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The Equivalent Standard Forms of a Class of Tropical Matrices and Centralizer Groups

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Abstract: In this paper, the equivalent standard forms of tropical idempotent strongly definite matrices are introduced. In particular, the observation of the equivalent standard forms of tropical idempotent normal matrices is given. An equivalence relation ρ on the set of all tropical idempotent normal matrices, which is relevant to their centralizer groups, is introduced and studied. It is proved that every ρ-class contains at least one strongly regular tropical idempotent normal matrix. Furthermore, a structural description of the centralizer groups of partial strongly regular tropical idempotent normal matrices is given.

Keywords: tropical semiring; tropical matrix; idempotent strongly definite matrix; idempotent normal matrix; centralizer group

MSC: 15A80; 20B99; 20H99

1. Introduction and Preliminaries

The *tropical semiring* $\overline{\mathbb{R}}$ is the set $\mathbb{R} \cup \{-\infty\}$ equipped with the operations of tropical addition $a \oplus b := max\{a, b\}$ and tropical multiplication $a \otimes b := a + b$, where 0 and $-\infty$ are the multiplicative neutral element and the additive neutral element, respectively. The completed tropical semiring $\overline{\overline{\mathbb{R}}}$ is the tropical semiring augmented with an extra element $+\infty$ (see [1]). Note that, by definition,

$$(-\infty) \otimes (+\infty) = (+\infty) \otimes (-\infty) = -\infty.$$

Let $M_n(\overline{\mathbb{R}})$ denote the set of $n \times n$ matrices with entries in $\overline{\mathbb{R}}$. As in conventional linear algebra, we can extend the operations $\oplus$ and $\otimes$ on the tropical semiring $\overline{\mathbb{R}}$ to $M_n(\overline{\mathbb{R}})$. Indeed, if $A = (a_{ij}), B = (b_{ij}) \in M_n(\overline{\mathbb{R}})$, then we have

$$(A \oplus B)_{ij} = a_{ij} \oplus b_{ij},\ (A \otimes B)_{ij} = \bigoplus_{k=1}^{n} a_{ik} \otimes b_{kj},$$

for all $i, j \in [n](= \{1, 2, \ldots, n\})$, where $(A \oplus B)_{ij}$ and $(A \otimes B)_{ij}$ denote the (i, j)-th entries of the matrices $A \oplus B$ and $A \otimes B$, respectively. For brevity, we usually write AB in place of $A \otimes B$ for a product of matrices. It is easy to check that $(M_n(\overline{\mathbb{R}}), \oplus, \otimes)$ is an idempotent semiring. The additive neutral element of $M_n(\overline{\mathbb{R}})$ is the tropical $n \times n$ matrix whose entries are all $-\infty$, denoted by Z_n, and the multiplicative neutral element of $M_n(\overline{\mathbb{R}})$ is the tropical identity matrix whose diagonal entries are 0 and off-diagonal ones are $-\infty$, denoted by I_n. We are interested in studying the multiplicative structure of the tropical matrices. There is a series of papers in the literature studying the multiplicative structure of this semiring (see [2–5]).

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Recall that a tropical matrix $E \in M_n(\overline{\mathbb{R}})$ is said to be *idempotent* if $E^2 = E$ and a tropical matrix $A = (a_{ij}) \in M_n(\overline{\mathbb{R}})$ is said to be *normal* if $-\infty \leq a_{ij} \leq 0$ and $a_{ii} = 0$ for all $i, j \in [n]$ (see [1,6]). Since we never refer to classical matrices in this paper, the following matrices refer to tropical matrices. A matrix $A \in M_n(\overline{\mathbb{R}})$ is said to be *strongly regular* if the system $Ax = b$ has a unique solution for some $b \in \mathbb{R}^n$. It is well known that an $n \times n$ matrix $A = (a_{ij})$ is strongly regular if and only if it has a strong permanent (see [1] [Proposition 6.2.2])—that is, there exists a unique $\sigma \in S_n$ such that

$$maper(A) = a_{1,1^\sigma} \otimes a_{2,2^\sigma} \otimes \cdots \otimes a_{n,n^\sigma},$$

where $maper(A) = \bigoplus_{\sigma \in S_n} a_{1,1^\sigma} \otimes a_{2,2^\sigma} \otimes \cdots \otimes a_{n,n^\sigma}$ is called the *permanent* of A. Throughout this paper, M_n^I and M_n^{IN} stand for the set of all idempotent $n \times n$ matrices and all idempotent normal $n \times n$ matrices, respectively. M_n^{SR} stands for the set of all strongly regular idempotent normal matrices and M_n^{NSR} stands for the set of all idempotent normal matrices that are not strongly regular (that is, $M_n^{NSR} = M_n^{IN} \setminus M_n^{SR}$). For more details about idempotent normal matrices, the reader is referred to [2,7–11].

An $n \times n$ matrix is called *diagonal*—notation $diag(\lambda_1, \lambda_2, \cdots, \lambda_n)$—if its diagonal entries are $\lambda_1, \lambda_2, \cdots, \lambda_n \in \mathbb{R}$ and the off-diagonal entries are $-\infty$. A matrix is said to be a *permutation matrix* (*generalized permutation matrix*, respectively) if it is formed from the identity matrix (the diagonal matrix, respectively) by reordering its columns and/or rows. Let $GL_n(\overline{\mathbb{R}})$ and $P_n(\overline{\mathbb{R}})$ denote the set of all generalized permutation matrices and the set of all permutation matrices in $M_n(\overline{\mathbb{R}})$, respectively. It is easy to see that $GL_n(\overline{\mathbb{R}})$ and $P_n(\overline{\mathbb{R}})$ are subgroups of the semigroup $(M_n(\overline{\mathbb{R}}), \otimes)$. In fact, the position of generalized permutation matrices in max-algebra is slightly more special than in conventional linear algebra as they are the only matrices having an inverse (see ([1], Theorem 1.1.3)). Let $A = (a_{ij}) \in M_n(\overline{\mathbb{R}})$. Define

$$U_n(A) := \{P \in GL_n(\overline{\mathbb{R}}) \mid PA = AP\},\ P_n(A) := \{P_\sigma \in P_n(\overline{\mathbb{R}}) \mid P_\sigma A = AP_\sigma\},$$

where P_σ denotes an $n \times n$ permutation matrix whose i-th row is equal to the i^σ-th row of I_n for any $i \in [n]$. It is easy to check that $P_n(A) \leq U_n(A)$, $P_n(A) \cong G_A$, where

$$G_A := \{\sigma \in S_n \mid (\forall i, j \in [n])\ a_{i^\sigma, j^\sigma} = a_{ij}\}.$$

$U_n(A)$ will be called the *generalized centralizer group* of A and $P_n(A)$(or G_A) will be called the *centralizer group* of A (see [11]).

