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Global Existence for the Semi-Dissipative 2D Boussinesq Equations on Exterior Domains

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Abstract: This paper concerns the viscous Boussinesq equations without a dissipation term and their relation to the temperature equation related to the exterior of a ball with a smooth boundary. We first prove the global existence of weak solutions on the bounded domain $\widetilde{\Omega}$ via the Schauder fixed-point theorem. Then, we derive the uniform estimates to obtain the global existence of weak solutions on the unbounded domain Ω by utilizing the domain expansion method. Finally, we show that the equations have a unique classical solution for H^3 initial data by a series of regularity estimations.

Keywords: Boussinesq equations; Schauder fixed-point theorem; domain expansion method; classical solution

MSC: 35Q35; 76B03; 76D03

1. Introduction

The Boussinesq equations play a crucial role in the atmospheric sciences and ocean circulation [1], as well as other geophysical applications. They describe the motion of a fluid under small amplitude fluctuations in fluid dynamics. In recent decades, they have attracted widespread attention from many researchers; see [2–11] and the references therein. Precisely, the incompressible Boussinesq equations read as follows:

$$\begin{cases} \rho_t + \operatorname{div}(\rho \mathbf{u}) = 0, \\ (\rho \mathbf{u})_t + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) = \nu \Delta \mathbf{u} - \nabla p + \rho \vartheta \mathbf{e_2}, \\ \vartheta_t + \mathbf{u} \cdot \nabla \vartheta = 0, \\ \nabla \cdot \mathbf{u} = 0, \end{cases} \tag{1}$$

where $\mathbf{u}(\mathbf{x}, t), \rho(\mathbf{x}, t), \vartheta(\mathbf{x}, t)$, and $p(\mathbf{x}, t)$ denote the velocity, density, temperature, and pressure of the fluid, respectively. $\nu > 0$ is the viscosity coefficient for the fluid and $\mathbf{e_2} = (0, 1)$.

One of the most fundamental questions for Equation (1) is its well-posedness. For the case of density-dependent Boussinesq equations, when $\vartheta = 0$, Kim [2] established the local existence of weak solutions on an open bounded subset of $\mathbb{R}^3$; here, the initial vacuum is allowed. Furthermore, Choe and Kim [3] improved this result, proving the existence of strong solutions of the Navier–Stokes equations for nonhomogeneous incompressible fluids—not only on an open bounded subset of $\mathbb{R}^3$, but also with respect to the problem on the whole space $\mathbb{R}^3$. They proved the existence and uniqueness of local strong solutions to the initial value problem (for $\mathbb{R}^3$) and the initial boundary value problem for an open

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bounded subset of $\mathbb{R}^3$. When $\Omega = \mathbb{R}^2$, Zhong [4] studied the Cauchy problem for the Boussinesq equations where both ρ and ϑ are not constants. They showed that there exists a unique local strong solution providing the initial density and that the initial temperature decay is not too slow at infinity.

On the other hand, for the case of a homogeneous incompressible fluid, a great deal of the literature is concerned with the question at hand. Cannon and DiBeneo [5] studied the homogeneous Boussinesq equations with full viscosity. They found the existence and uniqueness of weak solutions in L^p. Furthermore, they improved the regularity of the solution when the initial data are smooth. In recent years, the result regarding the existence of such solutions has been generalized to cases of "partial viscosity" by Chae [6] and Hou-Li [7] (who did this around the same time). In [7], Hou and Li studied the viscous Boussinesq equations, i.e., where $\nu \neq 0$ and diffusion does not have to feature in the density equation. They used sharp and delicate energy estimates to prove the global existence and strong regularity of the viscous Boussinesq equations for general initial data in H^m with $m \geq 3$. In [6], Chae considered both the viscous Boussinesq equations and the nonzero diffusivity Boussinesq equations. He proved the global-in-time regularity for both cases. Moreover, Lai, Pan and Zhao [11] studied the viscous Boussinesq equations over a bounded domain with a smooth boundary. They showed that the equations have a unique classical solution for H^3 initial data. In addition, they proved that the kinetic energy is uniformly bounded in time.

It should be noted that most of the existing results mentioned above mainly focus on the whole space $\mathbb{R}^n (n \geq 2)$ or the bounded domains with constraints from boundaries. Here, our purpose is to study the global existence of solutions of the viscous Boussinesq equations on an unbounded domain—more precisely, the domain Ω, which represents the exterior of a ball $B_{R_1}(0)^c \subset \mathbb{R}^2$ with a smooth boundary. We also remark that thermal dissipation as a physical mechanism describing the attenuation of temperature fields over time is crucial for understanding energy balance and stability in fluid dynamics. Especially in the models we study, it directly impacts the global well-posedness of the equations, namely, the existence, uniqueness, and stability of solutions. Thus, in this paper, we consider the following incompressible Boussinesq equations:

$$\begin{cases} \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = \nu \Delta \mathbf{u} - \nabla p + \vartheta g(r) \mathbf{e_r}, \\ \vartheta_t + \mathbf{u} \cdot \nabla \vartheta = 0, \\ \nabla \cdot \mathbf{u} = 0, \end{cases} \tag{2}$$

where the unknowns $\mathbf{u} = (u, v)^T, \vartheta, p$ denote the velocity, temperature, pressure of the fluid at point x at time t, respectively. $\nu > 0$ is the viscosity coefficient, $g(r)$ is a function directed towards the center of the Earth, and $\mathbf{e_r} = (\frac{x}{r}, \frac{y}{r})$.

The boundary condition and the condition at infinity are given by

$$\begin{aligned} &\mathbf{u}(t, x) = \mathbf{0}, \ \text{ for } \ t > 0, x \in \partial\Omega, \\ &\mathbf{u}(t, x) \to \mathbf{u}_\infty \ \text{ as } \ |x| \to \infty, \end{aligned} \tag{3}$$

and the initial conditions

$$\mathbf{u}(0, x) = \mathbf{u}_0, \ \ \vartheta(0, x) = \vartheta_0. \tag{4}$$

Without loss of generality, we assume that $\mathbf{u}_\infty = \mathbf{0}$, thanks to the Galilean invariance of fluid mechanics.

We will first prove the global existence of weak solutions to Equations (2)–(4), and then improve the regularity of the solutions by using energy estimates under the initial and boundary conditions Equations (3) and (4). Now, we give the definition of weak solutions.

Definition 1. *$(\mathbf{u}, \vartheta)$ is called the global weak solution of problems (2)–(4), if for any $T > 0$, $\mathbf{u} \in L^\infty([0,T); L^2(\Omega)^2) \cap L^2([0,T); H_0^1(\Omega)^2)$, $\vartheta \in L^\infty([0,T); L^\infty(\Omega))$, satisfying*

$$\begin{aligned} &\int_0^T \int_\Omega \left[\mathbf{u} \cdot \mathbf{\Phi}_t + (\mathbf{u} \cdot \nabla \mathbf{\Phi}) \cdot \mathbf{u} - \nu \nabla \mathbf{u} : \nabla \mathbf{\Phi} + \vartheta g(r) \mathbf{e}_r \cdot \mathbf{\Phi}\right] dx d\tau \\ &+ \int_\Omega \mathbf{u}_0 \mathbf{\Phi}(0,x) dx = 0, \\ &\int_0^T \int_\Omega \left[\vartheta \Psi_t + (\mathbf{u} \cdot \nabla \Psi)\vartheta\right] dx d\tau + \int_\Omega \vartheta_0 \Psi(0,x) dx = 0, \end{aligned} \tag{5}$$

for any $\mathbf{\Phi} \in C_0^\infty(\Omega \times [0,T])^2$ satisfying $\mathbf{\Phi}(T,x) = 0$ and $\nabla \cdot \mathbf{\Phi} = 0$, and for any $\Psi \in C^\infty(\Omega \times [0,T])$ satisfying $\Psi(T,x) = 0$.

Our main result can be stated as follows:

Theorem 1. *For any initial $\mathbf{u}_0 \in H_0^1(\Omega)^2, \vartheta_0 \in L^1(\Omega) \cap L^\infty(\Omega)$, there exists a global weak solution $(\mathbf{u}, \vartheta)$ of (2)–(4), such that, for any $T > 0$, $\mathbf{u} \in L^\infty([0,T); L^2(\Omega)^2) \cap L^2([0,T); H_0^1(\Omega)^2)$ and $\vartheta \in L^\infty([0,T); L^1(\Omega) \cap L^\infty(\Omega))$.*

The existence of pressure p follows immediately from Equations (2) and (4) by a classical consideration. For more regular initial data, we can improve the regularity of weak solutions and thus obtain the unique strong solution.

Theorem 2. *If the initial $\mathbf{u}_0 \in H^3(\Omega)^2, \vartheta_0 \in H^3(\Omega)$, satisfying the compatibility conditions in (20), then there exists a unique solution $(\mathbf{u}, \vartheta)$ of Equations (2)–(4) globally in time, such that for any $T > 0$,*

$$\sup_{0 \le t \le T} (\|\mathbf{u}\|_{H^3(\Omega)}^2 + \|\vartheta\|_{H^3(\Omega)}^2) \le C, \quad \int_0^T \|\mathbf{u}\|_{H^4(\Omega)}^2 d\tau \le C. \tag{6}$$

2. Preliminaries

In this section, we recall some known facts and elementary inequalities that will be used frequently later. First of all, we require the following compatibility conditions:

$$\begin{cases} \Delta p_0 = \nabla \cdot [\vartheta_0 g(r) \mathbf{e}_r - \mathbf{u}_0 \cdot \nabla \mathbf{u}_0], \quad x \in \Omega, \\ \nabla p_0 \cdot \mathbf{n}|_{\partial\Omega} = [\nu \Delta \mathbf{u}_0 + \vartheta_0 g(r) \mathbf{e}_r] \cdot \mathbf{n}|_{\partial\Omega}, \end{cases} \tag{7}$$

where $\mathbf{n}$ is the unit outward normal to $\partial\Omega$.

