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Joint Optimal Policy for Maintenance, Spare Unit Selection and Inventory Control Under a Partially Observable Markov Decision Process †

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Abstract: This research investigates the joint optimization of maintenance and spare unit management for series systems composed of multiple heterogeneous units. With advancements in communication and sensing technologies, condition-based maintenance has gained attention as an integral aspect of spare unit management. Furthermore, the inherent interaction between maintenance activities and spare unit management underscores the necessity of their simultaneous optimization to enhance overall system performance. Based on uncertain information about the system's deterioration state and spare unit inventory, decision-makers determine actions related to spare units and maintenance, such as replacements, selection of spare units for corresponding units, order quantities and inventory levels. Within the framework of a partially observable Markov decision process, this research proposes an optimal joint policy for maintenance and spare unit management with the objective of minimizing total expected costs. The proposed policy is demonstrated through a three-state, two-unit series system. Sensitivity analyses and comparisons with benchmark policies are also conducted to evaluate the performance of the proposed policy and to investigate the impact of various cost parameters on the proposed decisions.

Keywords: condition-based maintenance; non-identical units; partial observation; series system; stochastic process

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1. Introduction

Modern society relies on a wide range of infrastructure systems, including long-operating power supply networks [1] and transportation systems [2], which are critical to sustaining societal functions. These systems deteriorate over time due to prolonged operation and environmental stresses. As they deteriorate, the likelihood of unexpected failures increases, which can disrupt essential services and significantly affect economic activities and public safety [3]. Ensuring the long-term reliability and safety of such systems is therefore a vital challenge that necessitates effective and efficient maintenance policies.

Maintenance activities can be broadly classified into corrective maintenance (CM) and preventive maintenance (PM). CM refers to actions taken to restore a system to operational condition after a failure has occurred, whereas PM is conducted proactively to reduce the

risk of failures by addressing potential issues before they lead to system downtime. PM is instrumental in enhancing system reliability and safety by minimizing the likelihood of accidents or operational disruptions caused by failures [4]. PM can be further divided into time-based maintenance (TBM) and condition-based maintenance (CBM), depending on the criteria used to determine maintenance timing. TBM schedules maintenance activities at predetermined intervals, typically based on the statistical failure characteristics of system components [5]. On the other hand, CBM leverages real-time monitoring and diagnostic data to continuously evaluate the system's condition, which serves as the criterion for maintenance decisions. This approach not only optimizes maintenance resources but also significantly enhances system reliability and operational safety.

Recent advancements in communication and sensing technologies have made continuous monitoring of system conditions increasingly feasible, facilitating the widespread adoption of CBM. Numerous studies have explored the optimization of CBM for a variety of systems, including single-unit systems [6], series systems [7], and k-out-of-n systems [8]. Olde Keizer et al. [9] provided a comprehensive review of CBM research, with a particular focus on structural and stochastic dependencies, serving as a valuable resource for understanding developments in this field.

In addition to maintenance activities, other operational aspects of system management have also been considered in previous research. For instance, load sharing [10], operational performance control [11], and inventory management of spare units for maintenance are critical areas of focus. In multi-unit systems, deteriorated units are commonly replaced with spare units to ensure continued operation, restoring the system to as-good-as-new [6–8]. As mentioned in review papers [12,13], most existing studies assume that sufficient spare units are always available, with no associated management costs, and that spare unit availability does not affect maintenance decisions. However, in practical scenarios, spare unit management—which encompasses ordering, inventory control, and selecting spare units for delivery to maintenance sites—incurs costs and closely interacts with maintenance decisions. To optimize the overall operation of the system, it is essential to jointly consider maintenance and spare unit management [12,13].

Several researchers have investigated the joint optimization of CBM and spare unit management. For instance, Wang et al. [14] optimized maintenance for a single-unit system by determining the ordering of spare units based on a threshold related to the system's deterioration state. In the context of multi-unit systems, Xie et al. [15] optimized maintenance policies under an (s, S) ordering policy, where spare units are ordered to restore the inventory level to S whenever it drops below s. The (s, S) policy, being one of the most widely used inventory management policies, has been further developed by many researchers. For example, Wang et al. [16] optimized the parameters of the (s, S) policy, including s, S, the inspection interval, and the replacement threshold, and evaluated the resulting optimal policy. Wang et al. [17] introduced a Markov embedding method to jointly optimize maintenance and spare unit ordering under an (s, S) policy, taking user priorities into account. Their study focused on optimizing the number of maintenance engineers and the parameters of the ordering policy. Barron [18] proposed a practical (s, S, M)-type policy that considers limited storage capacity but does not include maintenance in the optimization process. Olde Keizer et al. [19] formulated a maintenance optimization problem using Markov decision processes (MDPs), integrating spare unit ordering into the model. They demonstrated that joint optimization policies outperformed traditional (s, S) policies in terms of overall effectiveness.

Other studies, such as those by Gan et al. [20] and Salari and Makis [21], explored joint optimization for various system configurations, including two-unit series systems and parallel production systems. These models assumed no spare unit inventory, with

ordered units being immediately available for use. Finally, Zheng et al. [22] addressed the optimization of maintenance and spare unit management for k-out-of-n systems. Their study highlighted the impact of spare unit inventory information on maintenance costs by comparing policies that rely solely on system states with those that incorporate inventory considerations.

More recent studies have expanded the scope of CBM and spare unit management to address additional complexities, such as multiple suppliers [23,24], differences in spare unit quality [25] and distributed maintenance sites [26,27]. Zheng et al. [23] proposed a joint optimal policy for spare unit ordering and maintenance, considering spare unit suppliers with varying ordering costs and lead times. Their numerical studies demonstrated the impact of system conditions on supplier selection and highlighted the effectiveness of the proposed policy. Zheng et al. [25] proposed a joint optimization model for spare unit ordering and maintenance, considering differences in spare unit quality among suppliers, and also evaluated several heuristic calculation methods to improve calculation efficiency. There is also research dealing with complex and large-scale systems. Zheng et al. [24] introduced a hybrid deep reinforce learning algorithm to improve efficiency in deriving the optimal policy. The HDRI algorithm is an enhanced version of the double deep Q-network algorithm that uses different networks for action selection and action value estimation. This method proved effective in optimizing policies under complex scenarios. Wang and Djurdjanovic [26] proposed an optimization approach for maintenance and spare unit management in distributed systems. They derived the optimal policy using a genetic algorithm and analyzed its characteristics through sensitivity analysis and related techniques. Similarly, Zhu et al. [27] proposed a simultaneous optimization approach that integrates maintenance, spare unit management, and routing decisions for onshore wind power systems.

Most previous studies assume that system states are fully observable. However, in real-world applications, monitoring devices often experience uncertainties due to failures or external noise [28]. Such uncertainties, if overlooked, can result in inaccurate risk assessments and suboptimal maintenance decisions. To address these challenges, partially observable Markov decision processes (POMDPs) have been used to model sequential decision-making under partial observations. For example, Jiang et al. [29] investigated optimal maintenance policies for multi-unit systems with non-identical units under partial observation, and subsequent studies further elucidated the structural properties of these policies [30]. Eggertsson et al. [31] expanded on this by developing an optimization model for multi-unit systems monitored by a single sensor. In their model, maintenance decisions are based on partial information about the overall system state rather than individual unit conditions. Karabag et al. [32] incorporated spare unit selection into maintenance policies under partial observations. Their study demonstrated that the proposed approach significantly reduced total expected costs compared to benchmark policies. Similarly, Tang et al. [33] proposed a framework for jointly optimizing maintenance and spare unit ordering in single-unit systems while accounting for observational uncertainties. Their numerical analysis highlighted the unique characteristics and benefits of the resulting policy.

