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Laguerre-Based Frequency-Limited Balanced Truncation of Discrete-Time Systems

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Abstract: This paper introduces a novel model order reduction (MOR) method for linear discrete-time systems, focusing on frequency-limited balanced truncation (BT) techniques. By leveraging Laguerre functions, we develop two efficient MOR algorithms that avoid the computationally expensive generalized Lyapunov equation solvers used in traditional methods. These algorithms employ recursive formulas to calculate Laguerre expansion coefficients, which are then used to derive low-rank decomposition factors for frequency-limited controllability and observability Gramians. Additionally, we enhance the Laguerre-based low-rank MOR algorithm by incorporating a modified frequency-limited BT method, further improving its computational efficiency. Numerical simulations validate the effectiveness of the proposed approach, demonstrating significant reductions in computational complexity while maintaining accuracy in system approximation.

Keywords: model order reduction; frequency-limited; balanced truncation; Laguerre functions; discrete-time systems

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1. Introduction

Model order reduction (MOR) plays a critical role in alleviating the computational burden associated with the simulation and analysis of complex systems [1,2]. It has gained increasing importance over the years, particularly in applications involving complex systems modeling, circuit design, and very-large-scale integration (VLSI) circuits. The fundamental goal of MOR is to generate a reduced-order model (ROM) that accurately represents the essential dynamics and properties of the original system, including its input–output behavior and stability. Balanced truncation (BT) was initially introduced by Moore in [3], and is one of the most widely recognized and theoretically grounded MOR techniques. When applied to stable systems, BT not only ensures stability preservation but also provides a priori error bounds on the approximation. However, despite its theoretical robustness, BT faces significant challenges when applied to large-scale systems due to the high computational cost involved in solving generalized Lyapunov equations, which results in computational complexity of $\mathcal{O}(n^3)$ and a storage requirement of $\mathcal{O}(n^2)$. This necessitates the development of more computationally efficient approaches, particularly for large-scale applications [4].

Interestingly, the origin of many MOR methods can be traced back to singularly perturbed systems [1], where system dynamics are split into fast and slow components. These

methods exploit the small parameters in these systems to reduce model complexity while preserving the most critical system features. The challenge of handling these perturbations efficiently has led to various reduced-order methods, which are especially useful in practical applications; however, the computational burden associated with exact balancing in large systems remains a limitation, spurring the exploration of approximate techniques in MOR.

While the classical BT method aims to minimize the reduction error across the entire frequency spectrum, many practical applications require the ROM to achieve low approximation error within specific frequency ranges crucial to the system's operation [5,6]. This has led to the development of frequency-limited balanced truncation (FLBT), which restricts the BT procedure to a predefined frequency interval $\delta\omega = [\omega_1, \omega_2]$, where $0 \leq \omega_1 < \omega_2 \leq \pi$ [7]. FLBT has the potential to improve accuracy by focusing computational efforts on the frequency range of interest, allowing for higher accuracy with ROMs of the same order compared to traditional BT. In [8], a novel FLBT method for discrete-time systems (DTSs) was introduced in which the definition of standard Gramians was modified to include a frequency-limited integral formulation. These frequency-limited Gramians can then be used in the traditional BT procedure. While this method is effective in practice, a significant limitation is that the ROM obtained in this way may not always be stable, and there are no computable a priori error bounds. Several modified FLBT schemes have been proposed to overcome these shortcomings, aiming to ensure stable ROMs and provide error bounds [9–13]. However, most of these FLBT-related methods still require the resolution of large (generalized) Lyapunov equations, which are computationally expensive or even infeasible for large-scale systems.

To address these limitations, this paper presents a novel approach to low-rank model order reduction for frequency-limited systems, particularly for discrete-time systems (DTSs). Our approach builds upon recent advancements in low-rank balanced truncation techniques and the use of Laguerre functions to efficiently approximate the Gramians [14–16]. Specifically, we propose a Laguerre-based method that computes low-rank approximations for the frequency-limited Gramians in FLBT. The key advantage of this method is that it avoids the need to solve (generalized) Lyapunov equations for the full Gramians, which is a significant computational bottleneck in traditional FLBT. By leveraging recursive formulas to compute the Laguerre expansion coefficient vectors, we can directly construct the low-rank decomposition factors of the frequency-limited controllability and observability Gramians. These factors are then combined with the low-rank square root method (LRSRM) to obtain efficient frequency-limited MOR algorithms.

Furthermore, our approach is extended to the modified FLBT, where the frequency-limited and frequency-dependent Gramians are handled similarly, improving the applicability of the method to a broader class of systems. The effectiveness of the proposed method is demonstrated through a series of numerical examples, showing its superiority in terms of computational efficiency when compared to traditional FLBT methods.

The rest of this paper is organized as follows: Section 2 provides an introduction to Moore's balanced truncation for DTSs. Section 3 presents the Laguerre-based frequency-limited balanced truncation method, including low-rank Gramian approximations and algorithms, and also extends the method to modified FLBT while presenting corresponding algorithms and discussing their implementation. A numerical example is provided in Section 4 to validate the proposed approach. Finally, the paper concludes in Section 5 with a discussion of potential future research directions.

2. Moore's Balanced Truncation of Discrete-Time Systems

Consider an asymptotically stable discrete-time system (DTS) $\{A, B, C\}$

$$\begin{cases} \dot{x}(k+1) = Ax(k) + Bu(k) \\ y(k) = Cx(k) \end{cases} \tag{1}$$

with transfer function $G(z) = C(zI - A)^{-1}B$, where $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^m$ and $y(k) \in \mathbb{R}^p$ are the state, input, and output, respectively. The constant matrices are $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$. The controllability and observability Gramians in the frequency domain of the DTS (1) are defined as [8]

$$P = \frac{1}{2\pi}\int_{-\pi}^{+\pi}(I - e^{-j\omega}A)^{-1}BB^{\mathrm{T}}(I - e^{j\omega}A^{\mathrm{T}})^{-1}d\omega,$$

$$Q = \frac{1}{2\pi}\int_{-\pi}^{+\pi}(I - e^{j\omega}A^{\mathrm{T}})^{-1}C^{\mathrm{T}}C(I - e^{-j\omega}A)^{-1}d\omega,$$

which satisfy the Lyapunov equations

$$APA^{\mathrm{T}} - P + BB^{\mathrm{T}} = 0,$$

$$A^{\mathrm{T}}QA - Q + C^{\mathrm{T}}C = 0.$$

A realization of the DTS, denoted by $\{A, B, C\}$, is said to be internally balanced [3] if and only if the following conditions are met:

$$P = Q = \Sigma = \mathtt{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$$

where $\sigma_i \geq \sigma_{i+1}$, $i = 1, 2, \ldots, n-1$. After transformation (1) into an internally balanced structure, the resulting balanced system and Gramians can be partitioned accordingly:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}$$

where $A_{11}, \Sigma_1 \in \mathbb{R}^{r\times r}$ and $r < n$. Subsequently, a reduced-order system $\{A_{11}, B_1, C_1\}$ of order r can be derived through a direct truncation of the balanced system [3,17]. This truncated system serves as a commendable approximation of the original system, and is deemed asymptotically stable if the diagonal elements $\sigma_r > \sigma_{r+1}$. Additionally, a rigorous error bound exists for the balanced approximation method [18] provided by

$$\|G(e^{j\omega}) - G_r(e^{j\omega})\|_\infty \leq 2\mathtt{trace}(\Sigma_2), \tag{2}$$

where $G_r(e^{j\omega})$ represents the transfer function of the subsystem $\{A_{11}, B_1, C_1\}$ and $\mathtt{trace}(\cdot)$ denotes the trace of a matrix.

