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Even-Order Pascal Tensors Are Positive-Definite

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Abstract: In this paper, we show that even-order Pascal tensors are positive-definite, and odd-order Pascal tensors are strongly completely positive. The significance of these is that our induction proof method also holds for some other families of completely positive tensors, whose construction satisfies certain rules, such that the inherence property holds. We show that for all tensors in such a family, even-order tensors would be positive-definite, and odd-order tensors would be strongly completely positive, as long as the matrices in this family are positive-definite. In particular, we show that even-order generalized Pascal tensors would be positive-definite, and odd-order generalized Pascal tensors would be strongly completely positive, as long as generalized Pascal matrices are positive-definite. We also investigate even-order positive-definiteness and odd-order strong complete positivity for fractional Hadamard power tensors. Furthermore, we study determinants of Pascal tensors. We prove that the determinant of the mth-order two-dimensional symmetric Pascal tensor is equal to the mth power of the factorial of $m-1$.

Keywords: Pascal tensor; positive-definite tensor; completely positive tensor; strongly completely positive tensor; determinant

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1. Introduction

The symmetric Pascal matrix has the form $P = \left(\frac{(i+j-2)!}{(i-1)!(j-1)!}\right)$. Its entries are the entries of the Pascal triangle, i.e., the coefficients of binomial expansion. It has many interesting properties and applications in numerical analysis, filter design, image and signal processing, probability, combinatorics, numerical analysis, chaotic system, and electrical engineering [1–4]. In 2016, Luo and Qi extended symmetric Pascal matrices to symmetric Pascal tensors and proved that they are completely positive tensors. Symmetric Pascal tensors are also studied in [5]. An mth-order n-dimensional Pascal tensor has the form

$$\mathcal{P} = \left(\frac{(i_1 + \ldots i_m - m)!}{(i_1 - 1)! \ldots (i_m - 1)!}\right), \tag{1}$$

where $i_1, \ldots, i_m = 1, \ldots, n, m \geq 3, n \geq 2$. In this paper, for brevity, we simply call P a Pascal matrix, and $\mathcal{P}$ a Pascal tensor.

Besides being a completely positive tensor, does a Pascal tensor have any other interesting properties? Structured tensors are very important in tensor analysis and applications [5].

As Pascal tensors have a nice structure, the study of Pascal tensors may serve as a prototype for studying more complicated structured tensors. This is the motivation of this paper.

For even-order tensors, positive semi-definiteness, positive-definiteness and sum-of-squares (SOS) are the three most important properties. Pascal tensors are completely positive tensors. Even order completely positive tensors are SOS and positive semi-definite tensors. Hence, even-order Pascal tensors are SOS and positive semi-definite tensors. It is well known that Pascal matrices are positive-definite. Thus, a core problem for Pascal tensors is whether even-order Pascal tensors are positive-definite or not. To the best of our knowledge, this is an unsolved problem in structured tensor study. In the next section, by induction, we show that even-order Pascal tensors are positive-definite.

For odd-order tensors, strong complete positiveness is a property somewhat corresponding to positive-definiteness in the even-order case [6–8]. In Section 3, we show that odd-order Pascal tensors are strongly completely positive.

Actually, these two properties are valid for broader families of completely positive tensors. We state such a theorem in Section 4, and show that even-order generalized Pascal tensors are positive-definite, and odd-order generalized Pascal tensors are strongly completely positive, as long as generalized Pascal matrices are positive-definite. We also investigate these two properties for the fractional Hadamard power tensors.

In 2004, Edelman and Strang showed that the determinant of a Pascal matrix is always 1 [2]. This is an exceptionally elegant property. A natural question is what is the determinant of a Pascal tensor? The determinant of a symmetric tensor was introduced in [9]. Suppose that an mth-order n-dimensional real tensor is denoted as $\mathcal{A} = \left(a_{i_1 \dots i_m}\right)$, where $i_1, \dots, i_m = 1, \dots, n$. Let $\mathbf{x} \in \mathbb{R}^n$. Then, $\mathcal{A}\mathbf{x}^{m-1}$ is a vector in $\mathbb{R}^n$, whose ith component is

$$\sum_{i_2, \dots, i_m = 1, \dots, n} a_{i i_2 \dots i_m} x_{i_2} \dots x_{i_m}.$$

The determinant of $\mathcal{A}$, denoted as $\det(\mathcal{A})$, is the resultant of $\mathcal{A}\mathbf{x}^{m-1} = \mathbf{0}$. For the determinant of such a tensor, also see [5,10].

In Sections 3 and 6, we explore this problem. Denote the determinant of the mth-order n-dimensional Pascal tensor by $f(m,n)$. If we count matrices as second-order tensors, we have $f(2,n) \equiv 1$ for $n \geq 2$. In Section 3, we explore the value of $f(m,2)$. We multiply the Sylvester–Pascal matrix of the mth-order two-dimensional Pascal tensor by a certain pivoting matrix. Then, we take the LU decomposition of the reshaped Sylvester–Pascal matrix. By identifying the formulas of the diagonal entries of the upper triangular matrix of the LU decomposition, and considering the sign changes, we prove that $f(m,2) = [(m-1)!]^m$. In Section 6, we explore the value of $f(m,n)$ for $n \geq 3$. We make a conjecture that $f(m,n)$ is the mth power of an integer function, and is an integer factor of $f(m,n+1)$. In particular, we make a conjecture on the formula of $f(m,3)$ for $m \geq 3$. This poses a challenge to people.

Some final remarks are made in Section 7.