There are a series of papers in the literature that study tropical matrix groups. In 2011, Johnson and Kambites [5] studied the algebraic structure of the multiplicative semigroup of all tropical 2×2 matrices. They described completely the structures of maximal subgroups of this semigroup. In 2012, Shitov [3] gave a complete description of the subgroups of the multiplicative semigroup of tropical $n \times n$ matrices up to isomorphism. They showed that every group of tropical matrices is isomorphic to a subgroup of $GL_n(\overline{\mathbb{R}})$ and therefore embeds into the permutation wreath product $\mathbb{R} \wr S_n$. In 2018, Izhakian et al. [12] studied the structure of the maximal subgroups of finitary tropical $n \times n$ matrices. They showed that the maximal subgroup containing a tropical idempotent matrix E is isomorphic to $U_n(E)$. Moreover, the maximal subgroup is, up to isomorphism, exactly the direct product of G_E and $\mathbb{R}$. In 2018, Yang [4,13] studied the generalized centralizer groups of nonsingular tropical idempotent matrices. In particular, a decomposition of the generalized centralizer groups of nonsingular symmetric tropical idempotent matrices was given. In 2022, Deng et al. [11] studied the generalized centralizer groups and centralizer groups of tropical $n \times n$ matrices. They proved that the centralizer group of a tropical matrix

is isomorphic to the centralizer group of an idempotent normal matrix E. Moreover, the structure of the centralizer group of E is given when E is not strongly regular. In this paper, by means of the introduction of the equivalent standard form of idempotent strongly definite matrices, we obtain that the centralizer group of every not strongly regular idempotent normal matrix is equal to the centralizer group of some strongly regular idempotent normal matrix. Further, a structural description of the centralizer groups of partial strongly regular idempotent normal matrices is given. Our results generalize and enrich corresponding results about idempotent normal matrices and their centralizer groups (see [11]).

In the remainder of this section, we recall some notions and results related to the weighted digraph $D(A)$ of a matrix A (see [1]), which will be required later. Let $A = (a_{ij}) \in M_n(\overline{\mathbb{R}})$. The *weighted digraph associated with* A is $D(A) = (N(A), E(A), w)$, where the node set $N(A) = [n]$ and the edge set $E(A) = \{(i,j) \mid a_{ij} > -\infty\}$ with weights $w(i,j) = a_{ij}$ for all $(i,j) \in E(A)$. Suppose that $\pi = (i_1, i_2, \cdots, i_p)$ is a path in $D(A)$; then, the *weight* of π is defined to be $w(\pi, A) = a_{i_1,i_2} + a_{i_2,i_3} + \cdots + a_{i_{p-1},i_p}$ if $p > 1$ and $w(\pi, A) = -\infty$ if $p = 1$. The number $p-1$ is called the *length* of π, denoted by $\ell(\pi)$. Recall that a path $(i_1, i_2, \cdots, i_p)$ is called a *cycle* if $p > 1$ and $i_1 = i_p$, and it is called an *elementary cycle* if, moreover, $i_k \neq i_\ell$ for any $k, \ell \in [p-1]$ and $k \neq \ell$. The *maximum cycle mean* of A, denoted by $\lambda(A)$, is defined by

$$\lambda(A) = max_\sigma \mu(\sigma, A),$$

where the maximization is taken over all elementary cycles in $D(A)$ and

$$\mu(\sigma, A) = \frac{w(\sigma, A)}{\ell(\sigma)}$$

denotes the mean of a cycle σ.

A cycle in $D(A)$ is called *critical* if its cycle mean is equal to $\lambda(A)$. The nodes and the edges of $D(A)$ that belong to some critical cycles are called *critical*. The *critical digraph of* A is the digraph $C(A) = (N_c(A), E_c(A))$, where $N_c(A)$ and $E_c(A)$ denote the set of critical nodes and critical edges of $D(A)$, respectively. If $i, j \in N_c(A)$ belong to the same critical cycle, then i and j are called *equivalent* in $D(A)$ and we write $i \sim_A j$. Clearly, $\sim_A$ is an equivalence relation on $N_c(A)$.

A matrix A is called *definite* if $\lambda(A) = 0$. Thus, a matrix is definite if and only if all cycles in $D(A)$ are non-positive (i.e., its cycle mean is non-positive) and at least one has weight zero. A matrix $A = (a_{ij}) \in M_n(\overline{\mathbb{R}})$ is called *increasing* if $a_{ii} \geq 0$ for all $i \in [n]$, and is called *strongly definite* if it is definite and increasing. Since the diagonal entries of A are the weights of cycles (loops), we have that $a_{ii} = 0$ for all $i \in [n]$ if A is strongly definite. In the following, M_n^{ISD} stands for the set of all idempotent strongly definite matrices. For more details about idempotent strongly definite matrices, the reader is referred to ([1], Section 6.2).

In addition to this introduction and preliminaries, this paper comprises two sections. In Section 2, we give some characterizations of $\sim_E$ and introduce the equivalent standard forms of E, where $E \in M_n^{ISD}$. In particular, we give the observation of the equivalent standard forms of $E \in M_n^{IN}$. In Section 3, by using the centralizer groups of idempotent normal matrices we introduce an equivalence relation ρ on M_n^{IN}. We prove that every ρ-class contains at least one strongly regular idempotent normal matrix. Let $E \in M_n^{SR}$. A new idempotent normal matrix $E^{(e_t)}$ is constructed from E, where e_t is an off-diagonal entry of E. We give the equivalent conditions for which $E^{(e_t)}$ is not strongly regular. Further, a structural description of G_E is given when $E^{(e_t)}$ is not strongly regular.

For other notations and terminologies not given in this article, the reader is referred to the books [1,14,15].

2. The Equivalent Standard Forms of Idempotent Strongly Definite Matrices

Let $\overline{\mathbb{R}}^n$ denote the set of all n-tuples x with entries in $\overline{\mathbb{R}}$. We write x_i for the i-th component of x. For any $x, y \in \overline{\mathbb{R}}^n$, we define

$$\langle x \mid y \rangle := max\{\lambda \in \overline{\overline{\mathbb{R}}} \mid \lambda \otimes x \leq y\} = min_{i\in[n]}\{y_i - x_i\},$$

where $\lambda \otimes x = (\lambda + x_1, \lambda + x_2, \ldots, \lambda + x_n)$. We set $(-\infty) - a = -\infty, a - (-\infty) = +\infty$ and $(-\infty) - (-\infty) = +\infty$ for any $a \in \mathbb{R}$. The map $(x, y) \mapsto \langle x \mid y \rangle$ is a residuation operator in the sense of residuation theory [16], and is ubiquitous in tropical algebra. Notice that $\langle x \mid y \rangle = +\infty$ if and only if $x = -\infty$. $\langle x \mid y \rangle = -\infty$ if and only if there exists $j \in [n]$ such that $y_j = -\infty$ and $x_j \neq -\infty$.

As a consequence of ([7], Lemma 5.3), we have the following:

Lemma 1. *Suppose that $E = (e_{ij}) \in M_n^I$. Let $r_1, r_2, \ldots, r_n$ denote the rows of E and $c_1, c_2, \ldots, c_n$ denote the columns of E. Then,*

(i) $e_{ij} \leq min\{\langle r_j \mid r_i \rangle, \langle c_i \mid c_j \rangle\}$ *for any* $i, j \in [n]$;
(ii) *If* $e_{ii} = 0$, *then* $e_{ij} = \langle c_i \mid c_j \rangle$ *and* $e_{ji} = \langle r_i \mid r_j \rangle$ *for all* $j \in [n]$.

Proof. Part (i) is a direct consequence of ([7], Lemma 5.3).