In the extensive research on the joint optimization of condition-based maintenance and spare unit management, it has been a common assumption that observational data contains no errors and that systems are composed of identical units. However, research addressing joint optimization problems that consider observational uncertainty and systems composed of heterogeneous units remains limited. If errors in observations are not taken into account, risks cannot be correctly assessed and appropriate maintenance policies cannot be proposed. In addition, when spare units are different for each unit, inventory control must be performed for each spare unit, and inventory control policies that assume

identical units do not yield optimal solutions. To address these gaps, in our previous research [34], we aim to jointly optimize maintenance and spare unit management under uncertain observations for a series system composed of non-identical units. Furthermore, the decision-making in the obtained policy was analyzed. In this extended research, a more extensive analysis of the obtained policies is performed to evaluate their performance and effectiveness. Our series of research makes the following contributions:

- To our knowledge, our research is the first to comprehensively address the joint optimization of maintenance and spare unit management for systems composed of non-identical units. The proposed model bridges the identified gaps between theoretical research and practical applications, enhancing both its applicability and accuracy in real-world scenarios.
- Our research introduces a versatile approach that enables the derivation of joint optimal policies for complex systems under a POMDP framework.
- Our research conducts a detailed and thorough analysis, including comparisons with several benchmark policies and sensitivity analyses. The results highlight the effectiveness and distinctive characteristics of the proposed policy. Moreover, the insights derived from these analyses can inform the development of practical maintenance and spare unit management policies.

The remainder of this paper is organized as follows. Section 2 introduces the model and formulates the decision-making problem within the POMDP framework. Section 3 presents the optimal policy, illustrated with an example of a two-unit series system. Section 4 provides a detailed sensitivity analysis to explore the impact of key parameters on the proposed policy. This section also includes a comparison between the proposed policy and several traditional policies, highlighting the effectiveness of the proposed approach. Finally, Section 5 concludes the study with a summary of findings and suggestions for future research directions.

2. Problem Descriptions

2.1. Target System and Spare Units

This research focuses on a series system that deteriorates over time consisting of N non-identical units and an inventory of spare units used for replacement. The deterioration state of the system is observed by the monitoring device at equally spaced time epochs, with the duration between two consecutive observations referred to as one time interval τ. Each unit i has a state s_i that belongs to the state space $S_i = \{0, 1, \ldots, \tilde{s_i}\}$, where $s_i = 0$ indicates that the unit is as good as new, and $s_i = \tilde{s_i}$ indicates failure. The set of all units is denoted as $\mathcal{I} = \{1, 2, \ldots, N\}$. Each unit i has a unique defect threshold Δ_i, where $0 < \Delta_i < \tilde{s_i}$. If $s_i < \Delta_i$, the unit is considered normal; if $\Delta_i \leq s_i < \tilde{s_i}$, it is considered defective. The state transition probability $p_i(s' = s(t+\tau)|s = s(t))$ that the state of unit i transitions from $s = s(t)$ to $s' = s(t+\tau)$ in the observation interval τ (between time t and $t+\tau$). We assume that the deterioration process of each unit follows an independent Markov chain. The state transition probability matrix for unit i, denoted as $\mathbf{P}_i$, has $p_i(s' \mid s)$ as its s-th row and s'-th column element. The overall system state is represented as $\boldsymbol{s} = (s_1, s_2, \ldots, s_N)$, which is a vector of the states of all units. The state transition probability of the system $p(\boldsymbol{s}' \mid \boldsymbol{s})$ from $\boldsymbol{s}$ to $\boldsymbol{s}'$ is obtained as the simultaneous probability of each state transition probability $p_i(s'|s)$ by the following:

$$p(\boldsymbol{s}' \mid \boldsymbol{s}) = \prod_{i \in \mathcal{I}} p_i(s' \mid s). \tag{1}$$

When a unit deteriorates to a defective or failed state, it may be replaced with a corresponding spare unit. Let $I_i \in \{0, 1, \ldots, \tilde{I}_i\}$ represent the inventory level of spare units for unit i, where $\tilde{I}_i$ is the maximum inventory capacity. The inventory vector for all units is

$\boldsymbol{I} = (I_1, I_2, \ldots, I_N)$. The combined state space $\mathcal{S}$ of the system and the spare unit inventory is defined as follows:

$$\mathcal{S} = \{(s_1, s_2, \ldots, s_N, I_1, I_2, \ldots, I_N) \mid s_i \in S_i,\ I_i \in \{0, 1, \ldots, \tilde{I}_i\},\ i \in \mathcal{I}\}. \tag{2}$$

To account for the uncertainties in observed information, this research introduces a belief vector $\boldsymbol{\pi}$ to describe the system's state as a probability distribution. The state of the system is observed by a single monitoring device. It is assumed that this monitoring device cannot observe the status of each unit. The decision-maker obtains partial information about the system's state through the monitoring device at equally spaced time intervals. Let $\theta \in \Theta = \{0, 1, \ldots, \tilde{\theta}\}$ denote the observation from the monitoring device, where a larger value of θ indicates greater system deterioration. The observation conditional probability $\gamma(\theta \mid \boldsymbol{s})$ represents the likelihood that the monitoring device outputs the observation θ given the system state $\boldsymbol{s}$. This observation conditional probability captures the characteristics and accuracy of the monitoring device.

Let $\mathbb{S}$ represent the set of all possible system states $\boldsymbol{s}$. The belief vector $\boldsymbol{\pi}$ consists of probabilities $\pi_{\boldsymbol{s}}$ for the system being in each state $\boldsymbol{s} \in \mathbb{S}$. The set of all possible belief vectors, denoted as $\boldsymbol{\Pi}$, is defined as follows:

$$\boldsymbol{\Pi} = \left\{ \boldsymbol{\pi} \in \mathbb{R}_+^{|\mathbb{S}|} \mid \sum_{\boldsymbol{s} \in \mathbb{S}} \pi_{\boldsymbol{s}} = 1 \right\}, \tag{3}$$

where $\mathbb{R}_+$ is the set of non-negative real numbers. Given the current belief vector $\boldsymbol{\pi}$, the probability of observing θ at the beginning of the next time epoch is expressed as follows:

$$\sigma(\theta \mid \boldsymbol{\pi}) = \sum_{\boldsymbol{s}' \in \mathbb{S}} \sum_{\boldsymbol{s} \in \mathbb{S}} \pi_{\boldsymbol{s}} p(\boldsymbol{s}' \mid \boldsymbol{s}) \gamma(\theta \mid \boldsymbol{s}'), \tag{4}$$

where $p(\boldsymbol{s}' \mid \boldsymbol{s})$ is the probability of transitioning from state $\boldsymbol{s}$ to state $\boldsymbol{s}'$ as defined in Equation (1) and $\gamma(\theta|\boldsymbol{s}')$ is the conditional probability of observation θ when the state is $\boldsymbol{s}'$. When the observation θ is obtained, the belief vector $\boldsymbol{\pi}$ is updated to $\boldsymbol{\pi}'$ using Bayes' theorem. The updated probability $\pi'_{\boldsymbol{s}'}$ for being in state $\boldsymbol{s}'$ is given by

$$\pi'_{\boldsymbol{s}'} = T_{\boldsymbol{s}'}(\boldsymbol{\pi}, \theta) = \frac{\sum_{\boldsymbol{s} \in \mathbb{S}} \pi_{\boldsymbol{s}} p(\boldsymbol{s}' \mid \boldsymbol{s}) \gamma(\theta \mid \boldsymbol{s}')}{\sigma(\theta \mid \boldsymbol{\pi})}. \tag{5}$$

Here, $T_{\boldsymbol{s}'}(\boldsymbol{\pi}, \theta)$ represents the belief update function, which incorporates the observation θ and the state transition probabilities to refine the system's state estimation.