In this paper, $f, g, h, i, j, k, m, n, r$ and s are all positive integer values.

2. Even-Order Pascal Tensors Are Positive-Definite

Let $\mathcal{A} = \left(a_{i_1 \dots i_m}\right)$ be an mth-order n-dimensional symmetric tensor, where $m \geq 2$ is an even number. Then, $F_{\mathcal{A}}(\mathbf{x}) = \mathcal{A}\mathbf{x}^m$ defines a function $F_{\mathcal{A}} : \mathbb{R}^n \to \mathbb{R}$. If $F_{\mathcal{A}}(\mathbf{x}) \geq 0$ for $\mathbf{x} \in \mathbb{R}^n$, then $\mathcal{A}$ is called a positive semi-definite tensor. If $F_{\mathcal{A}}(\mathbf{x})$ can be expressed as SOS polynomials of $x_1, \dots, x_n$, then $\mathcal{A}$ is called an SOS tensor. If $F_{\mathcal{A}}(\mathbf{x}) > 0$ for $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{x} \neq \mathbf{0}$, then $\mathcal{A}$ is called a positive-definite tensor. An SOS tensor is a positive semi-definite tensor, but not vice versa. Similarly, a positive-definite tensor is a positive semi-definite

tensor, but not vice versa. These three properties are the most important properties of even-order symmetric tensors [5].

Let $\mathcal{A}$ be a symmetric tensor of order m and dimension n. If there are nonnegative vectors $\mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(r)} \in \mathbb{R}^n$ such that

$$\mathcal{A} = \left(\mathbf{u}^{(1)}\right)^m + \cdots + \left(\mathbf{u}^{(r)}\right)^m, \tag{2}$$

then $\mathcal{A}$ is called a completely positive tensor. Furthermore, if $\left\{\mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(r)}\right\}$ spans $\mathbb{R}^n$, then $\mathcal{A}$ is called a strongly completely positive tensor [5,7].

A complex number λ is called an eigenvalue of $\mathcal{A}$ with an eigenvector $\mathbf{x}$ if $\mathbf{x} \in \mathbb{C}^n$, $\mathbf{x} \neq \mathbf{0}$ and

$$\mathcal{A}\mathbf{x}^{m-1} = \lambda \mathbf{x}^{[m-1]},$$

where $\mathbf{x}^{[m-1]} \in \mathbb{C}^n$ and its ith component is x_i^{m-1}. If $\mathbf{x} \in \mathbb{R}^n$, then λ is real and called an H-eigenvalue of $\mathcal{A}$. An even-order symmetric tensor always has H-eigenvalues. An even-order symmetric tensor is positive semi-definite if and only if it has no negative H-eigenvalues. An even-order symmetric tensor is positive-definite if and only if it has no non-positive H-eigenvalues. The determinant of a symmetric tensor is equal to the product of all of its eigenvalues. These are the main contents of the spectral theory of symmetric tensors [5,9].

Theorem 1. *Let $\mathcal{P}$ be an even-order Pascal tensor with-order m and dimension n. Then, it is a positive semi-define tensor and an SOS tensor. Furthermore, it is a positive-definite tensor as long as its determinant $f(m,n) \neq 0$.*

Proof. By Proposition 6.19 of [5], the Pascal tensor $\mathcal{P}$ is a completely positive tensor. Now, assume that $\mathcal{P}$ has the form of (2). Then

$$F_{\mathcal{P}}(\mathbf{x}) = \sum_{k=1}^{r} \left(\left(\mathbf{u}^{(k)}\right)^\top \mathbf{x}\right)^m. \tag{3}$$

This shows an even-order Pascal tensor is an SOS tensor, hence a positive semi-definite tensor.

If, in addition, $f(m,n) \neq 0$, then $\mathcal{P}$ does not have zero H-eigenvalues. The last conclusion of this theorem follows from the fact that a positive semi-define tensor is positive-definite if and only if it does not have zero H-eigenvalues. This completes the proof. □

The last conclusion of this theorem shows the relation between the positive-definiteness problem and the determinant problem of Pascal tensors.

The following is a sufficient and necessary condition for an even-order completely positive tensor to be positive-definite.

Theorem 2. *Suppose that $\mathcal{A}$ is a completely positive tensor of order m and dimension n, where m is even. Then, $\mathcal{A}$ is positive-definite if and only if $\mathcal{A}$ is strongly completely positive.*

Proof. Suppose that $\mathcal{A}$ is completely positive with the form (2).

Suppose that $\mathcal{A}$ is strongly and completely positive. Then, for any $\mathbf{x} \neq \mathbf{0}$, in (3), there is at least one k such that

$$\left(\left(\mathbf{u}^{(k)}\right)^\top \mathbf{x}\right)^m > 0,$$

as $\left\{\mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(r)}\right\}$ spans $\mathbb{R}^n$. This shows that $\mathcal{A}$ is positive-definite.

On the other hand, suppose that $\mathcal{A}$ is not strongly completely positive, i.e., $\left\{\mathbf{u}^{(1)},\ldots,\mathbf{u}^{(r)}\right\}$ does not span $\mathbb{R}^n$. Then, we may find $\mathbf{x}\in\mathbb{R}^n$, $\mathbf{x}\neq\mathbf{0}$, such that $\left(\mathbf{u}^{(k)}\right)^\top\mathbf{x}=0$ for $k=1,\ldots,r$. This results $F_{\mathcal{A}}(\mathbf{x})=0$, i.e., $\mathcal{A}$ is not positive-definite. □

This theorem may also be the resulted of Theorem 3.9 of [7], and the main content of spectral theory of symmetric tensors stated before Theorem 1.

