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Computably Enumerable Semisimple Rings

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Abstract: The theory of semisimple rings plays a fundamental role in noncommutative algebra. We study the complexity of the problem of semisimple rings using the tools of computability theory. Following the general idea of computably enumerable (c.e. for short) universal algebras, we define a c.e. ring as the quotient ring of a computable ring modulo a c.e. congruence relation and view such rings as structures in the language of rings, together with a binary relation. We formalize the problem of being semisimple for a c.e. ring by the corresponding index set and prove that the index set of c.e. semisimple rings is Σ_3^0-complete. This reveals that the complexity of the definability of c.e. semisimple rings lies exactly in the Σ_3^0 of the arithmetic hierarchy. As applications of the complexity results on semisimple rings, we also obtain the optimal complexity results on other closely connected classes of rings, such as the small class of finite direct products of fields and the more general class of semiperfect rings.

Keywords: computability theory; universal algebra; arithmetic hierarchy; computably enumerable ring; semisimple ring

MSC: 03D15; 03D80; 68Q15; 68Q17

1. Introduction

Computability theory (or recursion theory) has been a useful tool for examining the effective contents of problems in various branches of mathematics (e.g., analysis, algebra, combinatorics, and so on) since the advent of the subject in the early 1930s. In computability theory, we often adopt equivalent models of computability, such as the Turing machine, λ-calculus, and unlimited register machine, to formalize the notion of computability. Partial computable functions are functions whose values can be computed by a Turing program or a program in one of the popular programming languages (e.g., C, C++, Java, Python, and so on). Computable functions are partial computable functions that are defined on all natural numbers. Therefore, computable functions are just "total" computable functions. Computably enumerable sets are domains of partial computable functions and computable sets are those possessing computable characteristic functions. For standard books on the subject, we refer to Cooper [1], Nies [2], and Soare [3,4].

For the early work on the effective contents of field theory in the1950s and of vector spaces in the 1970s, we refer to [5–10]. In [7], motivated by the study of the lattice $\mathcal{L}(\omega)$ of computably enumerable (the same as recursively enumerable) subsets of natural numbers, Metakides and Nerode investigated the lattice $\mathcal{L}(V_\infty)$ of computably enumerable (often abbreviated as c.e.) subspaces of V_∞, an infinite dimensional vector space with an infinite computable basis over a computable field; in particular, they used the finite injury priority method to construct a c.e. independent set, $I \subseteq V_\infty$, such that I cannot be extended to an infinite c.e. basis of V_∞. Khoussainov and his coauthors also applied computability theory

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to study the effective aspects of universal algebras in [11,12], where a universal algebra $\mathcal{A}$ of a given signature is defined to be computably enumerable if it can be written as the quotient algebra $\mathcal{A} := \mathcal{B}/E$, with $\mathcal{B}$ a computable algebra of the same signature and E a computably enumerable congruence relation on the algebra $\mathcal{B}$.

Computable structure theory mainly studies the algorithmic aspects of countable structures by methods of computability theory; see, for instance, [13,14]. For work on special algebraic structures like abelian groups, we refer to [15–18]. In [16], it was proven that the isomorphism problem for torsion-free abelian groups is Σ_1^1-complete under many–one reducibility. Riggs investigated, in [18], the decomposablity problem for torsion-free abelian groups and concluded that the problem is also Σ_1^1-complete under many–one reducibility. The study of algebraic structures from the logic point of view is also connected with other hot research topics in modern computational algebra, such as cryptography and computational number theory [19–24]. For instance, in [20], the authors proposed new algorithms for finding isomorphisms between two algebraic number fields. Pellet-Mary and Stehlé investigated, in [21], the hardness of the NTRU problem via a reduction-based approach; they compared the hardness of the NTRU problem with other related problems and provided answers to long-standing open questions on the lower bound and the upper bound of the NTRU problem. Very recently, in [23], the authors developed practical algorithms for computing the roots of polynomials over number fields.

Motivated by the study of algebraic structures like abelian groups and vector spaces by methods of computability theory, we have investigated the computational complexity of semiperfect rings by the index set method in [25]. More precisely, we first formalized the problem of being semiperfect for a computable ring by a subset of natural numbers, namely, the index set of computable semiperfect rings, and then endeavored to determine the exact complexity of the index set. Semisimple rings are typical examples of semiperfect rings. The results of [25] tell us that the index set of computable semisimple rings is Σ_3^0 and that this index set can be both Σ_2^0-hard and Π_2^0-hard within the index set of computable rings; however, it did not obtain the exact complexity result that the index set of computable semisimple rings is Σ_3^0-complete and left this open (see Conjecture 1, [25]).

In this paper, instead of concentrating on computable rings as in [25], we consider the bigger class of computably enumerable rings and continue to explore the complexity of the problem of computably enumerable semisimple rings. As applications of the main results on semisimple rings, we also study the complexity of other related classes of rings, such as the small class of finite direct products of fields and the more general class of semiperfect rings.

1.1. Computably Enumerable Rings

Fix the language of rings $\mathcal{L}_r = (+, \cdot, -, 0, 1)$ with $+, \cdot$ binary operations for the addition and multiplication of rings, $-$ a unary operation for the inverse of the addition, and $0, 1$ constants for the identity of the addition and multiplication, respectively. A ring is just an $\mathcal{L}_r$-structure, $\mathcal{R} := (R, +_R, -_R, \cdot_R, 0_R, 1_R)$, that satisfies the usual Π_1^0 axioms of rings. In the study of effective aspects of ring theory, we only consider countable rings and often encode the domains of rings into subsets of natural numbers. Roughly speaking, computable rings are just rings with computable domains and computable operations. The formal definition is as follows:

Definition 1. *A ring $(R, +_R, -_R, \cdot_R, 0_R, 1_R)$ with domain $R \subseteq \mathbb{N}$ is computable if domain R is a computable set and the operations $+_R$ and $\cdot_R$ are computable binary functions on R; the operation $-_R$ is a computable unary function on R. We often write the ring $(R, +_R, -_R, \cdot_R, 0_R, 1_R)$ as R.*

A congruence relation E on a ring R is an equivalence binary relation that respects the operations of rings in the usual sense. For a ring R and an equivalence relation E on R, let R/E be the set of all representatives of the relation E. When E is a congruence relation on R, it naturally induces a ring structure on the set R/E, called the quotient ring of R modulo the congruence relation E. More specifically, for two representatives $a, b \in R/E$,

(1) the addition $a +_{R/E} b$ is the unique $c \in R/E$ such that $(a +_R b)Ec$ in R;
(2) the minus $-_{R/E}a$ is the unique $c \in R/E$ such that $(a +_R c)E0_R$ in R;
(3) the multiplication $a \cdot_{R/E} b$ is the unique $c \in R/E$ such that $(a \cdot_R b)Ec$ in R.

Following the general idea of computably enumerable universal algebras in [11,12], we have the notion of computably enumerable rings.

Definition 2. *A ring S is computably enumerable (c.e. for short) if it is the quotient ring of a computable ring R modulo a computably enumerable congruence relation E. We write $S = R/E$.*

Computable rings are automatically computably enumerable. Indeed, they can be viewed as c.e. rings of the form R/E, with R being a computable ring and E the identity relation on R; that is, xEy if and only if $x = y$ in the ring R. In general, if $S = R/E$ is a c.e. ring, then x equals y in the c.e. ring $S = R/E$ if and only if xEy for two elements $x, y \in R$. We often view E as the equality relation on the ring S and write $=_S$ for E. That is, for any $x, y \in R$, we have $x =_S y$ if and only if xEy.

1.2. Semisimple Rings and Semiperfect Rings

Semisimple rings form an important class of noncommutative rings, and the classical theory of such rings is well-understood (see algebra texts [26,27]); for example, the well-known Wedderburn–Artin Theorem says that a ring is semisimple if and only if it is isomorphic to a finite direct product of matrix rings over division rings. Semisimple rings possess various equivalent characterizations in terms of the properties of rings and modules in classical algebra texts. We adopt the following module-theoretic definition.

Definition 3. *Let R be a ring R with identity and M a nonzero left R-module.*

(1) If $M = Rx = \{rx : r \in R\}$ for any nonzero $x \in M$, then M is called a simple R-module.
(2) If the left regular module ${}_RR$ of the ring R can be written as a finite direct sum of simple submodules, then R is called a semisimple ring.

Since finite direct products of matrix rings over division rings are semisimple, we have various examples of computable semisimple rings.

Example 1. *(1) The rational field $\mathbb{Q}$ is a computable semisimple ring. (2) For a prime number p, the finite field $\mathbb{Z}_p$ is a computable semisimple ring.*

Let $n \geq 1$. For a ring R, let $A = (a_{ij})$ be a $n \times n$ matrix with the (i, j)-entry $a_{ij} \in R$ for $1 \leq i, j \leq n$. Then

$$M_n(R) = \{A = (a_{ij}) : 1 \leq i, j \leq n, a_{ij} \in R\}$$

is the matrix ring of $n \times n$ matrices over the ring R under usual matrix operations, which are determined by the operations of R. If R is a computable ring, so is $M_n(R)$. Furthermore, two matrices, $A = (a_{ij})$ and $B = (b_{ij})$, are equal to each other in the matrix ring $M_n(R)$ if and only if $a_{ij} = b_{ij}$ in R for all $1 \leq i, j \leq n$. If R is a c.e. ring, so is $M_n(R)$.

Example 2. *Let $n_i \geq 1$ and let D_i be a computable division ring for $1 \leq i \leq s$. The direct product ring $M_{n_1}(D_1) \times M_{n_2}(D_2) \times \cdots \times M_{n_s}(D_s)$ is a computable semisimple ring.*

Before introducing semiperfect rings, we need to review local rings. Fields and division rings are typical examples of local rings. For more background on local rings and semiperfect rings, we refer to Chapters 7 and 8 in [26].

Definition 4. *A ring R is left local if for any two $x, y \in R$, either one of x, y is left invertible or their sum $x + y$ is not left invertible.*

An element e of a ring R is an idempotent if $e = e^2$. An idempotent $e \in R$ is called *local* if the ring $eRe = \{ere : r \in R\}$ is a left local ring with identity e.

Definition 5. *A ring with identity is semiperfect if the identity can be written as a finite sum of pairwise orthogonal local idempotents of the ring.*

Local rings are natural examples of semiperfect rings. On the other hand, semisimple rings are semiperfect. Indeed, let R be a semisimple ring with identity 1_R. Based on the decomposition of the regular module ${}_RR$ as a finite direct sum of simple R-modules, the identity 1_R can be written as $1_R = e_1 + \cdots + e_n$ for some $n \geq 1$ and some pairwise orthogonal idempotents $e_1, \ldots, e_n$, such that Re_i is a simple R-module for $1 \leq i \leq n$. By Schur's Lemma, each e_iRe_i is a division ring, which is left local. Furthermore, matrix rings over semiperfect rings are also semiperfect (see Corollary 23.9 in Lam's book [26]).

Example 3. *Let $n \geq 1$ and let R be a c.e. semiperfect ring. The matrix ring $M_n(R)$ is a c.e. semiperfect ring.*

1.3. Main Results

Consider the language $\mathcal{L} = (+, \cdot, -, 0, 1, E)$ of rings plus a binary relation. As models in the language $\mathcal{L}$, the c.e. ring $S = R/E$ can be viewed as an $\mathcal{L}$-structure

$$\mathcal{S} := (R, +_R, -_R, \cdot_R, 0_R, 1_R, =_S)$$

such that it meets the axioms of rings together with the axioms of congruence relations on rings. Note that all partial computable functions are often listed as $\varphi_0, \varphi_1, \ldots, \varphi_e, \ldots$, with φ_e being the e-th partial computable function computed by the e-th Turing program. Analogously, all partial computable $\mathcal{L}$-structures can be effectively listed as $\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_e, \ldots$, where

$$\mathcal{A}_e = (R_e, +_e, -_e, \cdot_e, 0_e, 1_e, =_e)$$

is the e-th partial computable $\mathcal{L}$-structure computed by the e-th Turing problem.

When examining the complexity of subclasses of c.e. rings, we must consider the complexity of the corresponding index sets. For example, to study the complexity of the problem of c.e. semisimple rings, we will look at the index set $\{e \in \mathbb{N} : \mathcal{A}_e$ is a c.e. semisimple ring$\}$. For more details on index sets of c.e. rings, see Section 2 below.

The aim of this research is to determine the optimal complexity of representative subclasses of c.e. rings. We first consider c.e. semisimple rings in Section 3 and show that the index set of such rings is Σ^0_3. Then, using the finite injury priority method in computability theory, we further prove the following main result:

- the index set of c.e. semisimple rings is Σ^0_3-complete.

This reveals that the complexity of the definability of c.e. semisimple rings lies exactly in Σ^0_3 of the arithmetic hierarchy.