To prove part (ii), let $e_{ii} = 0$. Suppose that $e_{ij} < \langle c_i \mid c_j \rangle$ for some $j \in [n]$. Then, $e_{ij} < \langle c_i \mid c_j \rangle \otimes e_{ii}$. Since $\langle x \mid y \rangle \otimes x \leq y$ for any $x, y \in \overline{\mathbb{R}}^n$, it is implied that $\langle c_i \mid c_j \rangle \otimes c_i \leq c_j$. Hence, $\langle c_i \mid c_j \rangle \otimes e_{ki} \leq e_{kj}$ for all $k \in [n]$. This contradicts $e_{ij} < \langle c_i \mid c_j \rangle \otimes e_{ii}$. Thus, $e_{ij} = \langle c_i \mid c_j \rangle$ for all $j \in [n]$. Similarly, $e_{ji} = \langle r_i \mid r_j \rangle$ for all $j \in [n]$. □

Furthermore, we have the following lemma.

Lemma 2. *Let $E = (e_{ij}) \in M_n^{ISD}$ and $i, j \in [n]$. Then, the following are equivalent:*

(i) $e_{ij} = e_{ji} = 0$;
(ii) $e_{ik} = e_{jk}$ *for all* $k \in [n]$;
(iii) $e_{ki} = e_{kj}$ *for all* $k \in [n]$.

Proof. We need only to prove the equivalence of (i) and (ii), since the equivalence of (i) and (iii) may be showed dually.

$(i) \Rightarrow (ii)$. Suppose that $e_{ij} = e_{ji} = 0$. Then, by Lemma 1 and $E \in M_n^{ISD}$, $e_{ij} = \langle r_j \mid r_i \rangle$ and $e_{ji} = \langle r_i \mid r_j \rangle$, where r_i and r_j denote the i-th and j-th row of E, respectively. Moveover, $\langle r_j \mid r_i \rangle = \langle r_i \mid r_j \rangle = 0$, and so $r_i = r_j$. That is, $e_{ik} = e_{jk}$ for all $k \in [n]$.

$(ii) \Rightarrow (i)$. Suppose that $e_{ik} = e_{jk}$ for all $k \in [n]$. Setting $k = i$, we have $e_{ji} = e_{ii} = 0$. Setting $k = j$, we have $e_{ij} = e_{jj} = 0$. Thus, $e_{ij} = e_{ji} = 0$. □

Let $E = (e_{ij}) \in M_n^{ISD}$. It is easily seen that the maximum cycle mean $\lambda(E) = 0$ and that every node of $D(E)$ is critical, since all the diagonal entries of E are equal to 0. That is, $N_c(E) = [n]$. So, $\sim_E$ is an equivalence relation on $[n]$.

Note that $M_n^{IN} \subseteq M_n^{ISD}$. It is clear that Lemma 2 generalizes the part results of ([11], Lemma 3.2), which asserts that if $E = (e_{ij}) \in M_n^{IN}$ and $i, j \in [n]$, then $i \sim_E j$ if and only if $e_{ij} = e_{ji} = 0$. Now, let $E = (e_{ij}) \in M_n^{ISD}$. If $e_{ij} = e_{ji} = 0$, then $i \sim_E j$ by $\lambda(E) = 0$. However, the inverse of this result is not true. In fact, for idempotent strongly definite matrices, we have the following:

Proposition 1. *Let $E = (e_{ij}) \in M_n^{ISD}$ and $i, j \in [n]$. Then, the following are equivalent:*

(i) $i \sim_E j$;
(ii) $e_{ij} + e_{ji} = 0$;
(iii) $e_{ik} - e_{jk} = e_{ij}$ *for all* $k \in [n]$;
(iv) $e_{kj} - e_{ki} = e_{ij}$ *for all* $k \in [n]$.

Proof. We need only to prove the equivalence of (i), (ii), and (iii), since the equivalence of (i), (ii), and (iv) may be showed dually.

$(i) \Rightarrow (ii)$. Suppose that $i \sim_E j$. Then, there is a critical cycle π containing both i and j in $D(E)$. We shall write $\pi = (i_1 i_2 \dots i_s \dots i_p)$, where $i_1 = i$ and $i_s = j$. Moreover, $w(\pi, E) = e_{i_1,i_2} + \dots + e_{i_{s-1},i_s} + e_{i_s,i_{s+1}} + \dots + e_{i_p,i_1} = 0$. Since E is idempotent, it follows that $e_{i_1,i_2} + \dots + e_{i_{s-1},i_s} \leq e_{ij}$ and $e_{i_s,i_{s+1}} + \dots + e_{i_p,i_1} \leq e_{ji}$. Thus, $w(\pi, E) \leq e_{ij} + e_{ji} \leq 0$, and so $e_{ij} + e_{ji} = 0$.

$(ii) \Rightarrow (iii)$. Suppose that $e_{ij} + e_{ji} = 0$. Then, by Lemma 1, $\langle r_j \mid r_i \rangle + \langle r_i \mid r_j \rangle = 0$, and so $\langle r_j \mid r_i \rangle = -\langle r_i \mid r_j \rangle$—i.e.,

$$min_{k\in[n]}\{e_{ik} - e_{jk}\} = max_{k\in[n]}\{e_{ik} - e_{jk}\}.$$

This means that $e_{ik} - e_{jk} = e_{i\ell} - e_{j\ell}$ for any $k, \ell \in [n]$. Putting $\ell = j$, we deduce that $e_{ik} - e_{jk} = e_{ij}$ for all $k \in [n]$.

$(iii) \Rightarrow (i)$. Suppose that $e_{ik} - e_{jk} = e_{ij}$ for all $k \in [n]$. Putting $k = i$, we obtain that $e_{ij} + e_{ji} = 0$. Thus, $\tau = (ij)$ is a critical cycle with length two in $D(E)$. Consequently, $i \sim_E j$. □

It is clear that the above proposition is a generalization of ([11], Lemma 3.2). As usual, Δ (resp. ∇) stands for the equality relation (resp. universal relation).

Proposition 2. *Let $E = (e_{ij}) \in M_n^{ISD}$. Then, E is not strongly regular $\Longleftrightarrow \sim_E \neq \Delta$.*

Proof. Suppose that E is not strongly regular. Then, there exists $\tau \in S_n \setminus \{1\}$ such that $e_{1,\tau(1)} \otimes e_{2,\tau(2)} \otimes \dots \otimes e_{n,\tau(n)} = 0$. Write τ as a product of non-trivial disjoint cycles, say, $\tau = \tau_1 \tau_2 \dots \tau_\ell$ $(\ell < n)$. Then, any such cycle $\tau_i (i \in [\ell])$, say, $\tau_i = (j_1 j_2 \cdots j_k)$, satisfies $e_{j_1,j_2} \otimes e_{j_2,j_3} \otimes \dots \otimes e_{j_k,j_1} = 0$. Since E is idempotent, it follows that $e_{j_1,j_k} \geq e_{j_1,j_2} \otimes \dots \otimes e_{j_{k-1},j_k} = -e_{j_k,j_1}$. Moreover, $e_{j_1,j_k} + e_{j_k,j_1} \geq 0$. By $e_{j_1,j_k} \otimes e_{j_k,j_1} \leq e_{j_1,j_1} = 0$, which implies that $e_{j_1,j_k} + e_{j_k,j_1} = 0$. It follows from Proposition 1 that $j_1 \sim_E j_k$, and so $\sim_E \neq \Delta$.