We now present the main result of this section.

Theorem 3. *Even-order Pascal tensors are positive-definite.*

Proof. Fix $n\geq 2$. We prove this by induction on m.

For $m=2$, we know that Pascal matrices are positive-definite.

Suppose that the mth-order n-dimensional Pascal tensor is positive-definite, where m is even. Consider the $(m+2)$th-order n-dimensional Pascal tensor $\mathcal{P}$. Since $\mathcal{P}$ is a completely positive tensor, we have nonnegative vectors $\mathbf{u}^{(1)},\ldots,\mathbf{u}^{(r)}\in\mathbb{R}^n$ such that

$$\mathcal{P}=\left(\mathbf{u}^{(1)}\right)^{m+2}+\cdots+\left(\mathbf{u}^{(r)}\right)^{m+2}. \tag{4}$$

Now, fix the first two indices of the entries of $\mathcal{P}$ as $i_1=i_2\equiv 1$. Then, we obtain an mth-order n-dimensional sub-tensor of $\mathcal{P}$. Denote it as $\mathcal{P}_{11}$. Then, we have

$$\mathcal{P}_{11}=\left(u_1^{(1)}\right)^2\left(\mathbf{u}^{(1)}\right)^m+\cdots+\left(u_1^{(r)}\right)^2\left(\mathbf{u}^{(r)}\right)^m,$$

i.e.,

$$\mathcal{P}_{11}=\left(\mathbf{v}^{(1)}\right)^m+\cdots+\left(\mathbf{v}^{(r)}\right)^m, \tag{5}$$

where

$$\mathbf{v}^{(k)}=\left(u_1^{(k)}\right)^{\frac{2}{m}}\mathbf{u}^{(k)},$$

are nonnegative vectors, for $k=1,\ldots,r$. By the structure of Pascal tensors, $\mathcal{P}_{11}$ is nothing else, but the mth-order n-dimensional Pascal tensor. Then, (5) is its form as a completely positive tensor. By our induction assumption, $\mathcal{P}_{11}$ is positive-definite. By Theorem 2, $\mathcal{P}_{11}$ is strongly completely positive. Thus, $\left\{\mathbf{v}^{(1)},\ldots,\mathbf{v}^{(r)}\right\}$ spans $\mathbb{R}^n$. Then, $\left\{\mathbf{u}^{(1)},\ldots,\mathbf{u}^{(r)}\right\}$ also spans $\mathbb{R}^n$. This implies that $\mathcal{P}$ is strongly and completely positive. By Theorem 2, $\mathcal{P}$ is positive-definite. This completes our induction proof. □

3. Pascal Tensors Are Strongly Completely Positive

We have the following theorem.

Theorem 4. *Pascal tensors are strongly completely positive.*

Proof. By Theorems 2 and 3, even-order Pascal tensors are completely positive. We now consider odd-order Pascal tensors.

Suppose that $\mathcal{P}$ is the $(m+1)$th-order n-dimensional Pascal tensor, where $m,n\geq 2$ and m is even. Assume that it is not strongly or completely positive. By Theorem 3.9 of [7], $\mathcal{P}$ must have a zero H-eigenvalue. Thus, we have $\mathbf{x}\in\mathbb{R}^n$, $\mathbf{x}\neq\mathbf{0}$, such that $\mathcal{P}\mathbf{x}^m=\mathbf{0}$. This implies that $\mathcal{P}_1\mathbf{x}^m=0$, where $\mathcal{P}_1$ is the mth-order n-dimensional tensor obtained by fixing the first index of the entries of $\mathcal{P}$ as $i_1\equiv 1$. By the structure of Pascal tensors, $\mathcal{P}_1$ is exactly the mth-order n-dimensional Pascal tensor. Then, by the definition of even-order positive-definite tensors, $\mathcal{P}_1\mathbf{x}^m=0$ and $\mathbf{x}\neq\mathbf{0}$ implies that $\mathcal{P}_1$ is not positive-definite. This conflicts with Theorem 3. Hence, $\mathcal{P}$ must be strongly completely positive. □

4. An Inherence Property for Some Completely Positive Tensor Families

Actually, Pascal tensors are prototypes of some families of completely positive tensors. The study in Sections 2 and 3 shows that since Pascal matrices are positive-definite and the construction of Pascal tensors satisfies certain rules, even-order Pascal tensors are positive-definite, and odd-order Pascal tensors are strongly completely positive. The significance is that this inherence property may hold for some other completely positive tensor families. In this section, we show that for a certain completely positive tensor family, if its construction satisfies certain rules, then even-order tensors in this family would be positive-definite, and odd-order tensors are strongly completely positive, as long as matrices in this family are positive-definite.

Actually, such an inherence property reminds us of strong Hankel tensors [5]. In general, even-order Hankel tensors may not be positive-definite. However, if the Hankel matrix, generated by the same vector that generates a Hankel tensor, is positive semi-definite, then that Hankel tensor is called a strong Hankel tensor. We know that even-order strong Hankel tensors are positive semi-definite and SOS tensors [5].

We have the following inherence theorem for completely positive tensors.

Theorem 5. *(Inherence Theorem) Suppose that we have a family C of completely positive tensors. Assume that C satisfies the following three properties.*

1. *For each pair of m and n with $m, n \geq 2$, there is at least one member in C.*
2. *For $m = 2$, all the member matrices in C are positive-definite.*
3. *If we fix the first index of the entries of an mth-order n-dimensional tensor $\mathcal{A}$ in C as $i_1 = 1$, then we have a sub-tensor $\mathcal{A}_1$ of $\mathcal{A}$. Then, $\mathcal{A}_1 \in C$.*

Then, all the tensors in C are strongly completely positive, and all the even-order tensors in C are positive-definite.

