S \in \Theta$, we obtain $s \in \Theta$ by (26). Hence, $s \to_S \iota_S \in \Theta$ by (8).

(ii) Suppose that Θ is an O-closed ordered kernel of ς and $\iota_T \leq_T \varsigma(s)$ for $s \in S$. As the proof of (i), we can obtain $s \in \Theta$, and so $\iota_S \leq_S s$ by (10). Hence, $s \to_S \iota_S \in \Theta$ by (9) since $\iota_S \in \Theta$ and $\iota_S \leq_S \iota_S$. □

Corollary 3. *Let* $\boldsymbol{S} := (S, \to_S, \iota_S, \leq_S)$ *and* $\boldsymbol{T} := (T, \to_T, \iota_T, \leq_T)$ *be OBCI-algebras, and* $\varsigma : S \to T$ *be a homomorphism.*

(i) *If an ordered kernel* Θ *of* ς *is closed, then it satisfies* (21).

(ii) *If an ordered kernel* Θ *of* ς *is O-closed and satisfies* (10)*, then it further satisfies* (21).

The following example illustrates the fact that the assertion (30) does not hold if the ordered kernel Θ of ς in Proposition 6 is not (O-)closed.

Example 6. *Consider the ordered kernel* $\Theta = \{1, \frac{3}{4}, \frac{1}{2}\}$ *in Example 3. Since* $\iota \leq_\iota 1$ *but* $1 \to \iota = 0 \notin \Theta$, Θ *does not satisfy* (30).

We provide conditions for ordered kernels to be (O-)closed. Unlike Corollary 3, it will be verified that ordered kernels need additional conditions to be (O-)closed.

Let $\mathbf{S} := (S, \to_S, \iota_S, \leq_S)$ and $\mathbf{T} := (T, \to_T, \iota_T, \leq_T)$ be OBCI-algebras and $\varsigma : S \to T$ be a homomorphism. The following example provides an ordered kernel Θ of ς not satisfying the assertion (29).

Example 7. *The ordered kernel* $\Theta = \{1, \frac{3}{4}, \frac{1}{2}\}$ *in Example 3 does not satisfy* (29) *since* $\frac{1}{2} \in \Theta$ *but* $\iota_T = \frac{3}{4} \not\leq_\iota \varsigma(\frac{1}{2}) = \frac{1}{2}$.

Theorem 11. *Let* $\mathbf{S} := (S, \to_S, \iota_S, \leq_S)$ *and* $\mathbf{T} := (T, \to_T, \iota_T, \leq_T)$ *be OBCI-algebras, and* $\varsigma : S \to T$ *be a homomorphism. If an ordered kernel* Θ *of* ς *satisfies the assertions* (29) *and* (30)*, then it is closed.*

Proof. Let Θ be an ordered kernel of ς that satisfies (29) and (30). It suffices to verify that Θ is a subalgebra of $\mathbf{S}$. Let $s, t \in S$ be such that $s, t \in \Theta$. Then, $\iota_T \leq_T \varsigma(s)$ and $\iota_T \leq_T \varsigma(t)$ by (29), and so $s \to_S \iota_S \in \Theta$ and $t \to_S \iota_S \in \Theta$ by (30). Note that using (1) and (15), we have

$$\iota_T \leq_T \varsigma(s \to_S \iota_S) \to_T \varsigma((\iota_S \to_S t) \to_S (s \to_S t)).$$

Then, $(\iota_S \to_S t) \to_S (s \to_S t) \in \Theta$ by (26), and so

$$\iota_T \leq_T \varsigma(\iota_S \to_S t) \to_T \varsigma(s \to_S t)$$

by (29) and (15). Thus,

$$\iota_T \leq_T \varsigma(t) \to_T \varsigma(s \to_S t)$$

by (7), and so $s \to_S t \in \Theta$ by (26). Therefore, Θ is a subalgebra of $\mathbf{S}$, and the proof is completed. □

Now, given an ordered kernel Θ of ς, consider the following assertion.

$$(\forall s, t \in S)(s \to t \in \Theta \Rightarrow \varsigma(s) \leq_T \varsigma(t)). \tag{31}$$

The following example provides an ordered kernel Θ of ς not satisfying (31).

Example 8. *The ordered kernel* $\Theta = \{1, \frac{3}{4}, \frac{1}{2}\}$ *in Example 3 does not satisfy* (31) *since* $\frac{1}{2} \to \frac{1}{4} = \frac{1}{2} \in \Theta$ *but* $\frac{1}{2} = \varsigma(\frac{1}{2}) \not\leq_\iota \varsigma(\frac{1}{4}) = \frac{1}{4}$.

Theorem 12. *Let* $\mathbf{S} := (S, \to_S, \iota_S, \leq_S)$ *and* $\mathbf{T} := (T, \to_T, \iota_T, \leq_T)$ *be OBCI-algebras, and* $\varsigma : S \to T$ *be a homomorphism. If an ordered kernel* Θ *of* ς *satisfies the assertions* (30) *and* (31)*, then it is O-closed.*

Proof. Let Θ be an ordered kernel of ς that satisfies (30) and (31). It suffices to verify that Θ is an ordered subalgebra of $\mathbf{S}$. Let $s, t \in S$ be such that $s, t \in \Theta$, $\iota_S \leq_S s$ and $\iota_S \leq_S t$. Then, since $\iota_S \to_S s$ and $\iota_S \to_S t$ by (7), we have $\varsigma(\iota_S) \leq_T \varsigma(s)$ and $\varsigma(\iota_S) \leq_T \varsigma(t)$ by (31), and so $\iota_T \leq_T \varsigma(s)$ and $\iota_T \leq_T \varsigma(t)$ by (6) and (17). Thus, $s \to_S \iota_S \in \Theta$ and $t \to_S \iota_S \in \Theta$ by (30). Then, as the proof in Theorem 11, we obtain

$$(\iota_S \to_S t) \to_S (s \to_S t) \in \Theta,$$

and so

$$\iota_T \leq_T \varsigma(\iota_S \to_S t) \to_T \varsigma(s \to_S t)$$

by (31) and (5). Hence,

$$\iota_T \leq_T \varsigma(t) \to_T \varsigma(s \to_S t)$$

by (7), and so $s \to_S t \in \Theta$ by (26). Therefore, Θ is an ordered subalgebra of $\mathbf{S}$, and the proof is completed. □

One interesting fact is that we can verify the equivalence between (29) and (31).