2.2. Actions and Costs

In the system described above, the decision-maker determines actions for maintenance and spare unit management, including the order quantity and selection of spare units brought to the maintenance site, based on beliefs that represent partial information about the deterioration and the inventory. At the beginning of each observation epoch, the decision-maker selects an action a from the action space $\mathcal{A} = \{0, 1\}$, where $a = 1$ denotes maintenance, and $a = 0$ denotes continue operation. As shown in Figure 1, if maintenance is selected, the spare units to bring to the site are determined simultaneously based on partial monitoring observations. The selection is represented by a binary vector $\boldsymbol{g} = (g_1, g_2, \ldots, g_N)$, where $g_i = 1$ indicates bringing a spare unit for unit i, and $g_i = 0$ indicates not bringing one. The set of all possible selections is expressed as follows:

$$\mathbf{G} = \{(g_1, g_2, \ldots, g_N) \mid g_i \in \{0, 1\}, i \in \mathcal{I}\}. \tag{6}$$

After spare units are delivered to the site, a complete inspection determines the exact states of all units. Units which are not in normal state ($s_i \geq \Delta_i$) are replaced, and any unused spare units are returned to inventory. If continue operation is selected, the system operates for one interval without maintenance. Following Karabag et al. [32], this research assumes that the time for delivery, inspection, and replacement is negligible compared to the system's operating time, allowing the system to resume operation immediately.

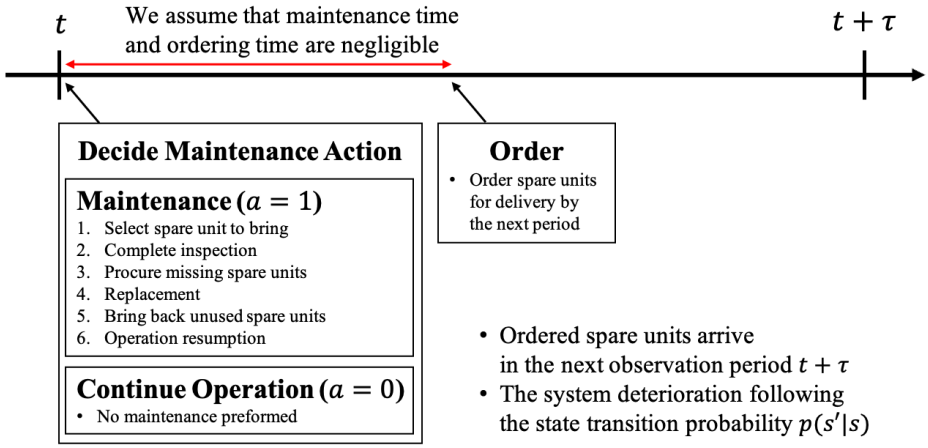


Figure 1. Overview of decision-making and action flow in one epoch.

At the beginning of each epoch, the decision-maker also determines the order quantity $\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots, \omega_N)$ for spare units. Each ω_i must not exceed the ordering limit $\tilde{\omega}_i$ and the amount of space in inventory. The amount of space in inventory can be obtained as the gap between inventory limit $\tilde{I}_i$ and the current inventory I_i. The set of quantities ordered for spare units given the inventory $\boldsymbol{I}$ is given as follows:

$$\boldsymbol{\Omega}_{(\boldsymbol{I})} = \{(\omega_1, \omega_2, \ldots, \omega_N) \mid \omega_i \in \{0, 1, \ldots, \min(\tilde{\omega}_i, \tilde{I}_i - I_i)\}, i \in \mathcal{I}\}. \tag{7}$$

The order is based on complete information if maintenance is performed, as the exact states are known after inspection. However, decisions rely on partial observations if continue operation is selected as an action. Ordering time is assumed to be negligible, and all decisions (maintenance, continue operation, and order in Figure 1) are considered to occur at the same time of each epoch.

The model considers three types of costs: operation cost, maintenance cost, and spare unit management cost. The operation cost $C^o(\boldsymbol{s})$ is incurred per epoch based on the system's deterioration state $\boldsymbol{s}$. Maintenance costs include a fixed maintenance cost $C^f(\boldsymbol{s})$ and a variable maintenance cost $C^v(\boldsymbol{I}, \boldsymbol{s}, \boldsymbol{g})$. We assume that fixed maintenance costs are incurred based on the state of the system and include labor costs and losses due to system downtime. The fixed maintenance cost $C^f(\boldsymbol{s})$ is defined as follows:

$$C^f(\boldsymbol{s}) = \begin{cases} c^{im} & \text{if} \quad s_i < \Delta_i \text{ for } \forall i \in \mathcal{I}, \\ c^{cm} & \text{if} \quad s_i = \tilde{s}_i \text{ for } \exists\, i \in \mathcal{I}, \\ c^{pm} & \text{otherwise.} \end{cases} \tag{8}$$

Here, CM cost c^{cm} is incurred if even one unit is failed, and inspection cost c^{im} is incurred when all units are normal, as no replacement is required in this case. In all other cases, PM cost c^{pm} is incurred.

The variable maintenance cost includes replacement costs per unit and transportation costs for spare units brought to the maintenance site. If a spare unit to be brought to the maintenance site is out of stock, an emergency procurement is performed and an emergency

procurement cost c_i^{eo} is incurred. The spare unit is brought to the site and the unit is replaced based on the inspection results. A replacement cost c_i^r is incurred for each unit to be replaced. However, if the spare unit required for replacement is not brought to the site, emergency bring-in is performed and an emergency bring-in cost c^{eb} is incurred. In addition, if the spare unit is out of stock, an emergency procurement is performed and an emergency procurement cost c_i^{eo} is incurred. Finally, spare units not used for replacement are returned to inventory. At this time, a return cost c_i^b is incurred. The variable maintenance cost $C^v(\boldsymbol{I}, \boldsymbol{s}, \boldsymbol{g})$ depends on inventory levels, system state, and spare unit selection:

$$\begin{aligned} C^v(\boldsymbol{I}, \boldsymbol{s}, \boldsymbol{g}) = \sum_{i \in \mathcal{I}} c_i^{eo} g_i \mathbf{1}_{\{I_i = 0\}} + \sum_{i \in \mathcal{I}} c_i^r \mathbf{1}_{\{s_i \geq \Delta_i\}} + c^{eb} \sum_{i \in \mathcal{I}} (1 - g_i) \mathbf{1}_{\{s_i \geq \Delta_i\}} \\ + \sum_{i \in \mathcal{I}} c_i^{eo} (1 - g_i) \mathbf{1}_{\{I_i = 0 \wedge s_i \geq \Delta_i\}} + c^b \sum_{i \in \mathcal{I}} g_i \mathbf{1}_{\{s_i < \Delta_i\}}. \end{aligned} \tag{9}$$