We next consider c.e. local rings in Section 4 and show that the index set of c.e. local rings is Π^0_2. Together with the result in [25] that the index set of computable local rings is Π^0_2-complete, we have the same complexity result on c.e. local rings.

As applications of the complexity results on semisimple rings and on local rings, in Section 4 we continue to explore the complexity results on other closely connected classes of rings and obtain the following results:

(1) the index set of c.e. fields is Π^0_2-complete;
(2) the index set of c.e. finite direct products of fields is Σ^0_3-complete;
(3) the index set of c.e. local rings is Π^0_2-complete;
(4) the index set of c.e. semiperfect rings is Σ^0_3-complete.

The remaining parts are organized as follows. In Section 2, we review the basic notions of computability theory and give the precise definition of the index set of c.e. rings meeting a specified property. Then, we prove the main result of c.e. semisimple rings in Section 3 and apply it to study the complexity of other related classes of rings in Section 4. Finally, we conclude this research in Section 5.

2. Preliminaries

For basic notions of computability theory, such as partial computable functions, computable functions, computably enumerable sets, computable sets, Σ^0_n sets, Π^0_n sets, m-reducibility, and m-completeness, we refer to standard books such as [1–4,13].

2.1. Basics of Computability Theory

A function $f : A \to \mathbb{N}$ with domain $A \subseteq \mathbb{N}$ is partial computable if the value $f(x)$ of $x \in A$ can be computed by a Turing program. That is, for any input $x \in A$, the program terminates after running finitely many steps, returning the value $f(x)$ of the function f at the argument x. A partial computable function is total computable (or simply, computable) if it has domain $A = \mathbb{N}$. A set is computable if it has a computable characteristic function. Based on the listing of all Turing programs, partial computable functions can be effectively listed as $\varphi_0, \varphi_1, \ldots, \varphi_e, \ldots$, with φ_e being the function computed by the e-th Turing program. A set is computably enumerable (often abbreviated as c.e.) if it is the domain of a partial computable function. All c.e. sets can be effectively listed, and they are often enumerated as $W_0, W_1, \ldots, W_e, \ldots$, with W_e being the domain of φ_e, i.e., $W_e = \{x \in \mathbb{N} : \varphi_e(x) \text{ converges}\}$.

Proposition 1. *For a set $A \subseteq \mathbb{N}$, A is computable if and only if both A and $\overline{A}$, the complement of A, are c.e. sets.*

Let $\langle \cdot, \cdot \rangle : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a fixed effective bijection, which sends a pair of numbers into a number. For $n \geq 3$, we can inductively define an effective bijection $\langle \cdot, \cdots, \cdot \rangle : \mathbb{N}^n \to \mathbb{N}$ that sends n-tuples of numbers to numbers. In the following, such effective bijections are fixed.

Arithmetic sets are sets that can be defined by arithmetic formulas (i.e., formulas with number quantifiers but no set quantifiers). Such sets are often divided into subclasses based on the appearance of number quantifiers. As a basic step, Σ^0_0 (the same as Π^0_0) sets are those that can be defined by formulas with bounded number quantifies. Inductively, for $n \geq 1$, a set A is Σ^0_n if there is a Π^0_{n-1} set B, such that

$$A = \{a \in \mathbb{N} : \exists x \in \mathbb{N}[\langle a, x \rangle \in B]\};$$

a set A is Π^0_n if there is a Σ^0_{n-1} set B, such that

$$A = \{a \in \mathbb{N} : \forall x \in \mathbb{N}[\langle a, x \rangle \in B]\}.$$

We now have the following hierarchy on arithmetic sets, known as the arithmetic hierarchy:

$$\Sigma_0^0 = \Pi_0^0 \subseteq \Sigma_1^0 \cap \Pi_1^0 = \Delta_1^0 \subseteq \Sigma_1^0 \cup \Pi_1^0 \subseteq \Sigma_2^0 \cap \Pi_2^0 = \Delta_2^0 \subseteq \Sigma_2^0 \cup \Pi_2^0 \subseteq \cdots .$$

For example, c.e. sets are Σ_1^0; computable sets are both Σ_1^0 and Π_1^0, that is, they are Δ_1^0. Furthermore, a Π_2^0 set can be expressed as

$$\{a \in \mathbb{N} : \forall x \exists y [\langle a, x, y \rangle \in B]\}$$

for some computable set B; a Σ_3^0 set can be written as

$$\{a \in \mathbb{N} : \exists x \forall y \exists z [\langle a, x, y, z \rangle \in B]\}$$

for some computable set B.

Let $A, B \subseteq \mathbb{N}$ be two sets. If there is a computable function f such that $x \in A \Leftrightarrow f(x) \in B$, then A is said to be *many–one reducible* (or *m-reducible*) to B. For a hierarchy Γ (e.g., Π_2^0, Σ_3^0), the following Γ-complete sets are viewed as the hardest Γ-sets in the sense of many–one reducibility.

Definition 6. *A set B is called m-complete Γ (or Γ-complete) if B is a Γ-set, and any other Γ-set A is m-reducible to B.*

The following examples of arithmetic complete sets are well-known in computability theory. They are widely used to measure the complexity of problems in other branches of mathematics.

(1) The Halting set $K = \{e \in \mathbb{N} : \varphi_e(e) \text{ converges}\}$ is Σ_1^0-complete.
(2) The index set of total computable functions $\text{Tot} = \{e \in \mathbb{N} : \varphi_e \text{ is total}\}$ is Π_2^0-complete.
(3) The index set finite c.e. set $\text{Fin} = \{e \in \mathbb{N} : W_e \text{ is finite}\}$ is Σ_2^0-complete.
(4) The index set cofinite c.e. set $\text{Cof} = \{e \in \mathbb{N} : W_e \text{ is cofinite}\}$ is Σ_3^0-complete.

2.2. The Index Set of Computably Enumerable Rings

Fix the language $\mathcal{L} = (+, \cdot, -, 0, 1, E)$ of rings together with a binary relation E. Similar to partial computable functions on natural numbers, partial computable $\mathcal{L}$-structures can be effectively listed as $\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_e, \ldots$. Now, a computably enumerable (c.e.) ring $S = R/E$ is just an $\mathcal{L}$-structure $\mathcal{S} := (R, +_R, -_R, \cdot_R, 0_R, 1_R, E)$, such that $(R, +_R, -_R, \cdot_R, 0_R, 1_R)$ is a computable ring and E is a c.e. congruence relation on the computable ring that serves as the equality relation $=_S$. In other words, elements of R are viewed as elements of S under the equality relation: $x =_S y$ if and only if xEy for all $x, y \in R$.

For all $e \in \mathbb{N}$, the partial computable $\mathcal{L}$-structure $\mathcal{A}_e = (R_e, +_e, -_e, \cdot_e, 0_e, 1_e, =_e)$ is a c.e. ring if $(R_e, +_e, -_e, \cdot_e, 0_e, 1_e)$ is a computable ring and $=_e$ is a c.e. congruence relation on the computable ring R_e. Note that the property of total computable functions is Π_2^0 and that both the axioms of rings and the axioms of congruence relations on rings are Π_1^0. The property that $\mathcal{A}_e$ is a c.e. ring is Π_2^0. That is, the index set $\{e \in \mathbb{N} : \mathcal{A}_e \text{ is a c.e. ring}\}$ of c.e. rings is Π_2^0.

We now encode the subclass of c.e. rings, satisfying a specified property $\mathcal{P}$ by the following index set.

Definition 7. *The index set of c.e. rings with a property $\mathcal{P}$ is the set*

$$\{e \in \mathbb{N} : \mathcal{A}_e \textit{ is a c.e. ring satisfying } \mathcal{P}\}.$$

To illustrate the notion of index sets, consider the property of being semisimple for a c.e. ring. Then, the index set of c.e. semisimple rings is just the set

$$\{e \in \mathbb{N} : \mathcal{A}_e \text{ is a c.e. semisimple ring}\}.$$

Definition 8 (Page 34, [3]). *A sequence $\langle X_e : e \in \mathbb{N}\rangle$ of c.e. subsets of natural numbers is uniformly computably enumerable if there is a computable function f such that, for all e,*

$$X_e = W_{f(e)} = \{x \in \mathbb{N} : \varphi_{f(e)}(x) \text{ converges}\}.$$

In computability theory, we often build c.e. sets with specified algorithmic properties based on the standard enumeration of c.e. sets $\langle W_{e,s} : e, s \in \mathbb{N}\rangle$ with $W_{e,0} = \emptyset$ and $|W_{e,s} \setminus W_{e,s-1}| \leq 1$ for $s \geq 1$; that is, at most, one number goes into W_e at each stage s. However, in our study of c.e. rings, we need a kind of "delayed" enumeration of c.e. sets in Lemma 1 below. For a detailed proof of the lemma, refer to [25].

Lemma 1 ([25]). *The sequence of c.e. sets $\langle W_e : e \in \mathbb{N}\rangle$ has an enumeration $\langle A_{e,s} : e, s \in \mathbb{N}\rangle$, such that, for all $e, s \in \mathbb{N}$, $A_{e,s} \subseteq W_{e,s}$ and $A_e = \bigcup_{s\in\mathbb{N}} A_{e,s} = W_e$; furthermore, if $W_e \neq \emptyset$ and s_n is the n-th stage at which a number enumerates into A_e for $n \geq 1$, then*

- *either no number goes into A_e after stage s_n, or the next stage s_{n+1} at which a number goes into A_e satisfies $s_{n+1} > s_n + n + 1$.*

Compared with the standard enumeration $\langle W_{e,s} : e, s \in \mathbb{N}\rangle$ of c.e. sets, Lemma 1 is a kind of "delayed" enumeration in the sense that there are at least $(n+1)$ many stages at which no number enumerates into A_e after the n-th stage s_n at which a number enumerates into A_e. The enumeration $\langle A_{e,s} : e, s \in \mathbb{N}\rangle$ of c.e. sets is crucial for us to construct the sequence of c.e. rings in Theorem 1 of Section 3.

3. Semisimple Rings

In this section, we study the complexity of c.e. semisimple rings. We first obtain the following upper bound.

Proposition 2. *The index set $\{e \in \mathbb{N} : \mathcal{A}_e$ is a c.e. semisimple ring$\}$ is Σ^0_3.*

Proof. Fix e. Let $\mathcal{A}_e = (R_e, +_e, -_e, \cdot_e, 0_e, 1_e, =_e)$ be the e-th partial computable $\mathcal{L}$-structure. It is a c.e. ring if $(R_e, +_e, -_e, \cdot_e, 0_e, 1_e)$ is a computable ring and $=_e$ is a c.e. congruence relation on the ring R_e. First, the property that $\mathcal{A}_e$ is a c.e. ring is Π^0_2.

Now suppose that $\mathcal{A}_e$ is a c.e. ring. As a quotient ring of the computable ring R_e modulo the c.e. relation $=_e$, the ring $\mathcal{A}_e$ is semisimple if and only if there are $x_1, \ldots, x_n \in R_e$, such that, for $1 \leq i, j \leq n$,

(1) $x_i \neq_e 0_e$;
(2) $x_i =_e x_i^2$, $x_i \cdot_e x_j =_e 0_e$ for $i \neq j$, and $1_e =_e x_1 +_e \cdots +_e x_n$;
(3) $\mathcal{A}_e x_i$ is a simple $\mathcal{A}_e$-module; that is, the following holds:

$$\forall a \in R_e \, \exists b \in R_e \, [a \cdot_e x_i =_e 0_R \vee x_i =_e b \cdot_e a \cdot_e x_i].$$

As R_e is a computable ring and $=_e$ is a Σ^0_1 relation on R_e, property (1) is Π^0_1, property (2) is Σ^0_1, and property (3) is Π^0_2. Therefore, being semisimple for a c.e. ring is a Σ^0_3 property on e. Together with the Π^0_2 property of c.e. rings, the index set of c.e. semisimple rings is Σ^0_3. □

We next prove that the Σ^0_3 complexity of semisimple rings cannot be weakened.

Theorem 1. *The index set of c.e. semisimple rings is Σ^0_3-complete.*

Proof. Since the index set of cofinite c.e. sets $\mathrm{Cof} = \{e \in \mathbb{N} : W_e \text{ is cofinite}\}$ is Σ^0_3-complete, we will establish an effective reduction from Cof to the index set of c.e. semisimple rings. We first develop requirements and strategies for building the effective reduction in Section 3.1, and then provide the formal construction and verification in Section 3.2. □

3.1. Requirements and Strategies

The commutative polynomial ring $S := \mathbb{Q}[x_i : i \in \mathbb{N}]$, with indeterminates $\{x_i : i \in \mathbb{N}\}$ over the rational field $\mathbb{Q}$ is a computable ring under a fixed coding into natural numbers. Let I be the ideal of S generated by $\{x_i - x_i^2 : i \in \mathbb{N}\}$. Then, I is a computable ideal and the quotient ring $R = S/I$ is a computable ring, such that x_i is an idempotent of R for all $i \in \mathbb{N}$. Elements of R are finite $\mathbb{Q}$-linear sums of monomials of the form $x_{i_1}x_{i_2}\cdots x_{i_k}$ with $i_1 < i_2 < \cdots < i_k$. To prove the theorem, we build a uniformly c.e. sequence of ideals $\langle I_e : e \in \mathbb{N}\rangle$ of R, such that the c.e. ring $S_e = R/I_e$ meets the following requirements:

$\mathcal{P}_e$: If $e \in \mathrm{Cof}$, then $S_e = R/I_e$ is a semisimple ring.
$\mathcal{N}_e$: If $e \notin \mathrm{Cof}$, then $S_e = R/I_e$ is not a semisimple ring.