Proposition 7. *Let* $\boldsymbol{S} := (S, \to_S, \iota_S, \leq_S)$ *and* $\boldsymbol{T} := (T, \to_T, \iota_T, \leq_T)$ *be OBCI-algebras, and* $\varsigma : S \to T$ *be a homomorphism. Then, for a set* $\Theta \subseteq S$*, the assertions* (29) *and* (31) *are equivalent to each other.*

Proof. (29) $\Longrightarrow$ (31): Let (29) hold and $s \to t \in \Theta \subseteq S$. Then, $\iota_T \leq_T \varsigma(s \to_S t)$ by (29), and so

$$\iota_T \leq_T \varsigma(s \to_S t) = \varsigma(s) \to_T \varsigma(t)$$

by (15). Hence, $\varsigma(s) \leq_T \varsigma(t)$ by (5).

(31) $\Longrightarrow$ (29): Let (31) hold and $s \in \Theta \subseteq S$. Then $\iota_S \to_S s$ by (7), and so $\varsigma(\iota_S) \leq_T \varsigma(s)$ by (31). Hence, $\iota_T \leq_T \varsigma(s)$ by (17) and (6). □

Corollary 4. *Let* $\boldsymbol{S} := (S, \to_S, \iota_S, \leq_S)$ *and* $\boldsymbol{T} := (T, \to_T, \iota_T, \leq_T)$ *be OBCI-algebras, and* $\varsigma : S \to T$ *be a homomorphism. The following are equivalent:*

(1) *If an ordered kernel* Θ *of* ς *satisfies* (21) *and* (29)*, then it is (O-)closed.*
(2) *If an ordered kernel* Θ *of* ς *satisfies* (21) *and* (31)*, then it is (O-)closed.*

Let $\mathcal{OK}(\mathcal{OBCI}) + \begin{smallmatrix}(29)\\(30)\end{smallmatrix}$ be the class of ordered kernels satisfying (29) and (30) in OBCI-algebras. Let $\mathcal{SA}(\mathcal{OBCI})_{OK}$ and $\mathcal{OSA}(\mathcal{OBCI})_{OK}$ also be the class of subalgebras and the class of ordered subalgebras, respectively, in the same domain as ordered kernels in OBCI-algebras. Note that subalgebras and ordered subalgebras in the same domain as ordered kernels need not be ordered kernels. Then, as in (S1), we can summarize the relations between the related classes as follows.

$$(S6)\ \mathcal{OK}(\mathcal{OBCI}) + \begin{smallmatrix}(29)\\(30)\end{smallmatrix}, \mathcal{OSA}(\mathcal{OBCI})_{OK} + (10) \subsetneq \mathcal{SA}(\mathcal{OBCI})_{OK} \subsetneq \mathcal{OSA}(\mathcal{OBCI})_{OK}.$$

This shows that while kernels in OBCI-algebras require (21) and (22) to be (O-)closed (see (S1)), ordered kernels in OBCI-algebras require (29) and (30) to be (O-)closed (see (S6)).

We next consider (ordered) subalgebras in the domain of a homomorphism ς to be ordered kernels. First, the following examples show that (ordered) subalgebras need not be ordered kernels.

Example 9. *Let us take a set* $\{\iota, \sigma\}$ *in Example 2. Then, it is clear that* $\{\iota, \sigma\}$ *forms a subalgebra of* $\boldsymbol{S}$*. However, it does not form an ordered kernel of* ς *since* $\iota \in \{\iota, \sigma\}$ *and* $\iota \leq \iota \to 1 = 1$ *but* $1 \notin \{\iota, \sigma\}$*.*

Example 10. *Let us take a set* $\{\iota, 0\}$ *in Example 2. Then, it is clear that* $\{\iota, 0\}$ *forms an ordered subalgebra of* $\boldsymbol{S}$*. However, it does not form an ordered kernel of* ς *since* $\iota \in \{\iota, 0\}$ *and* $\iota \leq \iota \to 1 = 1$ *but* $1 \notin \{\iota, 0\}$*.*

We then consider conditions for (ordered) subalgebras of $\mathbf{S}$ to be ordered kernels of ς.

Theorem 13. *Let* $\varsigma : S \to T$ *be a homomorphism from an OBCI-algebra* $\boldsymbol{S} := (S, \to_S, \iota_S, \leq_S)$ *to an OBCI-algebra* $\boldsymbol{T} := (T, \to_T, \iota_T, \leq_T)$ *satisfying* (24)*. If* Θ *is a subalgebra of* $\boldsymbol{S}$ *and satisfies the assertions* (14) *and*

$$(\forall s, t \in S)((s \to t) \to s, s \to (s \to t) \in \Theta \ \Rightarrow\ t \in \Theta, \tag{32}$$

then it is an ordered kernel of ς*.*

Proof. Let Θ be a subalgebra of $\mathbf{S}$ and satisfy (14) and (32). Since $\iota_S \leq_S \iota_S$, one has $\iota_S \in \Theta$ by (14). Let $s, t \in S$ be such that $\iota_T \leq_T \varsigma(s) \to_T \varsigma(t)$ and $s \in \Theta$. By (15), one has $\iota_T \leq_T \varsigma(s \to_S t)$, and so $\iota_S \leq_S s \to_S t$ by (24). Then, one further has $s \to_S t \in \Theta$ by (14), and so $(s \to t) \to s, s \to (s \to t) \in \Theta$ by (8). Hence, $t \in \Theta$ by (32). Therefore Θ is an ordered kernel of ς. □

Theorem 14. *Let $\varsigma : S \to T$ be a homomorphism from an OBCI-algebra $\mathbf{S} := (S, \to_S, \iota_S, \leq_S)$ to an OBCI-algebra $\mathbf{T} := (T, \to_T, \iota_T, \leq_T)$ satisfying (24). If Θ is an ordered subalgebra of $\mathbf{S}$ and satisfies the assertions (14), (10) and (32), then it is an ordered kernel of ς.*

Proof. The claim follows from Theorem 13 and Proposition 2. □

Corollary 5. *Let $\mathbf{S} := (S, \to_S, \iota_S, \leq_S)$ and $\mathbf{T} := (T, \to_T, \iota_T, \leq_T)$ be OBCI-algebras, and $\varsigma : S \to T$ be a homomorphism satisfying (24). The following are equivalent:*