Here, $\mathbf{1}_{\{\cdot\}}$ is the indicator function, returning 1 if the given condition is satisfied, and 0 otherwise. The spare unit management cost $C^I(\boldsymbol{I}, \boldsymbol{\omega})$ is composed of holding and procurement costs. Each spare unit stored in inventory incurs a holding cost each epoch. Next, procurement costs consist of ordering costs and spare unit prices. Ordering costs include labor and other costs incurred in ordering spare units, and we assume that these costs are constant for all types and quantities of spare units ordered. The spare unit management cost $C^I(\boldsymbol{I}, \boldsymbol{\omega})$ is defined as follows:

$$C^I(\boldsymbol{I}, \boldsymbol{\omega}) = \sum_{i \in \mathcal{I}} c_i^h I_i + c^l \mathbf{1}_{\{\omega_i > 0 \text{ for } \exists i\}} + \sum_{i \in \mathcal{I}} c_i^p \omega_i, \tag{10}$$

where c_i^h, c^l, and c_i^p are the holding cost, ordering cost, and unit price for spare units, respectively.

2.3. Decision Process

A Markov decision process (MDP) models sequential decision-making in stochastic systems where the state of the system is fully observable. An MDP consists of a state space S, an action space A, state transition probabilities P, and rewards or costs R. Decisions are made based on the current state $s \in S$, and the system transitions to a new state s' with probability $P(s'|s, a)$, yielding a reward or cost $r(s')$.

A POMDP extends the MDP to situations where the true system state cannot be directly observed. Instead, decision-makers rely on observations Θ and an observation function Γ that links observations to underlying states. To make decisions under this partial observability, a belief state π is introduced as a key factor, representing a probability distribution over all possible states in S. The belief is updated using Bayes' theorem as new observations are made, and decisions are based on this updated belief.

In this research, we use the POMDP framework to optimize decision-making in systems with partial observability, enabling the development of robust policies that account for observational uncertainty.

2.4. Optimal Joint Policy of Maintenance and Spare Unit Management

This research aims to minimize the total expected cost for maintenance and spare unit management over an infinite time horizon. To achieve this, POMDP is employed for optimizing the decision-making process. The total expected cost $V(\boldsymbol{\pi}, \boldsymbol{I})$ for a given belief $\boldsymbol{\pi}$ and spare unit inventory $\boldsymbol{I}$ is formulated as follows:

$$V(\boldsymbol{\pi}, \boldsymbol{I}) = \min \begin{cases} V^C(\boldsymbol{\pi}, \boldsymbol{I}) & \text{if} \quad a = 0, \\ V^M(\boldsymbol{\pi}, \boldsymbol{I}) & \text{if} \quad a = 1, \end{cases} \tag{11}$$

where $a = 0$ corresponds to continued operation, and $a = 1$ corresponds to maintenance.

The cost functions $V^C(\boldsymbol{\pi}, \boldsymbol{I})$ and $V^M(\boldsymbol{\pi}, \boldsymbol{I})$ are defined as follows:

$$V^C(\boldsymbol{\pi}, \boldsymbol{I}) = \sum_{\boldsymbol{s} \in \mathbb{S}} \pi_{\boldsymbol{s}} C^o(\boldsymbol{s}) + \min_{\boldsymbol{\omega} \in \boldsymbol{\Omega}} \left\{ C^I(\boldsymbol{I}, \boldsymbol{\omega}) + \beta \sum_{\theta \in \Theta} \sigma(\theta \mid \boldsymbol{\pi}) V(T(\boldsymbol{\pi}, \theta), \boldsymbol{I} + \boldsymbol{\omega}) \right\}, \tag{12}$$

$$V^M(\boldsymbol{\pi}, \boldsymbol{I}) = \sum_{\boldsymbol{s} \in \mathbb{S}} \pi_{\boldsymbol{s}} C^f(\boldsymbol{s}) + \min_{\boldsymbol{g} \in \mathbf{G}} \left\{ \sum_{\boldsymbol{s} \in \mathbb{S}} \pi_{\boldsymbol{s}} C^v(\boldsymbol{I}, \boldsymbol{s}, \boldsymbol{g}) + \sum_{\boldsymbol{s} \in \mathbb{S}} \pi_{\boldsymbol{s}} V^C(M(\boldsymbol{s}), U(\boldsymbol{I}, \boldsymbol{s}, \boldsymbol{g})) \right\}. \tag{13}$$

Here, $V^C(\boldsymbol{\pi}, \boldsymbol{I})$ represents the total expected cost under continue operation, while $V^M(\boldsymbol{\pi}, \boldsymbol{I})$ represents the total expected cost under maintenance. The parameter β is the discount factor for one epoch. The belief update after maintenance, denoted by $M(\boldsymbol{s})$, represents the updated system state after replacement actions. Let $\pi_{\hat{\boldsymbol{s}}}$ be the $\hat{\boldsymbol{s}}$-th element of $M(\boldsymbol{s})$ which indicates the probability of state $\hat{\boldsymbol{s}}$. Since replacements are performed for all units with $s_i \geq \Delta_i$, $\pi_{\hat{\boldsymbol{s}}}$ is defined as follows:

$$\pi_{\hat{\boldsymbol{s}}} = \begin{cases} 1 & \text{if} \quad \hat{\boldsymbol{s}} = \boldsymbol{s} \circ \boldsymbol{\mu}_{\{s_i < \Delta_i\}}, \\ 0 & \text{otherwise.} \end{cases} \quad (i \in \mathcal{I}). \tag{14}$$

The inventory update function, $U(\boldsymbol{I}, \boldsymbol{s}, \boldsymbol{g})$, is given by

$$U(\boldsymbol{I}, \boldsymbol{s}, \boldsymbol{g}) = \boldsymbol{I} - \boldsymbol{\mu}_{\{I_i > 0 \wedge s_i \geq \Delta_i\}} + \boldsymbol{g} \circ \boldsymbol{\mu}_{\{I_i = 0 \wedge s_i < \Delta_i\}}, \tag{15}$$

where the second term represents spare units removed from inventory, and the third term accounts for spare units returned to inventory after maintenance. Here, $\mu_{\{\cdot\}}$ is a column vector where the element is 1 if the condition is satisfied and 0 otherwise.

The optimal policy is determined by solving Equation (11) to identify the optimal action a^*, the optimal order quantity $\boldsymbol{\omega}^*$, and the optimal spare unit selection $\boldsymbol{g}^*$ based on the current belief $\boldsymbol{\pi}$ and inventory $\boldsymbol{I}$.

2.5. Value Iteration

We adopt the value iteration algorithm to solve for the optimal value function and optimal policy for an infinite horizon. The value iteration algorithm is one of the best-known methods for finding the optimal value function and optimal policy for an infinite horizon [35,36]. The value iteration algorithm is shown below.

Step 1. Initialization

First, initialize the value function and parameters. Set the convergence condition $\epsilon > 0$ and set the value function at iteration 0 for all states $\boldsymbol{\pi}, \boldsymbol{I}$ to $V_n(\boldsymbol{\pi}, \boldsymbol{I}) = 0$. Set the iteration counter to $n = 1$.

