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Circular Chromatic Number of Signed Planar Graphs Without Cycles of Length 4 to 9

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Abstract: Given a signed graph (G,σ) and a positive real number r, if there exists a vertex mapping $c : V(G) \to [0,r)$ satisfying that for every positive edge wx, $1 \leq |c(w) - c(x)| \leq r-1$ and for every negative edge wx, $|c(w) - c(x)| \leq \frac{r}{2} - 1$ or $|c(w) - c(x)| \geq \frac{r}{2} + 1$, then (G,σ) admits a circular r-coloring. We use $\chi_c(G,\sigma)$ to represent the circular chromatic number of (G,σ), which is the minimum r, such that a circular r-coloring of (G,σ) exists. This paper proves that $\chi_c(G,\sigma) < 4$, where (G,σ) is a simple signed planar graph containing no cycles of length 4 to 9. Moreover, we establish an upper bound for the chromatic number of such a graph to be $4 - \frac{2}{\lfloor \frac{v(G)+1}{2} \rfloor}$.

Keywords: signed planar graphs; circular colorings; strongly connected orientations

MSC: 05C15; 05C22

1. Introduction

A graph G is considered to have a *proper s-coloring* if there is a mapping $c : V(G) \to \{1,2,\ldots,s\}$ satisfying that for every edge $wx \in E(G)$, the vertices w and x are assigned different colors, which means that $c(w) \neq c(x)$. The study of coloring problems on planar graphs can be traced back to the four-color theorem, which asserts that each planar graph has a proper 4-coloring. In 1958, Grötzsch [1] proposed a more specific result: every triangle-free planar graph has proper 3-coloring. Based on the above results, we are devoted to seek sufficient conditions such that planar graphs can be properly colored with, at most, 3 colors. In 1988, Vince [2] introduced the concept of circular coloring, which can be regarded as a refinement of proper coloring of graphs. More results about circular coloring in planar graphs are presented in [2,3].

Recently, Naserasr et al. [4] proposed the concept of circular coloring of signed graphs, which can be viewed as an extension of the concept of circular coloring in graphs. Let (G,σ) be a signed graph that every edge has a mapping $\sigma : E(G) \to \{+,-\}$.

Definition 1 ([4]). *Let (G,σ) be a signed graph and s,t be two integers with s being even and $s \geq 2t > 0$. A* circular $\frac{s}{t}$-coloring *of (G,σ) is a vertex mapping $c : V(G) \to \{0,1,\ldots,s-1\}$ satisfying that for every edge wx, when $\sigma(wx) = +$,*

$$t \leq |c(w) - c(x)| \leq s - t,$$

and when $\sigma(wx) = -$,

$$|c(w) - c(x)| \leq \frac{s}{2} - t \text{ or } |c(w) - c(x)| \geq \frac{s}{2} + t.$$

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The circular chromatic number of (G,σ) *is defined as*

$$\chi_c(G,\sigma) := \inf\{\frac{s}{t} : (G,\sigma) \text{ has a circular } \frac{s}{t}\text{-coloring}\}.$$

In [4], Naserasr et al. proposed that each simple signed bipartite planar graph (G,σ) with the maximum degree of all vertices in one partite set being smaller than or equal to 2 satisfies that $\chi_c(G,\sigma) \leq \frac{16}{5}$, which is a rephrased version of the Four-Color Theorem. Furthermore, they showed that when (G,σ) is a simple signed bipartite planar graph, $\chi_c(G,\sigma) \leq 4 - \frac{4}{\lfloor \frac{v(G)+2}{2} \rfloor}$. The upper bound is tight when $v(G) \geq 2$. Recently, Li et al. [5] provided some results about the circular chromatic number concerning a signed graph with a high girth. The *girth* of a graph G refers to the length of the shortest cycle of G, and we denote it as $g(G)$.

Theorem 1 ([5]). *Let* (G,σ) *be a signed planar graph and* k *be an integer with* $k \geq 2$.

(a) *When* $g(G) \geq 3k-1$, *we have* $\chi_c(G,\sigma) \leq \frac{2k}{k-1}$.

(b) *When* $g(G) \geq 3k$, *we have* $\chi_c(G,\sigma) < \frac{2k}{k-1}$.

(c) *When* $g(G) \geq 3k+1$, *we have* $\chi_c(G,\sigma) \leq \frac{4k+2}{2k-1}$.

Thus, we can see that $\chi_c(G,\sigma) < 4$ when $g(G) \geq 6$. Restricted to signed planar graphs, we propose the following result. Here, we mainly focus on considering the chromatic number for a simple signed planar graph containing no cycles of length $4, 5, 6, 7, 8,$ *and* 9.

Theorem 2. *For a simple signed planar graph without cycles of length from* 4 *to* 9, *the chromatic number is strictly less than* 4.