(1) *Θ is a subalgebra of $\mathbf{S}$ and satisfies (14), (10) and (32).*
(2) *Θ is an ordered subalgebra of $\mathbf{S}$ and satisfies (14), (10) and (32).*

Let $\mathcal{SA}(\mathcal{OBCI})_{OK} + {}^{(24),(14)}_{(32)}$, $\mathcal{SA}(\mathcal{OBCI})_{OK} + {}^{(24),(14)}_{(32),(10)}$ and $\mathcal{OSA}(\mathcal{OBCI})_{OK} + {}^{(24),(14)}_{(32),(10)}$ be the class of subalgebras satisfying (14) and (32), the class of subalgebras satisfying (14), (32) and (10) the class of ordered subalgebras satisfying (14), (32) and (10), respectively, with homomorphisms satisfying (24) in the same domain as ordered kernels in OBCI-algebras. Then, as the other direction of (S6), we can summarize the relations between the related classes as follows.

$$(S7)\ \mathcal{SA}(\mathcal{OBCI})_{OK}+{}^{(24),(14)}_{(32),(10)} = \mathcal{OSA}(\mathcal{OBCI})_{OK}+{}^{(24),(14)}_{(32),(10)} \subsetneq$$

$$\mathcal{SA}(\mathcal{OBCI})_{OK}+{}^{(24),(14)}_{(32)} \subsetneq \mathcal{OK}(\mathcal{OBCI}).$$

3.3. (Ordered) Filters with Ordered Kernels

If ς is a homomorphism, then the kernel $\mathrm{kr}(\varsigma)$ is a filter of $\mathbf{S}$ in itself (see Theorem 3). However, ordered kernels of ς need not be its filters, as the following example shows.

Example 11. *Let us take the ordered kernel $\{\iota_S, s\}$ in Example 1. Then, since $s \to_S t = s$, one has that $s \to_S t, s \in \{\iota_S, s\}$ but $t \notin \{\iota_S, s\}$. Hence, $\{\iota_S, s\}$ is not a filter of $\mathbf{S}$.*

We first consider conditions for (ordered) kernels to be (ordered) filters in contrast to the related results in [14]. The following theorem shows that the assertion (29) is a condition for an ordered kernel of ς to be a filter of $\mathbf{S}$.

Theorem 15. *Let $\varsigma : S \to T$ be a homomorphism from an OBCI-algebra $\mathbf{S} := (S, \to_S, \iota_S, \leq_S)$ to an OBCI-algebra $\mathbf{T} := (T, \to_T, \iota_T, \leq_T)$. If Θ is an ordered kernel of ς and satisfies (29), then it is a filter of $\mathbf{S}$.*

Proof. Let Θ be an ordered kernel of ς that satisfies (29). Let $s, t \in S$ be such that $s \to_S t \in \Theta$ and $s \in \Theta$. Then, $\iota_T \leq_T \varsigma(s \to_S t)$ by (29), and so

$$\iota_T \leq_T \varsigma(s \to_S t) = \varsigma(s) \to_T \varsigma(t)$$

by (15). Hence, $t \in \Theta$ by (26). Therefore, Θ is a filter of $\mathbf{S}$. □

Corollary 6. *Let* $\mathbf{S} := (S, \to_S, \iota_S, \leq_S)$ *and* $\mathbf{T} := (T, \to_T, \iota_T, \leq_T)$ *be OBCI-algebras, and* $\varsigma : S \to T$ *be a homomorphism. The following are equivalent:*

(1) *If* Θ *is an ordered kernel of* ς *and satisfies* (28)*, then it is a filter of* $\mathbf{S}$*.*
(2) *If* Θ *is an ordered kernel of* ς *and satisfies* (29)*, then it is a filter of* $\mathbf{S}$*.*
(3) *If* Θ *is an ordered kernel of* ς *and satisfies* (31)*, then it is a filter of* $\mathbf{S}$*.*

Interestingly, (23) introduced in Theorem 4 is also a condition for ordered kernels to be ordered filters. The following example first illustrates that ordered kernels need not satisfy (23).

Example 12. *Let Table 6 provide a set* $S := \{1, \frac{1}{2}, 0\}$ *with the binary operation "* $\to_S$ *", and* $\leq_S$ *be the natural order in* S*.*

Table 6. Table for "$\to_S$".

$\to_S$	**1**	$\frac{1}{2}$	**0**
1	1	0	0
$\frac{1}{2}$	1	$\frac{1}{2}$	0
0	1	1	1

Then, $\mathbf{S} := (S, \to_S, \frac{1}{2}, \leq_S)$ *is an OBCI-algebra. Define a map*

$$\varsigma : S \to S,\ k \mapsto \begin{cases} 1 & \text{if } k = 0, \\ \frac{1}{2} & \text{if } k = \frac{1}{2}, \\ 0 & \text{if } k = 1. \end{cases} \tag{33}$$

It is routine to verify that ς *is a homomorphism. However,* ς *does not satisfy* (23) *since for* $1, \frac{1}{2} \in S$*, we have* $\iota_S = \frac{1}{2} \leq_S 1$ *but* $\iota_T = \frac{1}{2} \not\leq_S 0 = \varsigma(1)$*.*

Theorem 16. *Let* $\varsigma : S \to T$ *be a homomorphism from an OBCI-algebra* $\mathbf{S} := (S, \to_S, \iota_S, \leq_S)$ *to an OBCI-algebra* $\mathbf{T} := (T, \to_T, \iota_T, \leq_T)$ *and satisfy* (23)*. Every ordered kernel of* ς *is an ordered filter of* $\mathbf{S}$*.*

Proof. Let Θ be an ordered kernel of a homomorphism ς that satisfies (23), and $s, t \in S$ be such that $s \in \Theta$ and $\iota_S \leq_S s \to_S t$. Then,

$$\iota_T \leq_T \varsigma(s \to_S t) = \varsigma(s) \to_T \varsigma(t)$$

by (23) and (15). Hence, $t \in \Theta$ by (26). Therefore, Θ is an ordered filter of $\mathbf{S}$. □

Let $\mathcal{F}(\mathcal{OBCI})_{OK}$, $\mathcal{OF}(\mathcal{OBCI})_{OK}$, $\mathcal{OK}(\mathcal{OBCI})$ + (29) and $\mathcal{OK}(\mathcal{OBCI})$ + (23) be the class of filters, the class of ordered filters, the class of ordered kernels with homomorphisms satisfying (29), and the class of ordered kernels with homomorphisms satisfying (23), respectively, in the same domain as ordered kernels in OBCI-algebras. Then, related to (S2), (S3) and (S5), we can summarize the relations between the classes as follows.

$$(S8)\ \mathcal{K}(\mathcal{OBCI}) = \mathcal{OK}(\mathcal{OBCI}) + (28) = \mathcal{OK}(\mathcal{OBCI}) + (29) \subsetneq \mathcal{F}(\mathcal{OBCI})_{OK}.$$

$$(S9)\ \mathcal{K}(\mathcal{OBCI}) + (23) \subsetneq \mathcal{OK}(\mathcal{OBCI}) + (23) \subsetneq \mathcal{OF}(\mathcal{OBCI})_{OK}.$$

We next consider (ordered) filters in the domain of a homomorphism ς to be ordered kernels. First, the following examples show that (ordered) filters need not be ordered kernels.