Next, we state the following regularity results in Stokes equations, which are useful for further higher-order estimates (see Ref. [12], Proposition 2.3).

Lemma 1. *Let Ω be an open set in $\mathbb{R}^2$ with ℓ^r boundary, where $r = \max\{m+2, 2\}$, $m \ge -1$ is an integer. Consider the Stokes problem*

$$\begin{cases} -\nu \Delta \mathbf{u} + \nabla p = f, \\ \nabla \cdot \mathbf{u} = 0, \\ \mathbf{u}|_{\partial\Omega} = 0. \end{cases} \tag{8}$$

If $f \in W^{m,\alpha}(\Omega)$, then $\mathbf{u} \in W^{m+2,\alpha}(\Omega)$, $p \in W^{m+1,\alpha}(\Omega)$ and there exists a constant c_0 such that

$$\|\mathbf{u}\|_{W^{m+2,\alpha}(\Omega)} + \|p\|_{W^{m+1,\alpha}(\Omega)} \leq c_0 \|f\|_{W^{m,\alpha}(\Omega)}$$

for any $\alpha \in (1, \infty)$. c_0 is a positive constant depending only on α, ν, m, and Ω but independent of R.

Finally, we need the following Sobolev embeddings and Gagliardo–Nirenberg inequalities (here, the uniform constants are independent of the size of the domain):

Lemma 2. *Let Ω be a domain in $\mathbb{R}^2$ with smooth boundary C^1. Then, the following embedding inequalities hold:*

$$\begin{aligned} &H^1(\Omega) \hookrightarrow L^q(\Omega), \ \ \forall q \in [2, \infty); \\ &W^{1,q}(\Omega) \hookrightarrow L^\infty(\Omega), \ \ \forall q \in\in (2, \infty). \end{aligned} \tag{9}$$

Lemma 3. *There exists a constant $C > 0$, such that for any $\mathbf{u} \in H_0^1(\Omega) \cap H^2(\Omega)$, it holds that*

$$\|\mathbf{u}\|_{L^4(\Omega)}^2 \leq C \|\mathbf{u}\|_{L^2(\Omega)} \|\nabla \mathbf{u}\|_{L^2(\Omega)}. \tag{10}$$

3. Proofs of Theorems

In this section, we prove the results announced in the introduction. Before proving our main results, we first explain the notation and conventions used throughout this paper.

For $R > 0$, set

$$B_R = \{x \in \mathbb{R}^2 \mid |x| < R\}.$$

For $1 \leq q \leq \infty$ and integer $k \geq 0$, the standard Sobolev spaces are denoted by

$$L^q = L^q(\Omega), \ \ W^{k,q} = W^{k,q}(\Omega), \ \ H^k = W^{k,2}(\Omega).$$

Without confusion, we also write $L^q(\tilde{\Omega}), W^{k,q}(\tilde{\Omega})$, and $W^{k,2}(\tilde{\Omega})$ by $L^q, W^{k,q}$, and H^k, respectively.

3.1. Proof of Theorem 1

First, we consider a bounded domain $\tilde{\Omega} = \Omega \cap B_R$ with $R > R_1$. We construct approximate solutions via the Schauder fixed-point theorem (see [13]), derive uniform bounds, and thus obtain solutions by passing to the limit. Then, the existence for the unbounded space Ω follows in a straightforward way from the a priori estimates by the classical domain expansion technique. We first prove Theorem 1 for bounded domains.

3.1.1. Global Existence of Weak Solution on Bounded Domain

In this subsection, we use the same method as in [11,14] to prove Proposition 1 by a fixed-point argument. To implement this nethod, we fix any $T > 0$ and consider problems (2)–(4) in $\tilde{\Omega} \times [0, T]$.

Let B be the closed convex set in

$$X = L^\infty([0, T); L^2(\tilde{\Omega})^2) \cap L^2([0, T); H_0^1(\tilde{\Omega})^2),$$

defined by

$$B = \left\{ \mathbf{u} = (u, v) \in X \middle| \quad \nabla \cdot \mathbf{u} = 0, a.e., \text{on } \tilde{\Omega} \times (0, T), \|\mathbf{u}\|_X \leq R_0 \right\}, \tag{11}$$

where R_0 will be determined later. The norm $\|\cdot\|_X$ is defined by

$$\|\mathbf{u}\|_X^2 = \sup_{0\le t\le T}\|\mathbf{u}\|^2 + \int_0^T \|\mathbf{u}\|_{H_0^1}^2 d\tau.$$

For simplicity, we use $\|\cdot\|$ to denote $\|\cdot\|_{L^2}$ in this paper.

Proposition 1. *Let $\tilde{\Omega} = \Omega \cap B_R(R > R_1)$, and assume $\mathbf{u}_0 \in H_0^1(\tilde{\Omega})^2, \vartheta_0 \in L^\infty(\tilde{\Omega})$. If $g(r) \in L^\infty(\tilde{\Omega})$, then there exists a global weak solution $(\mathbf{u}, \vartheta)$ of Equations (2)–(4), such that for any $T > 0$,*

$$\begin{aligned} &\|\vartheta\|_{L^q} \le \|\vartheta_0\|_{L^q} \text{ for } 1 \le q \le \infty, \\ &\|\mathbf{u}\|^2 \le e^{\|g(r)\|_{L^\infty(\tilde{\Omega})}T}[\|\mathbf{u}_0\|^2 + \|g(r)\|_{L^\infty}\|\vartheta_0\|^2 T], \\ &\int_0^T \|\nabla \mathbf{u}\|^2 d\tau \le C, \end{aligned} \tag{12}$$

where the positive constant C depends only on $\|g(r)\|_{L^\infty}, \nu, \|\mathbf{u}_0\|, \|\vartheta_0\|$, and T, but is independent of R (the radius of ball $B_R = \{x \in \mathbb{R}^2 | |x| < R\}$).

Proof. We first construct an approximate solution. This will be divided into three steps as follows:

Step 1. We define the mapping $F_\varepsilon : B \to B$.

For fixed $\varepsilon \in (0,1)$ and any $\mathbf{v} \in B$, we first mollify $\mathbf{v}$ using the standard procedure to obtain

$$\mathbf{v}_\varepsilon = \eta_{\frac{\varepsilon}{3}} * \bar{\mathbf{v}}_\varepsilon,$$

where $\bar{\mathbf{v}}_\varepsilon$ is the truncation of $\mathbf{v}$ in $\widetilde{\Omega}_\varepsilon = \{x \in \widetilde{\Omega} \mid dist(x, \partial\widetilde{\Omega}) > \varepsilon\}$ (extended by 0 to $\widetilde{\Omega}$), and $\eta_{\frac{\varepsilon}{3}}$ is the standard mollifier. It thus follows from [14] that

$$\begin{aligned} &\mathbf{v}_\varepsilon \in C([0,T]; C_0^\infty(\bar{\tilde{\Omega}})), \ \ \nabla \cdot \mathbf{v}_\varepsilon = 0, \\ &\|\mathbf{v}_\varepsilon\|_{C([0,T];L^2(\tilde{\Omega}))} \le C\|\mathbf{v}\|_{C([0,T];L^2(\tilde{\Omega}))}, \\ &\|\mathbf{v}_\varepsilon\|_{L^2([0,T];H_0^1(\tilde{\Omega}))} \le C\|\mathbf{v}\|_{L^2([0,T];H_0^1(\tilde{\Omega}))} \end{aligned} \tag{13}$$

for some constant $C > 0$ that is independent of both ε and R. Similarly, we regularize the initial data to obtain the smooth approximation ϑ_0^ε for ϑ_0 and $\mathbf{u}_0^\varepsilon$ for $\mathbf{u}_0$, respectively, such that

$$\begin{aligned} &\mathbf{u}_0^\varepsilon \in C_0^\infty(\bar{\tilde{\Omega}}), \ \nabla \cdot \mathbf{u}_0^\varepsilon = 0, \ \text{ and } \mathbf{u}_0^\varepsilon \to \mathbf{u}_0 \text{ in } H_0^1(\tilde{\Omega})) \text{ as } \varepsilon \to 0, \\ &\vartheta_0^\varepsilon \in C^\infty(\bar{\tilde{\Omega}}), \ \text{ and } \vartheta_0^\varepsilon \to \vartheta_0 \text{ in } L^\infty(\tilde{\Omega})) \text{ as } \varepsilon \to 0. \end{aligned} \tag{14}$$

Then, we solve the equation with smooth initial data

$$\begin{cases} \vartheta_t + \mathbf{v}_\varepsilon \cdot \nabla \vartheta = 0, \\ \vartheta(0) = \vartheta_0^\varepsilon(x), \end{cases} \tag{15}$$

and we denote the solution by ϑ^ε. Next, we solve the Navier–Stokes equation with smooth initial data

$$\begin{cases} \mathbf{u}_t + \mathbf{v}_\varepsilon \cdot \nabla \mathbf{u} = \nu\Delta\mathbf{u} - \nabla p + \vartheta^\varepsilon g(r)\mathbf{e_r}, \\ \nabla \cdot \mathbf{u} = 0, \\ \mathbf{u}|_{\partial\tilde{\Omega}} = 0, \ \ \mathbf{u}(0) = \mathbf{u}_0^\varepsilon, \end{cases} \tag{16}$$

and denote the solution by $\mathbf{u}^\varepsilon$. Then, we define the mapping $F_\varepsilon(\mathbf{v}) = \mathbf{u}^\varepsilon$. The solvability of (15) and (16) follows easily from [14]. Next, we prove that F_ε maps B onto B.