Proof. The results may be obtained through the same induction process as in Theorems 3 and 4. Hence, we omit the details for brevity. □

The followings are some examples to apply this theorem.

1. Generalized Pascal tensors. Let $\mathbf{c} = (c_1, \ldots, c_n) \in \mathbb{R}^n_+$ be a nonnegative vector and let the mth-order n-dimensional tensor $\mathcal{A} = \left(a_{i_1 \ldots i_m}\right)$ be defined by

$$a_{i_1 \ldots i_m} = \frac{\Gamma(c_{i_1} + c_{i_2} + \cdots + c_{i_m} + 1)}{\Gamma(c_{i_1} + 1)\Gamma(c_{i_2} + 1) \cdots \Gamma(c_{i_m} + 1)}, \tag{6}$$

then $\mathcal{A}$ is a generalized Pascal tensor and $\mathbf{c}$ is called a generating vector of the generalized Pascal tensor $\mathcal{A}$. If $c_i = i - 1$ for any $i = 1 \ldots, n$, then $\mathcal{A}$ is the Pascal tensor [5].

Theorem 6. *Suppose that $\mathbf{c}$ satisfies $c_1 = 0$ and the generalized Pascal matrix is positive-definite. Then, the generalized Pascal tensors are strongly completely pos. Even-order generalized Pascal tensors are positive-definite.*

Proof. By Proposition 6.19 in [5], generalized Pascal tensors are completely positive.

Because there are no points z at which $\Gamma(z) = 0$, the generalized Pascal tensor class is nonempty for any $m, n \geq 2$. Furthermore, by fixing $i_1 = 1$, we have $\mathcal{A}_1$ is an $(m-1)$th-order n dimensional generalized Pascal tensor with the same generating vector $\mathbf{c}$. Then, the results of this theorem follow directly from Theorem 5.

This completes the proof. □

2. The fractional Hadamard power tensors.

For any given nonnegative tensor $\mathcal{A} = (a_{i_1\ldots i_m})$, its fractional Hadamard power is defined by the form $\mathcal{A}^{\circ\alpha} = (a^{\alpha}_{i_1\ldots i_m})$. For factional Hadamard powers, the complete positivity may not be preserved, as shown by a counterexample in Qi and Luo [5].

Theorem 7. *Suppose that $\alpha > 0$ and $\mathbf{c}$ satisfy $c_1 = 0$ and the fractional Hadamard power of the generalized Pascal matrix is positive-definite. Then, the fractional Hadamard power of the generalized Pascal tensor $\mathcal{A}^{\circ\alpha}$ is strongly completely positive. If, in addition, m is even, then $\mathcal{A}^{\circ\alpha}$ is positive-definite.*

Proof. By Corollary 6.22 in [5], the fractional Hadamard power of the generalized Pascal tensor is completely positive. Then, the conclusions of this theorem follow from Theorems 5 and 6. □

Theorem 8. *Let $\mathcal{A} = (a_{i_1\ldots i_m}) \in S_{m,n}$ be a generalized Hilbert tensor defined by*

$$a_{i_1\ldots i_m} = \frac{1}{i_1 + \cdots + i_m - m + c},$$

where $c > 0$ is a positive number. Suppose that $\alpha > 0$ and c satisfy $c_1 = 0$ and the fractional Hadamard power of the generalized Hilbert matrix is positive-definite. Then, the fractional Hadamard power of the generalized Hilbert tensor $\mathcal{A}^{\circ\alpha}$ is strongly completely positive. If, in addition, m is even, then $\mathcal{A}^{\circ\alpha}$ is positive-definite.

Proof. The generalized Hilbert tensor is a Cauchy tensor with $c_i = i - 1 + \frac{c}{m}$ for any $i = 1, \ldots, n$. By Theorem 6.21 in [5], the fractional Hadamard powers of the generalized Hilbert tensors are completely positive.

It follows from the definition that the generalized Hilbert tensor is well defined for any $m, n \geq 2$. Furthermore, by fixing $i_1 = 1$, we have $\mathcal{A}_1$ is an $(m-1)$th-order n dimensional generalized Hilbert tensor with the same parameter c. Then, the results of this theorem follow directly from Theorem 5.

This completes the proof. □

5. Determinants: The Case That $n = 2$

By the Sylvester Formula [9,11] and (1), for $m \geq 3$, we have $f(m, 2) = \det(S_m)$, where

$$S_m = \begin{pmatrix}
1 & a_1 & a_2 & \cdots & a_{m-2} & a_{m-1} & 0 & 0 & \cdots & 0 \\
0 & 1 & a_1 & \cdots & a_{m-3} & a_{m-2} & a_{m-1} & 0 & \cdots & 0 \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\
0 & 0 & 0 & \cdots & 1 & a_1 & a_2 & a_3 & \cdots & a_{m-1} \\
1 & b_1 & b_2 & \cdots & b_{m-2} & b_{m-1} & 0 & 0 & \cdots & 0 \\
0 & 1 & b_1 & \cdots & b_{m-3} & b_{m-2} & b_{m-1} & 0 & \cdots & 0 \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\
0 & 0 & 0 & \cdots & 1 & b_1 & b_2 & b_3 & \cdots & b_{m-1}
\end{pmatrix},$$

$a_i = \binom{m-1}{i} i!$ and $b_i = \binom{m-1}{i}(i+1)!$ for $i = 1, \ldots, m-1$. We call the $2(m-1) \times 2(m-1)$ matrix S_m a Sylvester–Pascal matrix.