3. Laguerre-Based Frequency-Limited MOR of Discrete-Time Systems

The frequency-limited controllability and observability Gramians defined in [8] are respectively provided by

$$P_{fl} = \frac{1}{2\pi}\int_{\delta\omega}(I - e^{-j\omega}A)^{-1}BB^{\mathrm{T}}(I - e^{j\omega}A^{\mathrm{T}})^{-1}d\omega,$$

$$Q_{fl} = \frac{1}{2\pi}\int_{\delta\omega}(I - e^{j\omega}A^{\mathrm{T}})^{-1}C^{\mathrm{T}}C(I - e^{-j\omega}A)^{-1}d\omega,$$

where $\delta\omega = [\omega_1, \omega_2]$ represents the frequency range of operation, with $0 \leq \omega_1 < \omega_2 \leq \pi$. It is noteworthy that due to the symmetry of the Fourier transform, the integration spans both intervals $[\omega_1, \omega_2]$ and $[-\omega_2, -\omega_1]$, ensuring real-valued Gramians [19]. These Gramians, P_{fl} and Q_{fl}, satisfy the Lyapunov equations

$$AP_{fl}A^{\mathrm{T}} - P_{fl} + F_{\delta\omega}BB^{\mathrm{T}} + BB^{\mathrm{T}}F_{\delta\omega}^{*} = 0, \tag{3}$$

$$A^{\mathrm{T}}Q_{fl}A - Q_{fl} + F_{\delta\omega}^{*}C^{\mathrm{T}}C + C^{\mathrm{T}}CF_{\delta\omega} = 0, \tag{4}$$

where $F_{\delta\omega} = -\frac{\omega_2-\omega_1}{4\pi}I + \frac{1}{2\pi}\int_{\delta\omega}(I - e^{-j\omega}A)^{-1}d\omega$ and $F_{\delta\omega}^{*}$ denotes the conjugate transpose of $F_{\delta\omega}$.

3.1. Laguerre-Based Low-Rank Decompositions of Frequency-Limited Gramians P_{fl} and Q_{fl}

First, we define

$$H(j\omega) = \left(I - e^{-j\omega}A\right)^{-1}B$$

and express it in terms of frequency-domain Laguerre functions, as follows:

$$H(j\omega) = \left(I - e^{-j\omega}A\right)^{-1}B = \sum_{i=0}^{+\infty} f_i\Phi_i^{\alpha}(j\omega) \tag{5}$$

where $f_i \in \mathbb{R}^{n\times m}$ are the corresponding expansion coefficient matrices. The functions $\Phi_i^{\alpha}(j\omega)$ take the form

$$\Phi_i^{\alpha}(j\omega) = \frac{\sqrt{2\alpha}}{j\omega+\alpha}\left(\frac{j\omega-\alpha}{j\omega+\alpha}\right)^{i}, \quad i = 0, 1, \ldots,$$

representing the Laplace transforms of scaled Laguerre functions $\phi_i^{\alpha}(t)$, where α ($\alpha > 0$) serves as the scaling parameter and

$$\phi_i^{\alpha}(t) = \sqrt{2\alpha}e^{-\alpha t}l_i(2\alpha t),$$

with $l_i(t)$ denoting the i-th Laguerre polynomial as defined in [20]:

$$l_i(t) = \frac{e^t}{i!}\frac{d^i}{dt^i}(e^{-t}t^i), \quad i = 0, 1, \ldots.$$

To derive the expansion coefficient matrices f_i, let us assume that $e^{-j\omega}$ can be expanded as

$$e^{-j\omega} = \sum_{i=0}^{\infty} e_i\Phi_i^{\alpha}(j\omega) = \sum_{i=0}^{\infty} e_i\frac{\sqrt{2\alpha}}{j\omega+\alpha}\left(\frac{j\omega-\alpha}{j\omega+\alpha}\right)^{i}, \tag{6}$$

where e_i represents the scalar coefficient. Utilizing the bilinear transformation $u = \dfrac{j\omega-\alpha}{j\omega+\alpha}$, we obtain

$$e^{-j\omega} = \frac{1}{\sqrt{2\alpha}}\sum_{i=0}^{\infty} e_i(1-u)u^i \tag{7}$$

and

$$\sqrt{2\alpha}e^{-\alpha}e^{-\frac{2\alpha u}{1-u}}(1-u)^{-1} = \sum_{i=0}^{\infty} e_i u^i. \tag{8}$$

Referring to the generating function for Laguerre polynomials [21]

$$e^{-\frac{xt}{1-x}}(1-x)^{-1} = \sum_{i=0}^{\infty} l_i(t)x^i,$$

where $l_i(t) = \frac{e^t}{i!}\frac{d^i}{dt^i}(t^i e^{-t})$ is the i-th Laguerre polynomial, we deduce $e_i = \sqrt{2\alpha}e^{-\alpha}l_i(2\alpha)$.