Multiplying $(15)_1$ by $|\vartheta|^{q-2}\vartheta (2 \le q < \infty)$ and integrating the resulting equation over $\tilde{\Omega}$ by parts, we obtain

$$\|\vartheta\|_{L^q} = \|\vartheta_0^\varepsilon\|_{L^q} \le C\|\vartheta_0\|_{L^q}, \ \ \forall\, 0 \le t \le T, \tag{17}$$

i.e.,

$$\|\vartheta^\varepsilon\|_{L^q} = \|\vartheta_0^\varepsilon\|_{L^q} \le C\|\vartheta_0\|_{L^q}, \ \ \forall\, 0 \le t \le T, \tag{18}$$

where C is a constant independent of both ε and R.

Multiplying $(16)_1$ by $\mathbf{u}$ and integrating the resulting equation over $\tilde{\Omega}$ by parts and using Young's inequality, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|^2 + \nu\|\nabla\mathbf{u}\|^2 &= \int_{\tilde{\Omega}} \vartheta^\varepsilon g(r)\mathbf{e_r}\cdot\mathbf{u}dx \\ &\le \|g(r)\|_{L^\infty}\int_{\tilde{\Omega}} \vartheta^\varepsilon \mathbf{e_r}\cdot\mathbf{u}dx \\ &\le \|g(r)\|_{L^\infty}\left[\frac{1}{2}\|\vartheta^\varepsilon\|^2 + \frac{1}{2}\|\mathbf{u}\|^2\right]. \end{aligned} \tag{19}$$

By dropping $\nu\|\nabla\mathbf{u}\|^2$ from (19) and applying Gronwall's inequality to the resulting inequality, we find that

$$\begin{aligned} \|\mathbf{u}\|^2 &\le e^{\|g(r)\|_{L^\infty}T}[\|\mathbf{u}_0^\varepsilon\|^2 + \|g(r)\|_{L^\infty}\|\vartheta_0^\varepsilon\|^2 T] \\ &\le e^{\|g(r)\|_{L^\infty}T}[\|\mathbf{u}_0\|^2 + \|g(r)\|_{L^\infty}\|\vartheta_0\|^2 T] \\ &\le C, \end{aligned} \tag{20}$$

which also implies, after integrating (19) over $[0, T]$, that

$$\int_0^T \|\nabla\mathbf{u}\|^2 d\tau \le C(\|g(r)\|_{L^\infty}, \nu, \|\mathbf{u}_0\|, \|\vartheta_0\|, T),$$

i.e.,

$$\|\mathbf{u}^\varepsilon\|^2_{L^\infty([0,T];L^2(\tilde{\Omega}))} + \|\mathbf{u}^\varepsilon\|^2_{L^2([0,T];H_0^1(\tilde{\Omega}))} \le C^*. \tag{21}$$

Choosing $R_0^2 = C^*$, we find that $\mathbf{u}^\varepsilon \in B$, which means that F_ε maps B onto B for any $0 < \varepsilon < 1$. Here, C^* denotes a generic positive constant depending only on $\|g(r)\|_{L^\infty}, \nu, \|\mathbf{u}_0\|, \|\vartheta_0\|$, and the terminal time T, while being independent of ε and the size of the domain $\tilde{\Omega}$.

Step 2. We prove the compactness of F_ε.

Taking L^2 as the inner product of $(16)_1$ with $\mathbf{u}_t$, one has

$$\begin{aligned} \|\mathbf{u}_t\|^2 + \frac{\nu}{2}\frac{d}{dt}\|\nabla\mathbf{u}\|^2 &\le \int_{\tilde{\Omega}} |\mathbf{v}_\varepsilon||\nabla\mathbf{u}||\mathbf{u}_t|dx + \int_{\tilde{\Omega}} \vartheta^\varepsilon g(r)\mathbf{e_r}\mathbf{u}_t dx \\ &\le \frac{1}{4}\|\mathbf{u}_t\|^2 + \|\mathbf{v}_\varepsilon\cdot\nabla\mathbf{u}\|^2 + \frac{1}{4}\|\mathbf{u}_t\|^2 + \|\vartheta^\varepsilon g(r)\mathbf{e_r}\|^2 \\ &\le \frac{1}{2}\|\mathbf{u}_t\|^2 + \|\mathbf{v}_\varepsilon\|^2_{L^\infty}\|\nabla\mathbf{u}\|^2 + \|g(r)\|^2_{L^\infty}\|\vartheta_0^\varepsilon\|^2, \end{aligned} \tag{22}$$

which implies that

$$\frac{1}{2}\|\mathbf{u}_t\|^2+\frac{\nu}{2}\frac{d}{dt}\|\nabla\mathbf{u}\|^2\le\|\mathbf{v}_\varepsilon\|^2_{L^\infty}\|\nabla\mathbf{u}\|^2+\|g(r)\|^2_{L^\infty}\|\vartheta_0^\varepsilon\|^2. \tag{23}$$

Applying Gronwall's inequality to (23) and using (18), we have

$$\begin{aligned}\|\nabla\mathbf{u}\|^2&\le e^{\frac{2}{\nu}\|\mathbf{v}_\varepsilon\|^2_{L^\infty([0,T],L^\infty)}T}\Big[\|\nabla\mathbf{u}_0^\varepsilon\|^2+\frac{2}{\nu}\|g(r)\|^2_{L^\infty}\|\vartheta_0^\varepsilon\|^2T\Big]\\&\le e^{\frac{2}{\nu}\|\mathbf{v}_\varepsilon\|^2_{L^\infty([0,T],L^\infty)}T}\Big[\|\nabla\mathbf{u}_0\|^2+\frac{2}{\nu}\|g(r)\|^2_{L^\infty}\|\vartheta_0\|^2T\Big]\\&\le C.\end{aligned} \tag{24}$$

Then, with the help of (24), integration of (23) over $[0,T]$ gives that

$$\begin{aligned}&\int_0^T\|\mathbf{u}_t\|^2d\tau\\&\le\nu\|\nabla\mathbf{u}_0^\varepsilon\|^2+2\|\mathbf{v}_\varepsilon\|^2_{L^\infty([0,T],L^\infty)}\int_0^T\|\nabla\mathbf{u}\|^2d\tau+2\|g(r)\|^2_{L^\infty}\|\vartheta_0^\varepsilon\|^2T\\&\le\nu\|\nabla\mathbf{u}_0\|^2+2\|\mathbf{v}_\varepsilon\|^2_{L^\infty([0,T],L^\infty)}\int_0^T\|\nabla\mathbf{u}\|^2d\tau+2\|g(r)\|^2_{L^\infty}\|\vartheta_0\|^2T\\&\le C.\end{aligned} \tag{25}$$

On the other hand, $(\mathbf{u},p,\vartheta)$ satisfies the following Stokes system:

$$\begin{cases}-\nu\Delta\mathbf{u}+\nabla p=-\mathbf{u}_t-\mathbf{v}_\varepsilon\cdot\nabla\mathbf{u}+\vartheta^\varepsilon g(r)\mathbf{e_r},\ \ x\in\tilde{\Omega},\\\nabla\cdot\mathbf{u}=0,\ \ x\in\tilde{\Omega},\\\mathbf{u}|_{\partial\Omega}=0.\end{cases} \tag{26}$$

By the regularity results on (26) (see Lemma 1), we know that

$$\begin{aligned}\|\mathbf{u}\|^2_{H^2}&\le c_0\big[\|\mathbf{u}_t\|^2+\|\mathbf{v}_\varepsilon\cdot\nabla\mathbf{u}\|^2+\|\vartheta^\varepsilon g(r)\mathbf{e_r}\|^2\big]\\&\le c_0\big[\|\mathbf{u}_t\|^2+\|\mathbf{v}_\varepsilon\|^2_{L^\infty}\|\nabla\mathbf{u}\|^2+\|g(r)\|^2_{L^\infty}\|\vartheta^\varepsilon\|^2\big].\end{aligned} \tag{27}$$

Here, c_0 is a constant independent of both ε and R. It thus follows that (24) and (25) yield

$$\int_0^T\|\mathbf{u}\|^2_{H^2}d\tau\le C. \tag{28}$$

From (25) and (28), we know that F_ε is compact by the Compactness Theorem (see [2,15,16]).

Step 3. We prove the continuity of F_ε.

Let $F_\varepsilon(\mathbf{v}_i)=\mathbf{u}_i^\varepsilon$, $i=1,2$; by definition, we know

$$\begin{cases}\vartheta_{it}^\varepsilon+\mathbf{v}_{i\varepsilon}\cdot\nabla\vartheta_i^\varepsilon=0,\\\mathbf{u}_{it}^\varepsilon+\mathbf{v}_{i\varepsilon}\cdot\nabla\mathbf{u}_i^\varepsilon=\nu\Delta\mathbf{u}_i^\varepsilon-\nabla p_i^\varepsilon+\vartheta_i^\varepsilon g(r)\mathbf{e_r},\\\nabla\cdot\mathbf{u}_i^\varepsilon=0,\ \ \mathbf{u}_i^\varepsilon\,|_{\partial\Omega}=0,\\\mathbf{u}_i^\varepsilon(0)=\mathbf{u}_0^\varepsilon,\ \ \vartheta_i^\varepsilon(0)=\vartheta_0^\varepsilon,\end{cases} \tag{29}$$

Subtracting the equation for $i = 2$ from the one for $i = 1$, we arrive at

$$\begin{cases} \theta_t^\varepsilon + \mathbf{v}_{1\varepsilon} \cdot \nabla \theta^\varepsilon + W_\varepsilon \cdot \nabla \vartheta_2^\varepsilon = 0, \\ \chi_t^\varepsilon + \mathbf{v}_{1\varepsilon} \cdot \nabla \chi^\varepsilon + W_\varepsilon \cdot \nabla \mathbf{u}_2^\varepsilon = \nu \Delta \chi^\varepsilon - \nabla Q^\varepsilon + \theta^\varepsilon g(r) \mathbf{e_r}, \\ \nabla \cdot \chi^\varepsilon = 0, \ \ \chi^\varepsilon \mid_{\partial\Omega} = 0, \\ \chi^\varepsilon(0) = 0, \ \ \theta^\varepsilon(0) = 0, \end{cases} \tag{30}$$

where $\theta^\varepsilon = \vartheta_1^\varepsilon - \vartheta_2^\varepsilon$, $W_\varepsilon = \mathbf{v}_{1\varepsilon} - \mathbf{v}_{2\varepsilon}$, $\chi^\varepsilon = \mathbf{u}_1^\varepsilon - \mathbf{u}_2^\varepsilon$, and $Q^\varepsilon = p_1^\varepsilon - p_2^\varepsilon$.