By direct computation, we have

$$f(3, 2) = \begin{vmatrix} 1 & 2 & 2 & 0 \\ 0 & 1 & 2 & 2 \\ 1 & 4 & 6 & 0 \\ 0 & 1 & 4 & 6 \end{vmatrix} = 8 = (2!)^3,$$

$$f(4,2)=\begin{vmatrix} 1 & 3 & 6 & 6 & 0 & 0 \\ 0 & 1 & 3 & 6 & 6 & 0 \\ 0 & 0 & 1 & 3 & 6 & 6 \\ 1 & 6 & 18 & 24 & 0 & 0 \\ 0 & 1 & 6 & 18 & 24 & 0 \\ 0 & 0 & 1 & 6 & 18 & 24 \end{vmatrix} = 6^4 = (3!)^4$$

and

$$f(5,2)=\begin{vmatrix} 1 & 3 & 6 & 6 & 0 & 0 \\ 0 & 1 & 3 & 6 & 6 & 0 \\ 0 & 0 & 1 & 3 & 6 & 6 \\ 1 & 6 & 18 & 24 & 0 & 0 \\ 0 & 1 & 6 & 18 & 24 & 0 \\ 0 & 0 & 1 & 6 & 18 & 24 \end{vmatrix} = 24^5 = (4!)^5$$

From the above fact, we may think that $f(m,2) = [(m-1)!]^m$.

Aiming at proving this, we consider the LU decomposition of the Sylvester–Pascal matrix. We first define the row permutation matrix P_m to reshape S_m as follows,

$$T_m \equiv P_m S_m = \begin{pmatrix} 1 & b_1 & b_2 & \cdots & b_{m-2} & b_{m-1} & 0 & 0 & \cdots & 0 \\ 1 & a_1 & a_2 & \cdots & a_{m-2} & a_{m-1} & 0 & 0 & \cdots & 0 \\ 0 & 1 & b_1 & \cdots & b_{m-3} & b_{m-2} & b_{m-1} & 0 & \cdots & 0 \\ 0 & 1 & a_1 & \cdots & a_{m-3} & a_{m-2} & a_{m-1} & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 & b_1 & b_2 & b_3 & \cdots & b_{m-1} \\ 0 & 0 & 0 & \cdots & 1 & a_1 & a_2 & a_3 & \cdots & a_{m-1} \end{pmatrix},$$

where $a_i = \binom{m-1}{i} i!$ and $b_i = \binom{m-1}{i}(i+1)!$ for $i = 1, \ldots, m-1$. Then, the leading principal minors are not zeros and the LU decomposition of T_m always exists. To move the m-th row to the first row while keeping the order of the first to $(m-1)$-th rows unchanged, it requires $(m-1)$-th row switches, and to move the $(m+1)$-th row to the third row while keeping the order of the third to m-th rows unchanged, it requires $(m-2)$-th row switches. We repeat this process. It requires totally $\frac{m(m-1)}{2}$ times of row permutations to formulate PS_m stated above. Therefore, $\det(S_m) = (-1)^{\frac{m(m-1)}{2}} \det(T_m)$.

Let

$$T_m = L_m U_m,$$

where L_m is a lower triangular matrix with all 1 diagonal entries. Then, $f(m,2)$ is equal to the product of the diagonal entries of the upper triangular matrix U_m multiplied by $(-1)^{\frac{m(m-1)}{2}}$. Here are the forms of U_m for $m = 3, 4, 5$:

$$U_3 = \begin{bmatrix} 1 & 4 & 6 & 0 \\ 0 & -2 & -4 & 0 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 2 \end{bmatrix},$$

$$U_4 = \begin{bmatrix} 1 & 6 & 18 & 24 & 0 & 0 \\ 0 & -3 & -12 & -18 & 0 & 0 \\ 0 & 0 & 2 & 12 & 24 & 0 \\ 0 & 0 & 0 & 6 & 18 & 0 \\ 0 & 0 & 0 & 0 & 6 & 24 \\ 0 & 0 & 0 & 0 & 0 & -6 \end{bmatrix}$$

and

$$U_5 = \begin{bmatrix} 1 & 8 & 36 & 96 & 120 & 0 & 0 & 0 \\ 0 & -4 & -24 & -72 & -96 & 0 & 0 & 0 \\ 0 & 0 & 2 & 8 & 72 & 120 & 0 & 0 \\ 0 & 0 & 0 & 12 & 72 & 144 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 48 & 120 & 0 \\ 0 & 0 & 0 & 0 & 0 & -24 & -96 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 24 & 120 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 24 \end{bmatrix}.$$

We see that $\det(T_m) = \det(U_m)$ is equal to the product of the diagonal entries of U_m. Then, we have the following claim.

Claim 5.1. Suppose the LU decomposition of $T_m = L_m U_m$, where $L_m = (l_{ij}) \in \mathbb{R}^{2(m-1)\times 2(m-1)}$ and $U_m = (u_{ij}) \in \mathbb{R}^{2(m-1)\times 2(m-1)}$. Then, we have $u_{2i-1,2i-1} = i!$ and $u_{2i,2i} = (-1)^i \binom{m-1}{i} i!$ for $i = 1, \ldots, m-1$.

By the above discussion, we see that this conclusion is true for $m = 2, 3, 4, 5$. Assume that $m \geq 6$.