Example 13. *Let us take a set* $\{\iota, \sigma\}$ *in Example 2. Then, it is clear that* $\{\iota, \sigma\}$ *forms a filter of* $\mathbf{S}$. *However, it does not form an ordered kernel of* ς *since* $\iota \in \{\iota, \sigma\}$ *and* $\iota \leq \iota \to 1 = 1$ *but* $1 \notin \{\iota, \sigma\}$.

Example 14. *Let us take a set* $\{1, \frac{3}{4}, \frac{1}{4}\}$ *in Example 3. Then, it is clear that* $\{1, \frac{3}{4}, \frac{1}{4}\}$ *forms an ordered filter of* $\mathbf{S}$. *However, it does not form an ordered kernel of* ς *since* $\frac{1}{4} \in \{1, \frac{3}{4}, \frac{1}{4}\}$ *and* $\iota \leq \frac{1}{4} \to \frac{1}{2} = \frac{3}{4}$ *but* $\frac{1}{2} \notin \{1, \frac{3}{4}, \frac{1}{4}\}$.

We then consider conditions for (ordered) filters of $\mathbf{S}$ to be ordered kernels of ς.

Theorem 17. *Let* $\varsigma : S \to T$ *be a homomorphism from an OBCI-algebra* $\mathbf{S} := (S, \to_S, \iota_S, \leq_S)$ *to an OBCI-algebra* $\mathbf{T} := (T, \to_T, \iota_T, \leq_T)$ *satisfying* (24). *If* Θ *is a filter of* $\mathbf{S}$ *and satisfies* (14), *then it is an ordered kernel of* ς.

Proof. Let Θ be a filter of $\mathbf{S}$ and satisfy (14). Let $s, t \in S$ be such that $\iota_T \leq_T \varsigma(s) \to_T \varsigma(t)$ and $s \in \Theta$. By (15), one has $\iota_T \leq_T \varsigma(s \to_S t)$, and so $\iota_S \leq_S s \to_S t$ by (24). Then, one further has $s \to_S t \in \Theta$ by (14). Hence, $t \in \Theta$ by (12). Therefore Θ is an ordered kernel of ς. □

Theorem 18. *Let* $\varsigma : S \to T$ *be a homomorphism from an OBCI-algebra* $\mathbf{S} := (S, \to_S, \iota_S, \leq_S)$ *to an OBCI-algebra* $\mathbf{T} := (T, \to_T, \iota_T, \leq_T)$ *satisfying* (24). *If* Θ *is an ordered filter of* $\mathbf{S}$, *then it is an ordered kernel of* ς.

Proof. Let Θ be an ordered filter of $\mathbf{S}$. Let $s, t \in S$ be such that $\iota_T \leq_T \varsigma(s) \to_T \varsigma(t)$ and $s \in \Theta$. As above, by (15) and (24), one has $\iota_S \leq_S s \to_S t$. Hence, $t \in \Theta$ by (13). Therefore, Θ is an ordered kernel of ς. □

Corollary 7. *Let* $\mathbf{S} := (S, \to_S, \iota_S, \leq_S)$ *and* $\mathbf{T} := (T, \to_T, \iota_T, \leq_T)$ *be OBCI-algebras, and* $\varsigma : S \to T$ *be a homomorphism satisfying* (24) *and* (23).

(i) *Let* $\Theta, \subseteq S$, *satisfy* (10). *Then, the following are equivalent:*
   (1) Θ *is a filter of* $\mathbf{S}$ *and satisfies* (14).
   (2) Θ *is an ordered kernel of* ς *and satisfies* (29).

(ii) *The following are equivalent:*
   (1) Θ *is an ordered filter of* $\mathbf{S}$.
   (2) Θ *is an ordered kernel of* ς.

Proof. (i): For (1) to (2), let Θ be a filter of $\mathbf{S}$ and satisfy (14). It suffices to show that Θ satisfies (29). We assume $s \in \Theta$ and prove $\iota_T \leq_T \varsigma(s)$. Let $s \in \Theta$. By (10), one has $\iota_S \leq_S s$, and so $\iota_T \leq_T \varsigma(s)$ by (23). For (2) to (1), let Θ be an ordered kernel of ς and satisfy (29). It suffices to show that Θ satisfies (14). We assume $\iota_S \leq_S s$ and prove $s \in \Theta$. Let $\iota_S \leq_S s$. By (29), one has $\iota_T \leq_T \varsigma(s)$, and so $\iota_T \leq_T \varsigma(\iota_S \to_S s) = \varsigma(\iota_S) \to_T \varsigma(s)$ by (7) and (15). Then, since $\iota_S \in \Theta$, one has $s \in \Theta$ by (26).

(ii) The claim follows from Theorems 16 and 18. □

Let $\mathcal{F}(\mathcal{OBCI})_{OK} + \overset{(14)}{(24)}$ and $\mathcal{OF}(\mathcal{OBCI})_{OK} + (24)$ be the class of filters satisfying (14) with homomorphisms satisfying (24) and the class of ordered filters with homomorphisms satisfying (24), respectively, in the same domain as ordered kernels in OBCI-algebras. Then, as the other directions of (S8) and (S9), we can summarize the relations between the related classes as follows.

$$(S10)\ \mathcal{F}(\mathcal{OBCI})_{OK} \supsetneq \mathcal{F}(\mathcal{OBCI})_{OK} + \overset{(14)}{(24)} \subsetneq \mathcal{OK}(\mathcal{OBCI}).$$

$$(S11)\ \mathcal{OF}(\mathcal{OBCI})_{OK} \supsetneq \mathcal{OF}(\mathcal{OBCI})_{OK} + (24) \subsetneq \mathcal{OK}(\mathcal{OBCI}).$$

As one can see in Theorem 5, we need not introduce any condition to consider relations between filters if ς is a homomorphism. We finally introduce new results on relations between ordered filters, between filters and ordered filters and between ordered filters and filters.