Taking the L^2 inner products of $(30)_1$ with θ^ε, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\theta^\varepsilon\|^2 = -\int_{\tilde{\Omega}} W_\varepsilon \cdot \nabla \vartheta_2^\varepsilon \theta^\varepsilon dx. \tag{31}$$

This holds as $\vartheta_2^\varepsilon \in C([0,T]; C^\infty(\bar{\tilde{\Omega}}))$ (see [11,14]). So, (31) becomes

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\theta^\varepsilon\|^2 &\le \|\nabla \vartheta_2^\varepsilon\|_{L^\infty} \|W_\varepsilon\| \|\theta^\varepsilon\| \\ &\le \frac{C(t)}{2}(\|W_\varepsilon\|^2 + \|\theta^\varepsilon\|^2). \end{aligned} \tag{32}$$

Here, $C(t)$ denotes a generic positive constant depending on t, and $\int_0^T C(\tau)d\tau \le C < \infty$. By Gronwall's inequality, we have

$$\begin{aligned} \|\theta^\varepsilon\|^2 &\le Ce^{CT} \int_0^T \|W_\varepsilon\|^2 d\tau \\ &\le C\|W_\varepsilon\|^2_{L^\infty([0,T];L^2(\tilde{\Omega}))}. \end{aligned} \tag{33}$$

Taking the L^2 inner products of $(30)_2$ with χ^ε, by using the Hölder inequality and Sobolev embedding inequality, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\chi^\varepsilon\|^2 + \nu\|\nabla\chi^\varepsilon\|^2 \\ &= -\int_{\tilde{\Omega}} (W_\varepsilon \cdot \nabla \mathbf{u}_2^\varepsilon) \cdot \chi^\varepsilon dx + \int_{\tilde{\Omega}} \theta^\varepsilon g(r) \mathbf{e_r} \cdot \chi^\varepsilon dx \\ &\le \|W_\varepsilon\| \|\nabla \mathbf{u}_2^\varepsilon\|_{L^4} \|\chi^\varepsilon\|_{L^4} + \|g(r)\|_{L^\infty} \|\theta^\varepsilon\| \|\chi^\varepsilon\| \\ &\le \|W_\varepsilon\| \|\mathbf{u}_2^\varepsilon\|_{H^2} \|\nabla\chi^\varepsilon\| + \|g(r)\|_{L^\infty} \|\theta^\varepsilon\| \|\chi^\varepsilon\| \\ &\le C\|W_\varepsilon\|^2 \|\mathbf{u}_2^\varepsilon\|^2_{H^2} + \frac{\nu}{2}\|\nabla\chi^\varepsilon\|^2 + C\|g(r)\|^2_{L^\infty}\|\theta^\varepsilon\|^2 + \frac{1}{2}\|\chi^\varepsilon\|^2 \\ &\le C\|W_\varepsilon\|^2_{L^\infty([0,T];L^2(\tilde{\Omega}))}(\|\mathbf{u}_2^\varepsilon\|^2_{H^2} + \|g(r)\|^2_{L^\infty}) + \frac{\nu}{2}\|\nabla\chi^\varepsilon\|^2 + \frac{1}{2}\|\chi^\varepsilon\|^2, \end{aligned} \tag{34}$$

where we have used (33). From (34) and (27), we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\chi^\varepsilon\|^2 + \frac{\nu}{2}\|\nabla\chi^\varepsilon\|^2 \\ &\le C\|W_\varepsilon\|^2_{L^\infty([0,T];L^2(\tilde{\Omega}))}(\|\mathbf{u}_2^\varepsilon\|^2_{H^2} + \|g(r)\|^2_{L^\infty}) + \frac{1}{2}\|\chi^\varepsilon\|^2 \\ &\le \frac{C_1(t)}{2}\|W_\varepsilon\|^2_{L^\infty([0,T];L^2(\tilde{\Omega}))} + \frac{1}{2}\|\chi^\varepsilon\|^2, \end{aligned} \tag{35}$$

where $\int_0^T C_1(\tau)d\tau \le C$. By dropping $\frac{\nu}{2}\|\nabla\chi^\varepsilon\|^2$ from (35) and applying Gronwall's inequality to the resulting inequality, we find that

$$\|\chi^\varepsilon\|^2 \le C\|W_\varepsilon\|^2_{L^\infty([0,T];L^2(\tilde{\Omega}))}. \tag{36}$$

Integrating (35) over $[0, T]$ using (36), we have

$$\int_0^T \|\nabla \chi^\varepsilon\|^2 d\tau \le C\|W_\varepsilon\|^2_{L^\infty([0,T];L^2(\tilde{\Omega}))}. \tag{37}$$

Combining (36) and (37), we obtain

$$\|\chi^\varepsilon\|^2_{L^\infty([0,T];L^2(\tilde{\Omega}))} + \|\chi^\varepsilon\|^2_{L^2([0,T];H_0^1(\tilde{\Omega}))} \le C\|W_\varepsilon\|^2_{L^\infty([0,T];L^2(\tilde{\Omega}))}; \tag{38}$$

i.e.,

$$\|\mathbf{u}_1^\varepsilon - \mathbf{u}_2^\varepsilon\|_B^2 \le C\|\mathbf{v}_1 - \mathbf{v}_2\|_B^2,$$

where $\|\cdot\|_B^2 = \|\cdot\|^2_{L^\infty([0,T];L^2(\tilde{\Omega}))} + \|\cdot\|^2_{L^2([0,T];H_0^1(\tilde{\Omega}))}$, which implies that F_ε is continuous.

Therefore, the Schauder theorem implies that for any fixed ε, there exists $\mathbf{u}^\varepsilon \in B$ such that $F_\varepsilon(\mathbf{u}^\varepsilon) = \mathbf{u}^\varepsilon$. Thus, $\mathbf{u}^\varepsilon$ satisfies the following equations:

$$\begin{cases} \vartheta_t^\varepsilon + \mathbf{u}_\varepsilon \cdot \nabla \vartheta^\varepsilon = 0, \\ \mathbf{u}_t^\varepsilon + \mathbf{u}_\varepsilon \cdot \nabla \mathbf{u}^\varepsilon = \nu \Delta \mathbf{u}^\varepsilon - \nabla p^\varepsilon + \vartheta^\varepsilon g(r)\mathbf{e_r}, \\ \nabla \cdot \mathbf{u}^\varepsilon = 0, \\ \mathbf{u}^\varepsilon \mid_{\partial\Omega} = 0, \ \ \mathbf{u}^\varepsilon(0) = \mathbf{u}_0^\varepsilon, \ \ \vartheta^\varepsilon(0) = \vartheta_0^\varepsilon, \end{cases} \tag{39}$$

where $\mathbf{u}_\varepsilon$ is the regularization of $\mathbf{u}^\varepsilon$. We conclude that $(\mathbf{u}^\varepsilon, \vartheta^\varepsilon)$ is called the approximate solution.

Next, we verify that the approximate solution $(\mathbf{u}^\varepsilon, \vartheta^\varepsilon)$ converges to the weak solution $(\mathbf{u}, \vartheta)$, satisfying Equation (2). It is obvious that $(\mathbf{u}^\varepsilon, \vartheta^\varepsilon)$ satisfy the integral identities (5): i.e.,

$$\int_{\tilde{\Omega}} \vartheta_0^\varepsilon \Psi(0, x) dx + \int_0^T \int_{\tilde{\Omega}} (\vartheta^\varepsilon \Psi_t + \vartheta^\varepsilon \mathbf{u}_\varepsilon \cdot \nabla \Psi) dx dt = 0, \tag{40}$$

$$\begin{aligned} \int_{\tilde{\Omega}} \mathbf{u}_0^\varepsilon \mathbf{\Phi}(0, x) dx + \int_0^T \int_{\tilde{\Omega}} (\mathbf{u}^\varepsilon \mathbf{\Phi}_t + \mathbf{u}_\varepsilon \cdot (\mathbf{u}^\varepsilon \cdot \nabla \mathbf{\Phi}) - \nu \nabla \mathbf{u}^\varepsilon : \nabla \mathbf{\Phi} \\ + \vartheta^\varepsilon g(r)\mathbf{e_r} \cdot \mathbf{\Phi}) dx dt = 0 \end{aligned} \tag{41}$$

for each fixed $0 < \varepsilon < 1$, for any test function $\mathbf{\Phi} \in C_0^\infty(\Omega \times [0, T])^2$ satisfying $\mathbf{\Phi}(T, x) = 0$ and $\nabla \cdot \mathbf{\Phi} = 0$, and for any $\Psi \in C^\infty(\Omega \times [0, T])$ satisfying $\Psi(T, x) = 0$.