Assume that $\sim_E \neq \Delta$. Then, by Proposition 1, $e_{ij} + e_{ji} = 0$ for some $i, j \in [n]$ and $i \neq j$. It follows from ([7], Lemma 3.3) that $maper(E) = 0$. Hence, the transposition $(ij) \in S_n$ attains the permanent of E, as required. □

Let $G \leq S_n$. Recall that $\emptyset \neq \Gamma \subseteq [n]$ is called *a block* for G if for each $\sigma \in G$ either $\Gamma^\sigma = \Gamma$ or $\Gamma^\sigma \cap \Gamma = \emptyset$, where $\Gamma^\sigma = \{i^\sigma \mid i \in \Gamma\}$. Clearly, for each $\sigma \in G$, Γ^σ is also a block for G (see [14]). *A G-congruence* on $[n]$ is an equivalence relation $\sim$ on $[n]$ with the property that

$$i \sim j \Longleftrightarrow i^\sigma \sim j^\sigma \text{ for all } \sigma \in G.$$

Let G be transitive on $[n]$ and Γ a block for G. Then, $\{\Gamma^\sigma \mid \sigma \in G\}$ is a partition of $[n]$ and $|\Gamma^\sigma| = |\Gamma|$ for each $\sigma \in G$. $\{\Gamma^\sigma \mid \sigma \in G\}$ is called *a system of blocks* for G containing Γ. If $\sim$ is a G-congruence on $[n]$, then the equivalence classes of $\sim$ form a system of blocks for G (see ([14], Exercise 1.5.4)).

Proposition 3. *Let $E = (e_{ij}) \in M_n^{ISD}$ and $\{\Gamma_1, \Gamma_2, \ldots, \Gamma_m\}$ be the set of all classes of $\sim_E$. Then, $\sim_E$ is a G_E-congruence on $[n]$. Also, $\{\Gamma_1, \Gamma_2, \ldots, \Gamma_m\}$ is a system of blocks for G_E if G_E is transitive on $[n]$.*

Proof. Suppose that $\sigma \in G_E$. Then, for any $i, j \in [n]$,

$$\begin{array}{rcll} i \sim_E j & \iff & e_{ik} - e_{jk} = e_{ij} & \text{(by Proposition 1)} \\ & \iff & e_{i^\sigma,k^\sigma} - e_{j^\sigma,k^\sigma} = e_{i^\sigma,j^\sigma} & \text{(since } \sigma \in G_E\text{)} \\ & \iff & i^\sigma \sim_E j^\sigma. & \end{array}$$

This shows that $\sim_E$ is a G_E-congruence on $[n]$, and so $\{\Gamma_1, \Gamma_2, \ldots, \Gamma_m\}$ is a system of blocks for G_E if G_E acts transitively on $[n]$. □

Now, suppose that $E \in M_n^{ISD}$ and $\{\Gamma_1, \Gamma_2, \ldots, \Gamma_m\}$ is the set of all classes of $\sim_E$, where $1 \leq m \leq n$. If E is not strongly regular, then $m \neq n$ by Proposition 2. Without loss of generality, let $1 \leq |\Gamma_1| \leq |\Gamma_2| \leq \cdots \leq |\Gamma_m| \leq n$ and $|\Gamma_k| = t_k$ for any $k \in [m]$. That is, $1 \leq t_1 \leq t_2 \leq \cdots \leq t_m \leq n$ and $t_1 + t_2 + \cdots + t_m = n$. It follows from Proposition 1 that for any $i, j \in \Gamma_k$ and $k \in [m]$, $c_j = e_{ij} \otimes c_i$ and $r_i = e_{ij} \otimes r_j$.

Next, for each $k \in [m]$, we shall write

$$\Omega_k = \{\sum_{i=1}^{k-1} t_i + 1, \sum_{i=1}^{k-1} t_i + 2, \ldots, \sum_{i=1}^{k} t_i\}.$$

Take $\tau \in S_n$ such that $\Omega_k^\tau = \Gamma_k$ for each $k \in [m]$. Further, by permuting simultaneously the rows and columns of E with τ, we obtain the following block matrix, $\overline{E} = P_\tau E P_\tau^{-1}$:

$$\overline{E} = (\overline{e}_{ij})_{n \times n} = \begin{pmatrix} \overline{E}_1 & \overline{E}_{12} & \cdots & \overline{E}_{1m} \\ \overline{E}_{21} & \overline{E}_2 & \cdots & \overline{E}_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{E}_{m1} & \overline{E}_{m2} & \cdots & \overline{E}_m \end{pmatrix}, \tag{1}$$

where $1 \leq m \leq n$, for each $k \in [m]$, $\overline{E}_k$ is an antisymmetric $t_k \times t_k$ matrix, and for any $k, \ell \in [m]$ and $k \neq \ell$, $\overline{E}_{k\ell}$ is an $t_k \times t_\ell$ matrix with $c_j(\overline{E}_{k\ell}) = \overline{e}_{ij} \otimes c_i(\overline{E}_{k\ell})$ for any $i, j \in \Omega_\ell$ and $r_i(\overline{E}_{k\ell}) = \overline{e}_{ij} \otimes r_j(\overline{E}_{k\ell})$ for any $i, j \in \Omega_k$. In fact, since $\overline{E} = P_\tau E P_\tau^{-1}$, we have that $\overline{E} \in M_n^{ISD}$ and $\overline{e}_{ij} = e_{i^\tau,j^\tau}$ for any $i, j \in [n]$. Then, for each $k \in [m]$,

$$i, j \in \Omega_k \Leftrightarrow i^\tau, j^\tau \in \Gamma_k \Leftrightarrow i^\tau \sim_E j^\tau \Leftrightarrow e_{i^\tau,j^\tau} + e_{j^\tau,i^\tau} = 0 \Leftrightarrow \overline{e}_{ij} + \overline{e}_{ji} = 0 \Leftrightarrow i \sim_{\overline{E}} j.$$

This implies that $\{\Omega_1, \Omega_2, \ldots, \Omega_m\}$ is the set of all classes of $\sim_{\overline{E}}$ and $\overline{e}_{ij} = -\overline{e}_{ji}$ for any $i, j \in \Omega_k$. That is to say, $\overline{E}_k$ is an antisymmetric $t_k \times t_k$ matrix for each $k \in [m]$. Also, $c_j(\overline{E}) = \overline{e}_{ij} \otimes c_i(\overline{E})$ and $r_i(\overline{E}) = \overline{e}_{ij} \otimes r_j(\overline{E})$ for any $i, j \in \Omega_k$ and $k \in [m]$. Moreover, for any $k, \ell \in [m]$ with $k \neq \ell$, $\overline{E}_{k\ell}$ is an $t_k \times t_\ell$ matrix with $c_j(\overline{E}_{k\ell}) = \overline{e}_{ij} \otimes c_i(\overline{E}_{k\ell})$ for any $i, j \in \Omega_\ell$ and $r_i(\overline{E}_{k\ell}) = \overline{e}_{ij} \otimes r_j(\overline{E}_{k\ell})$ for any $i, j \in \Omega_k$.

Now, we shall call the block matrix $\overline{E}$ *the equivalent standard form* of $E \in M_n^{ISD}$. It is easy to see that every idempotent strongly definite $n \times n$ matrix E can be transformed in linear time by simultaneous permutations of the rows and columns to an equivalent standard form $\overline{E}$ as above.