Now, we provide the structure of the rest of this work. Next, we propose some fundamental notations and the concept of circular flows of signed graphs, which is the duality of circular colorings in signed graphs. Then, we introduce some results about the relationship between circular flows and strongly-connected orientations satisfying some boundary conditions. Moreover, using the preparations, we focus on a particular property associated with the wheel graph W_{10}. In Section 3, we provide the main structural theorem (Theorem 4) of this paper, which we use to prove Theorem 2. Subsequently, to prove Theorem 4, we mainly explore some properties of a minimum counterexample (G,σ), provide some forbidden configurations of (G,σ), and utilize the discharging method to derive a contradiction.

2. Preliminary

In this paper, graphs may include multiple edges but no loops. Given a planar graph $G = (V,E,F)$, let $v(G)$ ($e(G)$, or $f(G)$) represent the number of vertices (edges or faces, respectively). Given a vertex w of G, if $d(w) = t$ ($d(w) \geq t$, $d(w) \leq t$, respectively), then we call w a t-vertex (t^+-vertex, t^--vertex, respectively). For a planar graph, an edge e or a vertex w is incident to a face f when e or w lies on the boundary of f. Let f_1 and f_2 be two faces, if there is at least an edge lying on both f_1 and f_2, then we call f_1 and f_2 adjacent.

For orientation D on a graph G, let $d_D^+(w)$ and $d_D^-(w)$ represent the number of edges out from w and into w for each $w \in V(G)$. We call an orientation of a connected graph *strongly connected* when each edge is in a directed cycle.

In [6], Tutte proposed the dual relationship between integer flows and colorings, which link problems of flows and colorings. Therefore, we have a new perspective on the study of coloring problems. Recently, Li et al. [5] proposed the definition of circular flows of signed graphs, which is the dual concept of circular colorings in the signed graphs studied in [4].

Definition 2 ([5]). *Let (G,σ) be a signed graph and s,t be two integers with s being even and $s \geq 2t > 0$. A* circular $\frac{s}{t}$-flow *in (G,σ) is defined as a pair (D,f), where D is an orientation of G and f is a mapping satisfying that for each $e \in E(G)$,*

(1) when $\sigma(e) = +$, we have $|f(e)| \in \{t, t+1, \ldots, s-t\}$;
(2) when $\sigma(e) = -$, we have $|f(e)| \in \{0, 1, \ldots, \frac{s}{2}-t\} \cup \{\frac{s}{2}+t, \ldots, s-1\}$;
(3) the out-flow of x is equal to the in-flow of x for each $x \in V(G)$.

The circular flow index of (G,σ) is defined as follows:

$$\phi_c(G,\sigma) := \min\{\frac{s}{t} \colon (G,\sigma) \text{ has a circular } \frac{s}{t}\text{-flow}\}.$$

Thus, in the dual setting, Theorem 2 can be restated that the circular flow index is strictly less than 4 for a signed planar graph (G,σ) that is 3-edge-connected and has no edge-cut with size $\{4,5,6,7,8,9\}$. Here, we need to mention that if (G,σ) has an edge-cut with size 1 or 2, then the dual graph of (G,σ) contains a loop or multiple edges, which does not satisfy the simple graph condition of Theorem 2. It has been introduced in [7] that circular r-flow of signed graphs with $r < \frac{2k}{k-1}$ is related to graphs with a special strongly connected orientation. Thus, when proving that $\phi_c(G,\sigma) < 4$, we can transform the problem into seeking a special strongly connected orientation of $2G$, as shown in the following theorem. Let $p^+(w)$ represent the number of positive edges that are incident to a vertex w in (G,σ).

Theorem 3 ([7]). *Let (G,σ) be a signed graph. If $2G$ admits a strongly connected orientation D satisfying that for each $w \in V(2G)$, $d_D^+(w) - d_D^-(w) \equiv 4 \cdot p^+(w) \pmod 8$, then $\phi_c(G,\sigma) < 4$.*

The orientation introduced in Theorem 3 is a specific instance of a strongly connected orientation satisfying some boundary conditions. Now, we introduce the related definitions proposed in [8].

Definition 3 ([8]). *Given a graph G,*

(i) if there exists a function $\beta : V(G) \to \{0, \pm 1, \ldots, \pm 4\}$ satisfying that

$$\beta(w) \equiv d(w) \pmod 2 \text{ and } \sum_{w \in V(G)} \beta(w) \equiv 0 \pmod 8,$$

then β is a parity-compliant $\mathbb{Z}_8$-boundary, *abbreviated as* $\mathbb{Z}_8$-pc-boundary.
(ii) An orientation D of G is said to be an $(8,\beta)$-orientation *if $d_D^+(w) - d_D^-(w) \equiv \beta(w) \pmod 8$ for every $w \in V(G)$ based on a $\mathbb{Z}_8$-pc-boundary β of G.*

Based on the above definition, we propose two sets of graphs that are frequently used in this paper.