Substituting the bilinear transformation into (5), we arrive at

$$H(j\omega) = \frac{1}{\sqrt{2\alpha}}\sum_{i=0}^{\infty} f_i(1-u)u^i. \tag{9}$$

From (5), we know that

$$(I - e^{-j\omega}A)H(j\omega) = B. \tag{10}$$

By substituting (7) and (9) into (10), we obtain

$$\Big(I - A\frac{1}{\sqrt{2\alpha}}\sum_{i=0}^{\infty} e_i(1-u)u^i\Big)\Big(\frac{1}{\sqrt{2\alpha}}\sum_{i=0}^{\infty} f_i(1-u)u^i\Big) = B,$$

which simplifies to

$$\left(I - Iu - \frac{1}{\sqrt{2\alpha}}A\sum_{i=0}^{\infty} e_i(u^i - 2u^{i+1} + u^{i+2})\right)\sum_{i=0}^{\infty} f_i u^i = \sqrt{2\alpha}B.$$

Equating coefficients of the term u^i for $i = 0, 1, \ldots, N-1$, the Laguerre expansion coefficient matrices f_i of $H(j\omega)$ satisfy the following linear system:

$$\begin{bmatrix} L_0 & & & & \\ L_1 & L_0 & & & \\ L_2 & L_1 & L_0 & & \\ \vdots & \ddots & \ddots & \ddots & \\ L_{N-1} & \cdots & L_2 & L_1 & L_0 \end{bmatrix}\begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_{N-1} \end{bmatrix} = \begin{bmatrix} \sqrt{2\alpha}B \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{11}$$

where the components of the coefficient matrix are as follows:

$$\begin{aligned} L_0 &= I - \sqrt{\frac{1}{2\alpha}}e_0 A, \\ L_1 &= -I - \sqrt{\frac{1}{2\alpha}}(e_1 - 2e_0)A, \\ L_2 &= -\sqrt{\frac{1}{2\alpha}}(e_2 - 2e_1 + e_0)A, \\ L_i &= -\sqrt{\frac{1}{2\alpha}}(e_i - 2e_{i-1} + e_{i-2})A, \quad i \geq 3. \end{aligned}$$

Assuming L_0 is invertible, the Laguerre coefficients f_i can be calculated iteratively as follows:

$$\begin{aligned} f_0 &= \sqrt{2\alpha}L_0^{-1}B, \\ f_1 &= -L_0^{-1}L_1 f_0, \\ f_i &= -L_0^{-1}(L_i f_0 + L_{i-1}f_1 + \cdots + L_1 f_{i-1}), \quad i \geq 2. \end{aligned} \tag{12}$$

This iterative approach is enabled by the block lower triangular structure of the coefficient matrix.

Then, the frequency-limited controllability Gramian P_{fl} can be rewritten as

$$P_{fl} = \frac{1}{2\pi}\int_{\delta\omega}(H(j\omega))(H(j\omega))^* d\omega. \tag{13}$$

Assuming that $H(j\omega)$ can be expressed using Laguerre functions as follows

$$H(j\omega) = \sum_{i=0}^{\infty} f_i \Phi_i^{\alpha}(j\omega) \approx \sum_{i=0}^{N-1} f_i \Phi_i^{\alpha}(j\omega) \tag{14}$$

and substituting (14) into the controllability Gramian P_{fl} in (13), we obtain

$$\begin{aligned} P_{fl} &\approx \frac{1}{2\pi} \int_{\delta\omega} \left(\sum_{i=0}^{N-1} f_i \Phi_i^{\alpha}(j\omega) \right) \left(\sum_{i=0}^{N-1} f_i \Phi_i^{\alpha}(j\omega) \right)^{\mathrm{H}} d\omega \\ &= \begin{bmatrix} f_0 & f_1 & \cdots & f_{N-1} \end{bmatrix} \mathcal{M}_{\delta\omega} \begin{bmatrix} f_0^{\mathrm{T}} \\ f_1^{\mathrm{T}} \\ \vdots \\ f_{N-1}^{\mathrm{T}} \end{bmatrix}, \end{aligned}$$

where $\mathcal{M}_{\delta\omega} = M_{\delta\omega} \otimes I_m$, I_m is the identity matrix of dimension m, and

$$M_{\delta\omega} = \frac{1}{2\pi} \int_{\delta\omega} \begin{bmatrix} \Phi_0^{\alpha}(j\omega) \\ \Phi_1^{\alpha}(j\omega) \\ \vdots \\ \Phi_{N-1}^{\alpha}(j\omega) \end{bmatrix} \begin{bmatrix} \overline{\Phi_0^{\alpha}(j\omega)} & \overline{\Phi_1^{\alpha}(j\omega)} & \cdots & \overline{\Phi_{N-1}^{\alpha}(j\omega)} \end{bmatrix} d\omega.$$

Here, " $\otimes$ " denotes the Kronecker product. If $M_{\delta\omega}$ can be decomposed as $M_{\delta\omega} = R_{\delta\omega} R_{\delta\omega}^{\mathrm{T}}$, where $R_{\delta\omega} \in \mathbb{R}^{N\times N}$, then

$$\mathcal{M}_{\delta\omega} = (R_{\delta\omega} R_{\delta\omega}^{\mathrm{T}}) \otimes I_m = (R_{\delta\omega} \otimes I_m)(R_{\delta\omega} \otimes I_m)^{\mathrm{T}}.$$

Finally, the frequency-limited controllability Gramian P_{fl} can be approximated by the above structure

$$P_{fl} \approx F(R_{\delta\omega} \otimes I_m)(R_{\delta\omega} \otimes I_m)^{\mathrm{T}} F^{\mathrm{T}} = F_{fl} F_{fl}^{\mathrm{T}}, \tag{15}$$

where $F = \begin{bmatrix} f_0 & f_1 & \cdots & f_{N-1} \end{bmatrix} \in \mathbb{R}^{n\times Nm}$ is calculated iteratively using (12) and where $F_{fl} = F(R_{\delta\omega} \otimes I_m)$. Similarly, a low-rank approximation of the frequency-limited observability Gramian Q_{fl} can be expressed as

$$Q_{fl} \approx G_{fl} G_{fl}^{\mathrm{T}}, \tag{16}$$

where $G_{fl} = G(R_{\delta\omega} \otimes I_p)$, $G = \begin{bmatrix} g_0 & g_1 & \cdots & g_{N-1} \end{bmatrix} \in \mathbb{R}^{n\times Np}$, and g_i are the Laguerre expansion coefficient matrices of $(I - e^{j\omega} A^{\mathrm{T}})^{-1} C^{\mathrm{T}}$, which can be calculated iteratively as described by

$$\begin{aligned} g_0 &= \sqrt{2\alpha} L_0^{-\mathrm{T}} C^{\mathrm{T}}, \\ g_1 &= -L_0^{-\mathrm{T}} L_1^{\mathrm{T}} g_0, \\ g_i &= -L_0^{-\mathrm{T}} (L_i^{\mathrm{T}} g_0 + L_{i-1}^{\mathrm{T}} g_1 + \cdots + L_1^{\mathrm{T}} g_{i-1}), \quad i \geq 2. \end{aligned}$$