Fix an enumeration $A_{e,s}(e, s \in \mathbb{N})$ of all c.e. sets as in Lemma 1 above. Here, $A_{e,s}$ is the finite set that contains numbers enumerated into $A_e = W_e$ by stage s. Then, the complement $\overline{A}_{e,s}$ of $A_{e,s}$ can be effectively listed as $\overline{A}_{e,s} = \{a^s_{e,0} < a^s_{e,1} < \cdots < a^s_{e,i} < \cdots\}$, and the complement of A_e is $\overline{A}_e = \bigcap_{s\in\mathbb{N}} \overline{A}_{e,s} = \overline{W}_e$.

To satisfy the requirement $\mathcal{P}_e$, we first divide it into infinitely many subrequirements:

$\mathcal{P}_{\langle e,i\rangle}$: If i is the least number such that $\lim_s a^s_{e,i} = +\infty$, then S_e is a semisimple ring.

The requirement $\mathcal{P}_e$ will be satisfied via all $\mathcal{P}_{\langle e,i\rangle}$-strategies $(i \in \mathbb{N})$ together, and the interactions among multiple $\mathcal{P}_{\langle e,i\rangle}$-strategies are finite injury priority arguments, where the priority of subrequirements of $\mathcal{P}_e$ is listed as

$$\mathcal{P}_{\langle e,0\rangle} \prec \mathcal{P}_{\langle e,1\rangle} \prec \cdots \prec \mathcal{P}_{\langle e,i\rangle} \prec \cdots.$$

That is, each $\mathcal{P}_{\langle e,i\rangle}$ can only be injured by $\mathcal{P}_{\langle e,j\rangle}$-requirements with $j < i$. However, for two different $\mathcal{P}_e$, $\mathcal{P}_{e'}$-requirements, there are no injuries between $\mathcal{P}_{\langle e,i\rangle}$ and $\mathcal{P}_{\langle e',i'\rangle}$-strategies for any $i, i' \in \mathbb{N}$.

Definition 9. *Let $s \geq 1$. For any $e, i \leq s$, $\mathcal{P}_{\langle e,i\rangle}$ requires attention at stage s if i is the least number, such that $a^s_{e,i} > a^{s-1}_{e,i}$.*

In the following, for two polynomials $f, g \in R$, we write $f =_{S_e} g$ when $f - g \in I_e$. In other words, f and g are equal to each other in the c.e. ring S_e.

A basic $\mathcal{P}_{\langle e,i\rangle}$-strategy proceeds as follows.

(1) If $\mathcal{P}_{\langle e,i\rangle}$ requires attention at some stage, let s_1 be the first stage at which $\mathcal{P}_{\langle e,i\rangle}$ requires attention. Add generators into I_e as follows:
   (1.1) At stage $t < s_1$, $\mathcal{P}_{\langle e,i\rangle}$ does not require attention; add $x_t - x_{t+1}$ into $I_{e,t}$.
   (1.2) At stage s_1, $\mathcal{P}_{\langle e,i\rangle}$ first requires attention; add $1_R - (x_{s_1} + x_{s_1+1})$ into I_{e,s_1}.
   (1.3) At stage $t > s_1 + 1$, if $\mathcal{P}_{\langle e,i\rangle}$ requires attention, add x_t into $I_{e,t}$; otherwise, add $x_t - x_{t+1}$ into $I_{e,t}$.

(2) Otherwise, at any stage t, $\mathcal{P}_{\langle e,i\rangle}$ does not require attention, add $x_t - x_{t+1}$ into $I_{e,t}$.

If there are infinitely many stages t at which $\mathcal{P}_{\langle e,i\rangle}$ requires attention, then we have $x_0 =_{S_e} x_1 =_{S_e} \cdots =_{S_e} x_{s_1}$, and for any $t > s_1 + 1$, we have $x_t =_{S_e} 0_R$. As $x_{s_1}x_{s_1+1} = (x_{s_1} + x_{s_1+1} - 1_R)x_{s_1} \in I_e$, we have $x_{s_1}x_{s_1+1} =_{S_e} 0_R$. Then $S_e = S_e x_{s_1} \oplus S_e x_{s_1+1}$ is a direct

sum of simple S_e-modules with $1_R =_{S_e} x_{s_1} + x_{s_1+1}$, $S_e x_{s_1} = \mathbb{Q}x_{s_1}$, and $S_e x_{s_1+1} = \mathbb{Q}x_{s_1+1}$. S_e is a semisimple ring. In this case, since S_e is a commutative ring, $S_e x_{s_1}$ and $S_e x_{s_1+1}$ are also rings with identity x_{s_1} and x_{s_1+1}, respectively; furthermore, $S_e x_{s_1} \cong S_e x_{s_1+1} \cong \mathbb{Q}$ as rings, and S_e is isomorphic to the product field $\mathbb{Q} \times \mathbb{Q}$.

If there are finitely many stages t at which $\mathcal{P}_{\langle e,i\rangle}$ requires attention, suppose that we have enumerated $1_R - (x_{s_1} + x_{s_1+1})$ into I_e, then there exists a large enough number t_e, such that $S_e x_{s_1} \neq S_e x_{t_e} x_{s_1}$ and $S_e x_{s_1+1} \neq S_e x_{t_e} x_{s_1+1}$ as S_e-modules. Although $S_e = S_e x_{s_1} \oplus S_e x_{s_1+1}$, the direct summands $S_e x_{s_1}$ and $S_e x_{s_1+1}$ are not simple S_e-modules. S_e is not a semisimple ring.

Due to the interactions between $\mathcal{P}_{\langle e,i\rangle}$ and $\mathcal{P}_{\langle e,j\rangle}$-strategies, we need to modify the primitive $\mathcal{P}_{\langle e,i\rangle}$-strategy above in the formal construction.

3.2. Construction and Verification

Based on the enumeration of c.e. sets $\langle A_e : e \in \mathbb{N}\rangle$, we enumerate a sequence of c.e. ideals $\langle I_e : e \in \mathbb{N}\rangle$ of the computable ring R stage by stage. For all e, s, we add a finite set of generators into I_e at each stage s, and let $I_{e,s}$ be the ideal of R generated by the generators of I_e that have been added by the end of stage s. Then, $I_{e,s} \subseteq I_{e,s+1}$ for all s, and $I_e = \bigcup_s I_{e,s}$.

Construction

Stage 0. For all e, no $\mathcal{P}_{\langle e,-\rangle}$-requirement requires attention; add $x_0 - x_1$ into $I_{e,0}$.

Declare that $\mathcal{P}_{\langle e,i\rangle}$ is invalid for all e, i.

Stage $s \geq 1$. For all $e \geq s+1$, no $\mathcal{P}_{\langle e,-\rangle}$-requirement requires attention at any stage $t \leq s$; add $x_s - x_{s+1}$ into $I_{e,s}$.

For each $e \leq s$, add generators into $I_{e,s}$ depending on whether some $\mathcal{P}_{\langle e,i\rangle}$ requires attention at stage s.

Case 1. No $\mathcal{P}_{\langle e,i\rangle}$ requires attention at current stage s for any $i \leq s$.

Case 1.1. If no $\mathcal{P}_{\langle e,-\rangle}$-requirements required attention before, add $x_s - x_{s+1}$ into $I_{e,s}$.

Case 1.2. Otherwise, let $s_1 < \cdots < s_m$ be all stages at which $\mathcal{P}_{\langle e,-\rangle}$-requirements required attention before. Then, at each stage s_j with $1 \leq j \leq m$, some number enumerates into A_{e,s_j}. By Lemma 1, for all $1 \leq j \leq m-1$, we have $s_j + j + 1 < s_{j+1}$.

Add generators into $I_{e,s}$ according to the following m subcases:

(1.2.1) If $m = 1$, there was exactly one $\mathcal{P}_{\langle e,i\rangle}$ that required attention before stage s; add $x_s - x_{s+2}$ into $I_{e,s}$.

(1.2.2) If $m = 2$, then there were exactly two $\mathcal{P}_{\langle e,-\rangle}$-requirements that required attention before stage s. We have added the following generators into I_e.

- At stage $t \in [0, s_1)$, we have $x_t - x_{t+1} \in I_{e,t}$. Set $a_1 = 0$. Then, $x_0 =_{S_e} x_1 =_{S_e} \cdots =_{S_e} x_{s_1} =_{S_e} x_{s_1+a_1}$.
- At stage $t \in [s_1+1, s_2)$, we have $x_t - x_{t+2} \in I_{e,t}$. Let a_2 be the unique number in $\{0,1\}$, satisfying:

$$x_{s_1+1} =_{S_e} x_{s_1+3} =_{S_e} \cdots =_{S_e} x_{s_2+(a_2+1)_{\mathrm{mod}2}};$$
$$x_{s_1+2} =_{S_e} x_{s_1+4} =_{S_e} \cdots =_{S_e} x_{s_2+a_2};$$

where $(a_2+1)_{\mathrm{mod}2}$ equals the remainder of a_2+1 divided by 2. That is,

$$(a_2+1)_{\mathrm{mod}2} = \begin{cases} 0, & \text{if } a_2 = 1; \\ 1, & \text{if } a_2 = 0. \end{cases}$$

In general, we use the symbol $a_{\mathrm{mod}p}$ to denote the remainder of a divided by p for $a, p \in \mathbb{N}$ with $p \geq 1$. So, $a_{\mathrm{mod}p} \in \{0, 1, \ldots, p-1\}$.

- Now, at stage s with $s > s_m = s_2$, no $\mathcal{P}_{\langle e,-\rangle}$-requirement requires attention; proceed as follows:
  - (i) If $s = s_2 + 1$, add $x_{s_2+(a_2+1)_{\mathrm{mod}2}} - x_{s_2+1+3}$ into $I_{e,s}$.
    Now we have $x_{s_1+1} =_{S_e} x_{s_2+(a_2+1)_{\mathrm{mod}2}} =_{S_e} x_{s_2+1+3}$.
  - (ii) Otherwise, $s \geq s_2 + 2$, add $x_s - x_{s+3}$ into $I_{e,s}$.

(1.2.3) If $m = 3$, then there were exactly three $\mathcal{P}_{\langle e,-\rangle}$-requirements that required attention before stage s. We have added $x_{s_2+(a_2+1)_{\mathrm{mod}2}} - x_{s_2+1+3}$ and $x_t - x_{t+3}$ into I_e for $t \in [s_2 + 2, s_3)$. Define a_3 to be the unique number in $\{0, 1, 2\}$, satisfying:

$$x_{s_1+1} =_{S_e} x_{s_2+(a_2+1)_{\mathrm{mod}2}} =_{S_e} x_{s_2+4} =_{S_e} \cdots =_{S_e} x_{s_3+(a_3+1)_{\mathrm{mod}3}};$$
$$x_{s_2+2} =_{S_e} x_{s_2+5} =_{S_e} \cdots =_{S_e} x_{s_3+(a_3+2)_{\mathrm{mod}3}};$$
$$x_{s_2+3} =_{S_e} x_{s_2+6} =_{S_e} \cdots =_{S_e} x_{s_3+a_3}.$$

- Now at stage s with $s > s_m = s_3$, no $\mathcal{P}_{\langle e,-\rangle}$-requirement requires attention; proceed as follows:
  - (i) If $s = s_3 + 1$, add $x_{s_3+(a_3+1)_{\mathrm{mod}3}} - x_{s_3+1+4}$ into $I_{e,s}$.
    Now we have $x_{s_1+1} =_{S_e} x_{s_2+(a_2+1)_{\mathrm{mod}2}} =_{S_e} x_{s_3+(a_3+1)_{\mathrm{mod}3}} =_{S_e} x_{s_3+1+4}$.
  - (ii) If $s = s_3 + 2$, add $x_{s_3+(a_3+2)_{\mathrm{mod}3}} - x_{s_3+2+4}$ into $I_{e,s}$.
    Now we have $x_{s_2+2} =_{S_e} x_{s_3+(a_3+2)_{\mathrm{mod}3}} =_{S_e} x_{s_3+2+4}$.
  - (iii) Otherwise, $s \geq s_3 + 3$, add $x_s - x_{s+4}$ into $I_{e,s}$.