Step 2. Calculate Expected Cost

Calculate the expected cost using Equation (11)–(13) for all states $\boldsymbol{\pi}, \boldsymbol{I}$. Also, store the action $\psi_n(\boldsymbol{\pi}, \boldsymbol{I})$ that has the lowest expected cost for all states:

$$\psi_n(\boldsymbol{\pi}, \boldsymbol{I}) = \underset{a \in A, \boldsymbol{g} \in \boldsymbol{G}, \boldsymbol{\omega} \in \Omega_{(\boldsymbol{I})}}{\arg\min} \left\{ V_n^C(\boldsymbol{\pi}, \boldsymbol{I}), V_n^M(\boldsymbol{\pi}, \boldsymbol{I}) \right\}.$$

Step 3. Calculate Errors

Next, the maximum and minimum errors are determined as follows:

$$M_n = \max\{V_n(\boldsymbol{\pi}, \boldsymbol{I}), V_{n-1}(\boldsymbol{\pi}, \boldsymbol{I})\},$$
$$m_n = \min\{V_n(\boldsymbol{\pi}, \boldsymbol{I}), V_{n-1}(\boldsymbol{\pi}, \boldsymbol{I})\}.$$

If $|M_n - m_n| < \epsilon$, the iteration is stopped and the action stored in Step 2. is the optimal action. Otherwise, add 1 to the iteration counter n and repeat Step 2. and Step 3.

3. Numerical Example

3.1. Parameter Settings

This numerical example considers a series system consisting of two units. Each unit has three states, denoted as $s_1, s_2 \in \{0(\text{normal}), 1(\text{defective}), 2(\text{failed})\}$, with a defect threshold of $\Delta_1 = \Delta_2 = 1$.

Next, we assume that monitoring devices provide observations of three states ($\theta \in \{0(\text{normal}), 1(\text{defective}), 2(\text{failed})\}$). The probability that a monitoring device outputs an observation that is different from the true state of the system (false alarm) is assumed to be 10%. However, the proportion of false alarms was determined to be equal. The transition probability matrices and the observation conditional probability $\gamma(\theta \mid \boldsymbol{s})$ are given by Equation (16) and Table 1, respectively.

$$\mathbf{P}_1 = \mathbf{P}_2 = \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0 & 0.8 & 0.2 \\ 0 & 0 & 1 \end{bmatrix} \tag{16}$$

We set the costs for maintenance to be shown in Table 2 with reference to Karabag et al. [32]. Spare unit ordering cost and holding cost were set with reference to the ratios between parameters in the numerical example in Zheng et al. [23]. The emergency bring-in cost comprises additional transportation costs, while the emergency procurement cost includes the cost for express delivery. Finally, we also set the discount rate to $\beta = 0.9$, and the convergence condition to $\epsilon = 0.01$.

Table 1. Observation conditional probability for each system state $\boldsymbol{s}$.

$\gamma(\boldsymbol{\theta} \mid \boldsymbol{s})$		$\boldsymbol{s}$								
		(0, 0)	**(0, 1)**	**(0, 2)**	**(1, 0)**	**(1, 1)**	**(1, 2)**	**(2, 0)**	**(2, 1)**	**(2, 2)**
θ	0	0.90	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05
	1	0.05	0.90	0.05	0.90	0.90	0.05	0.05	0.05	0.05
	2	0.05	0.05	0.90	0.05	0.05	0.90	0.90	0.90	0.90

Table 2. Maintenance and spare unit management costs ($i \in \mathcal{I}$).

	Operation cost			Fixed maintenance cost		
Parameter	c^n	c^d	c^f	c^{cm}	c^{pm}	c^{im}
Value	0	0	1000	300	100	100
	Variable maintenance cost			Spare unit management cost		
Parameter	c_i^r	c_i^{eo}	c^{eb}	c^b	c^l	c_i^p	c_i^h
Value	100	330	60	30	10	100	20

3.2. Proposed Policy

The proposed policies for various spare unit inventories $\boldsymbol{I}$ are illustrated in Figure 2. The horizontal axis corresponds to π_0^2, representing the probability of unit 2 being in a normal state, while the vertical axis corresponds to π_1^2, representing the probability of unit 2 being in a defective state. As shown in Figure 3, the colors of the symbols in the figure

indicate maintenance actions (a), the shapes of the symbols indicate the selection of spare units to be brought in (g), and the background patterns indicate orders (ω).

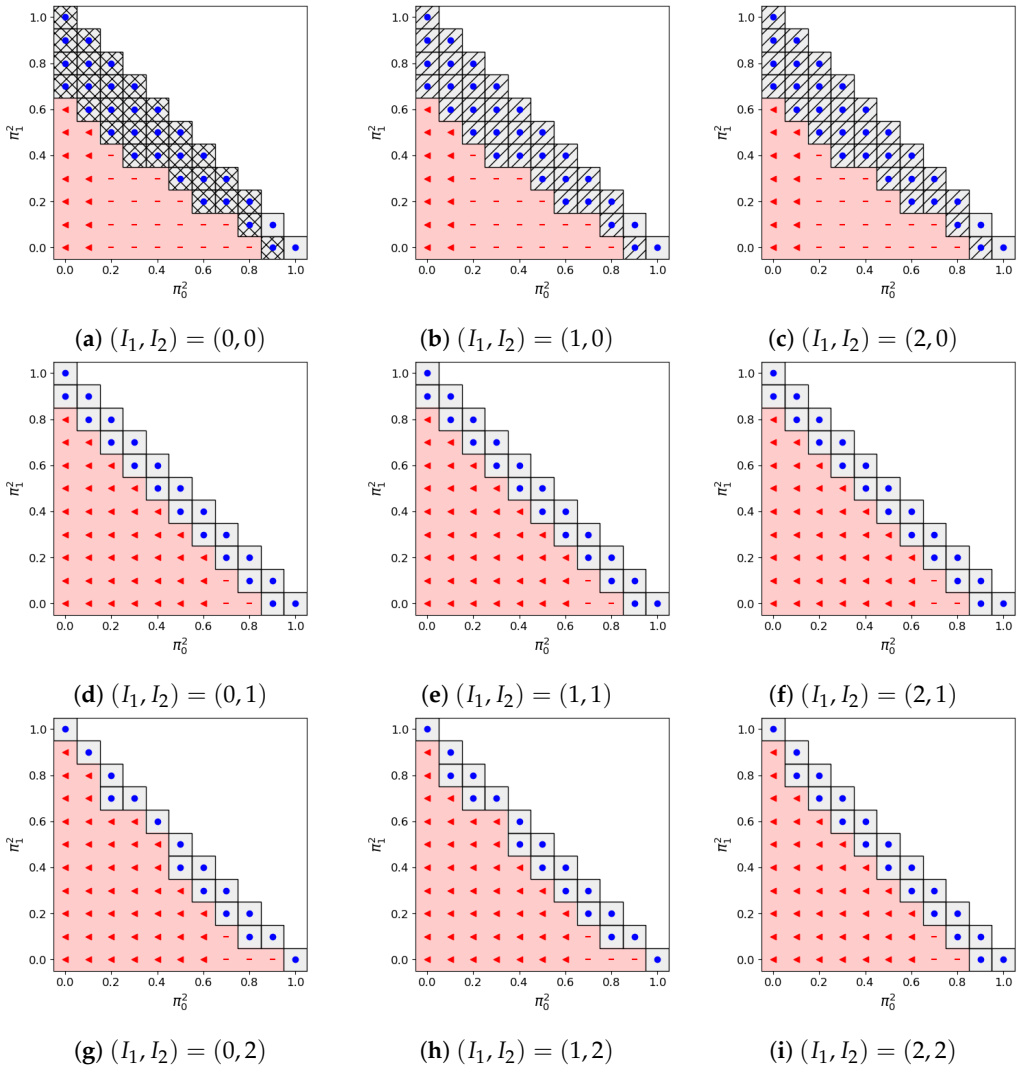


Figure 2. Proposed policy for each spare unit inventory ($\boldsymbol{\pi}^1 = (1,0,0)$).