Remark 1. *In Equations (15) and (16), the decomposition $M_{\delta\omega} = R_{\delta\omega} R_{\delta\omega}^{\mathrm{T}}$ is crucial. It is evident that $M_{\delta\omega}$ is a real symmetric matrix. If it is positive definite or positive semi-definite, then the Cheolesky decomposition can be applied. Otherwise, the decomposition $M_{\delta\omega} = R_{\delta\omega} \Lambda R_{\delta\omega}^{\mathrm{T}}$ is necessary, where Λ is a diagonal matrix with entries ± 1 along the main diagonal. In the case where $M_{\delta\omega} = (R_{\delta\omega}\Lambda_1)(R_{\delta\omega}\Lambda_1)^{\mathrm{T}} = \tilde{R}_{\delta\omega} \tilde{R}_{\delta\omega}^{\mathrm{T}}$ with $\Lambda_1 = diag(1, 1, \cdots, i, i)$, the matrix $G_{fl}^{\mathrm{T}} F_{fl}$ may become complex. To handle this indefiniteness, we apply the method proposed in [22] to achieve a semi-definite factorization. For simplicity, let $P_{fl} \approx \tilde{P}_{fl} = F(M_{\delta\omega} \otimes I_m) F^{\mathrm{T}}$. Even though $M_{\delta\omega}$*

is indefinite, we assume that $\tilde{P}_{fl} \succeq 0$, *as* $P_{fl} \succ 0$. *This assumption is practical only if* $\tilde{P}_{fl}$ *is a sufficiently accurate approximation of* P_{fl}. *The next step is to compute a QR-decomposition* $U_1 R = F$, *followed by a spectral decomposition* $U_2 \tilde{\Lambda} U_2^{\mathrm{T}} = R(M_{\delta\omega} \otimes I_m)R^{\mathrm{T}}$. *The approximation solution* $\tilde{P}_{fl}$ *can then be expressed as a semi-definite factorization* $\tilde{P}_{fl} = \tilde{F}_{fl}\tilde{F}_{fl}^{\mathrm{T}}$, *where* $\tilde{F}_{fl} = U_1 U_2 \tilde{\Lambda}^{\frac{1}{2}}$.

3.2. Frequency-Limited Balanced Truncation of Discrete-Time Systems

Utilizing the low-rank approximations of the frequency-limited Gramians P_{fl} and Q_{fl} as details in Equations (15) and (16), respectively, we can establish the ROM through LRSRM, employing the low-rank decomposition factors F_{fl} and G_{fl}. The reduction procedure is outlined in Algorithm 1.

Algorithm 1 Frequency-limited low-rank square root method for discrete-time systems (FL-LRSRM)

Input: Original system: $\{A, B, C, D\}$, frequency-interval $\Omega = [\omega_1, \omega_2]$, N, α, tol;
Output: ROM: $\{\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}\}$;
1. Compute low-rank factors F_{fl} and G_{fl} from (15) and (16), respectively;
2. Compute the SVD: $G_{fl}^{\mathrm{T}} F_{fl} = U\Sigma V^{\mathrm{T}}$, $U_r = U(:, 1:r)$, $V_r = V(:, 1:r)$, and $\Sigma_r = \Sigma(1:r, 1:r)$;
3. Compute projection matrices: $T_r = F_{fl} V_r \Sigma_r^{-\frac{1}{2}}$, $\quad S_r = \Sigma_r^{-\frac{1}{2}} U_r{}^{\mathrm{T}} G_{fl}^{\mathrm{T}}$;
4. Construct the ROM: $\tilde{A} = S_r A T_r$, $\tilde{B} = S_r B$, $\tilde{C} = C T_r$, $\tilde{D} = D$.

Given a tolerance "tol", the order r of the ROM is adaptively determined as the smallest integer satisfying the condition

$$\delta = 2 \sum_{j=r+1}^{r_N} \tilde{\sigma}_j \leq \text{tol},$$

where $r_N = \operatorname{rank}(G_{fl}^{\mathrm{T}} F_{fl})$, $\{\tilde{\sigma}_1, \tilde{\sigma}_1, \cdots, \tilde{\sigma}_{r_N}\}$ represent the singular values of $G_{fl}^{\mathrm{T}} F_{fl}$ in descending order. New frequency-limited techniques proposed in [23] ensure the stability of the ROM and provide error bounds for the frequency response. These results suggest that frequency-limited singular values offer a reliable criterion for selecting an appropriate reduced-order r, making δ a valuable guide for adaptively determining the ROM order.

The choice of the Laguerre parameter α significantly affects both the stability and accuracy of the ROM. Insights into selecting α for traditional or infinite-frequency Laguerre-based MOR methods in linear systems can be found in [24,25]. While strategies for guiding the selection of α exist, their application to frequency-limited or time-limited MOR methods as well as to more general cases requires further investigation. Our numerical results demonstrate the effectiveness of choosing $\alpha = \frac{\omega_2 + \omega_1}{2}$, where $0 \leq \omega_1 < \omega_2 \leq \pi$.

In Algorithm 1, apart from Step 1, the SVD of an $Np \times Nm$-dimensional matrix is only needed once, with computational complexity of $\mathcal{O}(\max\{m^2, p^2\}N^2 n)$. The overall complexity mainly depends on the computation of low-rank factors F_{fl} and G_{fl} in Step 1, which is highly influenced by the structure and sparsity of matrix A. Using a sparse matrix with a direct method typically leads to a computational complexity of $\mathcal{O}(c^2 Nn)$ [26], where c is the average number of non-zero elements per row/column. This complexity can be significantly reduced compared to $\mathcal{O}(n^3)$ when effective iterative solvers are applied, especially in sparse cases [27].

3.3. Stability Preservation and Modified Frequency-Limited Balanced Truncation

As noted in [9], the Lyapunov Equations (3) and (4) typically involve indefinite inhomogeneities, raising concerns about preserving stability in the ROM. Then, a modified

version of frequency-limited BT is introduced to ensure stability preservation. To better understand this, let us examine the eigenvalue decompositions of the inhomogeneities in (3) and (4):

$$F_{\delta\omega}BB^{\mathrm{T}} + BB^{\mathrm{T}}F_{\delta\omega}^{*} = Q_B S_B Q_B^{\mathrm{T}}, \quad S_B = \mathtt{diag}(\lambda_1^B, \lambda_2^B, \cdots, \lambda_{2m}^B, 0, \cdots, 0) \in \mathbb{R}^{n\times n},$$

$$F_{\delta\omega}^{*}C^{\mathrm{T}}C + C^{\mathrm{T}}CF_{\delta\omega} = Q_C S_C Q_C^{\mathrm{T}}, \quad S_C = \mathtt{diag}(\lambda_1^C, \lambda_2^C, \cdots, \lambda_{2p}^C, 0, \cdots, 0) \in \mathbb{R}^{n\times n},$$

with $Q_B^{\mathrm{T}}Q_B = Q_C^{\mathrm{T}}Q_C = I$, $Q_B, Q_C \in \mathbb{R}^{n\times n}$. Assuming that