(1.2.m) In general, if there were exactly m many $\mathcal{P}_{\langle e,-\rangle}$-requirements that required attention before stage s with $m \geq 3$, we have $s_1 < s_2 < s_3 \leq \cdots \leq s_m < s$. For each $t \in [s_j + 1, s_{j+1})$ with $2 \leq j \leq m - 1$,

- if $t = s_j + k$ for $1 \leq k \leq j - 1$, we have $x_{s_j+(a_j+k)_{\mathrm{mod}j}} - x_{s_j+k+(j+1)} \in I_{e,t}$;
- if $t \in [s_j + j, s_{j+1})$, we have $x_t - x_{t+(j+1)} \in I_{e,t}$.

Let $a_{j+1} \in \{0, 1, \ldots, j\}$ be the number with $x_{s_j+j+1} =_{S_e} x_{s_{j+1}+a_{j+1}}$. Then, for $1 \leq k \leq j - 1$, we have

$$x_{s_k+k} =_{S_e} x_{s_j+(a_j+k)_{\mathrm{mod}j}} =_{S_e} x_{s_j+k+(j+1)} =_{S_e} \cdots =_{S_e} x_{s_{j+1}+(a_{j+1}+k)_{\mathrm{mod}(j+1)}};$$
$$x_{s_j+j} =_{S_e} x_{s_j+j+(j+1)} =_{S_e} \cdots =_{S_e} x_{s_{j+1}+(a_{j+1}+j)_{\mathrm{mod}(j+1)}};$$
$$x_{s_j+j+1} =_{S_e} x_{s_j+j+1+(j+1)} =_{S_e} \cdots =_{S_e} x_{s_{j+1}+a_{j+1}}.$$

- Now, at stage s with $s > s_m \geq s_3$, no $\mathcal{P}_{\langle e,-\rangle}$-requirement requires attention, proceed as follows:
  - (i) If $s = s_m + k$ with $1 \leq k \leq m - 1$, add $x_{s_m+(a_m+k)_{\mathrm{mod}m}} - x_{s_m+k+(m+1)}$ into $I_{e,s}$.
  - (ii) Otherwise, $s \geq s_m + m$, add $x_s - x_{s+(m+1)}$ into $I_{e,s}$.

This ends the construction of $I_{e,s}$ at Case 1 where no $\mathcal{P}_{\langle e,-\rangle}$ requires attention.

Case 2. $\mathcal{P}_{\langle e,i\rangle}$ requires attention for some $i \leq s$ at stage s.

Case 2.1. If s is the first stage at which some $\mathcal{P}_{\langle e,i\rangle}$, $\mathcal{P}_{\langle e,i_1\rangle}$ for example, requires attention, add $1_R - (x_s + x_{s+1})$ into $I_{e,s}$. *Declare that $\mathcal{P}_{\langle e,i_1\rangle}$ is valid.*

Case 2.2. Otherwise, suppose that $s_1 < \cdots < s_m$ were all stages before stage s at which some $\mathcal{P}_{\langle e,-\rangle}$-requirement required attention. Now, the current stage $s = s_{m+1}$, which is the $(m + 1)$-th stage at which some $\mathcal{P}_{\langle e,-\rangle}$-requirement requires attention. Let $\mathcal{P}_{\langle e,i_j\rangle}$ be the requirement that required attention at stage s_j for $1 \leq j \leq m + 1$.

Add generators into $I_{e,s}$ according to the following m subcases:

(2.2.1) If $m = 1$, then $s = s_2$ is the second stage at which some $\mathcal{P}_{\langle e,i\rangle}$, namely, $\mathcal{P}_{\langle e,i_2\rangle}$, requires attention. We have obtained the following $=_{S_e}$-equalities by stage $s-1$:

$$x_0 =_{S_e} x_1 =_{S_e} \cdots =_{S_e} x_{s_1} =_{S_e} x_{s_1+a_1};$$
$$1_R =_{S_e} x_{s_1+a_1} + x_{s_1+1};$$
$$x_{s_1+1} =_{S_e} x_{s_1+3} =_{S_e} \cdots =_{S_e} x_{s+(a_2+1)_{\mathrm{mod}2}};$$
$$x_{s_1+2} =_{S_e} x_{s_1+4} =_{S_e} \cdots =_{S_e} x_{s+a_2}.$$

- Now, $\mathcal{P}_{\langle e,i_2\rangle}$ requires attention at stage s, and it acts as follows:
  - (i) If $i_2 > i_1$, the lower priority $\mathcal{P}_{\langle e,i_2\rangle}$ currently requires attention.
    - $*$ Add $1_R - (x_{s+a_2} + x_{s+2})$ into $I_{e,s}$.
      Declare that $\mathcal{P}_{\langle e,i_2\rangle}$ is valid. $\mathcal{P}_{\langle e,i_2\rangle}$ remains valid unless it is injured.
  - (ii) If $i_2 < i_1$, the higher priority $\mathcal{P}_{\langle e,i_2\rangle}$ currently requires attention.
    - $*$ Add $x_{s+(a_2+1)_{\mathrm{mod}2}}$ into $I_{e,s}$.
      Now, $x_{s_1+1} =_{S_e} 0_R$, $x_{s_1} =_{S_e} 1_R$, the generator $1_R - (x_{s_1} + x_{s_1+1})$ appointed for $\mathcal{P}_{\langle e,i_1\rangle}$ is destroyed. In this case, $\mathcal{P}_{\langle e,i_2\rangle}$ injures $\mathcal{P}_{\langle e,i_1\rangle}$.
      Declare that $\mathcal{P}_{\langle e,i_1\rangle}$ is invalid.
    - $**$ Add $1_R - (x_{s+a_2} + x_{s+2})$ into $I_{e,s}$.
      Declare that $\mathcal{P}_{\langle e,i_2\rangle}$ is valid. $\mathcal{P}_{\langle e,i_2\rangle}$ remains valid unless it is injured.
  - (iii) If $i_2 = i_1$, the valid $\mathcal{P}_{\langle e,i_1\rangle}$ requires attention again at current stage s. We have added the generator $1_R - (x_{s_1+a_1} + x_{s_1+1})$ for $\mathcal{P}_{\langle e,i_1\rangle}$. So, we do not need to add a new generator for $\mathcal{P}_{\langle e,i_2\rangle} = \mathcal{P}_{\langle e,i_1\rangle}$.
    - $*$ Add $1_R - (x_{s+a_2} + x_{s+2})$ and x_{s+2} into $I_{e,s}$.
      Now $x_{s_1+2} =_{S_e} x_{s_1+4} =_{S_e} \cdots =_{S_e} x_{s+a_2} =_{S_e} 1_R$, and $x_{s+2} =_{S_e} 0_R$.

(2.2.m) In general, if $m \geq 2$, then $s = s_{m+1}$ is the $(m+1)$-th stage at which some $\mathcal{P}_{\langle e,i\rangle}$, namely, $\mathcal{P}_{\langle e,i_{m+1}\rangle}$, require attention. Together with constructions during stages in $[s_m + 1, s)$ (see Case 1.2 above) where no $\mathcal{P}_{\langle e,-\rangle}$-requirements required attention, we have the following $=_{S_e}$-equalities:

- $x_0 =_{S_e} x_1 =_{S_e} \cdots =_{S_e} x_{s_1} =_{S_e} x_{s_1+a_1}$;
  $x_{s_1+1} =_{S_e} x_{s_1+3} =_{S_e} \cdots =_{S_e} x_{s_2+(a_2+1)_{\mathrm{mod}2}}$;
  $x_{s_1+2} =_{S_e} x_{s_1+4} =_{S_e} \cdots =_{S_e} x_{s_2+a_2}$ with $a_2 \in \{0,1\}$.
- For $1 \leq k \leq m-1$,

$$x_{s_k+k+1} =_{S_e} x_{s_k+k+1+(k+1)} =_{S_e} \cdots =_{S_e} x_{s_{k+1}+a_{k+1}};$$
$$x_{s_k+k} =_{S_e} x_{s_m+(a_m+k)_{\mathrm{mod}m}} =_{S_e} x_{s_m+k+(m+1)} \cdots =_{S_e} x_{s+(a_{m+1}+k)_{\mathrm{mod}(m+1)}};$$

  with $a_{k+1} \in \{0,1,\ldots,k\}$. We also have

$$x_{s_m+m} =_{S_e} x_{s_m+m+(m+1)} =_{S_e} \cdots =_{S_e} x_{s+(a_{m+1}+m)_{\mathrm{mod}(m+1)}};$$
$$x_{s_m+m+1} =_{S_e} x_{s_m+m+1+(m+1)} =_{S_e} \cdots =_{S_e} x_{s+a_{m+1}};$$

  with $s = s_{m+1}$ and $a_{m+1} \in \{0,1,\ldots,m\}$.

Therefore, for $0 \leq t \leq s + m + 1 = s_{m+1} + m + 1$, we have

$$x_t \in_{S_e} \{x_{s_k+a_k} : 1 \leq k \leq m+1\} \cup \{x_{s_k+k} : 1 \leq k \leq m+1\},$$

where, for an element $y \in R$ and a set $Y \subseteq R$, the symbol $y \in_{S_e} Y$ means that $y =_{S_e} z$ for some $z \in Y$.

When a valid $\mathcal{P}_{\langle e,i\rangle}$ was injured by some $\mathcal{P}_{\langle e,j\rangle}$ with $j < i$, the higher priority $\mathcal{P}_{\langle e,j\rangle}$ became valid at the same stage. So, there are valid $\mathcal{P}_{\langle e,-\rangle}$-requirements at current stage s. Let $\mathcal{P}_{\langle e,i_{n_1}\rangle} \prec \mathcal{P}_{\langle e,i_{n_2}\rangle} \prec \cdots \prec \mathcal{P}_{\langle e,i_{n_l}\rangle}$ with $1 \leq l \leq m$ be all such valid requirements. Note that a single $\mathcal{P}_{\langle e,i\rangle}$ may require attention at multiple stages. For all $1 \leq j \leq l$, suppose that $\mathcal{P}_{\langle e,i_{n_j}\rangle}$ was valid at stage s_{n_j} with the appointed generator $1_R - (x_{s_{n_j}+a_{n_j}} + x_{s_{n_j}+n_j})$. We have $s_{n_1} < s_{n_2} < \cdots < s_{n_l} \leq s_m < s$ because of the priority of the valid requirements.

- Now, $\mathcal{P}_{\langle e,i_{m+1}\rangle}$ requires attention at stage $s = s_{m+1}$, and it acts as follows:

  (i) If $i_{m+1} > i_{n_l}$, then $\mathcal{P}_{\langle e,i_{n_1}\rangle} \prec \cdots \prec \mathcal{P}_{\langle e,i_{n_l}\rangle} \prec \mathcal{P}_{\langle e,i_{m+1}\rangle}$. Now, the lower priority $\mathcal{P}_{\langle e,i_{m+1}\rangle}$ requires attention at stage $s = s_{m+1}$.

  $*$ Add $1_R - (x_{s+a_{m+1}} + x_{s+m+1})$ into $I_{e,s}$.
  Declare that $\mathcal{P}_{\langle e,i_{m+1}\rangle}$ is valid. $\mathcal{P}_{\langle e,i_{m+1}\rangle}$ remains valid unless it is injured.

  (ii) If $i_{n_j} < i_{m+1} < i_{n_{j+1}}$ for some $1 \leq j \leq l-1$, then $\mathcal{P}_{\langle e,i_{n_1}\rangle} \prec \cdots \prec \mathcal{P}_{\langle e,i_{n_j}\rangle} \prec \mathcal{P}_{\langle e,i_{m+1}\rangle} \prec \mathcal{P}_{\langle e,i_{n_{j+1}}\rangle} \prec \cdots \prec \mathcal{P}_{\langle e,i_{n_l}\rangle}$.

  $*$ Add $x_{s+(a_{m+1}+n_k)_{\mathrm{mod}(m+1)}}$ into $I_{e,s}$ for all $j+1 \leq k \leq l$.
  By $x_{s_{n_k}+n_k} =_{S_e} x_{s+(a_{m+1}+n_k)_{\mathrm{mod}(m+1)}}$, we have $x_{s_{n_k}+n_k} =_{S_e} 0_R$ and $x_{s_{n_k}+a_{n_k}} =_{S_e} 1_R$. The generator $1_R - (x_{s_{n_k}+a_{n_k}} + x_{s_{n_k}+n_k})$ appointed for $\mathcal{P}_{\langle e,i_{n_k}\rangle}$ is destroyed. In this case, $\mathcal{P}_{\langle e,i_{m+1}\rangle}$ injures $\mathcal{P}_{\langle e,i_{n_k}\rangle}$.
  Declare that $\mathcal{P}_{\langle e,i_{n_k}\rangle}$ is invalid.