With the aid of (18), from (21) and from the definition of $\mathbf{u}_\varepsilon$, we know that the sequence $(\mathbf{u}^\varepsilon, \vartheta^\varepsilon)$ converges up to the extraction of subsequences. To some, $(\mathbf{u}, \vartheta)$ in the obvious weak sequence—that is,

$$\begin{aligned} &\mathbf{u}^\varepsilon \rightharpoonup^* \mathbf{u} \ \text{ in } L^\infty([0, T); L^2(\tilde{\Omega})^2), \\ &\mathbf{u}_\varepsilon \rightharpoonup^* \mathbf{u} \ \text{ in } L^\infty([0, T); L^2(\tilde{\Omega})^2), \\ &\mathbf{u}^\varepsilon \rightharpoonup \mathbf{u} \ \text{ in } L^2([0, T); H_0^1(\tilde{\Omega})^2), \\ &\mathbf{u}_\varepsilon \rightharpoonup \mathbf{u} \ \text{ in } L^2([0, T); H_0^1(\tilde{\Omega})^2), \\ &\vartheta^\varepsilon \rightharpoonup^* \vartheta \ \text{ in } L^\infty([0, T); L^q(\tilde{\Omega})), \ \forall 1 \le q \le \infty. \end{aligned} \tag{42}$$

By the same method in [12,14], we can prove that

$$\begin{aligned} &\mathbf{u}^\varepsilon \to \mathbf{u} \ \text{ in } L^2([0, T); L^2(\tilde{\Omega})^2), \\ &\mathbf{u}_\varepsilon \to \mathbf{u} \ \text{ in } L^2([0, T); L^2(\tilde{\Omega})^2). \end{aligned} \tag{43}$$

Therefore, we can easily deduce that the limit $(\mathbf{u}, \vartheta)$ is a weak solution of the original Equations (2)–(4) and satisfies the following regularity estimates:

$$\begin{aligned} &\|\vartheta\|_{L^q} \le \|\vartheta_0\|_{L^q}, \ \ \forall 1 \le q \le \infty, \\ &\|\mathbf{u}\|^2 \le e^{\|g(r)\|_{L^\infty(\tilde{\Omega})} T}[\|\mathbf{u}_0\|^2 + \|g(r)\|_{L^\infty}\|\vartheta_0\|^2 T] \le C, \\ &\int_0^T \|\nabla \mathbf{u}\|^2 d\tau \le C. \end{aligned} \tag{44}$$

We conclude the argument by noticing that T is arbitrary; thus, we have finished the proof of Proposition 1. □

3.1.2. Global Existence of Weak Solution on Unbounded Domain

Since the estimate (12) is independent of R, the remaining case of Theorem 1 can be proved by means of a standard domain expansion technique.

Proposition 2. *Assume* $\mathbf{u}_0 \in H_0^1(\Omega)^2, \vartheta_0 \in L^1(\Omega) \cap L^\infty(\Omega)$. *If* $g(r) \in L^\infty(\tilde{\Omega})$, *then there exists a global weak solution* $(\mathbf{u}, \vartheta)$ *of problems (2)–(4) such that for any* $T > 0$,

$$\mathbf{u} \in L^\infty([0,T); L^2(\Omega)^2) \cap L^2([0,T); H_0^1(\Omega)^2),$$

$$\vartheta \in L^\infty([0,T); L^1(\Omega) \cap L^\infty(\Omega)).$$

Proof. Setting $\Omega_R = \Omega \cap B_R$, let $\tilde{\mathbf{u}}_0^R \in H_0^1(\Omega_R)^2$ satisfy

$$-\Delta \tilde{\mathbf{u}}_0^R + \nabla p_0^R = -\Delta \mathbf{u}_0, \ \text{ in } \Omega_R. \tag{45}$$

Extending $\tilde{\mathbf{u}}_0^R$ to $\mathbb{R}^2$ by defining 0 outside Ω_R, we have (see [14], Appendix A)

$$\tilde{\mathbf{u}}_0^R \to \mathbf{u}_0 \ \text{ in } \ H_0^1(\Omega)^2, \text{ as } R \to \infty.$$

For $\mathbf{u}_0^R = \tilde{\mathbf{u}}_0^R * \eta_{R^{-1}}$, it holds that $\mathbf{u}_0^R \in C_0^\infty(\Omega_R)^2 \cap C_0^\infty(\Omega)^2$ and that

$$\mathbf{u}_0^R \to \mathbf{u}_0 \ \text{ in } \ H_0^1(\Omega)^2, \text{ as } R \to \infty.$$

Similarly, we choose $\vartheta_0^R \in C_0^\infty(\Omega_R) \cap C_0^\infty(\Omega)$ satisfying

$$\vartheta_0^R \to \vartheta_0 \ \text{ in } \ L^1(\Omega) \cap L^\infty(\Omega), \text{ as } R \to \infty.$$

Hence, by virtue of Proposition 1, the initial boundary value problems (2)–(4) with the initial data $(\mathbf{u}_0^R, \vartheta_0^R)$ have a weak solution $(\mathbf{u}^R, \vartheta^R)$ on $\Omega_R \times [0, \infty)$. Moreover, $(\mathbf{u}^R, \vartheta^R)$ satisfies the estimates obtained in Proposition 1; that is,

$$\begin{aligned} &\|\vartheta^R\|_{L^q} \le \|\vartheta_0^R\|_{L^q}, \ \ \forall 1 \le q < \infty, \\ &\|\mathbf{u}^R\|^2 \le e^{\|g(r)\|_{L^\infty(\tilde{\Omega})} T}[\|\mathbf{u}_0^R\|^2 + \|g(r)\|_{L^\infty}\|\vartheta_0^R\|^2 T] \le C, \\ &\int_0^T \|\nabla \mathbf{u}^R\|^2 d\tau \le C, \end{aligned} \tag{46}$$

where C is a constant independent of R. Thus, extending $(\mathbf{u}^R, \vartheta^R)$ by zero on $\mathbb{R}^2 \setminus \Omega_R$, we find that the sequence $(\mathbf{u}^R, \vartheta^R)$ converges up to the extraction of subsequences to some limit $(\mathbf{u}, \vartheta)$ in the obvious weak sense—that is , as $R \to \infty$, we have

$$\begin{aligned}&\mathbf{u}^R \rightharpoonup \mathbf{u} \text{ weakly star in } L^\infty([0,T);L^2(\Omega)^2),\\&\nabla\mathbf{u}^R \rightharpoonup \nabla\mathbf{u} \text{ weakly in } L^2([0,T);L^2(\Omega)^2),\\&\vartheta^R \rightharpoonup \vartheta \text{ weakly star in } L^1([0,T);L^1(\Omega)\cap L^\infty([0,T);L^1(\Omega)).\end{aligned} \tag{47}$$

Moreover, by (46), the limit $(\mathbf{u},\vartheta)$ also satisfies the estimate (12). Finally, letting $R\to\infty$, we can easily show that $(\mathbf{u},\vartheta)$ is a weak solution to the original Equations (2)–(4). The proof of the existence part of Theorem 1 is finished. □

3.2. Proof of Theorem 2

In this subsection, we shall prove the regularity and uniqueness results of the solution obtained in Theorem 1. To show Theorem 2, whose proof will be postponed to the end of this section, we begin with the following standard energy estimate for $(\mathbf{u},\vartheta)$, which are stated as a sequence of lemmas.

Lemma 4. *For initial data $(\mathbf{u}_0,\vartheta_0)$ satisfying the assumptions of Theorem 2, it holds that*

$$\sup_{0\le t\le T}(\|\mathbf{u}\|^2+\|\vartheta\|_{L^1\cap L^\infty})+\int_0^T\|\nabla\mathbf{u}\|^2d\tau\le C, \tag{48}$$

where (and in what follows) C denotes a generic positive constant depending only on $\|g(r)\|_{L^\infty}$, ν, $\|\mathbf{u}_0\|$, $\|\vartheta_0\|$, and the terminal time T.

Proof. 1. Define particle path

$$\begin{cases}\dfrac{d}{dt}X(x,t)=\mathbf{u}(X(x,t),t),\\ X(x,0)=x.\end{cases} \tag{49}$$

Thus, along the particle path, we obtain from $(2)_2$ that

$$\frac{d}{dt}\vartheta(X(x,t),t)=0,$$

which implies

$$\vartheta(X(x,t),t)=\vartheta_0. \tag{50}$$

2. Multiplying $(2)_1$ by $\mathbf{u}$ and then integrating the resulting equation over Ω, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|^2+\nu\|\nabla\mathbf{u}\|^2&=\int_\Omega\vartheta g(r)\mathbf{e_r}\mathbf{u}dx\\&\le\|g(r)\|_{L^\infty(\Omega)}\int_\Omega\vartheta\mathbf{e_r}\mathbf{u}dx\\&\le\|g(r)\|_{L^\infty(\Omega)}\left[\frac{1}{2}\|\vartheta\|^2+\frac{1}{2}\|\mathbf{u}\|^2\right].\end{aligned} \tag{51}$$

Thus, Gronwall's inequality leads to

$$\sup_{0\le t\le T}\|\mathbf{u}\|^2+\int_0^T\|\nabla\mathbf{u}\|^2d\tau\le C, \tag{52}$$

which, together with (50), yields (48) and completes the proof of Lemma 4. □

Lemma 5. *For initial data* $(\mathbf{u}_0, \vartheta_0)$ *satisfying the assumptions of Theorem 2, it holds that*

$$\sup_{0\le t\le T} \|\nabla \mathbf{u}\|^2 + \int_0^T [\|\mathbf{u}_t\|^2 + \|\nabla^2 \mathbf{u}\|^2] d\tau \le C. \tag{53}$$