Write $U_m = (u_{ij})$, $T_m = (t_{ij})$ and $L_m = (l_{ij})$ for $i, j = 1, \ldots, 2m-2$. Then, $l_{ii} = 1$, $l_{ij} = 0$ for $j > i$ and $u_{ij} = 0$ for $j < i$. By $T_m = L_m U_m$, for $i, j = 1, \ldots, 2m-2$, we have

$$t_{ij} = \sum_{k=1}^{2m-2} l_{ik} u_{kj} = \sum_{k=1}^{\min\{i,j\}} l_{ik} u_{kj}. \tag{7}$$

Letting $i = 1$ in (7), we have $u_{1j} = t_{1j}$ for $j = 1, \ldots, 2m-2$. This implies that $u_{11} = t_{11} = 1$.

Now, letting $i \geq 2$ and $j = 1$ in (7), we have $t_{i1} = l_{i1} u_{11} = l_{i1}$. However, $t_{21} = 1$ and $t_{i1} = 0$ for $i \geq 3$. Thus, $l_{21} = 1$ and $l_{i1} = 0$ for $i \geq 3$.

Consider $i = 2$ and $j \geq 2$ in (7). We have $t_{2j} = l_{21} u_{1j} + l_{22} u_{2j} = t_{1j} + u_{2j}$. This implies that

$$u_{2j} = t_{2j} - t_{1j} = \binom{m-1}{j-1} (j-1)!(1-j),$$

for $j = 2, \ldots, m$, and $u_{2j} = t_{2j} - t_{1j} = 0$ for $j = m+1, \ldots, 2m-2$. Then, $u_{22} = 1 - m$.

For $i \geq 3$ and $j = 2$, by (7), we have $t_{32} = l_{31} u_{12} + l_{32} u_{22} = l_{32}(1-m)$. Thus, $l_{32} = \frac{t_{32}}{1-m}$. Since $t_{32} = 1$, $l_{32} = \frac{-1}{m-1}$. Similarly, we have $l_{42} = \frac{-1}{m-1}$ and $l_{i2} = 0$ for any $i \geq 5$.

For $i = 3$ and $j \geq 3$, by (7), we have $t_{3j} = l_{31} u_{1j} + l_{32} u_{2j} + l_{33} u_{3j} = \frac{u_{2j}}{1-m} + u_{3j}$, i.e.,

$$u_{3j} = t_{3j} - \frac{u_{2j}}{1-m},$$

$$u_{33} = t_{33} - \frac{u_{23}}{1-m} = 2(m-1) + \frac{\binom{m-1}{2} 2! \times 2}{1-m} = 2.$$

For $i \geq 4$ and $j = 3$, we could derive $l_{43} = \frac{3-m}{2}$, $l_{53} = \frac{1}{2}$, $l_{63} = \frac{1}{2}$, and $l_{i3} = 0$ for any $i \geq 7$.

By repeating this process, we could derive that for $i = 1, \ldots, m-1$, $j = 1, \ldots, 2m-2$,

$$u_{2i-1,j} = \begin{cases} i!, & \text{if } j = 2i-1; \\ \binom{m-i}{j-2i+1} (j-i+1)!, & \text{if } 2i \leq j \leq m+i-1; \\ 0, & \text{otherwise,} \end{cases}$$

$$u_{2i,j} = \begin{cases} (-1)^i \binom{m-1}{j-i} \binom{j-i}{i} (j-i)!, & \text{if } 2i \leq j \leq m+i-1; \\ 0, & \text{otherwise,} \end{cases}$$

and for $i = 1, \ldots, 2m-2$,

$$l_{ij} = \begin{cases} 1 & \text{if } j = i; \\ \frac{|X_{ij}|}{u_{11} \ldots u_{jj}}, & \text{if } j+1 \le i \le 2j; \\ 0, & \text{otherwise,} \end{cases} \tag{8}$$

where X_{ij} is a $j \times j$ matrix,

$$X_{ij} = \begin{bmatrix} u_{11} & u_{12} & \ldots & u_{1,j-1} & u_{1j} \\ 0 & u_{22} & \ldots & u_{2,j-1} & u_{2j} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & u_{j-1,j-1} & u_{j-1,j} \\ t_{i1} & t_{i2} & \ldots & t_{i,j-1} & t_{ij} \end{bmatrix}.$$

By matrix multiplication, we have $L_m U_m = T_m$. Then, for $i = 1, \ldots, m-1$, $u_{2i-1,2i-1} = i!$ and $u_{2i,2i} = (-1)^i \binom{m-1}{i} i!$.

By (8), the expression of the lower-triangular matrix L is more complicated than that of U. We list the upper left corner of L as follows,

$$L = \begin{bmatrix} 1 & & & & & & & \\ 1 & 1 & & & & & & \\ & \frac{1}{u_{22}} & 1 & & & & & \\ & \frac{1}{u_{22}} & \frac{3-m}{u_{33}} & 1 & & & & \\ & & \frac{1}{u_{33}} & \frac{4-m}{u_{44}} & 1 & & & \\ & & \frac{1}{u_{33}} & \frac{5-2m}{u_{44}} & \frac{m^2-6m+11}{u_{55}} & 1 & & \\ & & & \frac{1}{u_{44}} & \frac{7-m}{u_{55}} & \frac{m^2-7m+18}{u_{66}} & \frac{(m-5)(-m^2+5m-10)}{u_{77}} & \ldots \\ & & & \frac{1}{u_{44}} & \frac{8-2m}{u_{55}} & \frac{3m^2-17m+26}{u_{66}} & \frac{m^2-11m+46}{u_{77}} & \ldots \\ & & & & \frac{1}{u_{55}} & \frac{10-2m}{u_{66}} & \frac{3m^2-25m+58}{u_{77}} & \ldots \\ & & & & \frac{1}{u_{55}} & \frac{11-3m}{u_{66}} & \frac{14-2m}{u_{77}} & \ldots \\ & & & & & \frac{1}{u_{66}} & \frac{15-3m}{u_{77}} & \ldots \\ & & & & & \frac{1}{u_{66}} & \frac{1}{u_{77}} & \ldots \\ & & & & & & \frac{1}{u_{77}} & \ldots \\ & & & & & & & \ldots \end{bmatrix}.$$