  $**$ Add $1_R - (x_{s+a_{m+1}} + x_{s+m+1})$ into $I_{e,s}$.
  Declare that $\mathcal{P}_{\langle e,i_{m+1}\rangle}$ is valid. $\mathcal{P}_{\langle e,i_{m+1}\rangle}$ remains valid unless it is injured.

  (iii) If $i_{m+1} = i_{n_j}$ for some $1 \leq j \leq l$, then $\mathcal{P}_{\langle e,i_{m+1}\rangle} = \mathcal{P}_{\langle e,i_{n_j}\rangle}$ was valid at stage $s_{n_j} (\leq s_m < s = s_{m+1})$. Now, $\mathcal{P}_{\langle e,i_{n_j}\rangle}$ requires attention again at stage $s = s_{m+1}$. We do not need to add a new generator for $\mathcal{P}_{\langle e,i_{m+1}\rangle} = \mathcal{P}_{\langle e,i_{n_j}\rangle}$ because we have already added the generator $1_R - (x_{s_{n_j}+a_{n_j}} + x_{s_{n_j}+n_j})$ for $\mathcal{P}_{\langle e,i_{n_j}\rangle}$. Act as follows:

  $*$ Add $x_{s+(a_{m+1}+n_k)_{\mathrm{mod}(m+1)}}$ into $I_{e,s}$ for all $j < k \leq l$ (if $j < l$).
  As in (ii), the generator $1_R - (x_{s_{n_k}+a_{n_k}} + x_{s_{n_k}+n_k})$ appointed for the requirement $\mathcal{P}_{\langle e,i_{n_k}\rangle}$ with lower priority is destroyed. $\mathcal{P}_{\langle e,i_{n_j}\rangle}$ injures $\mathcal{P}_{\langle e,i_{n_k}\rangle}$.
  Declare that $\mathcal{P}_{\langle e,i_{n_k}\rangle}$ is invalid.

  $**$ Add $1_R - (x_{s+a_{m+1}} + x_{s+m+1})$ and x_{s+m+1} into $I_{e,s}$.
  Now, $x_{s_m+m+1} =_{S_e} x_{s_m+m+1+(m+1)} =_{S_e} \cdots =_{S_e} x_{s+a_{m+1}} =_{S_e} 1_R$, and $x_{s+m+1} =_{S_e} 0_R$. Note that the generator $1_R - (x_{s_{n_j}+a_{n_j}} + x_{s_{n_j}+n_j})$ of $\mathcal{P}_{\langle e,i_{n_j}\rangle}$ remains valid.

  (iv) If $i_{m+1} < i_1$, then $\mathcal{P}_{\langle e,i_{m+1}\rangle} \prec \mathcal{P}_{\langle e,i_{n_1}\rangle} \prec \cdots \prec \mathcal{P}_{\langle e,i_{n_l}\rangle}$. Now, the highest priority $\mathcal{P}_{\langle e,i_{m+1}\rangle}$ requires attention at stage $s = s_{m+1}$.

  $*$ Add $x_{s+(a_{m+1}+n_k)_{\mathrm{mod}(m+1)}}$ into $I_{e,s}$ for all $1 \leq k \leq l$.
  Again, as in (ii), the generator $1_R - (x_{s_{n_k}+a_{n_k}} + x_{s_{n_k}+n_k})$ appointed for $\mathcal{P}_{\langle e,i_{n_k}\rangle}$ is destroyed. $\mathcal{P}_{\langle e,i_{m+1}\rangle}$ injures $\mathcal{P}_{\langle e,i_{n_k}\rangle}$.
  Declare that $\mathcal{P}_{\langle e,i_{n_k}\rangle}$ is invalid.

  $**$ Add $1_R - (x_{s+a_{m+1}} + x_{s+m+1})$ into $I_{e,s}$.
  Declare that $\mathcal{P}_{\langle e,i_{m+1}\rangle}$ is valid. $\mathcal{P}_{\langle e,i_{m+1}\rangle}$ remains valid unless it is injured.

This ends the construction.

Recall that the computable ring R is the quotient ring of the polynomial ring $S := \mathbb{Q}[x_i : i \in \mathbb{N}]$ modulo, the ideal I of S generated by $\{x_i - x_i^2 : i \in \mathbb{N}\}$.

Lemma 2. *For all $e, s \geq 0$, $I_{e,s}$ is a computable ideal of R.*

Proof. Fix $e, s \geq 0$. At each stage $t \leq s$ of the construction, if no $\mathcal{P}_{\langle e,-\rangle}$-requirements require attention, we add new generators into $I_{e,t}$ according to the number m of $\mathcal{P}_{\langle e,-\rangle}$-requirements that have required attention before; otherwise, some $\mathcal{P}_{\langle e,i\rangle}$ requires attention at stage t, and we add new generators into $I_{e,t}$ based on the relative priority of $\mathcal{P}_{\langle e,i\rangle}$ compared with the valid $\mathcal{P}_{\langle e,-\rangle}$-requirements existed at stage t. The generators of $I_{e,s}$ can be computed by the end of stage s of the construction. To see why $I_{e,s}$ is a computable ideal of R, we will initiate an algorithm to determine whether a given polynomial f of R can be expressed as a finite R-linear combinations of the generators of $I_{e,s}$.

For $s = 0$, $I_{e,0}$ is the ideal of R generated by $x_0 - x_1$. Given $f \in R$, calculate a polynomial $f_* \in R$ by replacing each appearance of x_1 in f (if any) with x_0. Then, $f \in I_{e,0} \Leftrightarrow f_* = 0_R$. So, $I_{e,0}$ is computable.

For $s \geq 1$, if no $\mathcal{P}_{\langle e,i\rangle}$ requires attention at any stage $t \leq s$, then $I_{e,s}$ is the ideal of R generated by $\{x_t - x_{t+1} : 0 \leq t \leq s\}$. Given $f \in R$, calculate a polynomial $f_* \in R$ by replacing each appearance of $x_t (1 \leq t \leq s+1)$ in f (if any) with x_0. Then, $f \in I_{e,s} \Leftrightarrow f_* = 0_R$, and $I_{e,s}$ is computable.

For $s \geq 1$, if there are stages $\leq s$ at which some $\mathcal{P}_{\langle e,i\rangle}$ require attention, let $s_1 < \cdots < s_m$ be all such stages. For all $1 \leq j \leq m$, assume that $\mathcal{P}_{\langle e,i_j\rangle}$ requires attention at stage s_j. Since a single $\mathcal{P}_{\langle e,i\rangle}$ may require attention at multiple stages, we may have $i_j = i_{j'}$ for different j, j'. As in construction, let $a_1 = 0$, and let $a_{j+1} (1 \leq j \leq m-1)$ be the unique number in $\{0, 1, \ldots, j\}$, such that $x_{s_j+j+1} - x_{s_{j+1}+a_{j+1}} \in I_{e,s}$.

Case 1. $m = 1$ with $s = s_1$, the first stage of the construction at which some requirement $\mathcal{P}_{\langle e,i_1\rangle}$ requires attention. In this case, we have added a new generator $1_R - (x_s + x_{s+1})$ into I_e at stage s, and $I_{e,s}$ is the ideal of R generated by $\{x_t - x_{t+1} : 0 \leq t \leq s-1\} \cup \{1_R - (x_s + x_{s+1})\}$. Elements of $I_{e,s}$ are of the form

$$f = \sum_{t=0}^{s-1} g_t(x_t - x_{t+1}) + h[1_R - (x_s + x_{s+1})]$$

with $g_t, h \in R$ for $0 \leq t < s$. By replacing each $x_t (1 \leq t \leq s)$ in f with x_0, we obtain $f' = h'[1_R - (x_0 + x_{s+1})]$, where h' is obtained from h by the same actions. Now, elements of $I_{e,s}$ are of the form $f' = h'[1_R - (x_0 + x_{s+1})]$.

Let $f \in R$. Determine whether $f \in I_{e,s}$ or not as follows:

(1) Substitute each $x_t (1 \leq t \leq s)$ by x_0 in f to obtain a polynomial $f' \in R$.

(2) Write $f' = f_0 + f_1 x_0 + f_2 x_{s+1} + f_3 x_0 x_{s+1}$, where for $0 \leq i \leq 3$, f_i does not contain x_0 or x_{s+1}. By $x_0 x_{s+1} = -x_0[1_R - (x_0 + x_{s+1})] \in I_{e,s}$, we have

$$f' \in I_{e,s} \Leftrightarrow f'' = f_0 + f_1 x_0 + f_2 x_{s+1} \in I_{e,s}.$$

In this case, $f'' = f_0 + f_1 x_0 + f_2 x_{s+1} = h[1_R - (x_0 + x_{s+1})]$ for some $h \in R$.

- Take $x_0 = 1_R, x_{s+1} = 0_R$; we have $f_0 + f_1 = 0_R$.
- Take $x_0 = 0_R, x_{s+1} = 1_R$; we have $f_0 + f_2 = 0_R$.

Therefore, $f'' = f_0 + f_1 x_0 + f_2 x_{s+1} \in I_{e,s} \Leftrightarrow f_1 = f_2 = -f_0$. In this case, $f'' = f_0[1_R - (x_0 + x_{s+1})]$.

Case 2. $m = 1$ with $s_1 < s$. In the construction, only one requirement of the form $\mathcal{P}_{\langle e,-\rangle}$ required attention before stage s, and no $\mathcal{P}_{\langle e,-\rangle}$-requirements requires attention at

stage s. Then, the ideal $I_{e,s}$ of R is generated by $\{x_{t_0} - x_{t_0+1} : 0 \le t_0 \le s_1 - 1\} \cup \{1_R - (x_{s_1} + x_{s_1+1})\} \cup \{x_{t_1} - x_{t_1+2} : s_1 + 1 \le t_1 \le s\}$. Elements of $I_{e,s}$ can be expressed as

$$f = \sum_{t_0=0}^{s_1-1} g_{t_0}(x_{t_0} - x_{t_0+1}) + h[1_R - (x_{s_1} + x_{s_1+1})] + \sum_{t_1=s_1+1}^{s} g_{t_1}(x_{t_1} - x_{t_1+2})$$

with $g_{t_0}, g_{t_1}, h \in R$ for $0 \le t_0 < s_1$, $s_1 < t_1 \le s$. By replacing each $x_{t_0}(1 \le t_0 \le s_1)$ in f with x_0 and then replacing each $x_{s_1+2k+1}(s_1 + 3 \le s_1 + 2k + 1 \le s + 2)$ with x_{s_1+1} and each $x_{s_1+2k+2}(s_1 + 4 \le s_1 + 2k + 2 \le s + 2)$ with x_{s_1+2}, we obtain $f' = h'[1_R - (x_0 + x_{s_1+1})]$. Now, elements of $I_{e,s}$ are of the form $f' = h'[1_R - (x_0 + x_{s_1+1})]$ with neither $x_{t_0}(1 \le t_0 \le s_1)$ nor $x_{t_1}(s_1 + 3 \le t_1 \le s + 2)$ occurring in h'. Similar to Case 1, one can determine whether a polynomial $f \in R$ is of the form $h'[1_R - (x_0 + x_{s_1+1})]$. So, $I_{e,s}$ is computable.

Case 3. $m = 2$ with $s_1 < s = s_2$, the second stage of the construction at which a requirement of the form $\mathcal{P}_{\langle e,-\rangle}$ requires attention. In this case, there was exactly one requirement, namely, $\mathcal{P}_{\langle e,i_1\rangle}$, that required attention before stage s, and the requirement $\mathcal{P}_{\langle e,i_2\rangle}$ requires attention at current stage s. According to the strategy for $\mathcal{P}_{\langle e,i_2\rangle}$ at stage s, there are three cases depending on the priority of the requirement $\mathcal{P}_{\langle e,i_2\rangle}$.

Case 3.1. If $i_1 < i_2$, then the priority of $\mathcal{P}_{\langle e,i_1\rangle}$ is higher than $\mathcal{P}_{\langle e,i_2\rangle}$. In this case, $\mathcal{P}_{\langle e,i_2\rangle}$ is appointed a new generator $1_R - (x_{s+a_2} + x_{s+2})$, and the generator of $\mathcal{P}_{\langle e,i_1\rangle}$ is still valid. The ideal $I_{e,s}$ is generated by

$$\begin{aligned}&\{x_{t_0} - x_{t_0+1} : 0 \le t_0 \le s_1 - 1\} \cup \{1_R - (x_{s_1+a_1} + x_{s_1+1})\}\\ &\cup \{x_{t_1} - x_{t_1+2} : s_1 + 1 \le t_1 \le s - 1\} \cup \{1_R - (x_{s+a_2} + x_{s+2})\}.\end{aligned}$$

Similar to Case 2, we can reduce elements of $I_{e,s}$ to polynomials of the form $f' = h'_1[1_R - (x_0 + x_{s_1+1})] + h'_2[1_R - (x_{s_1+2} + x_{s+2})]$ such that h'_1, h'_2 do not contain $x_{t_0}(1 \le t_0 \le s_1)$ or $x_{t_1}(s_1 + 3 \le t_1 \le s + 1)$.