Symbol	Action	Selects spare units	Pattern	Orders spare units	Pattern	Orders spare units
●	Continue operation	-		Decided after maintenance		$\boldsymbol{\omega} = (1,1)$
–	Maintenance	$\mathbf{g} = (0,0)$		$\boldsymbol{\omega} = (0,0)$		$\boldsymbol{\omega} = (1,2)$
▶		$\mathbf{g} = (1,0)$		$\boldsymbol{\omega} = (0,1)$		$\boldsymbol{\omega} = (2,0)$
◀		$\mathbf{g} = (0,1)$		$\boldsymbol{\omega} = (0,2)$		$\boldsymbol{\omega} = (2,1)$
■		$\mathbf{g} = (1,1)$		$\boldsymbol{\omega} = (1,0)$		$\boldsymbol{\omega} = (2,2)$

Figure 3. Legend of the figures.

Figure 2a–c represent scenarios where there is no spare inventory for unit 2. In this situation, when the probability of being in a defective state (π_1^2) is high, the region of continue operation expands compared to scenarios with available inventory for unit 2 (Figure 2d–i). This suggests that delaying maintenance is preferable to avoid incurring emergency procurement costs. In other words, in the absence of spare inventory, the proposed policy prioritizes securing inventory before conducting maintenance.

Next, we examine the selection of spare units to bring to the maintenance site. When the probability of unit 2 being in a normal state (π_0^2) is high, spare units for unit 2 are not selected, as shown in Figure 2. In cases with no spare inventory, the region for choosing not

to bring in spare units expands, while it shrinks in cases with sufficient inventory. When inventory levels are sufficient, spare units are proactively utilized to minimize holding costs.

Finally, we focus on ordering. Orders are placed for spare units only when inventory is depleted. However, for unit 1, orders are postponed until deteriorated units are replaced if the probability of unit 2 being in a normal state is low (Figure 2d,g). This approach reduces holding and ordering costs, demonstrating that joint orders for units are more economical than ordering them separately.

Figure 4 illustrates the optimal policy for different beliefs about unit 1 ($\boldsymbol{\pi}^1$), assuming a spare unit inventory of $\boldsymbol{I} = (1, 1)$. When the probability of unit 1 being in a failed state (π_2^1) is high, continue operation is not selected, as shown in Figure 4f–i. Conversely, when the failure probability for both units (π_2^i) is sufficiently low, operation continuation is chosen; otherwise, maintenance is preferred. In this study, the operational continuation cost c^f in a failure state is set higher than other maintenance costs. Consequently, the optimal policy is conservative, favoring early replacement of units with a high failure probability.

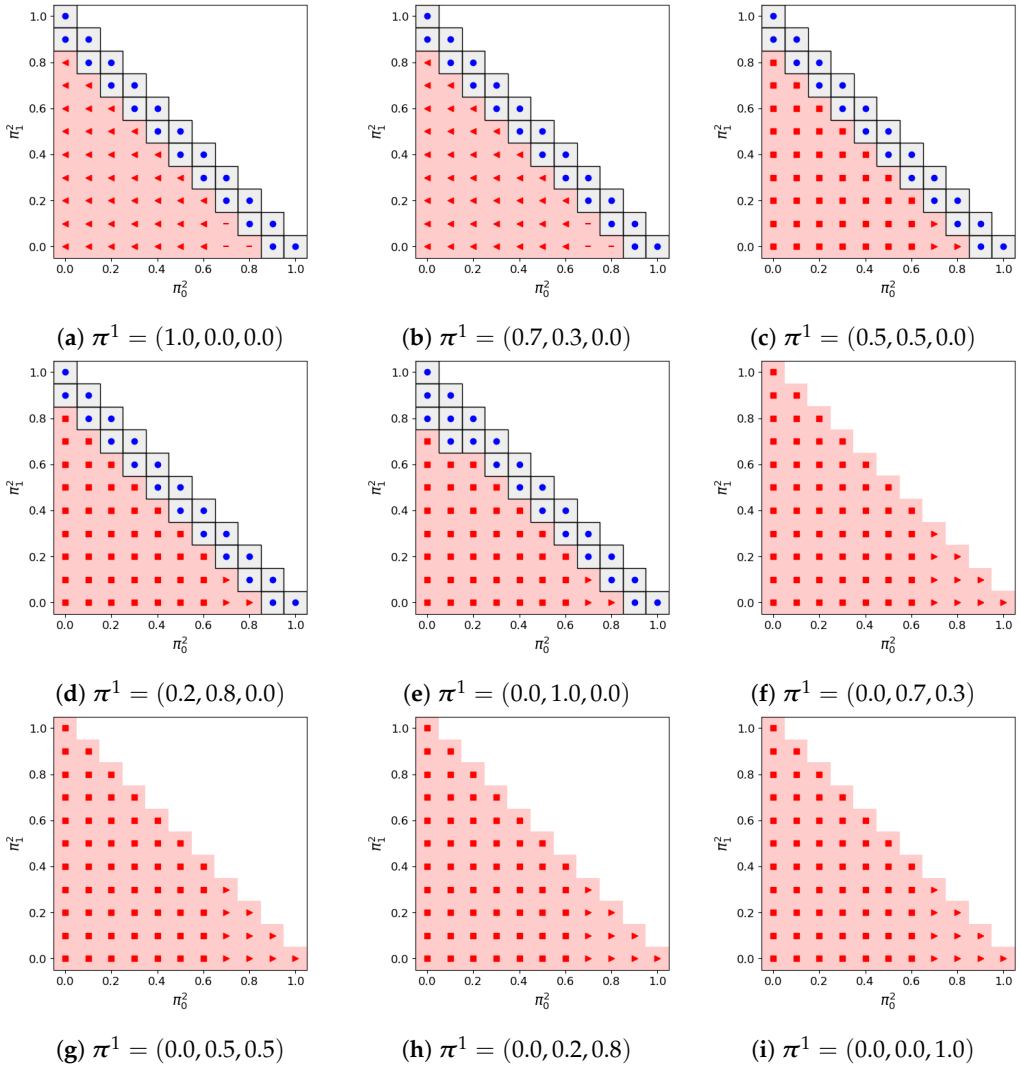


Figure 4. Proposed policy for each belief for state of unit 1 ($\boldsymbol{I} = (1, 1)$).

Next, we examine the selection of spare units to bring to the maintenance site. In the bottom-right region of each figure, where the probability of unit 2 being in a healthy state is high, the policy typically favors bringing only the spare unit for unit 1, even when unit 1 shows significant deterioration. However, as the deterioration of unit 1 progresses, the region favoring a single spare unit narrows, while the region where both spare units are

selected expands. This shift in policy is likely driven by economic dependencies, such as fixed maintenance costs, which play a significant role in determining spare unit selection.