Definition 4. *(i) Let $\mathcal{SZ}_4$ be the set of every connected graph admitting an $(8,\beta)$-orientation, where β is an arbitrary $\mathbb{Z}_8$-pc-boundary.*
(ii) Let $\mathcal{SC}_4$ be the set of every connected graph admitting a strongly connected $(8,\beta)$-orientation, where β is an arbitrary $\mathbb{Z}_8$-pc-boundary.

For a signed graph (G,σ), let β be a mapping of $2G$ satisfying that $\beta(w) \equiv 4 \cdot p^+(w) \pmod 8$ for every $w \in V(2G)$. Note that β is naturally a $\mathbb{Z}_8$-pc-boundary β in the graph $2G$. Thus, if we prove that $2G \in \mathcal{SC}_4$, then we can directly obtain the orientation on $2G$ stated in Theorem 3. In this work, we propose a general result (Theorem 4) about strongly connected orientations with an arbitrary boundary and we prove it in Theorem 3.

Note that contraction and lifting are two useful operations when exploring problems about flows and group connectivity. An operation of contracting an edge wx means that we identify the vertices w and x, and remove the resulting loop. Furthermore, we refer to the derived graph as G/H, which is formed by contracting all edges of a connected subgraph H of G. Let (ux, xw) denote an *edge-pair* at a vertex x for any two edges ux, xw. The operation of *lifting* an edge-pair (ux, xw) means that we delete ux, xw and create a new edge between u and w. Now, we introduce some results related to modifying boundaries when contracting a connected subgraph.

Lemma 1 ([9]). *Given a graph G and a $\mathbb{Z}_8$-pc-boundary β of G, let $G' := G/H$ where H is a connected subgraph of G, and let w_H be the new vertex that results from contraction. We establish a new function $\beta' : V(G') \to \{0, \pm 1, \ldots, \pm 4\}$, where*

$$\beta'(x) = \begin{cases} \sum\limits_{y \in V(H)} \beta(y) \pmod 8, & \text{if } x = w_H, \\ \beta(x), & \text{if } x \in V(G') \backslash \{w_H\}. \end{cases}$$

Thus, β' is a $\mathbb{Z}_8$-pc-boundary of G' and we call β' a corresponding $\mathbb{Z}_8$-pc-boundary of G'.

Lemma 2. *For a graph G, a vertex $x \in V(G)$, and a $\mathbb{Z}_8$-pc-boundary β of G, we do some lifting at x to form a new graph G'. When G' contains an induced subgraph H which is a member of $\mathcal{SZ}_4$ and G'/H has an $(8, \beta')$-orientation D_1, we can extend D_1 to obtain an $(8, \beta)$-orientation of G.*

Proof. Let $N := \{w \colon w \in V(H)$, and w has at least a neighbour in $G \setminus H\}$. We define a mapping β_1 on $V(H)$ as follows,

$$\beta_1(v) = \begin{cases} \beta(w) - \left(d^+_{D_1}(w) - d^-_{D_1}(w)\right) \pmod 8, & \text{if } w \in N, \\ \beta(v), & \text{otherwise.} \end{cases}$$

Note that β_1 is a $\mathbb{Z}_8$-pc-boundary of H. As $H \in \mathcal{SZ}_4$, we know that H admits an $(8, \beta_1)$-orientation D_2. We notice that some edges in H are obtained by doing some lifting at x. Now, we define an orientation D'_2 based on D_2. For each edge u_1u_2 in H formed by lifting at x and oriented from u_1 to u_2 under D_2, we define a directed path u_1xu_2 in D'_2; and for the other edges e in H, let $D'_2(e) = D_2(e)$. Note that $d^+_{D'_2}(u) = d^+_{D_2}(u)$ and $d^-_{D'_2}(u) = d^-_{D_2}(u)$ for each $u \in V(H)$.

Let $D = D_1 \cup D'_2$. We claim that D is an $(8, \beta)$-orientation of G.

- For each $w \in N$, $d^+_D(w) - d^-_D(w) = \left(d^+_{D'_2}(w) - d^-_{D'_2}(w)\right) + \left(d^+_{D_1}(w) - d^-_{D_1}(w)\right) \equiv \beta_1(w) + \left(d^+_{D_1}(w) - d^-_{D_1}(w)\right) \equiv \beta(w) \pmod 8$.
- For each $u \in V(H) \setminus N$, $d^+_D(u) - d^-_D(u) = d^+_{D'_2}(u) - d^-_{D'_2}(u) \equiv \beta_1(u) = \beta(u) \pmod 8$.
- For each $u \in V(G) \setminus V(H)$, $d^+_D(u) - d^-_D(u) = d^+_{D_1}(u) - d^-_{D_1}(u) \equiv \beta'(u) = \beta(u) \pmod 8$.

Hence, we complete the proof. □

More generally, it has been proved in [10,11] that contraction is preserved with respect to $\mathcal{SZ}_4$ and $\mathcal{SC}_4$.