Let $f \in R$. Determine whether $f \in I_{e,s}$ or not as follows.

(1) In f, by substituting each $x_{t_0}(1 \le t_0 \le s_1)$ with x_0, and then substituting each $x_{s_1+2k+1}(s_1 + 3 \le s_1 + 2k + 1 \le s + 1)$ with x_{s_1+1} and substituting each $x_{s_1+2k+2}(s_1 + 4 \le s_1 + 2k + 2 \le s + 1)$ with x_{s_1+2}, we obtain $f' \in R$.

(2) Let $u_1 = x_0, u_2 = x_{s_1+1}, u_3 = x_{s_1+2}, u_4 = x_{s+2}$. Write

$$f' = f_0 + \sum_{1\le i\le 4} f_i u_i + \sum_{1\le i<j\le 4} f_{ij} u_i u_j + \sum_{1\le i<j<k\le 4} f_{ijk} u_i u_j u_k + f_{1234} u_1 u_2 u_3 u_4,$$

where $f_0, f_i, f_{ij}, f_{ijk}, f_{1234}$ in f' above do not contain any u_1, u_2, u_3, u_4.
By $u_1u_2, u_3u_4 \in I_{e,s}$, we have

$$\begin{aligned}f' \in I_{e,s} \Leftrightarrow f'' =& f_0 + f_1u_1 + f_2u_2 + f_3u_3 + f_4u_4 +\\ & f_{13}u_1u_3 + f_{14}u_1u_4 + f_{23}u_2u_3 + f_{24}u_2u_4 \in I_{e,s}\\ \Leftrightarrow f'' =& h_1[1_R - (u_1 + u_2)] + h_2[1_R - (u_3 + u_4)]\end{aligned}$$

for some $h_1, h_2 \in R$.

- Take $u_1 = u_3 = 1_R, u_2 = u_4 = 0_R$; we have $f_0 + f_1 + f_3 + f_{13} = 0_R$.
- Take $u_1 = u_4 = 1_R, u_2 = u_3 = 0_R$; we have $f_0 + f_1 + f_4 + f_{14} = 0_R$.
- Take $u_2 = u_3 = 1_R, u_1 = u_4 = 0_R$; we have $f_0 + f_2 + f_3 + f_{23} = 0_R$.
- Take $u_2 = u_4 = 1_R, u_1 = u_3 = 0_R$; we have $f_0 + f_2 + f_4 + f_{24} = 0_R$.

Now we can check that

$$f'' \in I_{e,s} \Leftrightarrow f_{13} = -(f_0 + f_1 + f_3), f_{14} = -(f_0 + f_1 + f_4),$$
$$f_{23} = -(f_0 + f_2 + f_3), f_{24} = -(f_0 + f_2 + f_4).$$

Indeed, when the right-hand side conditions hold, we can calculate that

$$f'' = f_0[1_R - (u_1 + u_2)(u_3 + u_4)] + (f_1u_1 + f_2u_2)[1_R - (u_3 + u_4)] +$$
$$(f_3u_3 + f_4u_4)[1_R - (u_1 + u_2)] \in I_{e,s}.$$

So, $I_{e,s}$ is a computable ideal of R.

Case 3.2. If $i_2 < i_1$, then the priority of $\mathcal{P}_{\langle e,i_2\rangle}$ is higher than $\mathcal{P}_{\langle e,i_1\rangle}$. In this case, two new generators, $1_R - (x_{s+a_2} + x_{s+2})$ and $x_{s+(a_2+1)_{\mathrm{mod}2}}$, are added into I_e at stage s of the construction, and the ideal $I_{e,s}$ of R is generated by

$$\{x_{t_0} - x_{t_0+1} : 0 \le t_0 \le s_1 - 1\} \cup \{1_R - (x_{s_1+a_1} + x_{s_1+1})\}$$
$$\cup \{x_{t_1} - x_{t_1+2} : s_1 + 1 \le t_1 \le s - 1\} \cup \{x_{s+(a_2+1)_{\mathrm{mod}2}}, 1_R - (x_{s+a_2} + x_{s+2})\},$$

where $s + (a_2 + 1)_{\mathrm{mod}2}$ equals the greatest number of the form $s_1 + 2k + 1$ in $[s_1 + 3, s + 1]$, and $s + a_2$ equals the greatest number of the form $s_1 + 2k + 2$ in $[s_1 + 4, s + 1]$. In particular, $x_{s+(a_2+1)_{\mathrm{mod}2}} - x_{s_1+1}$ and $x_{s+a_2} - x_{s_1+2}$ belong to $I_{e,s}$. Then, we have $x_{s_1+1} = x_{s+(a_2+1)_{\mathrm{mod}2}} + (x_{s_1+1} - x_{s+(a_2+1)_{\mathrm{mod}2}}) \in I_{e,s}$, and thus, the generator $1_R - (x_{s_1+a_1} + x_{s_1+1})$ appointed for $\mathcal{P}_{\langle e,i_1\rangle}$ is injured. Elements of $I_{e,s}$ are of the form

$$f = \sum_{t_0=0}^{s_1-1} g_{t_0}(x_{t_0} - x_{t_0+1}) + h_1[1_R - (x_{s_1+a_1} + x_{s_1+1})] +$$
$$\sum_{t_1=s_1+1}^{s-1} g_{t_1}(x_{t_1} - x_{t_1+2}) + h_2 x_{s+(a_2+1)_{\mathrm{mod}2}} + h_3[1_R - (x_{s+a_2} + x_{s+2})]$$

with $g_{t_0}, g_{t_1}, h_1, h_2, h_3 \in R$ for $0 \le t_0 < s_1$, $s_1 < t_1 < s$. In the expression of f,

(1) replace each $x_{t_0}(0 \le t_0 \le s_1)$ with 1_R,
(2) replace each $x_{s_1+2k+1}(s_1 + 1 \le s_1 + 2k + 1 \le s + 1)$ with 0_R,
(3) replace each $x_{s_1+2k+2}(s_1 + 4 \le s_1 + 2k + 2 \le s + 1)$ with x_{s_1+2},

We obtain a polynomial $f' = h_3'[1_R - (x_{s_1+2} + x_{s+2})]$, with h_3' being the corresponding polynomial obtained from h_3. Similar to Case 1 above, $I_{e,s}$ is computable.

Case 3.3. If $i_2 = i_1$, then $\mathcal{P}_{\langle e,i_1\rangle} = \mathcal{P}_{\langle e,i_2\rangle}$ requires attention again at current stage s. In this case, two new generators, $1_R - (x_{s+a_2} + x_{s+2})$ and x_{s+2}, are also added into I_e to ensure that x_{s+2} is equal to the zero of the quotient ring $R_e = R/I_e$ and that x_{s+a_2}, and thus, all x_{s_1+2k} in $[x_{s_1+2}, x_{s+1}]$ are equal to the identity of the quotient ring $R_e = R/I_e$. However, the previous generator $1_R - (x_{s_1+a_1} + x_{s_1+1})$ appointed for the requirement $\mathcal{P}_{\langle e,i_1\rangle}$ is still valid at stage s. Now, $I_{e,s}$ is the ideal of R generated by

$$\{x_{t_0} - x_{t_0+1} : 0 \le t_0 \le s_1 - 1\} \cup \{1_R - (x_{s_1+a_1} + x_{s_1+1})\} \cup$$
$$\{x_{t_1} - x_{t_1+2} : s_1 + 1 \le t_1 \le s - 1\} \cup \{1_R - (x_{s+a_2} + x_{s+2}), x_{s+2}\}.$$

Elements of $I_{e,s}$ are of the form

$$f = \sum_{t_0=0}^{s_1-1} g_{t_0}(x_{t_0} - x_{t_0+1}) + h_1[1_R - (x_{s_1+a_1} + x_{s_1+1})] + \sum_{t_1=s_1+1}^{s-1} g_{t_1}(x_{t_1} - x_{t_1+2}) + h_2[1_R - (x_{s+a_2} + x_{s+2})] + h_3 x_{s+2}$$

with $g_{t_0}, g_{t_1}, h_1, h_2, h_3 \in R$ for $0 \leq t_0 < s_1, s_1 < t_1 < s$. In the expression of f,

(1) replace each $x_{t_0} (1 \leq t_0 \leq s_1)$ with x_0,
(2) replace each $x_{s_1+2k+1} (s_1 + 3 \leq s_1 + 2k + 1 \leq s + 1)$ with x_{s_1+1},
(3) replace each $x_{s_1+2k+2} (s_1 + 2 \leq s_1 + 2k + 2 \leq s + 1)$ with 1_R,
(4) replace x_{s+2} with 0_R,

Then, we obtain a polynomial $f' = h_1'[1_R - (x_0 + x_{s_1+1})]$, with h_1' being the corresponding polynomial obtained from h_1. As in Case 1 above, $I_{e,s}$ is computable.

Case 4. When $m \geq 2$ with $s_1 < s_2 < \cdots < s_m \leq s$, as in Case 3 above, elements of $I_{e,s}$ can be expressed as finite R-linear sums of generators added by stage s of the construction. For any polynomial $f \in R$, by reducing generators of the form $x_{t+a} - x_{t+a'}$ with $a < a'$ added at stages where no $\mathcal{P}_{\langle e,i \rangle}$ required attention, of the form x_{t+b} added at stages where some valid $\mathcal{P}_{\langle e,i \rangle}$ was injured by higher priority requirements, and of the form $1_R - (x_{t+c} + x_{t+c'})$ and $x_{t+c'}$ with $c < c'$ at stages where some valid $\mathcal{P}_{\langle e,i \rangle}$ required attention again, we obtain a polynomial f' with $f - f' \in I_{e,s}$, such that $f' \in I_{e,s}$ if and only if it is a finite R-linear sum of reduced generators of valid $\mathcal{P}_{\langle e,- \rangle}$-requirements. At the end of stage s of the construction, let $\mathcal{P}_{\langle e,i_{n_1} \rangle} \prec \mathcal{P}_{\langle e,i_{n_2} \rangle} \prec \cdots \prec \mathcal{P}_{\langle e,i_{n_l} \rangle}$ all be valid $\mathcal{P}_{\langle e,- \rangle}$-requirements. Also assume that $\mathcal{P}_{\langle e,i_{n_j} \rangle}$ was valid at a previous stage s_{n_j} with the appointed generator $1_R - (x_{s_{n_j}+a_{n_j}} + x_{s_{n_j}+n_j})$ for $1 \leq j \leq l$. Then, $s_{n_1} < s_{n_2} < \cdots < s_{n_l} \leq s$ because of the priority of $\mathcal{P}_{\langle e,- \rangle}$-requirements. As in Case 3.1 above, for $1 \leq j \leq l$, let $1_R - (u_{2j-1} + u_{2j})$ be the reduced generator for $\mathcal{P}_{\langle e,i_{n_j} \rangle}$ with $u_{2j-1} - x_{s_{n_j}+a_{n_j}}, u_{2j} - x_{s_{n_j}+n_j} \in I_{e,s}$. Then, we also have $u_{2j-1}u_{2j} \in I_{e,s}$ for $1 \leq j \leq l$. Now we have

$$f' \in I_{e,s} \Leftrightarrow f' = \sum_{j=1}^{l} h_j[1_R - (u_{2j-1} + u_{2j})]$$

for some $h_j \in R$. Furthermore, for $1 \leq k \leq l$, $1 \leq j_1 < \cdots < j_k \leq l$, since we have $1_R - (u_{2j_1-1} + u_{2j_1}), \ldots, 1_R - (u_{2j_k-1} + u_{2j_k}) \in I_{e,s}$, we can check that

$$1_R - \prod_{t=1}^{k} (u_{2j_t-1} + u_{2j_t}) \in I_{e,s}.$$

Determine whether $f' \in I_{e,s}$ or not as follows:

(1) Write

$$f' = f_0 + \sum_{k=1}^{2l} \sum_{1 \leq i_1 < \cdots < i_k \leq 2l} f_{i_1 \ldots i_k} u_{i_1} \cdots u_{i_k},$$

where f_0 and $f_{i_1 \ldots i_k}$ in the expression do not contain any u_i for $1 \leq i \leq 2l$. By $u_{2j-1}u_{2j} \in I_{e,s}$ for $1 \leq j \leq l$, we have

$$f' \in I_{e,s} \Leftrightarrow f'' = f_0 + \sum_{k=1}^{l-1} \sum_{\substack{1\le j_1<\cdots<j_k\le l \\ 1\le t\le k, i_{j_t}\in\{2j_t-1,2j_t\}}} f_{i_{j_1}\ldots i_{j_k}} u_{i_{j_1}} \cdots u_{i_{j_k}} + \sum_{\substack{1\le t\le l \\ i_t\in\{2t-1,2t\}}} f_{i_1\cdots i_l} u_{i_1} \cdots u_{i_l} \in I_{e,s}.$$