In particular, let $E = (e_{ij}) \in M_n^{IN}$. Then, E is equivalent to the following equivalent standard form $\overline{E}$:

$$\overline{E} = (\overline{e}_{ij})_{n\times n} = \begin{pmatrix} 0 & \overline{E}_{12} & \cdots & \overline{E}_{1m} \\ \overline{E}_{21} & 0 & \cdots & \overline{E}_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{E}_{m1} & \overline{E}_{m2} & \cdots & 0 \end{pmatrix}, \tag{2}$$

where $1 \leq m \leq n$ and $\overline{E}_{k\ell}$ is an $t_k \times t_\ell$ matrix with all entries $\widehat{e}_{k\ell} \leq 0$, for any $k, \ell \in [m]$ with $k \neq \ell$. In fact, since $\overline{e}_{ij} = -\overline{e}_{ji}$ for any $i, j \in \Omega_k$ and $k \in [m]$, it follows that $\overline{e}_{ij} = 0$ for any $i, j \in \Omega_k$ and $k \in [m]$ by $E \in M_n^{IN}$. Thus, all diagonal blocks of $\overline{E}$ are 0. Moreover, for any $k, \ell \in [m]$ with $k \neq \ell$, $c_j(\overline{E}_{k\ell}) = c_i(\overline{E}_{k\ell})$ for any $i, j \in \Omega_\ell$ and $r_j(\overline{E}_{k\ell}) = r_i(\overline{E}_{k\ell})$ for any $i, j \in \Omega_k$. Thus, all entries of $\overline{E}_{k\ell}$ are equal.

Notice that if $\widehat{e}_{k\ell} = 0$ for some $k, \ell \in [m]$ with $k \neq \ell$, then $\widehat{e}_{\ell k} < 0$. In fact, suppose that $\widehat{e}_{\ell k} = 0$. Then, $\overline{E}_{k\ell} = 0$ and $\overline{E}_{\ell k} = 0$. That is, $\overline{e}_{ij} = \overline{e}_{ji} = 0$ for any $i \in \Omega_k$ and $j \in \Omega_\ell$. This follows from ([11], Lemma 3.2) that $i \sim_{\overline{E}} j$ for any $i \in \Omega_k$ and $j \in \Omega_\ell$. This contradicts the fact that Ω_k and Ω_ℓ are two different $\sim_{\overline{E}}$-classes.

Remark 1. *In general, the equivalent standard forms of E are not unique. For instance, let*

$$E = \begin{pmatrix} 0 & -2 & -1 & -1 & -2 \\ -1 & 0 & -2 & -2 & 0 \\ -2 & -1 & 0 & 0 & -1 \\ -2 & -1 & 0 & 0 & -1 \\ -1 & 0 & -2 & -2 & 0 \end{pmatrix}. \tag{3}$$

It is easy to see that $E \in M_n^{IN} \subseteq M_n^{ISD}$. *If we take* $\tau_1 = (35) \in S_5$, *then* $\overline{E} = P_{\tau_1} E P_{\tau_1}^{-1}$ *is an equivalent standard form of E as follows:*

$$\overline{E} = \begin{pmatrix} 0 & -2 & -2 & -1 & -1 \\ -1 & 0 & 0 & -2 & -2 \\ -1 & 0 & 0 & -2 & -2 \\ -2 & -1 & -1 & 0 & 0 \\ -2 & -1 & -1 & 0 & 0 \end{pmatrix}. \tag{4}$$

If we take $\tau_2 = (24) \in S_5$, *then* $\overline{\overline{E}} = P_{\tau_2} E P_{\tau_2}^{-1}$ *is also an equivalent standard form of E as follows:*

$$\overline{\overline{E}} = \begin{pmatrix} 0 & -1 & -1 & -2 & -2 \\ -2 & 0 & 0 & -1 & -1 \\ -2 & 0 & 0 & -1 & -1 \\ -1 & -2 & -2 & 0 & 0 \\ -1 & -2 & -2 & 0 & 0 \end{pmatrix}. \tag{5}$$

In fact, for a general idempotent strongly definite matrix E, the diagonal blocks of the equivalent standard form of E are determined uniquely up to a simultaneous permutation of the rows and columns. Any such form is essentially determined by the critical components of $C(E)$. The form forms an interesting correspondence with the Frobenius normal form of E, which is essentially determined by the strongly connected components of $D(E)$ (see [1]).

3. The Centralizer Groups of Idempotent Normal Matrices

In this section, we shall study the centralizer groups of tropical matrices. By ([11], Proposition 3.1), we know that the centralizer group of every tropical matrix equals the centralizer group of some idempotent normal matrix. Based on this fact, we need only to consider the centralizer groups of idempotent normal matrices.

Define a binary relation ρ on M_n^{IN} by

$$A \,\rho\, B \Longleftrightarrow G_A \cong G_B.$$

Clearly, ρ is an equivalence relation on M_n^{IN}. It is easy to check that $G_{I_n} = S_n$, and $G_A = S_n$ if and only if all off-diagonal entries of A are $a \leq 0$, where $A \in M_n^{IN}$. Furthermore, we have immediately the following result.

Lemma 3. *$I_n\rho = \{A = (a_{ij}) \in M_n^{IN} \mid a_{ij} = a$ for any $i, j \in [n]$ with $i \neq j, a \leq 0\}$, where $I_n\rho$ denotes the ρ-class containing I_n.*

Let $[-\infty, 0]$ denote the set $\{a \in \overline{\mathbb{R}} \mid -\infty \leq a \leq 0\}$. Then, there exists a natural bijection between $I_n\rho$ and $[-\infty, 0]$. That is, A corresponds to a, where a is the off-diagonal entries of A.

Suppose that $E \in M_n^{IN}$ and $\overline{E}$ is the equivalent standard form of E related to $\tau \in S_n$, i.e., $\overline{E} = P_\tau E P_{\tau^{-1}}$. Then, $G_{\overline{E}} = G_{P_\tau E P_{\tau^{-1}}} = \tau G_E \tau^{-1}$. Further, $G_{\overline{E}} \cong G_E$. Therefore, in order to study the centralizer group G_E, we need only to consider $G_{\overline{E}}$ up to isomorphism.

Now, let $\overline{E} = (\overline{e}_{ij}) \in M_n^{NSR}$ be in the form (2). By replacing all diagonal blocks $t_k \times t_k$ matrix 0 of $\overline{E}$ with $t_k \times t_k$ matrix A, where $A \in I_{t_k}\rho$ and all off-diagonal entries are $a < 0$, we can obtain the following block matrix $\widetilde{E}$.

$$\widetilde{E} = (\widetilde{e}_{ij})_{n\times n} = \begin{pmatrix} A & \overline{E}_{12} & \cdots & \overline{E}_{1m} \\ \overline{E}_{21} & A & \cdots & \overline{E}_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{E}_{m1} & \overline{E}_{m2} & \cdots & A \end{pmatrix}, \tag{6}$$

where $1 \leq m < n$, $\overline{E}_{k\ell}$ is an $t_k \times t_\ell$ matrix with all entries $\widehat{e}_{k\ell} \leq 0$, and $k, \ell \in [m]$ with $k \neq \ell$.