Proposition 1 ([10,11]). *For a graph G and its connected subgraph G', if G' and G/G' belong to $\mathcal{SZ}_4$ (or $\mathcal{SC}_4$), then $G \in \mathcal{SZ}_4$ (or $\mathcal{SC}_4$).*

Furthermore, we introduce another set of graphs that can be contractible related to $\mathcal{SC}_4$, which is proposed in [11].

Definition 5. *(i) A graph G is a* supergraph *of a graph H if $H \subsetneq G$, and there are two vertices w, x of H satisfying that $G - E(H)$ has a (w, x)-path.*

(ii) Given a graph H, let G be a supergraph of H. Given an arbitrary $\mathbb{Z}_8$-pc-boundary β of G and the corresponding $\mathbb{Z}_8$-pc-boundary β' of G/H, we call H weakly contractible *with respect to $\mathcal{SC}_4$ if G can have a strongly-connected $(8, \beta)$-orientation by extending a strongly connected $(8, \beta')$-orientation on G/H.*

(iii) We define $\mathcal{W}_4$ as the set of all graphs that are weakly contractible with respect to $\mathcal{SC}_4$.

According to Definition 5, we know that graphs in $\mathcal{W}_4$ have nice "contractible" properties concerning $\mathcal{SC}_4$.

Proposition 2 ([11]). *For a graph G and its connected subgraph G', when $G' \in \mathcal{W}_4$ and $G/G' \in \mathcal{SC}_4$, it follows that $G \in \mathcal{SC}_4$.*

It has been introduced in [11] that graphs in $\mathcal{W}_4$ are closed to graphs in $\mathcal{SC}_4$.

Lemma 3 ([11]). *A graph H is in $\mathcal{W}_4$ if and only if $H + wx$ belongs to $\mathcal{SC}_4$, where w and x are two distinct vertices of H.*

According to the definition of graphs in $\mathcal{SC}_4$, together with Hamiltonian properties, we can easily derive the next lemma.

Lemma 4 ([11]). *A graph belongs to $\mathcal{SC}_4$ if it has a Hamiltonian cycle C such that the resulting graph formed by deleting the edges of C belongs to $\mathcal{SZ}_4$.*

Graphs on at most four vertices of $\mathcal{SZ}_4$ are characterized in [9]. Applying the above lemmas, we obtain some graphs belonging to $\mathcal{W}_4$ and $\mathcal{SC}_4$.

Observation 1.
- [9] Observe that $\{3K_2\} \cup \{T_{2,2,2}, T_{1,2,3}\} \subset \mathcal{SZ}_4$.
- Moreover, $\{4K_2\} \cup \{T_{2,3,3}\} \subset \mathcal{W}_4$.

Now, we begin to prove a property of a wheel W_{10}, which is a forbidden configuration related to the proof of a structural result (Theorem 4 in Section 3) in this article. By applying the relationship between the graphs in $\mathcal{W}_4$ and $\mathcal{SC}_4$, we can find more graphs in $\mathcal{W}_4$, such as the graph $2W_{10}$.

Lemma 5. *The graph $2W_{10} \in \mathcal{W}_4$.*

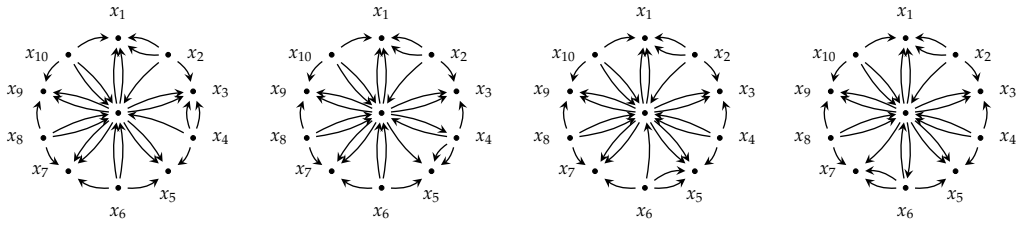
Proof. According to Lemma 3, we can finish the proof by verifying that $2W_{10} + wv \in \mathcal{SC}_4$, where w, v are two arbitrary vertices of $V(W_{10})$. Let $V(W_{10}) = \{x, x_1, x_2, \ldots, x_{10}\}$, where x is the central vertex of W_{10} and each x_i is on the 10-cycle of W_{10}.