$$\mathtt{rank}\begin{bmatrix} B & F_{\delta\omega}B \end{bmatrix} = r_B \le 2m \quad \text{and} \quad \mathtt{rank}\begin{bmatrix} C^{\mathrm{T}} & F_{\delta\omega}^{*}C^{\mathrm{T}} \end{bmatrix} = r_C \le 2p,$$

it is noteworthy that λ_i^B and λ_j^C are non-zero for $i = 1, 2, \ldots, r_B$ and $j = 1, 2, \ldots, r_C$. However, it is important to acknowledge that λ_i^B and λ_j^C can both be negative or positive. Let $Q_{B,1} \in \mathbb{R}^{n\times r_B}$ and $Q_{C,1} \in \mathbb{R}^{n\times r_C}$ represent the first r_B and r_C columns of Q_B and Q_C, respectively. Now, let us delve into the modified frequency-limited Lyapunov equations

$$AP_{fl}^{\mathrm{mod}}A^{\mathrm{T}} - P_{fl}^{\mathrm{mod}} + B_{fl}^{\mathrm{mod}}B_{fl}^{\mathrm{mod}\,\mathrm{T}} = 0, \tag{17}$$

$$A^{\mathrm{T}}Q_{fl}^{\mathrm{mod}}A - Q_{fl}^{\mathrm{mod}} + C_{fl}^{\mathrm{mod}\,\mathrm{T}}C_{fl}^{\mathrm{mod}} = 0, \tag{18}$$

with $B_{fl}^{\mathrm{mod}} := Q_{B,1}\mathtt{diag}(|\lambda_1^B|, |\lambda_2^B|, \cdots, |\lambda_{r_B}^B|)^{\frac{1}{2}}$, $C_{fl}^{\mathrm{mod}} := \mathtt{diag}(|\lambda_1^C|, |\lambda_2^C|, \cdots, |\lambda_{r_C}^C|)^{\frac{1}{2}}Q_{C,1}^{\mathrm{T}}$. In essence, the negative values in S_B and S_C are simply negated. It is worth noting that computing the $r_B, r_C \ll n$ non-zero eigenvalues λ_i^B, λ_j^C and their corresponding eigenvectors can be accomplished inexpensively. Balancing and truncating based on these modified frequency-limited Gramians P_{fl}^{mod} and Q_{fl}^{mod} leads to the modified FLBT (FLBT-mod). This approach ensures that the reduced-order model is asymptotically stable under mild conditions. Similar to (2), the error bound

$$\|G(e^{j\omega}) - G_r^{\mathrm{mod}}(e^{j\omega})\|_\infty \le 2\|J_B\|\|J_C\| \sum_{j=r+1}^{n} \sigma_j^{\mathrm{mod}}$$

is established, where

$$J_B := \mathtt{diag}(|\lambda_1^B|, |\lambda_2^B|, \cdots, |\lambda_{r_B}^B|)^{-\frac{1}{2}}Q_{B,1}^{\mathrm{T}}B, \quad J_C := CQ_{C,1}\mathtt{diag}(|\lambda_1^C|, |\lambda_2^C|, \cdots, |\lambda_{r_C}^C|)^{-\frac{1}{2}}.$$

Here, $G(e^{j\omega})$ and $G_r^{\mathrm{mod}}(e^{j\omega})$ represent the transfer functions of the original DTS A, B, C and the reduced-order system obtained by balanced truncation for $A, B_{fl}^{\mathrm{mod}}, C_{fl}^{\mathrm{mod}}$, respectively, while σ_j^{mod} denotes the modified frequency-limited singular values, i.e., the square roots of the eigenvalues of $P_{fl}^{\mathrm{mod}}Q_{fl}^{\mathrm{mod}}$. The computation of low-rank factors of P_{fl}^{mod} and Q_{fl}^{mod} can be carried out in a similar way to the previous approach by using Algorithm 1. After an accurate approximation of, e.g., $F_{\delta\omega}B$ has been computed, B_{fl}^{mod} is constructed as above. Subsequently, we use P_{fl}^{mod} as an example to illustrate the Laguerre-based approach for approximating the matrix $F_{\delta\omega}B$ and obtaining the low-rank factor of P_{fl}^{mod}.

To obtain the frequency-limited Gramian P_{fl}^{mod} by solving the modified frequency-limited Lyapunov Equation (17), we note that only $F_{\delta\omega}B$ is required to set up the inhomogeneities in (3). From the integral expression of $F_{\delta\omega}$, we have

$$F_{\delta\omega}B = -\frac{\omega_2 - \omega_1}{4\pi}B + \frac{1}{2\pi}\int_{\delta\omega}(I - e^{-j\omega}A)^{-1}Bd\omega.$$

Substituting (5) into the above equation, we obtain

$$\begin{aligned} F_{\delta\omega}B &= -\frac{\omega_2-\omega_1}{4\pi}B + \frac{1}{2\pi}\int_{\delta\omega} H(j\omega)d\omega \\ &\approx -\frac{\omega_2-\omega_1}{4\pi}B + \frac{1}{2\pi}\int_{\delta\omega} \sum_{i=0}^{N-1} f_i\Phi_i^{\alpha}(j\omega)d\omega \\ &= -\frac{\omega_2-\omega_1}{4\pi}B + \frac{1}{2\pi}\int_{\delta\omega} \sum_{i=0}^{N-1} f_i\Phi_i^{\alpha}(j\omega)d\omega \\ &= -\frac{\omega_2-\omega_1}{4\pi}B + F(D_{\delta\omega}\otimes I_m), \end{aligned}$$

where

$$D_{\delta\omega} = \frac{1}{2\pi}\int_{\delta\omega}\begin{bmatrix} \Phi_0^{\alpha}(j\omega) & \Phi_1^{\alpha}(j\omega) & \cdots & \Phi_{N-1}^{\alpha}(j\omega) \end{bmatrix}^{\mathrm{T}} d\omega \in \mathbb{R}^{N\times 1}.$$

This expression demonstrates that approximating $F_{\delta\omega}B$ becomes highly efficient after we obtain the Laguerre expansion coefficient matrix F of $H(j\omega)$ through the recursive formula in (12), as it only involves calculating the integral of an N-dimensional vector, where $N \ll n$.