(2) $f'' \in I_{e,s} \Leftrightarrow f'' = \sum_{j=1}^{l} h_j[1_R - (u_{2j-1} + u_{2j})]$ for some $h_j \in R$.
Suppose that $f'' \in I_{e,s}$. Fix $i_t \in \{2t-1, 2t\}$ with $t = 1, \ldots, l$. In the expression of f'' in (1), by setting

$$u_b = \begin{cases} 1_R, & \text{if } b \in \{i_t : 1 \le t \le l\}; \\ 0_R, & \text{otherwise.} \end{cases}$$

we obtain

$$(*) : f_0 + \sum_{k=1}^{l-1} \sum_{1\le j_1<\cdots<j_k\le l} f_{i_{j_1}\ldots i_{j_k}} + f_{i_1\ldots i_l} = 0_R.$$

Then, $f'' \in I_{e,s} \Leftrightarrow (*)$ holds for any $i_t \in \{2t-1, 2t\}$ with $t = 1, \ldots, l$.
Indeed, suppose that $(*)$ holds for any $i_t \in \{2t-1, 2t\}$, $t = 1, \ldots, l$, by replacing $f_{i_1\ldots i_l}$ in the expression of f'' in (1) with

$$-(f_0 + \sum_{k=1}^{l-1} \sum_{1\le j_1<\cdots<j_k\le l} f_{i_{j_1}\ldots i_{j_k}}),$$

similar to Case 3.1 above, we can directly calculate that

$$f'' = f_0[1_R - (u_1+u_2)(u_3+u_4)\cdots(u_{2l-1}+u_{2l})] + \sum_{k=1}^{l-1} \sum_{\substack{1\le j_1<\cdots<j_k\le l \\ 1\le t\le k, i_{j_t}\in\{2j_t-1,2j_t\}}} f_{i_{j_1}\ldots i_{j_k}} u_{i_{j_1}} \cdots u_{i_{j_k}} [1_R - \prod_{\substack{1\le j\le l \\ j\notin\{j_t:1\le t\le k\}}} (u_{2j-1}+u_{2j})] \in I_{e,s}.$$

This completes the proof of Lemma 2. □

A sequence $\langle B_e : e \in \mathbb{N}\rangle$ of c.e. subsets of natural numbers is uniformly computably enumerable if and only if the coding set $B = \{\langle e, x\rangle \in \mathbb{N}\times\mathbb{N} : x \in B_e\}$ is a c.e. (equivalently, a Σ_1^0) subset of the pairs of natural numbers. Similarly, we have the notion of uniformly computably enumerable sequence of c.e. ideals and of c.e. congruence relations of a computable ring.

For all $e \in \mathbb{N}$, let $I_e = \bigcup_{s\in\mathbb{N}} I_{e,s}$. Given $e \in \mathbb{N}, f \in R$, the polynomial $f \in I_e$ if and only if there exists a stage s of the construction, such that $f \in I_{e,s}$. We can proceed as in Lemma 2 to see whether $f \in I_{e,s}$; that is, the relation $f \in I_{e,s}$ is computable on f, e, s. Then, $f \in I_e$ is a Σ_1^0-property on e, f. This means that the sequence $\langle I_e : e \in \mathbb{N}\rangle$ of ideals of the computable ring R is uniformly computably enumerable. Then, the sequence of c.e. relations $\langle =_{S_e} : e \in \mathbb{N}\rangle$ of R is also uniformly computably enumerable. As before, $f =_{S_e} g$ if and only if $f - g \in I_e$ for any two polynomials $f, g \in R$.

Recall that $\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_e, \ldots$ is an effective listing of partial computable structures in the language $\mathcal{L}$ of rings together with a binary relation. As $\mathcal{L}$-structures, the c.e. ring $S_e = (R, +_R, -_R, \cdot_R, 0_R, 1_R, =_{S_e})$. Now, the sequence $\langle S_e : e \in \mathbb{N}\rangle$ is a uniformly computably enumerable sequence of $\mathcal{L}$-structures. More specifically, there is a computable function $\alpha : \mathbb{N} \to \mathbb{N}$, such that $S_e = \mathcal{A}_{\alpha(e)}$, which is the $\alpha(e)$-th partial computable $\mathcal{L}$-structure.

Lemma 3. *For all e, $\mathcal{P}_e$ is satisfied.*

Proof. $\mathcal{P}_e$ says that if $e \in \mathrm{Cof}$, then S_e is a semisimple ring. Assume that $e \in \mathrm{Cof}$, and let i be the least number with $\lim_s a^s_{e,i} = +\infty$; it suffices to show that $\mathcal{P}_{\langle e,i\rangle}$ is satisfied. Since $\lim_s a^s_{e,j} < +\infty$ for all $j < i$ (if any), there is a least stage t_e after which $\mathcal{P}_{\langle e,j\rangle}$ does not require attention for any $j < i$. $\mathcal{P}_{\langle e,i\rangle}$ has the highest priority after stage t_e, and it will require attention at infinitely many stages after stage t_e; let $s_{n_l} < s_{n_{l+1}} < \cdots$ denote such stages, where $s_{n_{j'}}\,(j' \geq l)$ is the $n_{j'}$-th stage at which $\mathcal{P}_{\langle e,i_{n_{j'}}\rangle} = \mathcal{P}_{\langle e,i\rangle}$ requires attention.

Suppose that $\mathcal{P}_{\langle e,i\rangle}$ is invalid at stage t_e, then it will be valid at stage $s_{n_l} > t_e$. Let $\mathcal{P}_{\langle e,i_{n_1}\rangle} \prec \cdots \prec \mathcal{P}_{\langle e,i_{n_l}\rangle} = \mathcal{P}_{\langle e,i\rangle}$ be all valid $\mathcal{P}_{\langle e,-\rangle}$-requirements at stage s_{n_l}. For all $1 \leq j \leq l$, let s_{n_j} be the stage at which $\mathcal{P}_{\langle e,i_j\rangle}$ became valid. We have appointed the generator $1_R - (x_{s_{n_j}+a_{n_j}} + x_{s_{n_j}+n_j})$ for $\mathcal{P}_{\langle e,i_{n_j}\rangle}$ at stage s_{n_j}. Let C be the set of all strings of length l of the form $\sigma = \sigma(0)\sigma(1)\cdots\sigma(l-1)$ with $\sigma(j-1) \in \{s_{n_j} + a_{n_j}, s_{n_j} + n_j\}$ for all $1 \leq j \leq l$. Clearly, there are 2^l many such strings. Furthermore, we have

$$1_R =_{S_e} \prod_{j=1}^{l}(x_{s_{n_j}+a_{n_j}} + x_{s_{n_j}+n_j}) =_{S_e} \sum_{\sigma\in C} x_{\sigma(0)}x_{\sigma(1)}\cdots x_{\sigma(l-1)}.$$

By $x_{s_{n_j}+a_{n_j}}x_{s_{n_j}+n_j} =_{S_e} 0_R$ for all $1 \leq j \leq l$, S_e can be decomposed as

$$S_e = \bigoplus_{\sigma\in C} S_e x_{\sigma(0)}x_{\sigma(1)}\cdots x_{\sigma(l-1)}.$$

We show that for any $\sigma \in C$, $S_e x_{\sigma(0)}x_{\sigma(1)}\cdots x_{\sigma(l-1)}$ is a simple S_e-module. For all $j' \geq l$, at stage $s_{n_{j'}}$, the highest priority $\mathcal{P}_{\langle e,i_{j'}\rangle} = \mathcal{P}_{\langle e,i\rangle}$ requires attention; all valid $\mathcal{P}_{\langle e,-\rangle}$-requirements with a priority lower than $\mathcal{P}_{\langle e,i\rangle}$ (if any) are injured, so for all $t \leq s_{n_{j'}} + n_{j'}$, we have

$$x_t \in_{S_e} \{x_{s_{n_j}+a_{n_j}}, x_{s_{n_j}+n_j} : 1 \leq j \leq l\} \cup \{0_R, 1_R\},$$

where for any $y \in R$ and $Y \subseteq R$, $y \in_{S_e} Y$ means that $y =_{S_e} z$ for some $z \in Y$. For any $t \in \mathbb{N}$, if $x_t \neq_{S_e} 0_R, 1_R$, then $x_t =_{S_e} x_{s_{n_j}+a_{n_j}}$ or $x_{s_{n_j}+n_j}$ for some $1 \leq j \leq l$; furthermore, we have

$$x_{\sigma(0)}x_{\sigma(1)}\cdots x_{\sigma(l-1)}x_t =_{S_e} \begin{cases} x_{\sigma(0)}x_{\sigma(1)}\cdots x_{\sigma(l-1)}, & \text{if } x_t =_{S_e} x_{\sigma(j-1)}; \\ 0_R, & \text{otherwise.} \end{cases}$$

Therefore, for any $f \in S_e$, when $x_{\sigma(0)}\cdots x_{\sigma(l-1)}f \neq_{S_e} 0_R$, we have that

$$x_{\sigma(0)}\cdots x_{\sigma(l-1)}f =_{S_e} qx_{\sigma(0)}\cdots x_{\sigma(l-1)}$$

for some nonzero $q \in \mathbb{Q}$, and thus, $S_e x_{\sigma(0)}\cdots x_{\sigma(l-1)} = \mathbb{Q}x_{\sigma(0)}\cdots x_{\sigma(l-1)}$. This shows that $S_e x_{\sigma(0)}\cdots x_{\sigma(l-1)}$ is a simple S_e-module. S_e is a semisimple ring. □

Lemma 4. *For all e, $\mathcal{N}_e$ is satisfied.*

Proof. $\mathcal{N}_e$ says that if $e \notin \mathrm{Cof}$, then S_e is not a semisimple ring. Assume that $e \notin \mathrm{Cof}$. For all i, $\lim_s a^s_{e,i} < +\infty$, so $\mathcal{P}_{\langle e,i\rangle}$ requires attention finitely often. Assume that $1_R =_{S_e} f_1 + \cdots + f_m$, where $f_1, \ldots, f_m$ are mutually orthogonal nontrivial idempotents of S_e. Now, $g = 1_R - (f_1 + \cdots + f_m) \in I_e$. Let s_g be the least stage with $g \in I_{e,s_g}$, and g can be reduced as a finite R-linear sum of generators appointed for valid $\mathcal{P}_{\langle e,-\rangle}$-requirements by stage s_g. Let $\mathcal{P}_{\langle e,i_{n_1}\rangle} \prec \cdots \prec \mathcal{P}_{\langle e,i_{n_l}\rangle}$ be such requirements with the corresponding generators $1_R - (x_{s_{n_1}+a_{n_1}} + x_{s_{n_1}+n_1}), \ldots, 1_R - (x_{s_{n_l}+a_{n_l}} + x_{s_{n_l}+n_l})$, respectively.

For each x_t occurring in $f_i (1 \leq i \leq m)$ in the expression of g, we have $x_t =_{S_e} x_{s_{n_j}+a_{n_j}}$ or $x_{s_{n_j}+n_j}$ for some $1 \leq j \leq l$. If all $\mathcal{P}_{\langle e,i_{n_j}\rangle} (1 \leq j \leq l)$ are injured later, then f_i becomes a constant in S_e. This contradicts the nontriviality of f_i, i.e., $f_i \neq_{S_e} 0_R, 1_R$. So at least one $\mathcal{P}_{\langle e,i_{n_j}\rangle}$ with $1 \leq j \leq l$ will be valid forever, and we can further reduce $g = 1_R - (f_1 + \cdots + f_m)$ to $g' = 1_R - (f'_1 + \cdots + f'_m)$, which is an R-linear sum of generators of the form $1_R - (x_{s+c} + x_{s+c'})$ appointed for valid $\mathcal{P}_{\langle e,-\rangle}$-requirements that cannot be injured later. Without loss of generality, we assume that $\mathcal{P}_{\langle e,i_{n_j}\rangle} (1 \leq j \leq l)$ is not injured after stage s_g. Since each $\mathcal{P}_{\langle e,i_{n_j}\rangle}$ requires attention finitely often, there is a stage $t_e \geq s_g$ after which $\mathcal{P}_{\langle e,i_{n_j}\rangle}$ does not require attention for any $1 \leq j \leq l$.

Let M be the number of $\mathcal{P}_{\langle e,-\rangle}$-requirements that have required attention by stage t_e, then $t_e \geq s_M$, the M-th stage at which some $\mathcal{P}_{\langle e,-\rangle}$ requires attention.