When adding an edge wv to $2W_{10}$ with w and v being adjacent, we aim to find a Hamiltonian cycle C of $2W_{10} + wv$, verify that the resulting graph formed from deleting C belongs to $\mathcal{SZ}_4$, and obtain that $2W_{10} + wv \in \mathcal{SC}_4$ by Lemma 4. For the graph $2W_{10} + x_ix_{i+1}$ with $i \in [10]$, we choose the Hamiltonian cycle $x_ix_{i+1}x_{i+2}\ldots x_{i-2}xx_{i-1}x_i$. Let G_1 be the resulting graph by deleting this Hamiltonian cycle. Note that G_1 contains $T_{2,2,2}$, which belongs to $\mathcal{SZ}_4$. Let $G_2 = G_1/T_{2,2,2}$, and now, there is a $3K_2$ in G_2. We denote by G_i the graph formed from contracting a $3K_2$ of G_{i-1}, where $3 \leq i \leq 8$. Note that $G_8 = T_{2,2,2}$ which belongs to $\mathcal{SZ}_4$. According to Proposition 1, $G_1 \in \mathcal{SZ}_4$, and thus by Lemma 4, $2W_{10} + x_ix_{i+1} \in \mathcal{SC}_4$. For the graph $2W_{10} + xx_i$ with $1 \leq i \leq 10$, we prove that $2W_{10} + xx_i \in \mathcal{SC}_4$ by doing a similar consideration on deleting a Hamiltonian cycle $xx_{i-1}x_ix_{i+1}\ldots x_{i-2}x$.

Using symmetry, we consider the left cases when adding an edge x_1x_i to $2W_{10}$ with $3 \leq i \leq 6$. Now, we still consider the resulting graph formed by deleting a Hamiltonian cycle C

of $2W_{10} + x_1x_i$. Here, we choose $C = x_1x_ix_{i-1}\dots x_2xx_{i+1}x_{i+2}\dots x_{10}x_1$. Let $G_1 = 2W_{10} - C$ for every $3 \le i \le 6$. Note that $d_{G_1}(x_1) = d_{G_1}(x_i) = 5$, $d_{G_1}(x) = 18$, and $d_{G_1}(x_j) = 4$ where $j \neq 1, i$. Let β be an arbitrary $\mathbb{Z}_8$-pc-boundary of $2W_{10} + x_1x_i$.

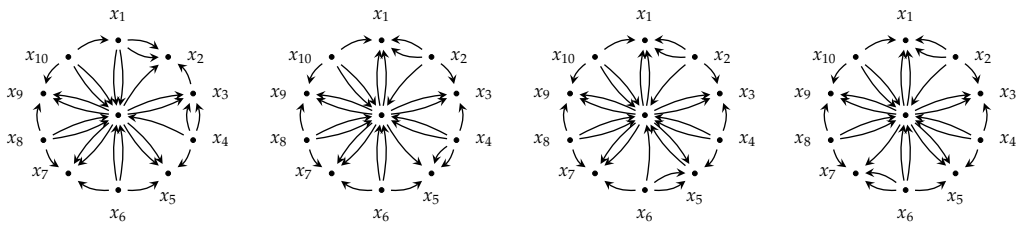
- When $\beta(x_j) \neq 4$ for $j \neq 1, i$, we lift an edge-pair (xx_j, x_jx_{j+1}), such that there are three parallel edges between x and x_{j+1}, and we denote the resulting graph as G_2. We observe that G_2 contains $3K_2$ belonging to $\mathcal{SZ}_4$. According to the construction of G_2, there is a sequence of graph $G_3, \dots, G_m$, such that each G_{i+1} is obtained from G_i by contracting a graph H, where $H \in \{3K_2, T_{2,2,2}\}$ and $2 \le i \le m-1$. Moreover, $G_m = 2K_2$ with the vertex-set $\{x_j, w\}$, where w is the new vertex formed from contraction. According to Lemma 1, we denote by β' the corresponding $\mathbb{Z}_8$-pc-boundary of G_m. As $\beta(x_j) \neq 4$, we know that $\beta'(x_j) = \beta(x_j) \neq 4$, and then $\beta'(w) \neq 4$. Thus, G_m admits an $(8, \beta')$-orientation. By Lemma 2, G_1 admits an $(8, \beta)$-orientation.
- When $\beta(x_1) \notin \{3, 5\}$ or $\beta(x_i) \notin \{3, 5\}$, the considerations are similar. Here, we give the proof of the former case. We lift an edge-pair (xx_1, x_1x_{10}) and denote by G_2 the resulting graph. Note that $\mu_{G_2}(xx_{10}) = 3$. Now, we can obtain a sequence of graphs $G_3, \dots, G_m$, such that every G_{i+1} is formed from G_i by contracting a graph H, where $H \in \{3K_2, T_{2,2,2}\}$, $2 \le i \le m-1$, and $G_m = T_{1,2,2}$ with the vertex-set $\{x_1, x_2, x\}$. As $\beta(x_1) \notin \{3, 5\}$, we know that $\beta'(x_1) = \beta(x_1) \notin \{3, 5\}$, where β' is a corresponding $\mathbb{Z}_8$-pc-boundary of G_m. Note that G_m admits an $(8, \beta')$-orientation. Thus, by Lemma 2, G_1 admits an $(8, \beta)$-orientation.
- Now, we consider the left cases that $\beta(x_1), \beta(x_i) \in \{3, 5\}$ and $\beta(x_j) = 4$ for $j \neq 1, i$. Let $(\beta(x_1), \beta(x_i), \beta(x))$ denote a triple.
  - When $\beta(x_1) = 3$, $\beta(x_i) = 3$, $\beta(x_j) = 4$ for $j \neq 1, i$, and $\beta(x) = 2$, we can find an $(8, \beta)$-orientation as depicted in Figure 1.