Proof. Multiplying $(2)_1$ by $\mathbf{u}_t$ and then integrating the resulting equation over Ω, by applying the Cauchy–Schwarz inequality, we have

$$\begin{aligned} \frac{\nu}{2}\frac{d}{dt}\|\nabla \mathbf{u}\|^2 + \|\mathbf{u}_t\|^2 &= -\int_\Omega \mathbf{u}\cdot\nabla\mathbf{u}\cdot\mathbf{u}_t dx + \int_\Omega \vartheta g(r)\mathbf{e_r}\cdot\mathbf{u}_t dx \\ &\le \frac{1}{2}\|\mathbf{u}_t\|^2 + \int_\Omega |\mathbf{u}|^2|\nabla\mathbf{u}|^2 dx + \int_\Omega |\vartheta|^2 |g(r)|^2 dx. \end{aligned} \tag{54}$$

The Hölder, Cauchy–Schwarz, and Gagliardo-Nirenberg inequalities (see [17]), together with (52), yield

$$\begin{aligned} \int_\Omega |\mathbf{u}|^2|\nabla\mathbf{u}|^2 dx &\le \|\mathbf{u}\|_{L^8}^2 \|\nabla\mathbf{u}\|_{L^{\frac{8}{3}}}^2 \\ &\le C\|\mathbf{u}\|^{\frac{1}{2}}\|\nabla\mathbf{u}\|^3\|\nabla\mathbf{u}\|_{H^1}^{\frac{1}{2}} \\ &\le \delta\|\mathbf{u}\|_{H^2}^2 + C(\delta)\|\nabla\mathbf{u}\|^4. \end{aligned} \tag{55}$$

For the third term on the right-hand side of (54), we obtain from (48) that

$$\int_\Omega |\vartheta|^2 g(r)^2 dx \le C. \tag{56}$$

Inserting (55) and (56) into (54) gives rise to

$$\frac{\nu}{2}\frac{d}{dt}\|\nabla\mathbf{u}\|^2 + \frac{1}{2}\|\mathbf{u}_t\|^2 \le \delta\|\mathbf{u}\|_{H^2}^2 + C(\delta)\|\nabla\mathbf{u}\|^4 + C. \tag{57}$$

To estimate the first term on the right-hand side of (57), we rewrite Equation $(2)_1$ as

$$-\nu\Delta\mathbf{u} + \nabla p = -\mathbf{u}_t - \mathbf{u}\cdot\nabla\mathbf{u} + \vartheta g(r)\mathbf{e_r}, \tag{58}$$

and applying the standard L^p-estimate to (58) (see Lemma 1) yields that for any $r \in (1,\infty)$,

$$\|\nabla^2\mathbf{u}\|_{L^r} + \|\nabla p\|_{L^r} \le C(\|\mathbf{u}_t\|_{L^r} + \|\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^r} + \|\vartheta g(r)\mathbf{e_r}\|_{L^r}), \tag{59}$$

where C is a generic constant independent of R. Then, it follows from (59), (48) and the Gagliardo–Nirenberg inequality that

$$\begin{aligned} \|\nabla^2\mathbf{u}\|_{L^2}^2 + \|\nabla p\|_{L^2}^2 &\le C(\|\mathbf{u}_t\|_{L^2}^2 + \|\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^2}^2 + \|\vartheta g(r)\mathbf{e_r}\|_{L^2}^2) \\ &\le C\|\mathbf{u}_t\|_{L^2}^2 + C\|\mathbf{u}\|_{L^4}^2\|\nabla\mathbf{u}\|_{L^4}^2 + C\|g(r)\|_{L^\infty}^2\|\vartheta\|_{L^2}^2 \\ &\le C\|\mathbf{u}_t\|_{L^2}^2 + C\|\mathbf{u}\|_{L^2}\|\nabla\mathbf{u}\|_{L^2}^2\|\nabla^2\mathbf{u}\|_{L^2} + C \\ &\le C\|\mathbf{u}_t\|_{L^2}^2 + \frac{1}{2}\|\nabla^2\mathbf{u}\|_{L^2}^2 + C\|\nabla\mathbf{u}\|_{L^2}^4 + C. \end{aligned} \tag{60}$$

Consequently, substituting (60) into (57) and choosing a δ suitably small, one has

$$\frac{\nu}{2}\frac{d}{dt}\|\nabla\mathbf{u}\|^2 + \frac{1}{4}\|\mathbf{u}_t\|^2 + \varepsilon\|\nabla^2\mathbf{u}\|^2 \le C\|\nabla\mathbf{u}\|^4 + C. \tag{61}$$

By dropping $\frac{1}{4}\|\mathbf{u}_t\|^2, \varepsilon\|\nabla^2\mathbf{u}\|^2$ from (61) and using Gronwall's inequality, we obtain

$$\|\nabla\mathbf{u}\|^2 \le e^{\frac{2C}{\nu}\int_0^T \|\nabla\mathbf{u}\|^2 d\tau}\Big[\|\nabla\mathbf{u}_0\|^2 + \frac{2C}{\nu}T\Big]. \tag{62}$$

Substituting this into (61) and integrating over $(0, T)$, we conclude that

$$\int_0^T [\|\mathbf{u}_t\|^2 + \|\nabla^2\mathbf{u}\|^2]d\tau \le C, \tag{63}$$

where we have used Lemma 4. This completes the proof of Lemma 5. □

Lemma 6. *For initial data* $(\mathbf{u}_0, \vartheta_0)$ *satisfying the assumptions of Theorem 2, it holds that*

$$\sup_{0\le t\le T}\|\mathbf{u}_t\|^2 \le C, \quad \int_0^T \|\nabla\mathbf{u}_t\|^2 d\tau \le C. \tag{64}$$

Proof. We now take the temporal derivative of $(2)_1$ to obtain

$$\mathbf{u}_{tt} + \mathbf{u}_t\cdot\nabla\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}_t = \nu\Delta\mathbf{u}_t - \nabla p_t + \vartheta_t g(r)\mathbf{e_r}. \tag{65}$$

Multiplying (65) by $\mathbf{u}_t$ and integrating the resulting equality by parts over Ω, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\mathbf{u}_t\|^2 + \nu\|\nabla\mathbf{u}_t\|^2 &= -\int_\Omega \mathbf{u}_t\cdot\nabla\mathbf{u}\cdot\mathbf{u}_t dx + \int_\Omega \vartheta_t g(r)\mathbf{e_r}\cdot\mathbf{u}_t dx\\ &\le \int_\Omega |\mathbf{u}_t|^2|\nabla\mathbf{u}|dx - |g(r)|_{L^\infty}\int_\Omega \mathbf{u}\cdot\nabla\vartheta u_{2t}dx\\ &\le \|\mathbf{u}_t\|_{L^4}^2\|\nabla\mathbf{u}\| + |g(r)|_{L^\infty}\int_\Omega \vartheta\mathbf{u}\cdot\nabla u_{2t}dx.\end{aligned} \tag{66}$$

With the help of Lemma 5 and the Gagliardo–Nirenberg inequality, we note that

$$\begin{aligned}\|\mathbf{u}_t\|_{L^4}^2\|\nabla\mathbf{u}\| &\le \|\mathbf{u}_t\|_{L^4}^2\\ &\le \|\nabla\mathbf{u}_t\|\|\mathbf{u}_t\|\\ &\le \frac{\nu}{4}\|\nabla\mathbf{u}_t\|^2 + C\|\mathbf{u}_t\|^2.\end{aligned} \tag{67}$$

On the other hand, with Lemma 4, we have

$$\begin{aligned}|g(r)|_{L^\infty}\int_{\tilde\Omega}\vartheta\mathbf{u}\cdot\nabla u_{2t}dx &\le |g(r)|_{L^\infty}\|\vartheta\|_{L^\infty}\|\mathbf{u}\|\|\nabla\mathbf{u}_t\|\\ &\le \frac{\nu}{4}\|\nabla\mathbf{u}_t\|^2 + C.\end{aligned} \tag{68}$$

Substituting (67), (68) into (66), we obtain

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}_t\|^2 + \frac{\nu}{2}\|\nabla\mathbf{u}_t\|^2 \le C\|\mathbf{u}_t\|^2 + C. \tag{69}$$

Using Gronwall's inequality, we obtain

$$\|\mathbf{u}_t\|^2 \le e^{2CT}\big[\|\mathbf{u}_t(0)\|^2 + 2CT\big]. \tag{70}$$

Next, we should estimate $\|\mathbf{u}_t(0)\|^2$. In fact, different from (54), we also have

$$\begin{aligned}\|\mathbf{u}_t\|^2 &= \nu\int_{\tilde{\Omega}}\Delta\mathbf{u}\cdot\mathbf{u}_t dx - \int_{\tilde{\Omega}}\mathbf{u}\cdot\nabla\mathbf{u}\cdot\mathbf{u}_t dx + \int_{\tilde{\Omega}}\vartheta g(r)\mathbf{e}_\mathbf{r}\cdot\mathbf{u}_t dx\\ &\le \frac{3}{4}\|\mathbf{u}_t\|^2 + C\|\Delta\mathbf{u}\|^2 + C\|\mathbf{u}\cdot\nabla\mathbf{u}\|^2 + C\|\vartheta\|^2\\ &\le \frac{3}{4}\|\mathbf{u}_t\|^2 + C\|\Delta\mathbf{u}\|^2 + C\|\mathbf{u}\|_{L^4}^2\|\nabla\mathbf{u}\|_{L^4}^2 + C\|\vartheta_0\|^2\\ &\le \frac{3}{4}\|\mathbf{u}_t\|^2 + C\|\Delta\mathbf{u}\|^2 + C\|\mathbf{u}\|\|\nabla\mathbf{u}\|^2\|\nabla^2\mathbf{u}\| + C\|\vartheta_0\|^2.\end{aligned} \tag{71}$$

