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Killing Vector Fields of Invariant Metrics on Five-Dimensional Solvable Lie Groups

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Abstract

In this paper we study the existence of Killing vector fields for right-invariant metrics on five-dimensional Lie groups. We begin by providing some explanation of the classification lists of the low-dimensional Lie algebras. Then we review some of the known results about Killing vector fields on Lie groups. We take as our invariant metric the sum of the squares of the right-invariant Maurer–Cartan one-forms, starting from a coordinate representation. A number of such metrics are uncovered that have one or more extra Killing vector fields, besides the left-invariant vector fields that are automatically Killing for a right-invariant metric. In each case the corresponding Lie algebra of Killing vector fields is found and identified to the extent possible on a standard list. The computations are facilitated by use of the symbolic manipulation package MAPLE.

Keywords: invariant metric; five-dimensional Lie algebra; Killing vector field

MSC: 17B30; 17B56; 22E15; 22E60

1. Introduction

In this paper we are concerned with the existence of Killing vector fields for right-invariant metrics on a five-dimensional Lie group. It is known that any left-invariant vector field is automatically Killing for a right-invariant metric. The question that we are addressing is whether there are additional Killing vector fields. In a previous paper [1], we have studied a similar problem for a Lie group G of dimension two to four. In that paper we were rather more ambitious; we started from an arbitrary right-invariant metric and used the automorphism group of the Lie algebra $\mathfrak{g}$ of G to reduce the metric. Even in dimension four, it is barely possible to give an exhaustive analysis. Some of the Lie algebras involved can depend on up to two parameters, and if the automorphism group of $\mathfrak{g}$ is small, it can lead to complicated conditions on the rank of certain submatrices that may depend on several parameters. In [1] we presented a worked example; the Lie algebra in question was $A_{4.1}$, whose numbering will be explained in Section 2 below. In that example, we started from a completely arbitrary right-invariant metric. The reader may find it helpful to consult that example.

Although, strictly speaking, the motion of a particle obeying the geodesics of the metric takes place on the Lie group, because the metric is right-invariant, we can effectively work at the Lie algebra level. We shall only consider Lie groups for which the associated Lie algebra is indecomposable in the sense that it is not a direct sum of lower-dimensional Lie algebras. According to the way one counts, there are approximately 40 classes of Lie algebra in dimension five, depending on up to three parameters. We begin in each case with a representation of the Lie group and from it derive a right-invariant metric

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Academic Editor: Vladimir Balan

Received: 17 October 2025
Revised: 3 December 2025
Accepted: 15 December 2025
Published: 17 December 2025

Citation: Thompson, G. Killing Vector Fields of Invariant Metrics on Five-Dimensional Solvable Lie Groups. *Mathematics* **2025**, *13*, 4019. https://doi.org/10.3390/math13244019

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and a basis for the left-invariant vector fields. A general right-invariant metric may be written in the form $\omega^t G\omega$, where ω is a "column vector" whose components comprise a basis for the right-invariant one-forms on and G is an arbitrary constant, symmetric non-singular matrix. Thus, in dimension five, a general right-invariant metric depends on fifteen parameters. On the other hand, the dimension of the automorphism groups of most five-dimensional Lie algebras is comparatively small, and so, even after normalizing the metric, many parameters remain, and we also have to take into account that the Lie algebras themselves may involve up to three parameters. As a typical example, consider the Lie algebra $A_{5.22}$ (the notation will be explained in Section 2) for which the non-zero Lie brackets are as follows:

$$[e_1, e_3] = e_2, [e_1, e_4] = e_3, [e_4, e_5] = e_5.$$

The Lie algebra of derivations is given by the following:

$$\begin{bmatrix} s_1 & 0 & 0 & 0 & 0 \\ s_2 & 2s_1 & s_3 & s_4 & 0 \\ s_5 & 0 & s_1 & s_3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & s_6 & s_7 \end{bmatrix},$$

with $s_1, s_2, s_3, s_4, s_5, s_6, s_7$ being arbitrary parameters. The automorphism group is obtained by exponentiating the derivation algebra and it too is of dimension seven. Even though $A_{5.22}$ itself contains no parameters, the reduced metric would depend on a best eight parameters, which is too many to be useful. Other cases yield even more parameters that cannot be normalized. The upshot is that in dimension five, the situation is much worse than in dimension four in terms of analyzing right-invariant metrics. Accordingly, in this work, out of necessity, we shall make the ansatz that our right-invariant metric is the sum of the squares of the Maurer–Cartan forms. Considering more general metrics quickly leads to intractable problems in linear algebra. In each case, the corresponding Lie algebra of Killing vector fields is identified, to the extent possible, as one on a standard list. These standard lists are outlined in Section 2.

Even though we have the metric explicitly in a system of local coordinates, we side-step completely the issue of writing down Killing's equations. Even in the lowest dimensions, Killing's equations are very difficult to solve by hand. Accordingly, we use the symbolic manipulation system MAPLE to find in each case a basis of the Killing vector fields. In most cases, MAPLE returns just the left-invariant vector fields, that we knew from the outset. There are, however, a few cases in which extra Killing vector fields occur. There are several Lie algebras that depend on parameters, and for which extra Killing fields arise when the parameters assume certain particular values. We shall not be able to explain exactly how we come up with such special values. To do so entails examining the rank of a certain 50×10 matrix that involves the parameters and then choosing them so that it has a rank less than ten. The reader can see an example of this process in the worked example of algebra $A_{4.1}$ in [1]. In addition, in the present work, there is one case, algebra $A_{5.40}$, where we do write down such a matrix, albeit in the Lie algebra frame, so that the reader may appreciate some of the complexities that are involved. Section 4 is devoted to this one case.