(**a**) adding an edge x_1x_3 (**b**) adding an edge x_1x_4 (**c**) adding an edge x_1x_5 (**d**) adding an edge x_1x_6

Figure 1. The triple $(\beta(x_1), \beta(x_3), \beta(x))$ is $(3, 3, 2)$.

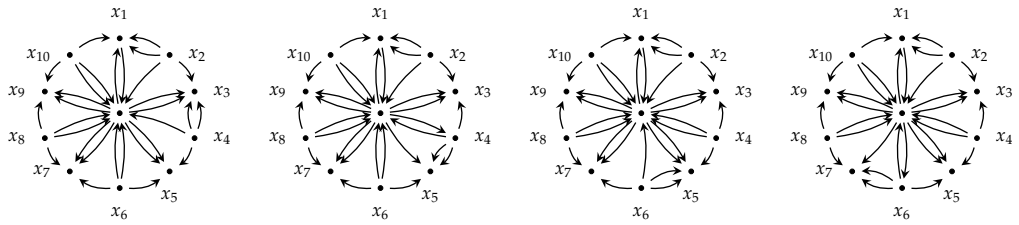
  - When $\beta(x_1) = 3$, $\beta(x_i) = 5$, $\beta(x_j) = 4$ for $j \neq 1, i$, and $\beta(x) = 0$, we can find an $(8, \beta)$-orientation as depicted in Figure 2.



(**a**) adding an edge x_1x_3 (**b**) adding an edge x_1x_4 (**c**) adding an edge x_1x_5 (**d**) adding an edge x_1x_6

Figure 2. The triple $(\beta(x_1), \beta(x_4), \beta(x))$ is $(3, 5, 0)$.

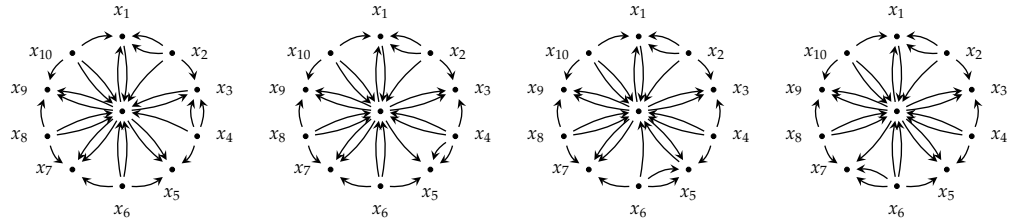
  - When $\beta(x_1) = 5$, $\beta(x_i) = 3$, $\beta(x_j) = 4$ for $j \neq 1, i$, and $\beta(x) = 0$, we can find an $(8, \beta)$-orientation as depicted in Figure 3.



(**a**) adding an edge x_1x_3 (**b**) adding an edge x_1x_4 (**c**) adding an edge x_1x_5 (**d**) adding an edge x_1x_6

Figure 3. The triple $(\beta(x_1), \beta(x_5), \beta(x))$ is $(5, 3, 0)$.

– When $\beta(x_1) = 5$, $\beta(x_i) = 5$, $\beta(x_j) = 4$ for $j \neq 1, i$, and $\beta(x) = 6$, we can find an $(8, \beta)$-orientation as depicted in Figure 4.



(**a**) adding an edge x_1x_3 (**b**) adding an edge x_1x_4 (**c**) adding an edge x_1x_5 (**d**) adding an edge x_1x_6

Figure 4. The triple $(\beta(x_1), \beta(x_6), \beta(x))$ is $(5, 5, 6)$.

Hence, we know that G_1 always has an $(8, \beta)$-orientation for any $\mathbb{Z}_8$-pc-boundary β, which implies that $G_1 \in \mathcal{SZ}_4$. According to Lemma 4, $2W_{10} + x_1x_i \in \mathcal{SC}_4$ for every $3 \leq i \leq 6$.

Therefore, we complete the proof. □

3. Main Results

In this section, we are devoted to proving Theorem 4. If Theorem 4 is true, then $2G$ admits a strongly connected orientation with a special boundary. Together with applying Theorem 3, we can obtain a result of circular flow in a signed planar graph. Considering the duality, we complete to show Theorem 2. When proving Theorem 4, we finish it mainly by contradiction and using the discharging method.