In the low-rank decomposition (15) of P_{fl}, with $\delta\omega = [0, \pi]$ and B_{fl}^{mod} replaced by B, we derive a low-rank decomposition of P_{fl}^{mod} as follows:

$$\begin{aligned} P_{fl}^{\mathrm{mod}} &= \frac{1}{2\pi}\int_{-\pi}^{+\pi}(I-e^{-j\omega}A)^{-1}B_{fl}^{\mathrm{mod}}B_{fl}^{\mathrm{mod}\,\mathrm{T}}(I-e^{j\omega}A^{\mathrm{T}})^{-1}d\omega \\ &\approx F_{fl}^{\mathrm{mod}}F_{fl}^{\mathrm{mod}\,\mathrm{T}}, \end{aligned} \tag{19}$$

where $F_{fl}^{\mathrm{mod}} = F^{\mathrm{mod}}(R_\pi \otimes I_m)$, $F^{\mathrm{mod}} = \begin{bmatrix} f_0^{\mathrm{mod}} & f_1^{\mathrm{mod}} & \cdots & f_{N-1}^{\mathrm{mod}} \end{bmatrix} \in \mathbb{R}^{n\times Nm}$ is calculated iteratively as follows:

$$\begin{aligned} f_0^{\mathrm{mod}} &= \sqrt{2\alpha}L_0^{-1}B_{fl}^{\mathrm{mod}}, \\ f_1^{\mathrm{mod}} &= -L_0^{-1}L_1 f_0^{\mathrm{mod}}, \\ f_i^{\mathrm{mod}} &= -L_0^{-1}(L_i f_0^{\mathrm{mod}} + L_{i-1}f_1^{\mathrm{mod}} + \cdots + L_1 f_{i-1}^{\mathrm{mod}}), \quad i \geq 2. \end{aligned}$$

Here, R_π is the decomposition factor of M_π, such that $M_\pi = R_\pi R_\pi^{\mathrm{T}}$ and

$$M_\pi = \frac{1}{2\pi}\int_{-\pi}^{+\pi}\begin{bmatrix} \Phi_0^{\alpha}(j\omega) \\ \Phi_1^{\alpha}(j\omega) \\ \vdots \\ \Phi_{N-1}^{\alpha}(j\omega) \end{bmatrix}\begin{bmatrix} \overline{\Phi_0^{\alpha}(j\omega)} & \overline{\Phi_1^{\alpha}(j\omega)} & \cdots & \overline{\Phi_{N-1}^{\alpha}(j\omega)} \end{bmatrix} d\omega.$$

Similarly, a low-rank approximation of the modified frequency-limited observability Gramian Q_{fl}^{mod} can be expressed as

$$Q_{fl}^{\mathrm{mod}} \approx G_{fl}^{\mathrm{mod}}G_{fl}^{\mathrm{mod}\,\mathrm{T}}, \tag{20}$$

where $G_{fl}^{\mathrm{mod}} = G^{\mathrm{mod}}(R_\pi \otimes I_p)$, $G^{\mathrm{mod}} = \begin{bmatrix} g_0^{\mathrm{mod}} & g_1^{\mathrm{mod}} & \cdots & g_{N-1}^{\mathrm{mod}} \end{bmatrix} \in \mathbb{R}^{n\times Np}$, and g_i^{mod} are calculated iteratively as follows:

$$
\begin{aligned}
g_0^{\text{mod}} &= \sqrt{2\alpha} L_0^{-\mathrm{T}} {C_{fl}^{\text{mod}}}^{\mathrm{T}}, \\
g_1^{\text{mod}} &= -L_0^{-\mathrm{T}} L_1^{\mathrm{T}} g_0^{\text{mod}}, \\
g_i^{\text{mod}} &= -L_0^{-\mathrm{T}} (L_i^{\mathrm{T}} g_0^{\text{mod}} + L_{i-1}^{\mathrm{T}} g_1^{\text{mod}} + \cdots + L_1^{\mathrm{T}} g_{i-1}^{\text{mod}}), \quad i \geq 2.
\end{aligned}
$$

Having obtained the low-rank decomposition factors F_{fl}^{mod} and G_{fl}^{mod}, we can then construct the modified Laguerre-based low-rank MOR method for linear discrete-time systems, as described by Algorithm 2.

Algorithm 2 Modified Laguerre-based frequency-limited MOR for linear discrete systems (Modified FL-LRSRM)

Input: Original system: $\{A, B, C, D\}$, frequency-interval $\Omega = [\omega_1, \omega_2]$, N, α, tol;
Output: ROM: $\{\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}\}$;
1. Compute low-rank factors F_{fl}^{mod} and G_{fl}^{mod} from (19) and (20), respectively;
2. Compute the SVD: ${G_{fl}^{\text{mod}}}^{\mathrm{T}} F_{fl}^{\text{mod}} = U \Sigma V^{\mathrm{T}}$, $U_r = U(:, 1:r)$, $V_r = V(:, 1:r)$, and $\Sigma_r = \Sigma(1:r, 1:r)$;
3. Compute projection matrices: $T_r = F_{fl}^{\text{mod}} V_r \Sigma_r^{-\frac{1}{2}}$, $S_r = \Sigma_r^{-\frac{1}{2}} {U_r}^{\mathrm{T}} {G_{fl}^{\text{mod}}}^{\mathrm{T}}$;
4. Construct the ROM: $\tilde{A} = S_r A T_r$, $\tilde{B} = S_r B$, $\tilde{C} = C T_r$, $\tilde{D} = D$.

4. Numerical Experiments

In this section, we evaluate the effectiveness of our algorithm using an Interconnect Layout in IC Chips [14,28]. We also compare our approach with Moore's BT [3] (Moore-BT) and Laguerre-based low-rank BT (LRBT) proposed in [14]. All experiments were conducted in MATLAB (R2017b) on a PC equipped with an Intel(R) Core(TM) i7-4712MQ CPU (2.30 GHz) and 12 GB RAM.

The interconnect layout of integrated circuit (IC) chips plays a critical role in signal propagation, especially at high frequencies, where transmission line effects become more pronounced. In the context of the interconnection structure described in Figure 1, which resembles a tree configuration, the signal transmission between various components of the IC is not purely resistive; instead, it behaves like a transmission line, meaning that we need to account for the inductive, capacitive, and resistive elements of the interconnections in order to accurately model the signal behavior. To effectively model this interconnection, we can break the transmission line into smaller segments, each of which can be approximated as an RLC circuit, where the resistance (R), inductance (L), and capacitance (C) are distributed across the length of the interconnect. The number of segments required to accurately model the transmission line depends on the delay per unit length and the rise time of the signal. The key relationship is provided by the following inequality:

$$
n \geq \frac{10 \tau d}{t_r}
$$

where n is the number of RLC segments used to model the transmission line, $\tau = \sqrt{LC}$ represents the delay per unit length of the transmission line (accounting for both inductance and capacitance), d is the length of the transmission line, and t_r is the rise time of the signal. This inequality ensures that the model has enough segments to accurately capture the transmission line's behavior. Specifically, it guarantees that the number of segments is large enough to resolve the propagation delay of the signal across the transmission line and take into account the transmission line effects on high-frequency signals.