4. Sensitivity Analysis

4.1. Benchmark Policies and Performance Indicators

In this section, the performance of the proposed policy is evaluated by comparing it to several benchmark policies: the preventive maintenance policy (PMP) and the corrective maintenance Policy (CMP). Under PMP, maintenance is performed when the defect or failure information is observed. Under CMP, maintenance is performed only when the failure information is observed. Both benchmark policies use an (s, Q) ordering policy, where $s = 0$ and $Q = 1$, meaning that one unit is ordered when the spare inventory is depleted. Additionally, PMP_f and CMP_f are variants of PMP and CMP that always bring both types of spare units to the maintenance site. The decision-making criteria for the proposed and benchmark policies are summarized in Table 3.

Table 3. Decision-making criteria for proposed and benchmark policies.

	Proposed Policy	**Benchmark Policies**			
		PMP	**CMP**	**PMP_f**	**CMP_f**
Criteria for maintenance	Expected cost	Observation			
Criteria for ordering	Expected cost	Inventory level			
Criteria for spare unit selection	Expected cost	Expected cost		Fixed: $\boldsymbol{g} = (1, 1)$	

To examine the impact of the proposed policy compared to the benchmark policies, a sensitivity analysis is conducted. The performance indicators used are the average total expected cost and the reduction rate achieved by the proposed policy. The average total expected cost (V_{avg}) represents the population mean of the total expected cost when optimal actions are applied for each state. It is calculated as follows:

$$V_{\text{avg}} = \frac{\sum_{\boldsymbol{I}} \sum_{\boldsymbol{\pi}} V(\boldsymbol{\pi}, \boldsymbol{I})}{\prod_{i \in \mathbf{I}} (\tilde{I}_i + 1) \times |\Pi|}. \tag{17}$$

The reduction rate (R^{BP}) measures the cost improvement of the proposed policy compared to a benchmark policy (BP). It is calculated using the average total expected cost of the benchmark policy, $V_{\text{avg}}^{\text{BP}}$, and the proposed policy, $V_{\text{avg}}^{\text{IP}}$, as follows:

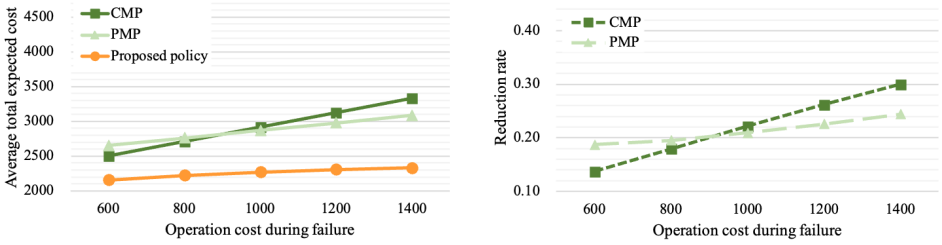
$$R^{\text{BP}} = \frac{V_{\text{avg}}^{\text{BP}} - V_{\text{avg}}^{\text{IP}}}{V_{\text{avg}}^{\text{BP}}}. \tag{18}$$

4.2. Results and Discussions

4.2.1. Operation Cost During Failure (c^f)

Figure 5 illustrates the impact of varying operation costs during failure (c^f) on the average total expected cost for the proposed policy and benchmark policies. The average total expected costs for all three policies increase as c^f rises. However, the benchmark policies, which base decisions solely on observational information, exhibit a significant increase in average costs as c^f rises, particularly in the case of the CMP. In contrast, the proposed policy demonstrates the least increase in costs compared to the benchmark policies, as it prioritizes actions to prevent the system from remaining in a failed state. Moreover, the cost reduction rates of the CMP and PMP improve as c^f increases. This result underscores the superior performance of the proposed policy, achieving consistently lower

expected costs compared to the benchmark policies under high c^f. These findings highlight the effectiveness of the proposed policy in scenarios where failure costs are substantial.

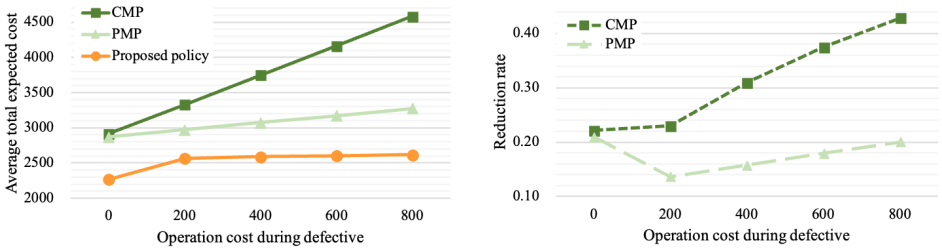


(**a**) Average total expected cost (**b**) Reduction rate

Figure 5. Impact of operation cost during failure on average expected cost and reduction rate.

4.2.2. Operation Cost During Defective (c^d)

Figure 6 illustrates the impact of varying operation costs during a defective state (c^d). Similar to the case of c^f, the average total expected costs for all three policies increase as c^d rises. However, the proposed policy exhibits only a minimal increase in cost. In particular, the CMP is significantly impacted by c^d because replacements are performed only after failure information is obtained. This result indicates that the proposed policy effectively mitigates the impact of higher penalties by performing early maintenance, thereby reducing the adverse effects associated with decreased system performance.

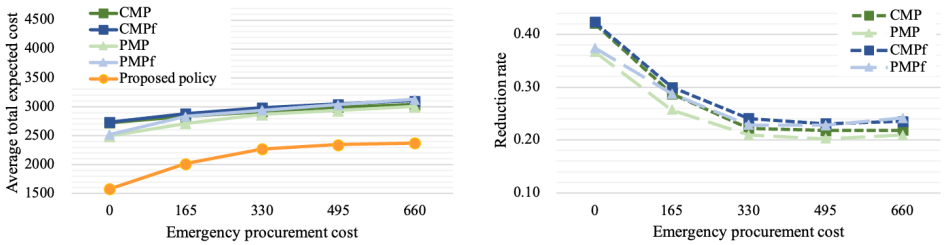


(**a**) Average total expected cost (**b**) Reduction rate

Figure 6. Impact of operation cost during defective on average expected cost and reduction rate.

4.2.3. Emergency Procurement Cost (c_i^{eo})

Figure 7 illustrates the average total expected costs and reduction rates for the proposed policy and benchmark policies under varying emergency procurement costs (c_i^{eo}). As c_i^{eo} increases, the increase rate in cost becomes smaller for all policies, including the proposed policy. This is because higher c_i^{eo} prompts a consistent inventory maintenance policy to avoid emergency procurement. Despite this trend, the proposed policy consistently achieves lower expected costs compared to the benchmark policies. Among the benchmark policies, PMP_f and CMP_f incur higher expected costs than PMP and CMP, respectively, due to their failure to account for inventory levels during maintenance actions. Specifically, PMP_f and CMP_f always incur emergency procurement costs during maintenance when inventory is unavailable. These results highlight the critical importance of incorporating inventory levels into decision-making to effectively reduce average total expected costs.

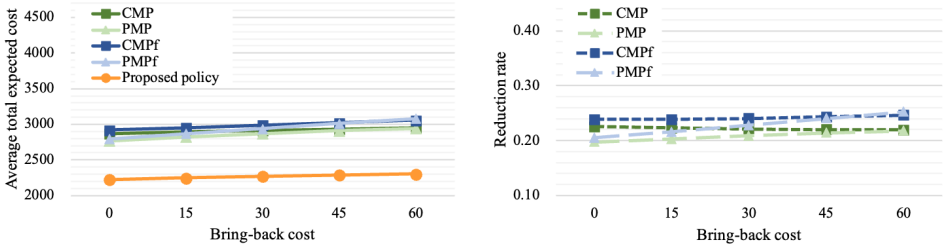


(**a**) Average total expected cost (**b**) Reduction rate

Figure 7. Impact of emergency procurement cost on average expected cost and reduction rate.