Theorem 4. *For a 3-edge-connected signed planar graph (G, σ), when G includes no edge-cut with size $\{4, 5, 6, 7, 8, 9\}$, we have $2G \in \mathcal{SC}_4$.*

Proof. Suppose that G is a minimum counterexample with respect to $v(G)$. Thus, $2G \notin \mathcal{SC}_4$. □

Claim 1. (i) *The counterexample G contains no $2K_2$.*
(ii) *The counterexample G contains no W_{10}.*

Proof. We prove (i) and (ii) of this claim in the same way. Suppose that G has $2K_2$ or W_{10} as a copy, by analyzing the graph formed from G by contracting $2K_2$ or W_{10}, we will conclude that $2G \in \mathcal{SC}_4$, which is a contradiction. According to Observation 1 and Lemma 5, we know that $\{4K_2, 2W_{10}\} \subseteq \mathcal{W}_4$. Here, we provide the detailed proof of (i).

If G contains $2K_2$, then $4K_2 \subseteq 2G$. Observe that $G/(2K_2)$ also meets the condition of Theorem 4, and $v(G/(2K_2)) < v(G)$. Hence, $2(G/(2K_2)) \in \mathcal{SC}_4$ according to the minimality of G. As $4K_2 \in \mathcal{W}_4$, together with Proposition 2, we obtain that $2G \in \mathcal{SC}_4$, which is a contradiction. □

Next, we shall obtain a contradiction of Theorem 4 utilizing the discharging method. We primarily focus on the charges of every vertex and each face in (G, σ). The condition of Theorem 4 implies that $d(w) = 3$ or $d(w) \geq 10$ for each vertex $w \in V(G)$. Before describing our discharging argument, we propose the following easy observation.

Observation 2. *Any two 3-vertices are not adjacent in* (G, σ).

Proof. It can be obtained by the fact that G has no edge-cut with size $\{4, 5, 6, 7, 8, 9\}$.

Discharging part. Initially, we assign a starting charge $ch_0(\cdot)$ to every face and vertex. We initialize $ch_0(f)$ as $2d(f) - 6$ for each face $f \in F(G)$, and define $ch_0(w) = d(w) - 6$ for each vertex $w \in V(G)$. Utilizing Euler's formula, we find that the aggregate charge across all faces and vertices is strictly smaller than 0.

$$\begin{aligned}\sum_{f \in F(G)} ch_0(f) + \sum_{w \in V(G)} ch_0(w) &= \sum_{f \in F(G)} (2d(f) - 6) + \sum_{w \in V(G)} (d(w) - 6) \\ &= (4e(G) - 6f(G)) + (2e(G) - 6v(G)) = -12 < 0.\end{aligned}$$

Next, we redistribute the charges of each face and vertex to ensure that they both end with charges that are not negative. Thus, it contradicts the above inequation. Let x be a 10^+-vertex and w be a 3-vertex. We call the vertex x *semi-poor* if wx is an edge lying on a 4^+-face. According to Claim 1(i), we know that G contains no 2-face. Note that only 3-vertex starts with a negative charge. To redistribute charges, we propose the discharging rules as follows.

Rule (A). Every 10^+-vertex gives a charge 1 to its adjacent 3-vertex.

Rule (B). Every 4^+-face sends 1 to its adjacent semi-poor 10^+-vertex.

Now, we consider the charge of each face f with $d(f) \geq 3$. For every 3-face f, it ends with charge $ch_1(f) = ch_0(f) = 2 \times 3 - 6 = 0$. For every 4^+-face f, by Observation 2, we know that there are at most $\frac{1}{2}\lfloor d(f) \rfloor$ 3-vertices incident to f. Thus, there are at most $\frac{1}{2}\lceil d(f) \rceil$ semi-poor 10^+-vertices lying on f. According to **Rule (B)**, it ends with charge $ch_1(f) \geq ch_0(f) - 1 \times (\frac{1}{2}\lceil d(f) \rceil) = 2d(f) - 6 - \frac{1}{2}\lceil d(f) \rceil \geq 0$. Hence, each face finishes with a non-negative charge.

For each 3-vertex w, it starts with $ch_0(w) = d(w) - 6 = -3$. By Observation 2, the vertex w is adjacent with 10^+-vertices. By **Rule (A)**, it ends with $ch_1(w) \geq -3 + 1 \times 3 = 0$. For every 10-vertex w, it is incident to at least a 4^+-face by the fact that G contains no W_{10} (Claim 1(ii)). When w is semi-poor, we consider the number of 3-vertices adjacent with w, and denote it as a. When $a \leq 5$, by **Rules (A)** and **(B)**, the final charge of w is $ch_1(w) \geq 4 - 1 \times 5 + 1 = 0$. If $a \geq 6$, then w ends with $ch_1(w) \geq 4 - a + (2a - 10) \geq 0$. For every 11^+-vertex w, we let m represent the number of 3-vertices adjacent with w. If $m \leq \frac{1}{2}\lfloor d(w) \rfloor$, then w ends with $ch_1(w) \geq ch_0(w) - m \geq (d(w) - 6) - \frac{1}{2}\lfloor d(w) \rfloor \geq 0$. If $m > \frac{1}{2}\lfloor d(w) \rfloor$, then w is incident to at least one 4^+-face. According to **Rules (A)** and **(B)**, the vertex w ends with $ch_1(w) \geq ch_0(w) - m + (2m - d(w)) = m - 6 > \frac{1}{2}\lfloor d(w) \rfloor - 6 \geq 0$. Thus, every vertex finishes with a non-negative charge.