Applying Kirchhoff's current and voltage laws to each node of the RLC circuit yields

$$\Delta hC\frac{\mathrm{d}V_i}{\mathrm{d}t} = i_i - i_{i+1}, \quad \Delta hL\frac{\mathrm{d}i_i}{\mathrm{d}t} = V_{i-1} - V_i - \Delta hRi_i, \tag{21}$$

where $i = 1, 2, \cdots, n/2$, Δh denotes the length of each segment, and n is even. After performing semi-implicit Euler discretization with a length $\Delta t = 0.05$, we construct a discrete-time system of order 300 with $m = 1$ and $p = 2$.

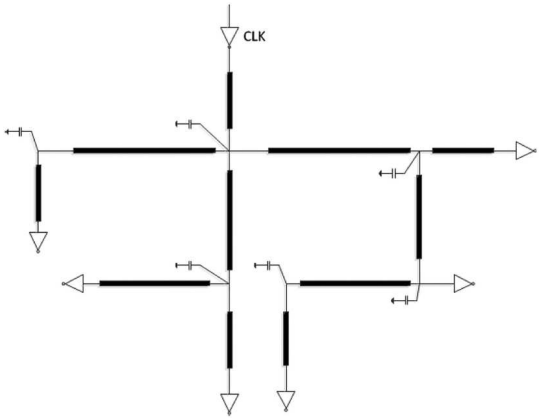


Figure 1. Interconnect layout in IC chips.

First, we examine the normalized eigenvalues (ratios to the largest eigenvalues) of the controllability Gramian P, the frequency-limited controllability Gramian P_{fl}, and its modified version P_{fl}^{mod}, as shown in Figure 2. The Gramians P, P_{fl} and P_{fl}^{mod} are accurately computed using MATLAB's `dlyap` routine, while the matrix-valued function $F(\delta\omega)B$ or B_{fl}^{mod} is handled using the `integral` routine. Eigenvalues smaller than 10^{-16} are discarded. As seen in the figure, the eigenvalues of P_{fl} decay slightly faster than those of P and P_{fl}^{mod}, with their decay patterns being nearly identical. These results are consistent with the theoretical analysis presented in [22], suggesting that the frequency-limited Gramians P_{fl} and Q_{fl} can be effectively approximated using low-rank factors.

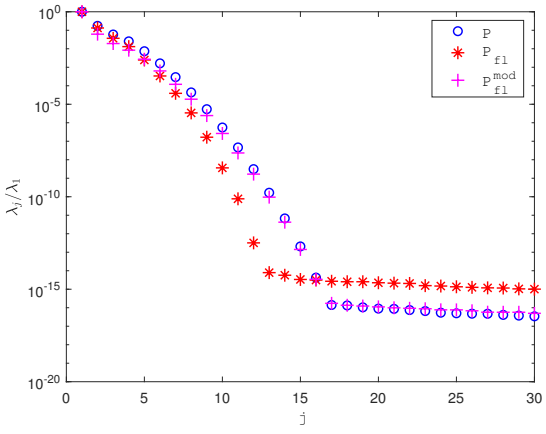
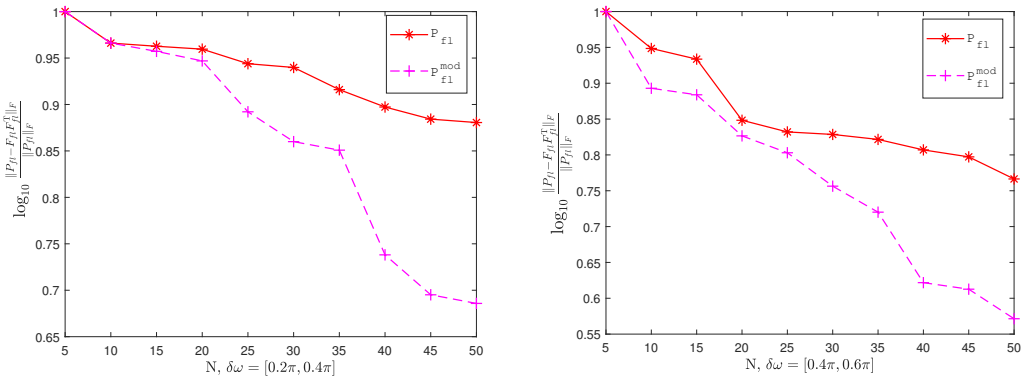


Figure 2. Normalized eigenvalues of the Gramians P, P_{fl}, and P_{fl}^{mod}.

Next, to evaluate the performance of low-rank approximation for P_{fl} and P_{fl}^{mod} using Laguerre functions, we plot the relative errors in a logarithmic scale

$$\log_{10}\frac{\|P_{fl} - F_{fl}F_{fl}^{\mathrm{T}}\|_{\mathrm{F}}}{\|P_{fl}\|_{\mathrm{F}}} \quad \text{and} \quad \log_{10}\frac{\|P_{fl}^{\mathrm{mod}} - F_{fl}^{\mathrm{mod}}F_{fl}^{\mathrm{mod}\,\mathrm{T}}\|_{\mathrm{F}}}{\|P_{fl}^{\mathrm{mod}}\|_{\mathrm{F}}}$$

with respect to the truncation term N in the frequency intervals $[0.2\pi, 0.4\pi]$ and $[0.4\pi, 0.6\pi]$, as shown in Figure 3. The results indicate an exponential decay in errors as the truncation term N increases. According to [20], the orthogonal polynomials expansion of a Lebesgue-measurable function converges uniformly, leading to optimal least-squares error approximation. Therefore, even a modest order of approximation can yield high accuracy.



(**a**) (**b**)

Figure 3. Relative errors in logarithmic sense of the Gramians P_{fl} and P_{fl}^{mod} via Laguerre functions: (**a**) relative errors in $[0.2\pi, 0.4\pi]$ and (**b**) relative errors in $[0.4\pi, 0.6\pi]$.

We also implemented Moore's BT and frequency-limited BT using the low-rank factors of the respective Gramians. For Moore's BT, the low-rank factors of the Gramians P and Q were computed using the `dlyapchol` routine in MATLAB. Algorithms 1 and 2 based on Laguerre functions were used to calculate the low-rank factors of the frequency-limited Gramians P_{fl} and Q_{fl} along with their modifications P_{fl}^{mod} and Q_{fl}^{mod}. The parameters are listed in Table 1 with the Laguerre parameter $\alpha = \frac{\omega_2+\omega_1}{2}$, where $0 \leq \omega_1 < \omega_2 \leq \pi$. Given a tolerance $tol = 10^{-8}$, the order r of the ROM is adaptively determined by the indicator $\delta \leq tol$.