Theorem 19. *Let $\varsigma : S \to T$ be injective and a map from an OBCI-algebra $\boldsymbol{S} := (S, \to_S, \iota_S, \leq_S)$ to an OBCI-algebra $\boldsymbol{T} := (T, \to_T, \iota_T, \leq_T)$ satisfying (22).*

(i) *Let ς be a homomorphism and satisfy (23), and Δ be an ordered filter of $\boldsymbol{T}$. As an ordered kernel of ς, $\varsigma^{-1}(\Delta)$ is an ordered filter of $\boldsymbol{S}$.*

(ii) *Let ς be surjective and Γ be an ordered filter of $\boldsymbol{S}$ as an ordered kernel of ς. $\varsigma(\Gamma)$ is an ordered filter of $\boldsymbol{T}$.*

Proof. (i) The claim follows from Theorem 16.

(ii) Assume that ς is surjective and Γ is an ordered filter of **S** as an ordered kernel of ς. Since $\iota_S \in \Gamma$, we have $\iota_T = \varsigma(\iota_S) \in \varsigma(\Gamma)$. Let $s, t \in T$ be such that $s \in \varsigma(\Gamma)$ and $\iota_T \leq_T s \to_T t$. Then, since ς is surjective, $\varsigma(s) = s$ for some $s \in \Gamma$ and $\varsigma(t) = t$ for some $t \in S$. Hence,

$$\iota_T \leq_T \varsigma(s) \to_T \varsigma(t) = s \to_T t$$

and so $t \in \Gamma$ by (26). This entails $t = \varsigma(t) \in \varsigma(\Gamma)$. Therefore, $\varsigma(\Gamma)$ is an ordered filter of **T**. □

Note that $\varsigma^{-1}(\Delta)$ in (i) and Γ in (ii) of Theorem 5 do not require any additional conditions to be an ordered filter of **S** and an ordered filter of **T**, respectively, whereas $\varsigma^{-1}(\Delta)$ in (i) and Γ in (ii) of Theorem 19 require the additional condition "an ordered kernel of ς" to be an ordered filter of **S** and an ordered filter of **T**, respectively. The following two theorems are new results on the relationship between filters and ordered filters, which are not investigated in [15].

Theorem 20. *Let $\varsigma : S \to T$ be injective and a homomorphism from an OBCI-algebra $\boldsymbol{S} := (S, \to_S, \iota_S, \leq_S)$ to an OBCI-algebra $\boldsymbol{T} := (T, \to_T, \iota_T, \leq_T)$ satisfying (22).*

(i) *Let ς satisfy (23) and Δ be a filter of $\boldsymbol{T}$. Then, $\varsigma^{-1}(\Delta)$, which contains the kernel $\mathrm{kr}(\varsigma)$ of ς, is an ordered filter of $\boldsymbol{S}$.*

(ii) *Let ς be surjective, and Γ be a filter of $\boldsymbol{S}$ and contain the kernel $\mathrm{kr}(\varsigma)$ of ς. Then, $\varsigma(\Gamma)$ is an ordered filter of $\boldsymbol{T}$.*

Proof. (i) Let ς satisfy (23), Δ be a filter of **T** and $\varsigma^{-1}(\Delta)$ contain the kernel $\mathrm{kr}(\varsigma)$ of ς. It is obvious that $\iota_S \in \varsigma^{-1}(\Delta)$ by (11) since ς satisfies (22) and $\iota_T \in \Delta$. Let $s, t \in S$ be such that $s \in \varsigma^{-1}(\Delta)$ and $\iota_S \leq_S s \to_S t$. Then, $\varsigma(s) \in \Delta$, and $\iota_T \leq_T \varsigma(s \to_S t)$ by (23). Thus, $s \to_S t \in \mathrm{kr}(\varsigma)$ by (19), and so $s \to_S t \in \varsigma^{-1}(\Delta)$. Then, $\varsigma(s \to_S t) \in \Delta$ and so

$$\varsigma(s) \to_T \varsigma(t) = \varsigma(s \to_S t) \in \Delta$$

by (15). It follows from (12) that $\varsigma(t) \in \Delta$. Hence, $t \in \varsigma^{-1}(\Delta)$, and therefore, $\varsigma^{-1}(\Delta)$ is an ordered filter of **S**.

(ii) Assume that ς is surjective, and Γ is a filter of **S** and contains the kernel of ς. Since $\iota_S \in \Gamma$, we have $\iota_T = \varsigma(\iota_S) \in \varsigma(\Gamma)$. Let $s, t \in T$ be such that $s \in \varsigma(\Gamma)$ and $\iota_T \leq_T s \to_T t$. Then, since ς is surjective, $\varsigma(s) = s$ for some $s \in \Gamma$ and $\varsigma(t) = t$ for some $t \in S$. Hence,

$$\iota_T \leq_T \varsigma(s) \to_T \varsigma(t) = \varsigma(s \to_S t)$$

by (15), and so $s \to_S t \in \mathrm{kr}(\varsigma)$ by (19). Then, since $\mathrm{kr}(\varsigma) \subseteq \Gamma$, we obtain $t \in \Gamma$ by (12). Therefore, $\varsigma(\Gamma)$ is an ordered filter of **T**. □

Theorem 21. *Let $\varsigma : S \to T$ be injective and a map from an OBCI-algebra $\boldsymbol{S} := (S, \to_S, \iota_S, \leq_S)$ to an OBCI-algebra $\boldsymbol{T} := (T, \to_T, \iota_T, \leq_T)$ satisfying (22).*

(i) *Let Δ be an ordered filter of **T**. As the kernel of ς, $\varsigma^{-1}(\Delta)$ is a filter of **S**.*

(ii) *Let ς be a surjective homomorphism, and Γ be an ordered filter of **S** and satisfy (10). Then, $\varsigma(\Gamma)$ is a filter of **T**.*

Proof. (i) Assume that Δ is an ordered filter of **T** and $\varsigma^{-1}(\Delta)$ is the kernel of ς. It is obvious that $\iota_S \in \varsigma^{-1}(\Delta)$ by (11) since (22) and $\iota_T \in \Delta$. Let $s, t \in S$ be such that $s \in \varsigma^{-1}(\Delta)$ and $s \to_S t \in \varsigma^{-1}(\Delta)$. Then, $\varsigma(s) \in \Delta$, and

$$\iota_T \leq_T \varsigma(s \to_S t) = \varsigma(s) \to_T \varsigma(t)$$

by (19) and (15). It follows from (13) that $\varsigma(t) \in \Delta$. Hence, $t \in \varsigma^{-1}(\Delta)$, and therefore, $\varsigma^{-1}(\Delta)$ is a filter of **S**.