For each $k \in [m]$, we let

$$d_k = \bigoplus_{\ell \in [m]\setminus\{k\}} (\widehat{e}_{k\ell} \otimes \widehat{e}_{\ell k}).$$

Note that for any $k, \ell \in [m]$ and $k \neq \ell$, $\widehat{e}_{k\ell} = 0$ implies $\widehat{e}_{\ell k} < 0$. Thus, $d_k < 0$ for all $k \in [m]$. Suppose that $T = \{e_1, e_2, \ldots, e_h\}$ is the set of all entries in $\overline{E}$, where $1 \leq h \leq m^2 - m + 1$. We shall write

$$\Sigma := \{a \leq 0 \mid a \notin T \text{ and } a > d_k \text{ for all } k \in [m]\}.$$

Since T is finite and $d_k < 0$ for all $k \in [m]$, we have that $\Sigma \neq \emptyset$ by the denseness of the real numbers.

Lemma 4. *Let $\overline{E} \in M_n^{NSR}$ be in the form (2) and $\widetilde{E}$ be defined as above. If $a \in \Sigma$, then $\widetilde{E} \in M_n^{SR}$.*

Proof. It is clear that $\widetilde{E}$ is a normal matrix. Now, we shall show that $\widetilde{E}$ is idempotent. By $a \in \Sigma$, it is implied that $a > d_k$ for all $k \in [m]$. Hence, for any $k \in [m]$,

$$\overline{E}_{k1}\overline{E}_{1k} \oplus \overline{E}_{k2}\overline{E}_{2k} \oplus \cdots \oplus A^2 \oplus \cdots \oplus \overline{E}_{km}\overline{E}_{mk} = A^2 = A.$$

For any $k, \ell \in [m]$ and $k \neq \ell$, without loss the generality, let $k < \ell$. Since $\overline{E}$ is idempotent, it follows that

$$\overline{E}_{k1}\overline{E}_{1\ell} \oplus \overline{E}_{k2}\overline{E}_{2\ell} \oplus \cdots \oplus 0\overline{E}_{k\ell} \oplus \cdots \oplus \overline{E}_{k\ell}0 \oplus \cdots \oplus \overline{E}_{km}\overline{E}_{m\ell} = \overline{E}_{k\ell}.$$

It is easy to check that $A\overline{E}_{k\ell} = \overline{E}_{k\ell}A = 0\overline{E}_{k\ell} = \overline{E}_{k\ell}0 = \overline{E}_{k\ell}$. Moreover, we have

$$\overline{E}_{k1}\overline{E}_{1\ell} \oplus \overline{E}_{k2}\overline{E}_{2\ell} \oplus \cdots \oplus A\overline{E}_{k\ell} \oplus \cdots \oplus \overline{E}_{k\ell}A \oplus \cdots \oplus \overline{E}_{km}\overline{E}_{m\ell} = \overline{E}_{k\ell}.$$

This shows that $\widetilde{E}$ is idempotent. That is, $\widetilde{E} \in M_n^{IN}$. It follows from the construction of $\widetilde{E}$ that either $\widetilde{e}_{ij} = 0$ or $\widetilde{e}_{ji} = 0$ for any $i, j \in [n]$ with $i \neq j$. Thus, by ([11], Lemma 3.2), $\sim_{\widetilde{E}} = \Delta$, and so $\widetilde{E} \in M_n^{SR}$ by ([11], Lemma 3.3). □

Furthermore, we have the following theorem.

Theorem 1. *Let $\overline{E} = (\overline{e}_{ij}) \in M_n^{NSR}$ be in the form (2) and $\widetilde{E} = (\widetilde{e}_{ij}) \in M_n^{SR}$ as above. Then, $G_{\overline{E}} = G_{\widetilde{E}}$. Moreover, $\widetilde{E} \in \overline{E}\rho$.*

Proof. Suppose that $\sigma \in G_{\overline{E}}$ and $i, j \in [n]$. If $i, j \in \Omega_k$ for some $k \in [m]$, then by Proposition 3, $i^\sigma, j^\sigma \in \Omega_\ell$ for some $\ell \in [m]$. Thus, $\widetilde{e}_{ii} = \widetilde{e}_{i^\sigma,i^\sigma} = 0$ and $\widetilde{e}_{ij} = \widetilde{e}_{i^\sigma,j^\sigma} = a$ for any $i, j \in \Omega_k$ with $i \neq j$.

If $i \in \Omega_k$ and $j \in \Omega_\ell$ for some $k, \ell \in [m]$ with $k \neq \ell$, then $i^\sigma \in \Omega_u$ and $j^\sigma \in \Omega_v$ for some $u, v \in [m]$ and $u \neq v$. In fact, suppose that $u = v$. Then, $i^\sigma, j^\sigma \in \Omega_u$. That is, $i^\sigma \sim_{\overline{E}} j^\sigma$. Since $\sim_{\overline{E}}$ is a $G_{\overline{E}}$-congruence, it implies that $(i^\sigma)^{\sigma^{-1}} \sim_{\overline{E}} (j^\sigma)^{\sigma^{-1}}$ by $\sigma^{-1} \in G_{\overline{E}}$. That is, $i \sim_{\overline{E}} j$ and so $\Omega_k = \Omega_\ell$, yielding a contradiction. Clearly, $\widetilde{e}_{ij} = \overline{e}_{ij}$ and $\widetilde{e}_{i^\sigma,j^\sigma} = \overline{e}_{i^\sigma,j^\sigma}$. Notice that $\overline{e}_{ij} = \overline{e}_{i^\sigma,j^\sigma}$. This implies that $\widetilde{e}_{ij} = \widetilde{e}_{i^\sigma,j^\sigma}$. We conclude that $\widetilde{e}_{ij} = \widetilde{e}_{i^\sigma,j^\sigma}$ for any $i, j \in [n]$. Consequently, $\sigma \in G_{\widetilde{E}}$. Thus, $G_{\overline{E}} \leq G_{\widetilde{E}}$.

On the other hand, suppose that $\sigma \in G_{\widetilde{E}}$. Then, $\widetilde{e}_{ij} = \widetilde{e}_{i^\sigma,j^\sigma}$ for any $i, j \in [n]$. If $i, j \in \Omega_k$ for some $k \in [m]$, then $\widetilde{e}_{i^\sigma,i^\sigma} = \widetilde{e}_{ii} = 0$ and $\widetilde{e}_{i^\sigma,j^\sigma} = \widetilde{e}_{ij} = a$ for $i \neq j$. By $a \in \Sigma$, it is implied that $a \notin T$. Therefore, $i^\sigma, j^\sigma \in \Omega_\ell$ for some $\ell \in [m]$, and so $\overline{e}_{i^\sigma,j^\sigma} = \overline{e}_{ij} = 0$.

If $i \in \Omega_k$ and $j \in \Omega_\ell$ for some $k, \ell \in [m]$ with $k \neq \ell$, then $\widetilde{e}_{i^\sigma,j^\sigma} = \widetilde{e}_{ij} = \overline{e}_{ij}$. Since $a \notin T$, it implies that $\overline{e}_{ij} \neq a$ and so $\widetilde{e}_{i^\sigma,j^\sigma} \neq a$. By $i \neq j$, it follows that $i^\sigma \neq j^\sigma$. Thus, there exist $u, v \in [m]$ and $u \neq v$ such that $i^\sigma \in \Omega_u$ and $j^\sigma \in \Omega_v$. Moreover, $\widetilde{e}_{i^\sigma,j^\sigma} = \overline{e}_{i^\sigma,j^\sigma}$. Consequently, $\overline{e}_{i^\sigma,j^\sigma} = \overline{e}_{ij}$. This shows that $\overline{e}_{i^\sigma,j^\sigma} = \overline{e}_{ij}$ for any $i, j \in [n]$. Hence, $\sigma \in G_{\overline{E}}$, and so $G_{\widetilde{E}} \leq G_{\overline{E}}$. Thus, $G_{\overline{E}} = G_{\widetilde{E}}$. This completes the proof. □

From the above theorem, we know that every ρ-class contains at least one strongly regular idempotent normal matrix. Therefore, to study the centralizer groups of idempotent normal matrices, we need only to consider the centralizer groups of strongly regular idempotent normal matrices up to isomorphism.