Taking $t\to 0^+$ in the above inequality, and using Lemma 4 and Lemma 5, we obtain

$$\|\mathbf{u}_t(0)\|^2 \le C\|\mathbf{u}(0)\|_{H^2}^2 + C. \tag{72}$$

Now, substituting the result into (70) and integrating (69) over $(0,T)$ give

$$\|\mathbf{u}_t\|^2 \le C,\quad \int_0^T\|\nabla\mathbf{u}_t\|^2 d\tau \le C, \tag{73}$$

where we have used Lemma 5. The proof of Lemma 6 is finished. □

Lemma 7. *For initial data $(\mathbf{u}_0,\vartheta_0)$ satisfying the assumptions of Theorem 2, it holds that*

$$\sup_{0\le t\le T}\|\mathbf{u}\|_{L^\infty}^2 \le C,\quad \int_0^T\|\nabla\mathbf{u}\|_{L^\infty}^2 d\tau \le C. \tag{74}$$

Proof. By Lemmas 5 and 6, we obtain from (60) that

$$\|\mathbf{u}\|_{H^2}^2 \le C(\|\mathbf{u}_t\|^2 + \|\nabla\mathbf{u}\|^4) + C \le C, \tag{75}$$

which implies, by Sobolev embedding, that

$$\|\mathbf{u}\|_{L^\infty}^2 \le C. \tag{76}$$

As an immediate consequence of (75) and (76) and the Gagliardo–Nirenberg inequality, we see that

$$\begin{aligned}\|\mathbf{u}\cdot\nabla\mathbf{u}\|_{H^1}^2 &= \int_{\tilde{\Omega}}(|\mathbf{u}|^2|\nabla\mathbf{u}|^2 + |\nabla\mathbf{u}|^4 + |\mathbf{u}|^2|\nabla^2\mathbf{u}|^2)dx\\ &\le \|\mathbf{u}\|_{L^\infty}^2\|\nabla\mathbf{u}\|^2 + \|\nabla\mathbf{u}\|_{L^4}^4 + \|\mathbf{u}\|_{L^\infty}^2\|\nabla^2\mathbf{u}\|^2\\ &\le \|\mathbf{u}\|_{L^\infty}^2\|\nabla\mathbf{u}\|^2 + \|\nabla\mathbf{u}\|^2\|\nabla^2\mathbf{u}\|^2 + \|\mathbf{u}\|_{L^\infty}^2\|\nabla^2\mathbf{u}\|^2\\ &\le (\|\mathbf{u}\|_{L^\infty}^2 + \|\nabla\mathbf{u}\|^2)\|\nabla^2\mathbf{u}\|^2 \le C,\end{aligned} \tag{77}$$

which implies, by Sobolev embedding, that

$$\|\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^q}^2 \le C,\ \ \forall 2\le q<\infty. \tag{78}$$

On the other hand, using the same Sobolev embedding, we known from (73) that

$$\int_0^T\|\mathbf{u}_t\|_{L^q}^2 d\tau \le C,\ \ \forall 2\le q<\infty. \tag{79}$$

Therefore, using (59) and (78)–(79), we obtain

$$\begin{aligned}\int_0^T \|\mathbf{u}\|^2_{W^{2,q}} d\tau &\le C\int_0^T (\|\mathbf{u}_t\|^2_{L^q} + \|\mathbf{u}\cdot\nabla\mathbf{u}\|^2_{L^q} + \|\vartheta g(r)\mathbf{e}_r\|^2_{L^q}) d\tau \\ &\le C.\end{aligned} \tag{80}$$

Thus, we have

$$\int_0^T \|\nabla\mathbf{u}\|^2_{L^\infty} d\tau \le C. \tag{81}$$

□

Lemma 8. *For initial data* $(\mathbf{u}_0, \vartheta_0)$ *satisfying the assumptions of Theorem 2, it holds that*

$$\sup_{0\le t\le T} \|\nabla\vartheta\|_{L^\infty} \le C. \tag{82}$$

Proof. Operating ∇ to $(2)_2$ and then multiplying $|\nabla\vartheta|^{q-2}\nabla\vartheta$ for $q \in [2,\infty)$ gives the following:

$$\begin{aligned}\frac{1}{q}\frac{d}{dt}\|\nabla\vartheta\|^q_{L^q} &= \int_{\tilde{\Omega}} \nabla(\mathbf{u}\cdot\nabla\vartheta)\cdot|\nabla\vartheta|^{q-2}\nabla\vartheta dx \\ &\le \|\nabla\mathbf{u}\|_{L^\infty}\|\nabla\vartheta\|^q_{L^q},\end{aligned} \tag{83}$$

which, along with Gronwall's inequality, leads to

$$\|\nabla\vartheta\|_{L^q} \le \|\nabla\vartheta_0\|_{L^q} e^{\int_0^T \|\nabla\mathbf{u}\|_{L^\infty} d\tau} \le C. \tag{84}$$

Letting $q \to \infty$, we obtain (82). This completes the proof of Lemma 8. □

Lemma 9. *For initial data* $(\mathbf{u}_0, \vartheta_0)$ *satisfying the assumptions of Theorem 2, it holds that*

$$\sup_{0\le t\le T} \|\nabla\mathbf{u}_t\|^2 \le C, \quad \int_0^T \|\mathbf{u}_{tt}\|^2 d\tau \le C. \tag{85}$$

Proof. Multiplying (65) by $\mathbf{u}_{tt}$ and integrating the resulting equality by parts over Ω, we obtain

$$\begin{aligned}&\|\mathbf{u}_{tt}\|^2 + \frac{\nu}{2}\frac{d}{dt}\|\nabla\mathbf{u}_t\|^2 \\ &= -\int_\Omega \mathbf{u}_t\cdot\nabla\mathbf{u}\cdot\mathbf{u}_{tt}dx + \int_\Omega \mathbf{u}\cdot\nabla\mathbf{u}_t\cdot\mathbf{u}_{tt}dx + \int_\Omega \vartheta_t g(r)\mathbf{e}_r\cdot\mathbf{u}_{tt}dx \\ &\le \int_\Omega |\mathbf{u}_t||\nabla\mathbf{u}||\mathbf{u}_{tt}|dx + \int_\Omega |\mathbf{u}||\nabla\mathbf{u}_t||\mathbf{u}_{tt}|dx + \int_\Omega \vartheta_t g(r)u_{2tt}dx \\ &= I_1 + I_2 + I_3.\end{aligned} \tag{86}$$

We estimate each term on the right-hand side of (86) as follows:

First, it follows from Young's inequality and Lemma 6 that

$$\begin{aligned} I_1 &= \int_{\tilde{\Omega}} |\mathbf{u}_t||\nabla \mathbf{u}||\mathbf{u}_{tt}|dx \\ &\leq \frac{1}{6}\|\mathbf{u}_{tt}\|^2 + C\int_{\tilde{\Omega}} |\mathbf{u}_t|^2|\nabla \mathbf{u}|^2 dx \\ &\leq \frac{1}{6}\|\mathbf{u}_{tt}\|^2 + C\|\nabla \mathbf{u}\|_{L^\infty}^2\|\mathbf{u}_t\|^2 \\ &\leq \frac{1}{6}\|\mathbf{u}_{tt}\|^2 + C\|\nabla \mathbf{u}\|_{L^\infty}^2. \end{aligned} \tag{87}$$

Then, Young's inequality combined with Lemma 7 leads to

$$\begin{aligned} I_2 &= \int_{\tilde{\Omega}} |\mathbf{u}||\nabla \mathbf{u}_t||\mathbf{u}_{tt}|dx \\ &\leq \frac{1}{6}\|\mathbf{u}_{tt}\|^2 + C\int_{\tilde{\Omega}} |\mathbf{u}|^2|\nabla \mathbf{u}_t|^2 dx \\ &\leq \frac{1}{6}\|\mathbf{u}_{tt}\|^2 + C\|\mathbf{u}\|_{L^\infty}^2\|\nabla \mathbf{u}_t\|^2 \\ &\leq \frac{1}{6}\|\mathbf{u}_{tt}\|^2 + C\|\nabla \mathbf{u}_t\|^2. \end{aligned} \tag{88}$$

Similarly, using Young's inequality, (2) and Lemmas 4 and 8 indicates that

$$\begin{aligned} I_3 &= \int_{\tilde{\Omega}} \vartheta_t u_{2tt} dx \\ &\leq \frac{1}{6}\|\mathbf{u}_{tt}\|^2 + C\int_{\tilde{\Omega}} |\vartheta_t|^2 dx \\ &\leq \frac{1}{6}\|\mathbf{u}_{tt}\|^2 + C\int_{\tilde{\Omega}} |\mathbf{u}\cdot\nabla\vartheta|^2 dx \\ &\leq \frac{1}{6}\|\mathbf{u}_{tt}\|^2 + C\|\nabla\vartheta\|_{L^\infty}^2\|\mathbf{u}_t\|^2 \\ &\leq \frac{1}{6}\|\mathbf{u}_{tt}\|^2 + C. \end{aligned} \tag{89}$$

Substituting (87)–(89) into (86), we obtain

$$\frac{1}{2}\|\mathbf{u}_{tt}\|^2 + \frac{\nu}{2}\frac{d}{dt}\|\nabla \mathbf{u}_t\|^2 \leq C\|\nabla \mathbf{u}_t\|^2 + C\|\nabla \mathbf{u}\|_{L^\infty}^2 + C. \tag{90}$$

We note that all the terms on the right-hand side of (90) are integrable in time due to Lemmas 6 and 7. Therefore, we integrate (90) in time over $[0, T]$ and after using the compatibility conditions (7) to obtain the estimates in (85). This completes the proof of Lemma 9. □