(ii) Let ς be a surjective homomorphism, Γ be an ordered filter of **S** and satisfy (10). Since $\iota_S \in \Gamma$, we have $\iota_T = \varsigma(\iota_S) \in \varsigma(\Gamma)$. Let $s, t \in T$ be such that $s \in \varsigma(\Gamma)$ and $s \to_T t \in \varsigma(\Gamma)$. Then, since ς is surjective, $\varsigma(s) = s$ for some $s \in \Gamma$ and $\varsigma(t) = t$ for some $t \in S$. Hence, $\varsigma(s) \in \varsigma(\Gamma)$ and $\varsigma(s) \to_T \varsigma(t) \in \varsigma(\Gamma)$. Then,

$$\varsigma(s) \to_T \varsigma(t) = \varsigma(s \to_S t) \in \varsigma(\Gamma)$$

by (15), and so $s \to_S t \in \Gamma$. Hence, $\iota_S \leq_S s \to_S t$ by (10), and so $t \in \Gamma$ by (13). This entails $t = \varsigma(t) \in \varsigma(\Gamma)$. Therefore, $\varsigma(\Gamma)$ is a filter of **T**. □

Associated with Theorems 7 and 8 above, we finally introduce new results with respect to ordered kernels.

Theorem 22. *Let $\varsigma : S \to T$ be bijective and a homomorphism from an OBCI-algebra $\boldsymbol{S} := (S, \to_S, \iota_S, \leq_S)$ to an OBCI-algebra $\boldsymbol{T} := (T, \to_T, \iota_T, \leq_T)$ satisfying (22).*

(i) *Take the following two sets:*

$$\mathcal{J} := \{\Gamma \mid \Gamma \text{ is a filter of } S \text{ as an ordered kernel of } \varsigma\},$$
$$\mathcal{K} := \{\Delta \mid \Delta \text{ is an ordered filter of } T\},$$

One can construct a bijective function ξ satisfying (25) as in Theorem 7.

(ii) *Take the following two sets:*

$$\mathcal{J} := \{\Gamma \mid \Gamma \text{ is an ordered filter of } S \text{ as an ordered kernel of } \varsigma\},$$
$$\mathcal{K} := \{\Delta \mid \Delta \text{ is an ordered filter of } T\},$$

One can construct a bijective function ξ satisfying (25) as in Theorem 7.

Proof. (i) First, it is verified that

$$(\forall \Gamma \in \mathcal{J})(\varsigma^{-1}(\varsigma(\Gamma)) = \Gamma). \tag{34}$$

Let $\Gamma \in \mathcal{J}$. Since it is obvious that $\Gamma \subseteq \varsigma^{-1}(\varsigma(\Gamma))$, we prove its converse. Suppose $s \in \varsigma^{-1}(\varsigma(\Gamma))$. Then, $\varsigma(s) \in \varsigma(\Gamma)$ and so there is $u \in \Gamma$ such that $\varsigma(s) = \varsigma(u)$. By (3), one has

$$\iota_T \leq_T \varsigma(u) \to_S \varsigma(s) = \varsigma(s) \to_S \varsigma(s).$$

Then, since Γ is an ordered kernel of ς, we obtain $s \in \Gamma$ by (26), which entails that $\varsigma^{-1}(\varsigma(\Gamma)) \subseteq \Gamma$. Hence, (34) holds true. Using Theorem 5 (ii), one can introduce the homomorphism $\xi : \mathcal{J} \to \mathcal{K}$, $\Gamma \mapsto \xi(\Gamma)$ provided by $\xi(\Gamma) = \varsigma(\Gamma)$ for all $\Gamma \in \mathcal{J}$. Suppose that $\xi(\Gamma_1) = \xi(\Gamma_2)$ for all $\Gamma_1, \Gamma_2 \in \mathcal{J}$. Then, $\varsigma(\Gamma_1) = \varsigma(\Gamma_2)$, which implies from (34) that

$$\Gamma_1 = \varsigma^{-1}(\varsigma(\Gamma_1)) = \varsigma^{-1}(\varsigma(\Gamma_2)) = \Gamma_2.$$

Hence, ξ is injective. Note that ς is a surjective homomorphism. It is clear that $\Delta = \varsigma(\varsigma^{-1}(\Delta))$ and $\varsigma^{-1}(\Delta)$ is a filter of $\mathbf{S} := (S, \to_S, \iota_S, \leq_S)$ as an ordered kernel of ς for all $\Delta \in \mathcal{K}$, i.e., $\varsigma^{-1}(\Delta) \in \mathcal{J}$, and $\xi(\varsigma^{-1}(\Delta)) = \Delta$. Hence, ξ is surjective. Therefore, the function ξ is bijective and $\xi^{-1}(\Delta) = \varsigma^{-1}(\Delta)$.

(ii) The proof is almost the same as (i). We just note that here we need to use Theorem 19 (ii) in order to consider the homomorphism $\xi : \mathcal{J} \to \mathcal{K}$, $\Gamma \mapsto \xi(\Gamma)$ given by $\xi(\Gamma) = \varsigma(\Gamma)$ for all $\Gamma \in \mathcal{J}$. □

Note that in order for ξ to be a bijective function, the Γ in Theorem 7 and the Γ in Theorem 8 are a filter of S containing $\mathrm{kr}(\varsigma)$ and an ordered filter of S containing $\mathrm{kr}(\varsigma)$, respectively, whereas the Γ in (i) and the Γ in (ii) of Theorem 22 are a filter of S as an ordered kernel of ς and an ordered filter of S as an ordered kernel of ς, respectively, so as for ξ to be a bijective function.