Now, let $E = (e_{ij}) \in M_n^{SR}$ and $O(E) = \{e_1, e_2, \ldots, e_r\}$ be the set of all off-diagonal entries of E, where $1 \leq r \leq n^2 - n$. By replacing only all diagonal entries 0 of E with some $e_t \in O(E)$, we can obtain an $n \times n$ matrix E^{e_t} as follows:

$$E^{e_t} = (e_{ij}^{e_t})_{n\times n} = \begin{pmatrix} e_t & e_{12} & \cdots & e_{1n} \\ e_{21} & e_t & \cdots & e_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ e_{n1} & e_{n2} & \cdots & e_t \end{pmatrix}. \tag{7}$$

It is easy to check that $G_E = G_{E^{e_t}}$. We define an equivalence relation $\sim^{E^{e_t}}$ on $[n]$ by

$$i \sim^{E^{e_t}} j \Longleftrightarrow e_{ik}^{e_t} = e_{jk}^{e_t} \text{ and } e_{ki}^{e_t} = e_{kj}^{e_t} \text{ for all } k \in [n].$$

That is, $i \sim^{E^{e_t}} j$ if and only if the i-th row and the j-th row of E^{e_t} are equal and so are the i-th column and the j-th column. In the following, we shall give the equivalent conditions of $\sim^{E^{e_t}} \neq \Delta$. Further, a structural description of the centralizer group G_E is obtained.

Suppose that $\{\Sigma_1, \Sigma_2, \ldots, \Sigma_s\}$ is the set of all $\sim^{E^{e_t}}$-classes for some $e_t \in O(E)$, where $1 \leq s \leq n$. Let $E^{e_t}[\Sigma_k]$ denote the principal submatrix of E^{e_t} where the row indices and the column indices are taken from Σ_k for each $k \in [s]$. It is clear that all entries of $E^{e_t}[\Sigma_k]$ are e_t. In fact, for any $i, j \in \Sigma_k$, $e_{ij}^{e_t} = e_{ii}^{e_t} = e_t$ by $i \sim^{E^{e_t}} j$. By replacing $E^{e_t}[\Sigma_k]$ with 0 for all $k \in [s]$, we can obtain an $n \times n$ matrix $E^{(e_t)} = (e_{ij}^{(e_t)})$. That is to say, $e_{ij}^{(e_t)} = 0$ for any $i, j \in \Sigma_k$ and $k \in [s]$, and $e_{ij}^{(e_t)} = e_{ij}^{e_t} = e_{ij}$ for any $i \in \Sigma_k$ and $j \in \Sigma_\ell$, where $k, \ell \in [s]$ and $k \neq \ell$. Moreover, we obtain a preliminary lemma.

Lemma 5. *Suppose that $E = (e_{ij}) \in M_n^{SR}$ and $E^{(e_t)}$ is defined as above. Then, $E^{(e_t)} \in M_n^{IN}$.*

Proof. Clearly, $E^{(e_t)}$ is a normal matrix. Now, we shall show that $E^{(e_t)}$ is idempotent, i.e., $e_{ij}^{(e_t)} = max_{k\in[n]}\{e_{ik}^{(e_t)} + e_{kj}^{(e_t)}\}$ for any $i, j \in [n]$.

Suppose that $i, j \in \Sigma_\ell$ for some $\ell \in [s]$. Then, $e_{ij}^{(e_t)} = 0$. If $k \in \Sigma_\ell$, then $e_{ik}^{(e_t)} + e_{kj}^{(e_t)} = 0$. If $k \notin \Sigma_\ell$, then $e_{ik}^{(e_t)} + e_{kj}^{(e_t)} = e_{ik} + e_{kj}$. Since E is idempotent, it implies that $e_{ik} + e_{kj} \leq e_{ij}$ for any $k \in [n]$, and so $e_{ik}^{(e_t)} + e_{kj}^{(e_t)} \leq e_{ij} \leq 0$. Thus, $max_{k\in[n]}\{e_{ik}^{(e_t)} + e_{kj}^{(e_t)}\} = e_{ij}^{(e_t)} = 0$.

Assume that $i \in \Sigma_{\ell_1}$ and $j \in \Sigma_{\ell_2}$, where $\ell_1, \ell_2 \in [s]$ and $\ell_1 \neq \ell_2$. Then, $e_{ij}^{(e_t)} = e_{ij}$. If $k \in \Sigma_{\ell_1}$, then $e_{ik}^{(e_t)} + e_{kj}^{(e_t)} = e_{kj}^{(e_t)} = e_{kj}$. Since $i, k \in \Sigma_{\ell_1}$, i.e., $i \sim^{E^{e_t}} k$, we have that $e_{kj} = e_{ij}$. Thus, $e_{ik}^{(e_t)} + e_{kj}^{(e_t)} = e_{ij}$. If $k \in \Sigma_{\ell_2}$, then $e_{ik}^{(e_t)} + e_{kj}^{(e_t)} = e_{ik}^{(e_t)} = e_{ik}$. Since $j, k \in \Sigma_{\ell_2}$, i.e., $j \sim^{E^{e_t}} k$, we have that $e_{ik} = e_{ij}$. Thus, $e_{ik}^{(e_t)} + e_{kj}^{(e_t)} = e_{ij}$. If $k \notin \Sigma_{\ell_1} \cup \Sigma_{\ell_2}$, then $e_{ik}^{(e_t)} + e_{kj}^{(e_t)} = e_{ik} + e_{kj}$. Since E is idempotent, it implies that $e_{ik} + e_{kj} \leq e_{ij}$, and so $e_{ik}^{(e_t)} + e_{kj}^{(e_t)} \leq e_{ij}$. Consequently, $max_{k\in[n]}\{e_{ik}^{(e_t)} + e_{kj}^{(e_t)}\} = e_{ij}^{(e_t)} = e_{ij}$. This shows that $e_{ij}^{(e_t)} = max_{k\in[n]}\{e_{ik}^{(e_t)} + e_{kj}^{(e_t)}\}$ for any $i, j \in [n]$, as required. □

Now, we have the following:

Theorem 2. *Let $E = (e_{ij}) \in M_n^{SR}$ and $E^{(e_t)}$ be defined as above. Then, $\sim^{E^{e_t}} = \sim_{E^{(e_t)}}$.*

Proof. Suppose that $i \sim^{E^{e_t}} j$ for any $i, j \in [n]$. Then, $i, j \in \Sigma_\ell$ for some $\ell \in [s]$. Thus, $e_{ij}^{(e_t)} = e_{ji}^{(e_t)} = 0$. By $E^{(e_t)} \in M_n^{IN}$ and ([11], Lemma 3.2), it follows that $i \sim_{E^{(e_t)}} j$. Thus, $\sim^{E^{e_t}} \subseteq \sim_{E^{(e_t)}}$.