Theorem 9. *For $m \ge 2$, we have $f(m,2) = [(m-1)!]^m$.*

Proof. Based on Claim 5.1, we have

$$\begin{aligned} f(m,2) &= \det(S_m) = (-1)^{\frac{m(m-1)}{2}} \times \det(T_m) = (-1)^{\frac{m(m-1)}{2}} \times \prod_{r=1}^{2m-2} u_{r,r} \\ &= (-1)^{\frac{m(m-1)}{2}} \times \prod_{i=1}^{m-1} (-1)^i \times \prod_{r=1}^{2m-2} |u_{r,r}| = \prod_{r=1}^{2m-2} |u_{r,r}|. \end{aligned}$$

Therefore,

$$\begin{aligned} & f(m,2) \\ = & \prod_{r=1}^{2m-2} |u_{r,r}| \\ = & \left[\prod_{k=1}^{m-2} u_{2k-1,2k-1}\left|u_{2m-2k-2,2m-2k-2}\right|\right] \times u_{2m-3,2m-3} \times \left|u_{2m-2,2m-2}\right| \\ = & \left[\prod_{k=1}^{m-2} k! \times \binom{m-1}{m-k-1} \times (m-k-1)!\right] \times [(m-1)!]^2 \\ = & [(m-1)!]^m. \end{aligned}$$

This completes the proof. □

6. Determinants: The General Case

The entries of the mth-order n-dimensional Pascal tensor are the coefficients of the n-th power of an m-term multinomial. The determinant of the mth-order n-dimensional Pascal tensor $\mathcal{P}$, $f(m,n)$, is equal to the multivariate resultant of $\mathcal{P}x^{m-1} = 0$. We may use the Macaulay formula [12,13] to calculate $f(m,n)$ for $m \geq 3$ and $n \geq 3$. The Macaulay formula converts the computation of a resultant to the computation of the quotient of two matrix determinants. Thus, theoretically, we even do not know if $f(m,n)$ is an integer for $m \geq 3$ and $n \geq 3$.

By computation via the Macaulay formula and (1), we have the following results

$$f(3,2) = \begin{vmatrix} 1 & 2 & 2 & 0 \\ 0 & 1 & 2 & 2 \\ 1 & 4 & 6 & 0 \\ 0 & 1 & 4 & 6 \end{vmatrix} / 1 = 8 = 2^3,$$

$$\begin{aligned} & f(3,3) \\ = & \begin{vmatrix} 1 & 2 & 2 & 2 & 6 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 2 & 0 & 2 & 6 & 6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 2 & 0 & 2 & 6 & 6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 2 & 2 & 0 & 0 & 2 & 6 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 & 2 & 0 & 0 & 2 & 6 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 & 2 & 0 & 0 & 2 & 6 & 6 \\ 0 & 1 & 0 & 4 & 6 & 0 & 6 & 24 & 30 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 4 & 6 & 0 & 6 & 24 & 30 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 4 & 6 & 0 & 0 & 6 & 24 & 30 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 4 & 6 & 0 & 0 & 6 & 24 & 30 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 4 & 6 & 0 & 0 & 6 & 24 & 30 \\ 0 & 1 & 0 & 6 & 12 & 0 & 12 & 60 & 90 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 6 & 12 & 0 & 12 & 60 & 90 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 6 & 12 & 0 & 0 & 12 & 60 & 90 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 6 & 12 & 0 & 0 & 12 & 60 & 90 \end{vmatrix} / \begin{vmatrix} 1 & 0 & 6 \\ 0 & 1 & 2 \\ 0 & 1 & 6 \end{vmatrix} \\ = & 72^3, \end{aligned}$$

and

$$f(3,4) = 72^3 \times 80^9 = (72 \times 512000)^3.$$

Then, we have a conjecture.

Conjecture 1. For $n \geq 2$, $f(3,n)$ is a cubic number.

Write $f(3,n) = [g(3,n)]^3$. We have $g(3,2) = 2$, $g(3,3) = 72$ and $g(3,4) = 72 \times 512000$. Then, $g(3,3) = g(3,2) \times 6^2$ and $g(3,4) = g(3,3) \times 80^3$. Thus, we have a further conjecture.

Conjecture 2. For $n \geq 2$, $g(3,n+1)$ can be divided by $g(3,n)$, i.e., $g(3,n) = g(3,n-1)h(3,n)$ for some positive integer valued function h.

According the last section, we have $f(m,2) = [(m-1)!]^m$. By computation, we know that

$$f(3,3) = \left[2^3 \times 3^2\right]^3,$$

$$f(4,3) = \left[2^5 \times 3^9 \times 5^2\right]^4,$$

$$f(5,3) = \left[2^{21} \times 3^{13} \times 5^4 \times 7^2\right]^5,$$

$$f(6,3) = \left[2^{27} \times 3^{21} \times 5^{15} \times 7^4\right]^6,$$

$$f(7,3) = \left[2^{44} \times 3^{40} \times 5^{19} \times 7^6 \times 11^2\right]^7,$$

$$f(8,3) = \left[2^{52} \times 3^{50} \times 5^{23} \times 7^{21} \times 11^4 \times 13^2\right]^8,$$

$$f(9,3) = \left[2^{105} \times 3^{62} \times 5^{29} \times 7^{25} \times 11^6 \times 13^4\right]^9.$$

Therefore, in general, we have the following conjecture.