Lemma 10. *For initial data* $(\mathbf{u}_0, \vartheta_0)$ *satisfying the assumptions of Theorem 2, it holds that*

$$\sup_{0\leq t\leq T}(\|\mathbf{u}\|_{H^3}^2 + \|\vartheta\|_{H^3}^2) \leq C, \quad \int_0^T \|\mathbf{u}\|_{H^4}^2 d\tau \leq C. \tag{91}$$

Proof. We deduce from (70), (77), (84), (85), and Lemma 1 that

$$\|\mathbf{u}\|_{H^3}^2 \leq C(\|\mathbf{u}_t\|_{H^1}^2 + \|\mathbf{u}\cdot\nabla\mathbf{u}\|_{H^1}^2 + \|\vartheta g(r)\mathbf{e}_\mathbf{r}\|_{H^1}^2) \leq C. \tag{92}$$

Thus, by the Sobolev inequality, we have

$$\|\mathbf{u}\|_{W^{2,r}}^2 \leq C\|\mathbf{u}\|_{H^3}^2 \leq C. \tag{93}$$

Then,

$$\|\nabla \mathbf{u}\|_{L^\infty}^2 \le C. \tag{94}$$

Now, it is clear that one needs a higher-order estimate on ϑ to complete the proof of this lemma. For this purpose, taking ∂_{xx} of $(2)_2$, we have

$$\vartheta_{xxt} + u_{xx}\partial_x\vartheta + 2u_x\partial_{xx}\vartheta + v_{xx}\partial_y\vartheta + 2v_x\partial_{xy}\vartheta + \mathbf{u}\cdot\nabla\vartheta_{xx} = 0. \tag{95}$$

For any $q \ge 2$, multiplying (95) by $|\partial_{xx}\vartheta|^{q-2}\partial_{xx}\vartheta$, integrating over Ω, and using Hölder's inequality, we obtain

$$\begin{aligned}\frac{1}{q}\frac{d}{dt}\int_\Omega |\vartheta_{xx}|^q dx &= -\int_\Omega \left[u_{xx}\vartheta_x + 2u_x\vartheta_{xx} + v_{xx}\vartheta_y + 2v_x\vartheta_{xy}\right]|\vartheta|_{xx}^{q-2}\vartheta_{xx}dx \\ &\le \|\nabla\vartheta\|_{L^\infty}\|\nabla^2\mathbf{u}\|_{L^q}\|\nabla^2\vartheta\|_{L^q}^{q-1} + 2\|\nabla\mathbf{u}\|_{L^\infty}\|\nabla^2\vartheta\|_{L^q}^q \\ &\le C(\|\nabla^2\vartheta\|_{L^q}^{q-1} + \|\nabla^2\vartheta\|_{L^q}^q),\end{aligned} \tag{96}$$

where (82), (93), and (94) are used. Similarly, one can show

$$\begin{aligned}\frac{1}{q}\frac{d}{dt}\int_{\tilde\Omega} |\vartheta_{xy}|^q dx \le C(\|\nabla^2\vartheta\|_{L^q}^{q-1} + \|\nabla^2\vartheta\|_{L^q}^q), \\ \frac{1}{q}\frac{d}{dt}\int_{\tilde\Omega} |\vartheta_{yy}|^q dx \le C(\|\nabla^2\vartheta\|_{L^q}^{q-1} + \|\nabla^2\vartheta\|_{L^q}^q).\end{aligned} \tag{97}$$

It follows that

$$\frac{d}{dt}\|\nabla^2\vartheta\|_{L^q} \le C(1 + \|\nabla^2\vartheta\|_{L^q}). \tag{98}$$

Applying Gronwall inequality to (75), one has

$$\|\nabla^2\vartheta\|_{L^q} \le e^{CT}(\|\nabla^2\vartheta_0\|_{L^q} + CT) \le C. \tag{99}$$

In a quite similar manner as in the derivation of (76), further estimates show that

$$\|\vartheta\|_{H^3}^2 \le C. \tag{100}$$

Furthermore, by (94) and (84) and Lemmas 6, 7 and 9, we have

$$\begin{aligned}\|\mathbf{u}_t\cdot\nabla\mathbf{u}\|^2 &= \int_{\tilde\Omega}|\mathbf{u}_t|^2|\nabla\mathbf{u}|^2dx \le \|\nabla\mathbf{u}\|_{L^\infty}^2\|\mathbf{u}_t\|^2 \le C, \\ \|\mathbf{u}\cdot\nabla\mathbf{u}_t\|^2 &= \int_{\tilde\Omega}|\mathbf{u}|^2|\nabla\mathbf{u}_t|^2dx \le \|\mathbf{u}\|_{L^\infty}^2\|\nabla\mathbf{u}_t\|^2 \le C, \\ \|\vartheta_t g(r)\mathbf{e}_r\|^2 &\le C\int_{\tilde\Omega}|\vartheta_t|^2dx \le C\int_{\tilde\Omega}|\mathbf{u}\cdot\nabla\vartheta|^2dx \le C,\end{aligned} \tag{101}$$

which, together with (85) and (101), gives

$$\begin{aligned}\int_0^T \|\mathbf{u}_t\|_{H^2}^2 d\tau &\le C\int_0^T [\|\mathbf{u}_{tt}\|^2 + \|\mathbf{u}_t\cdot\nabla\mathbf{u}\|^2 + \|\mathbf{u}\cdot\nabla\mathbf{u}_t\|^2 + \|\vartheta_t g(r)\mathbf{e}_r\|^2]d\tau \\ &\le C.\end{aligned} \tag{102}$$

In addition, the Sobolev inequality and (92) yield

$$\|\mathbf{u}\cdot\nabla\mathbf{u}\|_{H^2}^2 \le C. \tag{103}$$

Thus, we have

$$\int_0^T \|\mathbf{u}\|_{H^4}^2 d\tau \le C, \tag{104}$$

which completes the proof of Lemma 10. □

It remains only to prove the uniqueness of the strong solutions.

Proposition 3. *Let* $(\mathbf{u}_1, \vartheta_1, p_1)$*, and* $(\mathbf{u}_2, \vartheta_2, p_2)$ *be two solutions satisfying (2)–(4); then,*

$$\mathbf{u}_1 = \mathbf{u}_2, \ \ \vartheta_1 = \vartheta_2 \ \text{ and } \ p_1 = p_2. \tag{105}$$

Proof. Denote

$$\tilde{\mathbf{u}} = \mathbf{u}_1 - \mathbf{u}_2, \ \ \tilde{p} = p_1 - p_2, \ \ \tilde{\vartheta} = \vartheta_1 - \vartheta_2.$$

First, subtracting Equation $(2)_1$ satisfied by $(\mathbf{u}_1, \vartheta_1, p_1)$ and $(\mathbf{u}_2, \vartheta_2, p_2)$ gives

$$\tilde{\mathbf{u}}_t + \mathbf{u}_1 \cdot \nabla \tilde{\mathbf{u}} + \tilde{\mathbf{u}} \cdot \nabla \mathbf{u}_2 = \nu \Delta \tilde{\mathbf{u}} - \nabla \tilde{p} + \tilde{\vartheta} g(r) \mathbf{e}_\mathbf{r}. \tag{106}$$

Multiplying (106) by $\tilde{\mathbf{u}}$ and integrating by parts yields

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\tilde{\mathbf{u}}\|^2 + \nu\|\nabla\tilde{\mathbf{u}}\|^2 &= -\int_{\tilde{\Omega}} \tilde{\mathbf{u}} \cdot \nabla \mathbf{u}_2 \cdot \tilde{\mathbf{u}} dx + \int_{\tilde{\Omega}} \tilde{\vartheta} g(r) \mathbf{e}_\mathbf{r} \cdot \tilde{\mathbf{u}} dx \\ &\le \|\nabla \mathbf{u}_2\|_{L^\infty} \|\tilde{\mathbf{u}}\|^2 + C\|\tilde{\mathbf{u}}\|^2 + C\|\tilde{\vartheta}\|^2 \\ &\le C\|\tilde{\mathbf{u}}\|^2 + C\|\tilde{\vartheta}\|^2, \end{aligned} \tag{107}$$

due to Young's inequality and (94).

Next, subtracting Equation $(2)_2$ satisfied by $(\mathbf{u}_1, \vartheta_1, p_1)$ and $(\mathbf{u}_2, \vartheta_2, p_2)$ leads to

$$\tilde{\vartheta}_t + \mathbf{u}_1 \cdot \nabla \tilde{\vartheta} + \tilde{\mathbf{u}} \cdot \nabla \vartheta_2 = 0. \tag{108}$$

Multiplying (108) by $\tilde{\vartheta}$ and using Lemma 8, we obtain after integration by parts that

$$\frac{1}{2}\frac{d}{dt}\|\tilde{\vartheta}\|^2 = -\int_{\tilde{\Omega}} \tilde{\mathbf{u}} \cdot \nabla \vartheta_2 \tilde{\vartheta} dx \le C\|\tilde{\mathbf{u}}\|^2 + C\|\tilde{\vartheta}\|^2. \tag{109}$$

Combining (107) and (109), we finally have

$$\frac{1}{2}\frac{d}{dt}(\|\tilde{\mathbf{u}}\|^2 + \|\tilde{\vartheta}\|^2) + \nu\|\nabla\tilde{\mathbf{u}}\|^2 \le C(\|\tilde{\mathbf{u}}\|^2 + \|\tilde{\vartheta}\|^2), \tag{110}$$

which, together with Gronwall's inequality, implies that

$$\|\tilde{\mathbf{u}}\|^2 + \|\tilde{\vartheta}\|^2 \le e^{2CT}(\|\tilde{\mathbf{u}}_0\|^2 + \|\tilde{\vartheta}_0\|^2) = 0 \tag{111}$$

for any $t \ge 0$. The proof of Proposition 3 is completed. □

Thus, the proof of Theorem 2 is completed.

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