Therefore, we are done. □

4. Conclusions

In conclusion, we propose the proof of Theorem 2.

Proof of Theorem 2. Let (G^*, σ^*) denote the dual signed planar graph of (G, σ). Thus, G^* is 3-edge-connected and has no edge-cut with size $\{4, 5, 6, 7, 8, 9\}$. According to Theorem 4, we know that $2G^* \in \mathcal{SC}_4$, which means that $2G^*$ has a strongly connected $(8, \beta)$-orientation

satisfying the condition of Theorem 3. Hence, $\phi_c(G^*, \sigma^*) < 4$, and thus (G, σ) possesses a circular r-coloring, where $r < 4$. □

Remark 1. *Actually, we can also infer when a signed simple planar graph contains cycles without length from* 4 *to* 9*, the circular chromatic number does not exceed* $4 - \frac{2}{\lfloor \frac{n+1}{2} \rfloor}$*, where* n *is the number of vertices. It has been proposed in Corollary 1.7 of [12] that there exist an integer* t *and a cycle of length* s *in* (G, σ)*, such that* $\chi_c(G, \sigma) = 2s/t$*. According to Theorem 2, we have* $\chi_c(G, \sigma) < 4$*. Thus* $\frac{2s}{t} < 4$*, and then* $2s < 4t$*. According to the parity of* $2s$ *and* $4t$*, we have* $2s \leq 4t - 2$*. Hence,* $\chi_c(G, \sigma) \leq \frac{4t-2}{t} = 4 - \frac{2}{t}$*. Moreover, by the fact that* $2s \leq 2n$*, we have* $\chi_c(G, \sigma) \leq \frac{2n}{t}$*. Note that the sequence* $(4 - \frac{2}{t})_{t \in \mathbb{N}}$ *is increasing and when* n *is fixed, the sequence* $(\frac{2n}{t})_{t \in \mathbb{N}}$ *is decreasing. Hence,* $\chi_c(G, \sigma) \leq 4 - \frac{4}{n+1}$ *if* n *is odd and* $\chi_c(G, \sigma) \leq 4 - \frac{4}{n}$ *if* n *is even.*

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References

1. Grötzsch, H. Zur Theorie der diskreten Gebilde. VII. Ein Dreifarbensatz für dreikreisfreie Netze auf der Kugel. *Wiss. Z. Der-Martin-Luther-Univ.-Wittenberg.-Math.-Naturwissenschaftliche Reihe* **1958**, *8*, 109–120.
2. Vince, A. Star chromatic number. *J. Graph Theory* **1988**, *12*, 551–559. [CrossRef]
3. Zhu, X. Circular chromatic number: A survey. *Discret. Math.* **2001**, *229*, 371–410. [CrossRef]
4. Naserasr, R.; Wang, Z.; Zhu, X. Circular chromatic number of signed graphs. *Electron. J. Comb.* **2021**, *28*, P2.44. [CrossRef]
5. Li, J.; Naserasr, R.; Wang, Z.; Zhu, X. Circular flows in mono-directed signed graphs. *J. Graph Theory* **2024**, *106*, 686–710. [CrossRef]
6. Tutte, W.T. A contribution to the theory of chromatic polynomials. *Can. J. Math.* **1954**, *6*, 80–91. [CrossRef]
7. Cranston, D.W.; Li, J.; Wang, Z.; Wei, C. Planar graphs with homomorphisms to the 9-cycle. *arXiv* **2024** , arXiv:2402.02689.
8. Li, J.; Wu, Y.; Zhang, C.-Q. Circular flows via extended Tutte orientations. *J. Comb. Theory Ser. B* **2020**, *145*, 307–322. [CrossRef]
9. Li, J.; Shi, Y.; Wang, Z.; Wei, C. Homomorphisms to small negative even cycles. *Eur. J. Comb.* **2024**, *118*, 103941. [CrossRef]
10. Cranston, D.W.; Li, J. Circular flows in planar graphs. *Siam J. Discret. Math.* **2020**, *34*, 497–519. [CrossRef]
11. Han, M.; Lai, H.-J.; Li, J.; Wu, Y. Contractible graphs for flow index less than three. *Discret. Math.* **2020**, *343*, 112073. [CrossRef]
12. Kardoš, F.; Narboni, J.; Naserasr, R.; Wang, Z. Circular (4-ε)-coloring of some classes of signed graphs. *Siam J. Discret. Math.* **2023**, *37*, 1198–1211. [CrossRef]

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