3.4. Composite Functions with Ordered Kernels

Now, we address ordered kernels in composite functions. The following example illustrates that $\varsigma(\Theta)$ is not necessarily an ordered kernel of ξ in case Θ is an ordered kernel of ς.

Example 15. *Let* $\mathbf{S}$ *and* $\varsigma : S \to S$ *be the OBCI-algebra* $\mathbf{S}$ *and the map, respectively, in Example 2. Let Table 7 provide a set* $T := \{1, \frac{2}{3}, \frac{1}{3}, 0\}$ *with the binary operation "* $\to_T$ *".*

Table 7. Table for "$\to_T$".

$\to_T$	**1**	$\frac{2}{3}$	$\frac{1}{3}$	**0**
1	1	0	0	0
$\frac{2}{3}$	1	$\frac{2}{3}$	$\frac{1}{3}$	0
$\frac{1}{3}$	1	$\frac{2}{3}$	$\frac{2}{3}$	0
0	1	1	1	1

Let $\leq_T := \{(1,1), (\frac{2}{3}, \frac{2}{3}), (\frac{1}{3}, \frac{1}{3}), (0,0), (\frac{2}{3}, 1), (\frac{1}{3}, \frac{2}{3}), (0, \frac{1}{3})\}$. *Then,* $\boldsymbol{T} := (T, \to_T, \frac{2}{3}, \leq_T)$ *is an OBCI-algebra. Define a map* ξ *from* S *to* T *as follows:*

$$\xi : S \to T,\ k \mapsto \begin{cases} 1 & \text{if } k = 0, \\ \frac{2}{3} & \text{if } k = \sigma, \\ \frac{1}{3} & \text{if } k = \iota, \\ 0 & \text{if } k = 1. \end{cases} \tag{35}$$

Certainly, $\{1, \iota, \sigma\}$ *is an ordered kernel of* ς *and* $\varsigma(\{1, \iota, \sigma\}) = \{1, \iota, \sigma\}$. *However,* $\{1, \iota, \sigma\}$ *is not an ordered kernel of* ξ *since* $\iota_T = \frac{2}{3} \leq_T \xi(\iota) \to_T \xi(0) = \frac{1}{3} \to_T 1 = 1$ *but* $0 \notin \{1, \iota, \sigma\}$.

Notice that the map ξ in Example 15 is not a homomorphism because, e.g., $\xi(\sigma) \to_T \xi(\iota) = \frac{2}{3} \to_T \frac{1}{3} = \frac{1}{3} \neq \frac{2}{3} = \xi(\sigma) = \xi(\sigma \to_S \iota)$. Then, the following question naturally arises.

Does $\varsigma(\Theta)$ form an ordered kernel of ξ if ς and ξ are homomorphisms and Θ is an ordered kernel of ς?

Unfortunately, $\varsigma(\Theta)$ need not be an ordered kernel of ξ, as the following example shows.

Example 16. *Suppose that* $\boldsymbol{S}$ *and* $\boldsymbol{T}$ *are the OBCI-algebra* $\boldsymbol{S}$ *in Example 12 and* $\boldsymbol{T}$ *the OBCI-algebra* $\boldsymbol{S}$ *in Example 15, respectively. Define a map* ς *from* S *to* T *as follows:*

$$\varsigma : S \to T,\ k \mapsto \begin{cases} 1 & \text{if } k = 1, \\ \frac{1}{3} & \text{if } k = \frac{1}{2}, \\ 0 & \text{if } k = 0. \end{cases} \tag{36}$$

Let $\boldsymbol{U}$ *be the OBCI-algebra* $\boldsymbol{S}$ *in Example 3. Define a map* ξ *from* T *to* U *as follows:*

$$\xi : T \to U,\ k \mapsto \begin{cases} 1 & \text{if } k = 1, \\ \frac{3}{4} & \text{if } k = \frac{2}{3}, \\ \frac{1}{4} & \text{if } k = \frac{1}{3}, \\ 0 & \text{if } k = 0. \end{cases} \tag{37}$$

Clearly, both ς *and* ξ *satisfy* (15). *Also,* $\{1, \frac{1}{2}\}$ *is an ordered kernel of* ς. *However,* $\{1, \frac{1}{3}\}$ *does not form an ordered kernel of* ξ *since* $\varsigma(\{1, \frac{1}{2}\}) = \{1, \frac{1}{3}\}$ *and so* $\frac{2}{3} = \iota_T \notin \{1, \frac{1}{3}\} = \varsigma(\{1, \frac{1}{2}\})$.

We henceforth assume that $\mathbf{S} := (S, \to_S, \iota_S, \leq_S)$, $\mathbf{T} := (T, \to_T, \iota_T, \leq_T)$ and $\mathbf{U} := (U, \to_U, \iota_U, \leq_U)$ are OBCI-algebras such that $\iota_T = \varsigma(\iota_S)$ and $\iota_U = \varsigma(\iota_T)$, and $\varsigma : S \to T$ and $\xi : T \to U$ are injective maps from $\mathbf{S}$ to $\mathbf{T}$ and from $\mathbf{T}$ to $\mathbf{U}$, respectively.

Theorem 23. *If* Θ *is an ordered kernel of* $\mathbf{S}$ *on* ς *and* $\varsigma(\Theta)$ *is an ordered kernel of* $\mathbf{T}$ *on* ξ, *then* Θ *is an ordered kernel of* $\mathbf{S}$ *on* $\xi \circ \varsigma$.

Proof. Let Θ and $\varsigma(\Theta)$ be ordered kernels of $\mathbf{S}$ on ς and $\mathbf{T}$ on ξ, respectively. It is certain that $\iota_S = (\xi \circ \varsigma)^{-1}(\iota_U) \in \Theta$ since

$$(\xi \circ \varsigma)^{-1}(\iota_U) = \varsigma^{-1}(\xi^{-1}(\iota_U)) = \varsigma^{-1}(\iota_T) = \iota_S \in \Theta.$$

Let $s, t \in S$ be such that $s \in \Theta$ and $\iota_U \leq_U \xi \circ \varsigma(s) \to_U \xi \circ \varsigma(t)$. Then, $\varsigma(s) \in \varsigma(\Theta)$ and $\iota_U \leq_U \xi(\varsigma(s)) \to_U \xi(\varsigma(t))$. Hence,

$$\varsigma(t) \in \varsigma(\Theta)$$

by (26), and so $t \in \Theta$. This completes the proof. □

Corollary 8. *Let* ς *and* ξ *be homomorphisms. If* Θ *is an ordered kernel of* $\mathbf{S}$ *on* ς *and* $\varsigma(\Theta)$ *is an ordered kernel of* $\mathbf{T}$ *on* ξ, *then* Θ *is an ordered kernel of* $\mathbf{S}$ *on* $\xi \circ \varsigma$.