(1) If no $\mathcal{P}_{\langle e,-\rangle}$-requirements requires attention after stage t_e, at any stage $t \geq s_M + M$, we have added $x_t - x_{t+(M+1)}$ into I_e; in particular,

$$x_{s_M+M} =_{S_e} x_{s_M+M+(M+1)} =_{S_e} \cdots =_{S_e} x_{s_M+M+k(M+1)} =_{S_e} \cdots .$$

For any f_i in the expression of $g = 1_R - (f_1 + \cdots + f_m)$, $f_i x_{s_M+M} \neq_{S_e} 0_R$, and $S_e f_i x_{s_M+M}$ is a nonzero proper submodule of $S_e f_i$.

(2) Otherwise, there is at least one $\mathcal{P}_{\langle e,-\rangle}$-requirement that requires attention after stage t_e. Let $\mathcal{P}_{\langle e,\widetilde{i}\rangle}$ be such a requirement with $\widetilde{i}$ least, i.e., $\mathcal{P}_{\langle e,\widetilde{i}\rangle}$ has the highest priority after stage t_e. Let $s_{l'}$ be the least stage $> t_e$ at which $\mathcal{P}_{\langle e,\widetilde{i}\rangle}$ requires attention; $s_{l'}$ also denote the l'-th stage at which some $\mathcal{P}_{\langle e,-\rangle}$-requirement requires attention. Note that $\mathcal{P}_{\langle e,\widetilde{i}\rangle}$ has a priority lower than $\mathcal{P}_{\langle e,i_{n_j}\rangle}$ for all $1 \leq j \leq l$.

- If $\mathcal{P}_{\langle e,\widetilde{i}\rangle}$ was invalid at the end of stage $s_{l'} - 1$, then it becomes valid at stage $s_{l'}$, and we have $1_R - (x_{s_{l'}+a_{l'}} + x_{s_{l'}+l'}) \in I_{e,s_{l'}}$. In this case, the generator will never be injured, so $x_{s_{l'}+l'} \neq_{S_e} 0_R$. For any f_i in the expression of $g = 1_R - (f_1 + \cdots + f_m)$, $S_e f_i x_{s_{l'}+l'}$ is a nonzero proper submodule of $S_e f_i$.
- Otherwise, $\mathcal{P}_{\langle e,\widetilde{i}\rangle}$ was valid at some stage $s_{l''} < s_{l'}$ with an appointed generator $1_R - (x_{s_{l''}+a_{l''}} + x_{s_{l''}+l''})$. $\mathcal{P}_{\langle e,\widetilde{i}\rangle}$ has not been injured since stage $s_{l''}$, and we have $s_{n_1} < \cdots < s_{n_l} < s_{l''} < s_{l'}$. In this case, $S_e f_i x_{s_{l''}+l''}$ is a nonzero proper submodule of $S_e f_i$.

We have shown that $S_e f_i$ contains nonzero proper submodules. This implies that $S_e f_i$ is not a simple S_e-module. Therefore, $S_e = S_e f_1 \oplus \cdots \oplus S_e f_m$ is not a direct sum of simple S_e-modules. S_e is not a semisimple ring.

We have constructed a uniformly c.e. sequence $\langle S_e : e \in \mathbb{N}\rangle$ of c.e. rings, such that $e \in \mathrm{Cof} \Leftrightarrow S_e$ is a semisimple ring. Let $\alpha : \mathbb{N} \to \mathbb{N}$ be a computable function, such that $S_e = \mathcal{A}_{\alpha(e)}$ for all e, and let X denote the index set of c.e. semisimple rings. Then we have $e \in \mathrm{Cof} \Leftrightarrow \alpha(e) \in X$. So, the Σ^0_3-complete set Cof is m-reducible to X, and the latter set X is Σ^0_3-complete.

This completes the proof of Theorem 1. □

4. Applications

Based on the complexity results of semisimple rings developed in Section 3 above, we continue to study other related classes of rings.

Fix the effective listing $\mathcal{A}_0, \mathcal{A}_1 \ldots, \mathcal{A}_e, \ldots$ of all partial computable structures in the language of rings together with a binary relation. Recall that being a c.e. ring for $\mathcal{A}_e$ is a Π^0_2 property on e. In the following, if $S = (R, +_R, -_R, \cdot_R, 0_R, 1_R, =_S)$ is a c.e. ring with $(R, +_R, -_R, \cdot_R, 0_R, 1_R)$ a computable ring and $=_S$ a Σ^0_1 congruence relation on R, which is also the equality relation of the ring S, then we simply write $S = (R, =_S)$ for convenience.

4.1. Fields and Local Rings

We first consider the complexity of c.e. fields. A commutative ring R is a field if, for any nonzero $r \in R$, there is an element $s \in R$, such that $sr = 1_R$.

Proposition 3. *The index set* $\{e \in \mathbb{N} : \mathcal{A}_e$ *is a c.e. field*$\}$ *is* Π^0_2.

Proof. Let $S = (R, =_S)$ be a c.e. ring with R a computable ring and $=_S$ a Σ^0_1 equality relation. The ring S is a field if and only if the following two properties hold:

(1) $(\forall a, b \in R)\ [a \cdot_R b =_S b \cdot_R a]$, which is Π^0_2;
(2) $(\forall a \in R)(\exists b \in R)\ [a =_S 0_R \vee b \cdot_R a =_S 1_R]$, which is Π^0_2.

So, being a field for a c.e. ring is a Π^0_2 property. Together with the Π^0_2 property of c.e. rings, the index set of c.e. fields is Π^0_2. □

Fields are typical examples of local rings, where a ring R is left local if the sum of any two non-left invertible elements of R is still non-left invertible.

Proposition 4. *The index set* $\{e \in \mathbb{N} : \mathcal{A}_e$ *is a c.e. left local ring*$\}$ *is* Π^0_2.

Proof. A c.e. ring $S = (R, =_S)$ is left local if and only if, for any $a, b \in R$, one of the following (1)–(3) holds:

(1) $(\exists x \in R)[x \cdot_R a =_S 1_R]$;
(2) $(\exists y \in R)[y \cdot_R b =_S 1_R]$;
(3) $(\forall z \in R)(z \cdot_R (a +_R b) \neq_S 1_R)$.

Both (1) and (2) are Σ^0_1 properties because the bracket relations are Σ^0_1. (3) is a Π^0_1 property because the bracket relation is Π^0_1. So, the property of c.e. left local rings is Π^0_2, and the index set of c.e. left local rings is Π^0_2. □

In Theorem 2 [25], we have already obtained the optimal complexity results for computable local rings and computable fields. More precisely, we proved that the index set of computable local rings (respectively, computable fields) is Π^0_2-complete. Since computable rings are automatically computably enumerable, we immediately have corresponding results on c.e. fields and on c.e. local rings.

Corollary 1. *The index set* $\{e \in \mathbb{N} : \mathcal{A}_e$ *is a c.e. field*$\}$ *is* Π^0_2*-complete.*

Corollary 2. *The index set* $\{e \in \mathbb{N} : \mathcal{A}_e$ *is a c.e. left local ring*$\}$ *is* Π^0_2*-complete.*

4.2. Finite Direct Products of Fields

A ring R with identity 1_R is a finite direct product of fields if it can be decomposed as

$$R = e_1Re_1 \oplus e_2Re_2 \oplus \cdots \oplus e_nRe_n$$

for some pairwise orthogonal idempotents $e_1, e_2, \ldots, e_n \in R$, such that $1_R = e_1 +_R e_2 +_R \cdots +_R e_n$, and for $1 \leq i \leq n$, e_i is central in R (i.e., $x \cdot_R e_i = e_i \cdot_R x$ for all $x \in R$), and $e_iRe_i = \{e_i \cdot_R x \cdot_R e_i : x \in R\}$ is a field with identity e_i.

Proposition 5. *The index set* $\{e \in \mathbb{N} : \mathcal{A}_e$ *is a c.e. finite direct product of fields*$\}$ *is* Σ^0_3.

Proof. A c.e. ring $S = (R, =_S)$ is a finite direct product of fields if and only if there are elements $x_1, \ldots, x_n \in R$, such that the following properties hold: for $1 \leq i, j \leq n$,

(1) $x_i \neq_S 0_R$;

(2) $1_R =_S x_1 +_R \cdots +_R x_n$, $x_i =_S x_i^2$, $x_i \cdot_R x_j =_S 0_R$ for $i \neq j$;
(3) x_i is central in S, i.e., for all $y \in R$, $y \cdot_R x_i =_S x_i \cdot_R y$;
(4) $x_i S x_i$ is a field with identity x_i.

(1) is Π_1^0, (2) is Σ_1^0, (3) is Π_2^0, and by Proposition 3, (4) is also Π_2^0. So, being a finite direct product of fields for a c.e. ring is a Σ_3^0 property, and the index set of such c.e. rings is Σ_3^0. □

Based on the proof of Theorem 1 above, we have a corresponding result for c.e. finite direct product of fields.

Corollary 3. *The index set of c.e. finite direct product of fields is Σ_3^0-complete.*

Proof. Consider the uniformly c.e. sequence of c.e. rings $\langle S_e : e \in \mathbb{N} \rangle$ constructed in the proof of Theorem 1 above. By Lemma 3, when $e \in \mathrm{Cof}$, the ring S_e is isomorphic to a finite direct product of the rational field $\mathbb{Q}$. On the other hand, by Lemma 4, when $e \notin \mathrm{Cof}$, S_e is not a finite direct product of fields. Hence, the Σ_3^0-complete set Cof is m-reducible to the index set of c.e. finite direct product of fields, and the latter is Σ_3^0-complete. □

4.3. Semiperfect Rings

We now consider the complexity of the more general semiperfect rings. Recall that a ring R is semiperfect if the identity 1_R can be written as $1_R = e_1 +_R \cdots +_R e_n$ for some pairwise orthogonal idempotents $e_1, \ldots, e_n \in R$, such that $e_i R e_i$ is a left local ring with identity e_i for $1 \leq i \leq n$.

Similar to Proposition 5, it is not hard to see the following upper bounded on the complexity of semiperfect rings.

Proposition 6. *The index set $\{e \in \mathbb{N} : \mathcal{A}_e$ is a c.e. semiperfect ring$\}$ is Σ_3^0.*

Proof. A c.e. ring $S = (R, =_S)$ is semiperfect if and only if there are elements $x_1, \ldots, x_n \in R$, such that the following properties hold:

(1) $x_i \neq_S 0_R$;
(2) $1_R =_S x_1 +_R \cdots +_R x_n$, $x_i =_S x_i^2$, $x_i \cdot_R x_j =_S 0_R$ for $i \neq j$;
(3) $x_i S x_i$ is a left local ring with identity x_i, which is a Π_2^0 property by Proposition 4.

(1) is Π_1^0, (2) is Σ_1^0, and (3) is Π_2^0. Thus, the property of c.e. semiperfect rings is Σ_3^0, and the index set of c.e. semiperfect rings is Σ_3^0. □

Finally, Theorem 1 also implies the corresponding result of c.e. semiperfect rings.

Corollary 4. *The index set of c.e. semiperfect rings is Σ_3^0-complete.*

Proof. Let $\langle S_e : e \in \mathbb{N} \rangle$ be the sequence of c.e. rings in the proof of Theorem 1. It suffices to check that $e \in \mathrm{Cof}$ if and only if S_e is semiperfect for all e. For $e \in \mathrm{Cof}$, S_e is a semisimple ring, which is already semiperfect. For $e \notin \mathrm{Cof}$, suppose that $1_R =_{S_e} f_1 +_R \cdots +_R f_m$ is a sum of mutually orthogonal nontrivial idempotents $f_1, \ldots, f_m$ of S_e. For each $1 \leq i \leq m$, according to Lemma 4, take a large number M_i such that $x_{M_i} \neq_{S_e} x_n$ for all x_n appearing in f_i. Then, $g = x_{M_i} f_i$ is not invertible in the ring $f_i S_e f_i$ with identity f_i. Indeed, if g is invertible, let $f_i r f_i$ be the inverse of g in $f_i S_e f_i$, then $f_i =_{S_e} g f_i r f_i =_{S_e} x_{M_i} r f_i$ and x_{M_i} appears in f_i, which is a contradiction. Similarly, $h = (1_R - x_{M_i}) f_i$ is not invertible either. However, their sum $g + h =_{S_e} f_i$ is invertible in $f_i S_e f_i$. This shows that $f_i S_e f_i$ is not left local with identity f_i. S_e is not a semiperfect ring if $e \notin \mathrm{Cof}$. □

5. Conclusions

Based on the idea of computably enumerable (c.e.) universal algebras, we defined c.e. rings as quotient rings of computable rings modulo c.e. congruence relations. First, by identifying c.e. rings as structures in the expanded language of rings with an additional binary relation, we formalized the problem of c.e. rings by a corresponding index set. We studied the problem of c.e. semisimple rings and obtained the optimal complexity of the problem by proving that the index set of c.e. semisimple rings is Σ_3^0-complete. Finally, we applied the results on semisimple rings to other closely connected classes of rings; in particular, we obtained the same complexity results for more general semiperfect rings.

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