Conjecture 3. For $m \geq 3$ and $n \geq 2$, we have $f(m,n) = [g(m,n)]^m$, $g(m,2) = (m-1)!$, $g(m,n+1) = g(m,n)h(m,n+1)$, where g and h are positive integer valued functions.

Then, we have $g(m,2) = (m-1)!$,

$$h(3,3) = 6^2,$$

$$h(4,3) = \left(2^2 \times 3^4 \times 5\right)^2,$$

$$h(5,3) = \left(2^9 \times 3^6 \times 5^2 \times 7\right)^2,$$

$$h(6,3) = \left(2^{12} \times 3^{10} \times 5^7 \times 7^2\right)^2,$$

$$h(7,3) = \left(2^{20} \times 3^{19} \times 5^9 \times 7^3 \times 11\right)^2,$$

$$h(8,3) = \left(2^{24} \times 3^{24} \times 5^{11} \times 7^{10} \times 11^2 \times 13\right)^2,$$

$$h(9,3) = \left(2^{49} \times 3^{30} \times 5^{14} \times 7^{12} \times 11^3 \times 13^2\right)^2.$$

Conjecture 4. For $m \geq 3$, we have $f(m,3) = \left[(m-1)!(s(m))^2\right]^m$, where s is a positive integer valued function.

Here, we have

$$s(3) = 2 \times 3,$$

$$s(4) = 2^2 \times 3^4 \times 5,$$

$$s(5) = 2^9 \times 3^6 \times 5^2 \times 7,$$

$$s(6) = 2^{12} \times 3^{10} \times 5^7 \times 7^2,$$

$$s(7) = 2^{20} \times 3^{19} \times 5^9 \times 7^3 \times 11,$$

$$s(8) = 2^{24} \times 3^{24} \times 5^{11} \times 7^{10} \times 11^2 \times 13,$$
$$s(9) = 2^{49} \times 3^{30} \times 5^{14} \times 7^{12} \times 11^3 \times 13^2.$$

Based on these computational results, we have the following conjecture.

Conjecture 5. For $m \geq 3$, we have $s(m) = [(m-1)!]^{m-1} \times 3!! \times \cdots \times (2m-3)!!$, i.e.,

$$f(m,3) = \left\{ [(m-1)!]^{2m-3} \times [3!! \times \cdots \times (2m-3)!!]^2 \right\}^m. \tag{9}$$

The above computational results for $m = 3, \ldots, 9$, confirm this conjecture. It would be a big challenge to prove or to disprove (9).

7. Conclusions and Discussions

In this paper, we showed the even-order Pascal tensors are positive-definite and strongly completely positive. We also showed that odd-order Pascal tensors are strongly and completely positive. We further presented the determinants of two-dimensional Pascal tensors. These research topics are highly related to each other. These attempts not only increase our knowledge of Pascal tensors but also enhance our understanding of general structured tensors. This is the significance of our exploration.

In this paper, we concluded that an inherence property for Pascal tensors may work for some other completely positive tensor families, such as generalized Pascal tensors and fractional Hadamard power tensors. More study on such tensors and some other possible completely positive tensor classes would also be interesting.

Very recently, the concepts of positive semi-definiteness, SOS and positive-definiteness were extended to odd-order symmetric tensors in [14]. Let $\mathcal{A} = \left(a_{i_1 \ldots i_m}\right)$ be an mth-order n-dimensional symmetric tensor, where $m \geq 2$ is an odd number. Then, $G_{\mathcal{A}}(\mathbf{x}) = \mathcal{A}\mathbf{x}^{m-1}$ defines a function $G_{\mathcal{A}} : \mathbb{R}^n \to \mathbb{R}^n$. If $G_{\mathcal{A}}(\mathbf{x}) \geq \mathbf{0}$ for $\mathbf{x} \in \mathbb{R}^n$, then $\mathcal{A}$ is called a strongly positive semi-definite tensor. If each component of $G_{\mathcal{A}}(\mathbf{x})$ can be expressed as SOS polynomials of $x_1, \ldots, x_n$, then $\mathcal{A}$ is called a strongly SOS tensor. If If $G_{\mathcal{A}}(\mathbf{x}) > \mathbf{0}$ for $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{x} \neq \mathbf{0}$, then $\mathcal{A}$ is called a strongly positive-definite tensor. Odd order completely positive tensors are strongly positive semi-definite and SOS tensors. Thus, odd-order Pascal tensors are strongly positive semi-definite and SOS tensors. On the other hand, though several other odd-order completely positive tensor classes, such as odd-order Hilbert tensors and Lehmer tensors, have been proved to be strongly positive-definite, we still do not know if odd-order Pascal tensors are strongly positive-definite or not. Hence, this will be the next challenging question for Pascal tensors.

We explored the value of $f(m,n)$ for $n \geq 3$. There are still some conjectures left unanswered. We made a conjecture that $f(m,n)$ is the mth power of an integer function, and is an integer factor of $f(m,n+1)$. In particular, we made a conjecture on the formula of $f(m,3)$ for $m \geq 3$. It would be intriguing to provide responses to them.

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