On the other hand, assume that $i \sim_{E^{(e_t)}} j$ for any $i, j \in [n]$. It follows from ([11], Lemma 3.2) that $e_{ik}^{(e_t)} = e_{jk}^{(e_t)}$ and $e_{ki}^{(e_t)} = e_{kj}^{(e_t)}$ for all $k \in [n]$. Now, we shall show that $i \sim^{E^{e_t}} j$, i.e., $\Sigma(i) = \Sigma(j)$, where $\Sigma(i)$ and $\Sigma(j)$ denote the $\sim^{E^{e_t}}$-classes containing i and j, respectively.

Suppose that $\Sigma(i) \neq \Sigma(j)$. For any $k \in \Sigma(i)$, i.e., $i \sim^{E^{e_t}} k$, we have $e_{ik}^{(e_t)} = e_{ki}^{(e_t)} = 0$. Moreover, by ([11], Lemma 3.2), $i \sim_{E^{(e_t)}} k$. Since $i \sim_{E^{(e_t)}} j$, it implies that $k \sim_{E^{(e_t)}} j$ and so $e_{jk}^{(e_t)} = e_{kj}^{(e_t)} = 0$. Note that $\Sigma(k) \neq \Sigma(j)$, which implies that $e_{jk}^{(e_t)} = e_{jk}^{e_t} = e_{jk}$ and $e_{kj}^{(e_t)} = e_{kj}^{e_t} = e_{kj}$. Moreover, $e_{jk} = e_{kj} = 0$. It follows from ([11], Lemma 3.2) that $j \sim_E k$. Therefore, $E \in M_n^{NSR}$—a contradiction. Consequently, $\Sigma(i) = \Sigma(j)$. That is, $i \sim^{E^{e_t}} j$. Thus, $\sim_{E^{(e_t)}} \subseteq \sim^{E^{e_t}}$, and so $\sim^{E^{e_t}} = \sim_{E^{(e_t)}}$. □

Corollary 1. *Suppose that $E \in M_n^{SR}$ and $e_t \in O(E)$. Then,*

$$\sim^{E^{e_t}} \neq \Delta \Longleftrightarrow \sim_{E^{(e_t)}} \neq \Delta \Longleftrightarrow E^{(e_t)} \in M_n^{NSR}.$$

By $G_E = G_{E^{e_t}}$, ([11], Theorem 3.7, Lemma 4.4) and Corollary 1, we have immediately the following theorem.

Theorem 3. *Let $E = (e_{ij}) \in M_n^{SR}$. If there exists $e_t \in O(E)$ such that $E^{(e_t)} \in M_n^{NSR}$, then G_E is a split extension of $\widetilde{G}_{E^{e_t}}$ by $H_{E^{e_t}}$, i.e., $G_E = \widetilde{G}_{E^{e_t}} \rtimes H_{E^{e_t}}$, where*

$$\widetilde{G}_{E^{e_t}} := \{\sigma \in S_n \mid P_\sigma E^{e_t} = E^{e_t} P_\sigma = E^{e_t}\}$$

and

$$H_{E^{e_t}} := \{\sigma \in G_{E^{e_t}} \mid (\forall k \in [s])\ (\forall i, j \in \Sigma_k)\ i \leq j \Leftrightarrow i^\sigma \leq j^\sigma\}.$$

Example 1. *Consider the centralizer group of the following strongly regular idempotent normal matrix:*

$$E = \begin{pmatrix} 0 & d & d & x & x \\ d & 0 & d & x & x \\ d & d & 0 & x & x \\ y & y & y & 0 & u \\ y & y & y & u & 0 \end{pmatrix}, \tag{8}$$

where $x, y, d, u \in [-1.1, -0.9]$ are distinct. By ([12], Theorem 5.10), it follows that $E \in M_n^{IN}$. It is easy to check that $\widetilde{G}_{E^d} \cong S_3$ and $H_{E^d} \cong S_2$. Thus, $G_E = S_3 \rtimes S_2$.

Remark 2. *According to Corollary 1 and Theorem 3, we give a structural description of the centralizer groups of partial idempotent normal matrices. The characterization of the centralizer groups of the remaining idempotent normal matrices, which are strongly regular idempotent normal matrices E satisfying $E^{(e_t)} \in M_n^{SR}$ for all $e_t \in O(E)$, is still an unsolved problem. It is well known that G is a finite two-closed permutation group if and only if G equals to G_E, where E is idempotent normal matrix. The polycirculant conjecture, which is important in graph theory, asserts that every non-trivial finite transitive two-closed permutation group contains a fixed-point-free element of prime order. By Theorem 3, it is clear that the centralizer groups of partial idempotent normal matrices contains a fixed-point-free element of prime order. Furthermore, the main results of this paper may be helpful for further research on the polycirculant conjecture.*

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References

1. Butkovič, P. *Max-Linear Systems: Theory and Algorithms*; Springer: London, UK, 2010.
2. Johnson, M.; Kambites, M. Green's $\mathcal{J}$-order and the rank of tropical matrices. *J. Pure Appl. Algebra* **2013**, *217*, 280–292. [CrossRef]
3. Shitov, Y. Tropical matrices and group repersentations. *J. Algebra* **2012**, *370*, 1–4. [CrossRef]
4. Yang, L. Regular $\mathcal{D}$-classes of the semigroup of $n \times n$ tropical matrices. *Turk. J. Math.* **2018**, *42*, 2061–2070. [CrossRef]
5. Johnson, M.; Kambites, M. Multiplicative structrue of 2×2 tropical matrices. *Linear Algebra Appl.* **2011**, *435*, 1612–1625. [CrossRef]
6. Butkovič, P. Max-algebra: The linear algebra of combinatorics. *Linear Algebra Appl.* **2003**, *367*, 313–335. [CrossRef]
7. Johnson, M.; Kambites, M. Idempotent tropical matrices and finite metric spaces. *Adv. Geom.* **2014**, *14*, 253–276.

8. Izhakian, Z.; Johnson, M.; Kambites, M. Pure dimension and projectivity of tropical polytopes. *Adv. Math.* **2016**, *303*, 1236–1263. [CrossRef]
9. Linde, J.; Puente, M.J.d. Matrices commuting with a given normal tropical matrix. *Linear Algebra Appl.* **2015**, *482*, 101–121. [CrossRef]
10. Yu, B.M.; Zhao, X.Z.; Zeng, L.L. A congruence on the semiring of normal tropical matrices. *Linear Algebra Appl.* **2018**, *555*, 321–335. [CrossRef]
11. Deng, W.N.; Zhao, X.Z.; Cheng, Y.L.; Yu, B.M. On the groups associated with a tropical $n \times n$ matrix. *Linear Algebra Appl.* **2022**, *639*, 1–17. [CrossRef]
12. Izhakian, Z.; Johnson, M.; Kambites, M. Tropical matrix groups. *Semigroup Forum.* **2018**, *96*, 178–196. [CrossRef]
13. Yang, L. The tropical matrix groups with symmetric idempotents. *Discret. Dyn. Nat. Soc.* **2018**, *2018*, 1–9. [CrossRef]
14. Dixon, J.D.; Mortimer, B. *Permutation Groups*; Springer: New York, NY, USA, 1996.
15. Cameron, P.J. *Permutation Groups (London Mathematical Society Student Texts Vol 45)*; Cambridge University Press: Cambridge, UK, 1999.
16. Blyth, T.S.; Janowitz, M.F. *Residuation Theory*; Pergamon Press: Oxford, UK, 1972.

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