Theorem 24. *If* Θ *is an ordered kernel of* $\mathbf{S}$ *on* ς, $\xi^{-1}(\iota_U) = \iota_T$ *and* ξ^{-1} *is a homomorphism and satisfies the assertion*

$$(\forall s \in U)(\iota_U \leq_U s \ \Rightarrow\ \xi^{-1}(\iota_U) \leq_T \xi^{-1}(s)), \tag{38}$$

then Θ *is an ordered kernel of* $\mathbf{S}$ *on* $\xi \circ \varsigma$.

Proof. Let Θ be an ordered kernel of $\mathbf{S}$ on ς, $\xi^{-1}(\iota_U) = \iota_T$ and ξ^{-1} be a homomorphism and satisfy (38). As above, $\iota_S = (\xi \circ \varsigma)^{-1}(\iota_U) \in \Theta$. Let $s, t \in S$ be such that $s \in \Theta$ and $\iota_U \leq_U \xi \circ \varsigma(s) \to_U \xi \circ \varsigma(t)$. Then,

$$\xi^{-1}(\iota_U) \leq_T \xi^{-1}(\xi \circ \varsigma(s) \to_U \xi \circ \varsigma(t))$$

by (38), and so

$$\iota_T \leq_T \xi^{-1}(\xi \circ \varsigma(s)) \to_T \xi^{-1}(\xi \circ \varsigma(t))$$

by (15) and $\xi^{-1}(\iota_U) = \iota_T$. Since $\xi^{-1}(\xi \circ \varsigma(s)) \to_T \xi^{-1}(\xi \circ \varsigma(t)) = \iota_T \leq_T \varsigma(s) \to_T \varsigma(t)$ as above, we have

$$\iota_T \leq_T \varsigma(s) \to_T \varsigma(t),$$

and so $t \in \Theta$ by (26). This completes the proof. □

Corollary 9. *Let ς and ξ be homomorphisms. If Θ is an ordered kernel of $\mathbf{S}$ on ς, $\xi^{-1}(\iota_U) = \iota_T$ and ξ^{-1} is a homomorphism and satisfies the assertion* (38)*, then Θ is an ordered kernel of $\mathbf{S}$ on $\xi \circ \varsigma$.*

4. Conclusions

As the answer to a future work in [15], we introduced an ordered generalization of kernels. To be more exact, first, the notion of ordered kernels of OBCI-algebras was introduced. Properties of the ordered kernels related to (ordered) subalgebras, (ordered) filters and functional compositions were then discussed in the homomorphism environments.

Because of space limitations, we could not introduce more results such as direct products of ordered kernels and ordered kernels in ordered maps and ordered homomorphisms. We promise that we will provide such results in other manuscripts.

Some problems still remain open. First, related to specific (ordered) filters and (ordered) kernels, homomorphisms and ordered kernels have to be dealt with. Second, the study of homomorphisms and ordered kernels of OBCI-algebras needs be extended to related results such as intuitionistic fuzzy OBCI-algebras. In particular, one reviewer suggested a kernel defined by a relation. His idea is a new one in the sense that it is based on the equation of two elements under a homomorphism, while the kernel of BCI-algebras is defined based on identity (see (20)). We have to investigate his suggestion more exactly and address the relationship between this definition and the definition of ordered kernel.

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References

1. Jansana, R.; Moraschini, T. Relational Semantics, Ordered Algebras, and Quantifiers for Deductive Systems. In Proceedings of the Logic, Algebra and Truth Degrees, Bern, Switzerland, 28–31 August 2018.
2. Galatos, N.; Jipsen, P.; Kowalski, T.; Ono, H. *Residuated Lattices: An Algebraic Glimpse at Substructural Logics*; Elsevier: Amsterdam, The Netherlands, 2007.
3. Galatos, N.; Ono, H. Cut elimination and strong separation for substructural logics. *Ann. Pure Appl. Log.* **2010**, *161*, 1097–1133. [CrossRef]
4. Dunn, J.M. Partial gaggles applied to logics with restricted structural rules. In *Substructural Logics*; Schroeder-Heister, P., Došen, K., Eds.; Clarendon: Oxford, UK, 1993; pp. 63–108.
5. Dunn, J.M.; Hardegree, G. *Algebraic Methods in Philosophical Logic*; Oxford Univ Press: Oxford, UK, 2001.
6. Cintula, P. Weakly Implicative (Fuzzy) Logics I: Basic properties. *Arch. Math. Log.* **2006**, *45*, 673–704. [CrossRef]
7. Cintula, P.; Noguera, C. A general framework for mathematical fuzzy logic. In *Handbook of Mathe-Matical Fuzzy Logic, Volume 1*; Cintula, P., Hájek, P., Noguera, C., Eds.; College Publications: London, UK, 2011; pp. 103–207.
8. Cintula, P.; Noguera, C. Implicational (semilinear) logics I: A new hierarchy. *Arch. Math. Log.* **2010**, *49*, 417–446. [CrossRef]
9. Cintula, P.; Noguera, C. Implicational (semilinear) logics III: Completeness properties. *Arch. Math. Log.* **2018**, *57*, 391–420. [CrossRef]
10. Hu, Q.P.; Li, X. On BCH-algebras. *Math. Semin. Notes* **1983**, *11*, 313–320.
11. Hu, Q.P.; Li, X. On proper BCH-algebras. *Math. Japo.* **1985**, *30*, 659–661.
12. Neggers, J.; Kim, H.S. On B-algebras. *Mate. Vesn.* **2002**, *54*, 21–29.
13. Neggers, J.; Kim, H.S. A fundamental Theorem of B-homomorphism for B-algebras. *Int. Math. J.* **2002**, *2*, 215–219.
14. Yang, E.; Roh, E.H.; Jun, Y.B. An introduction to the theory of OBCI-algebras. *AIMS Math.* **2024**, *9*, 36336–36350. [CrossRef]
15. Yang, E.; Roh, E.H.; Jun, Y.B. Homomorphisms of OBCI-algebras. *Afr. Mat.* **2025**, *36*, 30. [CrossRef]

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