4.2.4. Bring-Back Cost (c^b)

Figure 8 shows the average total expected costs and reduction rates for the proposed policy and the benchmark policies under varying bring-back costs (c^b). Both the proposed policy and the benchmark policies CMP and PMP exhibit an increasing trend in expected costs as c^b rises. However, the benchmark policies PMP_f and CMP_f show a higher rate of cost increase with rising c^b. The proposed policy, along with the PMP and CMP, optimizes the number of spare units to be brought in. When c^b is high, these policies strategically avoid bringing in spare units, minimizing the impact of c^b. In contrast, PMP_f and CMP_f always bring back all spare units, making them significantly more sensitive to c^b. This underscores the importance of dynamically selecting the number of spare units based on system states and inventory levels to effectively minimize expected costs.



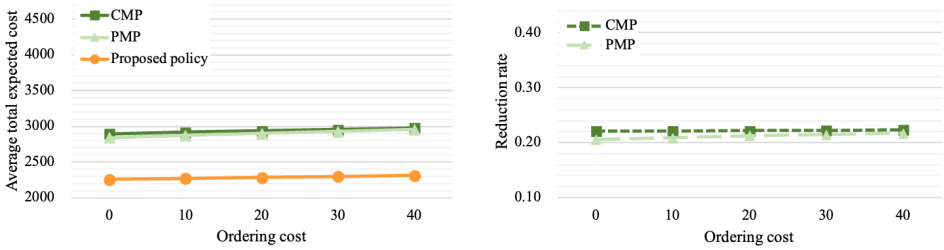
(**a**) Average total expected cost (**b**) Reduction rate

Figure 8. Impact of bring-back cost on average expected cost and reduction rate.

4.2.5. Ordering Cost (c^l)

Figure 9 illustrates the average total expected costs and reduction rates for the proposed policy and benchmark policies under varying ordering costs (c^l). For both the proposed policy and the benchmark policies, the rate of increase in expected costs is lower as c^l rises compared to the sensitivity analysis results for other costs. This outcome may be attributed to the relatively small value of c^l used in this analysis. However, considering the results of the sensitivity analysis on holding costs (c_i^h) as shown in Figure 10, it is also possible that c^l inherently has a limited impact on expected costs.

Next, focusing on the reduction rates for the benchmark policies, the reduction rate for CMP remains constant regardless of c^l. In contrast, the reduction rate for PMP increases slightly as c^l rises. This difference arises because the benchmark policies use sequential ordering. Consequently, the impact of c^l on expected costs is greater for PMP, which has a higher maintenance frequency.

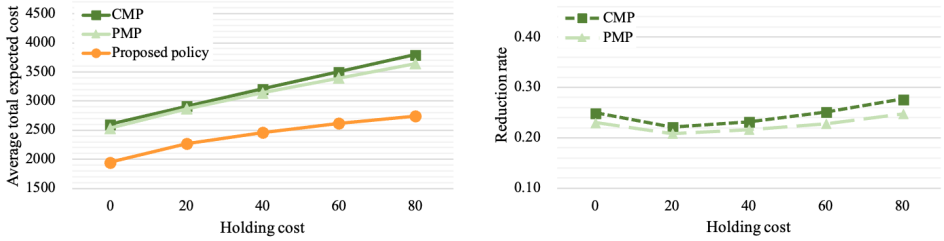


(**a**) Average total expected cost (**b**) Reduction rate

Figure 9. Impact of ordering cost on average expected cost and reduction rate.

4.2.6. Holding Cost (c_i^h)

Figure 10 illustrates the average total expected costs and reduction rates for the proposed policy and benchmark policies under varying holding costs (c_i^h). When c_i^h is high, the proposed policy achieves lower expected costs compared to the benchmark policies. This is because the proposed policy dynamically adjusts its actions to minimize inventory levels when c_i^h is high, while maintaining consistent inventory levels when c_i^h is low. In contrast, the benchmark policies, which follow the (s, S) policy, maintain inventory levels consistently, regardless of c_i^h. As a result, as c_i^h increases, the reduction rate of the proposed policy becomes more significant.



(**a**) Average total expected cost (**b**) Reduction rate

Figure 10. Impact of holding cost on average expected cost and reduction rate.

4.2.7. Conclusions of Sensitivity Analysis

Within the range of sensitivity analyses conducted, the proposed policy consistently achieved lower average expected costs compared to the benchmark policies. In particular, the reduction rate increases significantly when the operating costs during failure (c^f) and defective (c^d), as well as the holding costs, rise. These results indicate that the effectiveness of the proposed policy becomes more pronounced when the costs incurred in each epoch are higher. While PMP achieves lower expected costs than CMP under high (c^f) and (c^d), the average total expected costs increase as maintenance-related costs rise. Additionally, the costs associated with spare unit management, including ordering costs and emergency procurement costs, have a notable impact. These findings highlight that decision-making that accounts for both system conditions and spare unit inventory is more effective in reducing the average total expected cost compared to traditional preventive maintenance policies.

Furthermore, the results of the sensitivity analyses on emergency procurement costs (c_i^{eo}) and return costs (c^b) indicate that optimizing the selection of spare units to be deployed is also effective within the proposed model, which explicitly considers inventory management. This conclusion was drawn from the perspective of minimizing the average total expected cost.

5. Conclusions and Future Research

This research proposes an optimal joint policy for maintenance and spare unit management in a series system consisting of non-identical units under partial observation. The policy integrates maintenance decisions, spare unit ordering, and the selection of spare units to be brought to the maintenance site, based on partial information about the system's deterioration state. The optimization problem is formulated as a partially observable Markov decision process (POMDP) to minimize the expected cost. The behavior of the proposed joint policy is illustrated through a numerical example involving a series system with two units, demonstrating how the policy adapts to different beliefs and inventory levels. Furthermore, a detailed sensitivity analysis evaluates the performance of the proposed policy against several benchmark policies. The results confirm that the proposed policy consistently achieves lower average total expected cost compared to the benchmark policies, highlighting its potential effectiveness and adaptability under different operating conditions.

Future research can address several limitations and explore extensions to enhance the model's applicability and accuracy. First, while this study incorporates economic dependencies in maintenance and ordering decisions, real systems may exhibit correlations in deterioration among units or structural constraints in maintenance actions, such as cases where replacing one unit requires replacing another. Incorporating these stochastic and structural dependencies would provide a more comprehensive framework. Additionally, this research assumes immediate maintenance and constant lead times; however, in practical situations, maintenance and transportation times for spare units may vary and include randomness. Considering variable maintenance durations and random lead times would increase the realism of the model. Finally, the replacement thresholds in this study were pre-specified, but optimizing these thresholds as part of the decision-making process represents a promising avenue for future exploration.

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Abbreviations

The following abbreviations are used in this manuscript:

CM	Corrective maintenance
PM	Preventive maintenance
TBM	Time-based maintenance
CBM	Condition-based maintenance
MDP	Markov decision process
POMDP	Partially observable Markov decision process
PMP	Preventive maintenance policy
CMP	Corrective maintenance policy

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