Table 1. Settings of the relevant parameters.

$\delta\omega$	α	N	tol	r
$[0.2\pi, 0.4\pi]$	0.3π	10	10^{-8}	8
$[0.4\pi, 0.6\pi]$	0.5π	10	10^{-8}	6

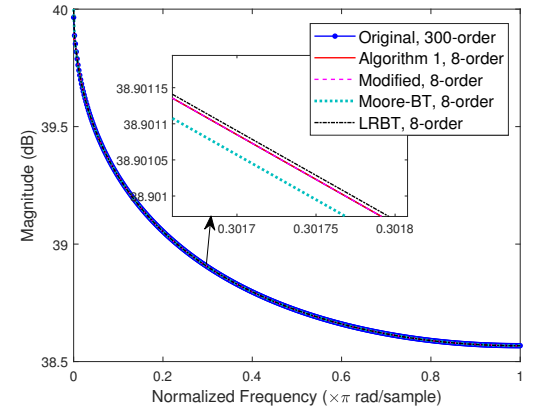
To assess the approximation quality of the constructed ROMs, we used the following relative error:

$$\mathcal{E}(\omega) = |G(e^{j\omega}) - G_r(e^{j\omega})|/|G(e^{j\omega})|, \quad \omega \in [0, \pi]$$

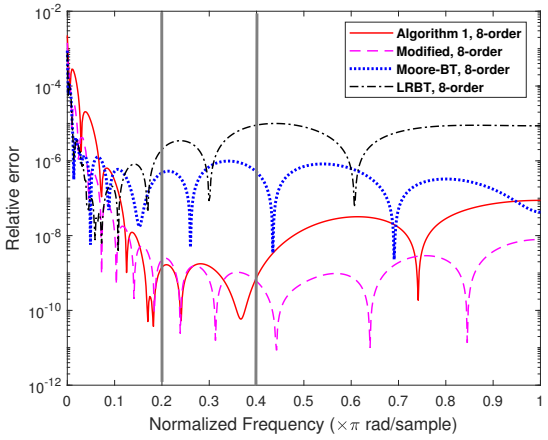
where $G(z)$ and $G_r(z)$ represent the transfer function matrices of the original system and the ROMs, respectively. The magnitude plots of $G(z)$, $G_r(z)$ and the relative errors $\mathcal{E}(\omega)$ obtained using the MATLAB `freqz` routine are shown in Figure 4, with the frequency interval boundaries ω_1 and ω_2 indicated by thick vertical lines.

Next, we compared the performance of our proposed low-rank MOR algorithms with traditional methods. Specifically, we evaluated Moore's BT, FLBT, and the proposed Laguerre-based algorithms in terms of computational time and approximation error.

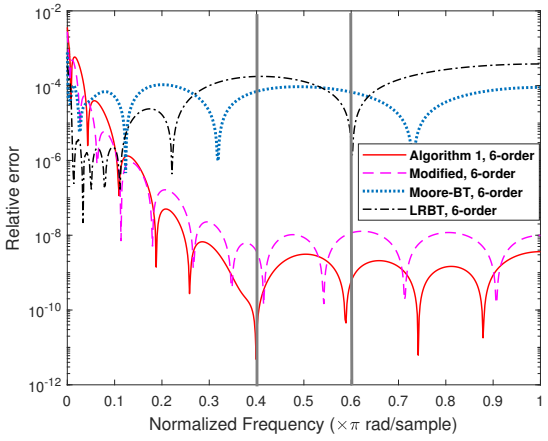
From Table 2, it is evident that the proposed Laguerre-based algorithms exhibit superior approximation accuracy compared to traditional methods such as Moore's BT and FLBT. Furthermore, the computational time is significantly reduced, especially when compared to methods that require the direct solution of Lyapunov equations. The proposed approach also uses less memory, making it more suitable for large-scale systems.



(**a**)



(**b**)



(**c**)

Figure 4. Results of the ROMs obtained by different BT versions: (**a**) magnitude plots of the original system and the ROMs, (**b**) relative errors $\mathcal{E}(\omega)$ of the ROMs with the frequency interval $\delta\omega = [0.2\pi, 0.4\pi]$, and (**c**) relative errors $\mathcal{E}(\omega)$ of the ROMs with the frequency interval $\delta\omega = [0.4\pi, 0.6\pi]$.

The simulation results show that all four ROMs demonstrate superior performance in approximating the frequency responses of the original system. In the considered frequency intervals, it is evident that the approximation achieved by Algorithm 1 and Algorithm 1 surpasses that of Moore's BT and LRBT with the same reduced order r. Compared to other frequency-limited BT methods, our proposed approach involves computing an N-dimensional integral matrix of Laguerre functions and its subsequent decomposition. This method offers the advantage of generating ROMs with high accuracy, particularly for small

and medium-sized systems, while avoiding the complexity of matrix-valued functions and the direct solution of related generalized Lyapunov equations. This makes it particularly suitable for frequency-limited MOR of larger-scale systems.

Table 2. The computational time and approximation error of the ROMs obtained by different methods.

Method	Time (Second)	Relative Error		Remark
		$\delta\omega = [0.2\pi, 0.4\pi]$	$\delta\omega = [0.4\pi, 0.6\pi]$	
Algorithm 1	1.754	2.025×10^{-8}	2.051×10^{-9}	Low-rank (frequency-limited)
Modified	1.739	1.391×10^{-9}	9.742×10^{-9}	Improved stability
Moore-BT	14.727	2.804×10^{-6}	6.362×10^{-5}	Standard
LRBT	0.899	7.213×10^{-6}	1.257×10^{-4}	Low-rank (infinite-frequency)

5. Conclusions

This paper presents a novel approach to frequency-limited MOR for linear discrete-time systems leveraging low-rank approximations of Gramians. We introduce two MOR algorithms based on Laguerre functions, which differ from traditional methods that rely on solving generalized Lyapunov equations. These algorithms utilize Laguerre expansion coefficient vectors computed using recursive formulas to derive low-rank factors of frequency-limited controllability and observability Gramians, thereby ensuring computational efficiency. Additionally, a Laguerre-based frequency-limited low-rank MOR algorithm is proposed in combination with the modified frequency-limited BT method.

The proposed algorithms are demonstrated to be effective and efficient through numerical examples. However, to further enhance the approach's performance, careful selection of the Laguerre parameter α is essential. Although there are established methods for parameter selection in frequency-free or infinite-frequency MOR in linear systems [24,25], the optimal choice of parameters for frequency-limited or more general cases remains an open question. Future research should focus on addressing this challenge in order to refine the proposed method and extend its applicability to more complex scenarios such as nonlinear and bilinear systems, which could ultimately provide more robust solutions for large-scale systems analysis and simulations.

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