The outline of this paper is as follows: Section 2 of the paper gives a brief description of the low-dimensional Lie algebras and how they are arranged in lists. Section 3 provides

theoretical results relevant to the existence of Killing vector fields and are not necessarily restricted to the context of invariant metrics. Section 4 considers one particular algebra $A_{5,40}$; in this case MAPLE is unable to provide a basis for the Killing vector fields. Accordingly, we take a different approach and use integrability conditions to show that there are no extra Killing vector fields. The theory developed in Section 4 has significance above and beyond $A_{5,40}$ in that it is the only way to prove that there are no extra Killing vector fields. In Section 5 we give a list of all the five-dimensional cases for which there are no additional Killing fields beyond the left-invariant vector fields. In each case we give a representation for the corresponding Lie group, a right-invariant metric, and a basis for the left-invariant vector fields. In Section 6 we exhibit the complementary cases, the extra Killing vector fields, and the Lie algebra that they form. In order to identify this Lie algebra, we have to perform a change of basis (COB). In so doing, e_1, e_2, e_3, e_4, e_5 will represent the given left-invariant vector fields in the same order, and $e_6, e_7, ...$ will denote the new Killing vector fields that are introduced, again in the same order. The change of basis consists of mapping $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7...\}$ to the list of basis vectors that follow COB in each case. Section 7 provides some perspective on the results obtained and indicates some future directions for research. Appendix A summarizes the results of Section 6.

We will say now a few words about recent contributions to this research area. In [2] the author studies bi-invariant metrics. Such metrics are of limited interest in the Riemannian context since they can only be products of compact simple Lie groups with a multiple of the Killing form on each factor. As such, they possess no extra Killing vector fields. In [3,4], invariant metrics of various signatures are studied on four-dimensional nilpotent Lie groups, of which there are three up to isomorphism. Perhaps more relevant to the current memoir are [5] and [6] and references contained in these sources. [5] is devoted to a comprehensive analysis of isometry groups of three-dimensional invariant metrics. It exhibits some alternative ways to construct Killing vector fields based on work by I Singer and uses an invariant called the index of symmetry. [6] is concerned with reducing metrics in dimension four using the automorphism group of the Lie algebra, much as was performed in [1]. However, they do also consider decomposable Lie algebras, which were not considered in [1], but fail to distinguish non-generic cases where the parameters in the Lie algebra assume special values.

2. Classification of Low-Dimensional Lie Algebras

In this Section we shall give a brief summary of the structure of real Lie algebras in general and explain how some of them are arranged in lists that are frequently referred to in this text. An important definition is that a Lie algebra L is said to be *decomposable* if it is a direct sum of lower-dimensional Lie algebras $L = L_1 \oplus L_2$, and *indecomposable* in the contrary case. In the first instance, our concern is always to work with Lie groups whose Lie algebras are indecomposable.

Any Lie algebra L is the semi-direct sum of a semi-simple subalgebra $\mathfrak{s}$ and a solvable ideal $\mathfrak{n}$: we write $L = \mathfrak{n} \ltimes \mathfrak{s}$. In case neither $\mathfrak{n}$ nor $\mathfrak{s}$ is zero, we say that L is a *Levi decomposition* Lie algebra, and in case the semi-direct sum is not direct, we say that L is a *non-trivial* Levi decomposition Lie algebra.

The classification of the *complex* semi-simple Lie algebras is very well known. Refer, for example, to [7]. First of all, a semi-simple Lie algebra is a direct sum of simple Lie algebras. A Lie algebra is *simple* if it possesses no proper non-trivial ideals. Apart from five exceptions, over $\mathbb{C}$, these algebras are isomorphic to the following classes of matrix algebras: orthogonal $\mathfrak{so}(n, \mathbb{C})$, special linear $\mathfrak{sl}(n, \mathbb{C})$, and symplectic $\mathfrak{sp}(2n, \mathbb{C})$. Over $\mathbb{R}$, the situation becomes much more complicated. As regards the current work, we shall have to refer to $\mathfrak{sl}(2, \mathbb{R})$ (2×2 trace-free matrices), $\mathfrak{so}(n)$ real-orthogonal matrices ($n \times n$

skew-symmetric matrices), and $\mathfrak{so}(n,1)$ pseudo-orthogonal matrices ($(n+1)\times(n+1)$ skew-symmetric matrices relative to a Lorentz inner product). In [8], $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{so}(3)$ are denoted by $A_{(3,8)}$ and $A_{(3,9)}$, respectively. They are the only simple, and for that matter semi-simple, Lie algebras in dimensions up to, and including, five.

The classification problem for real, solvable, indecomposable Lie algebras are much messier than the semi-simple case. At the outset, it may help to point out that any *complex* solvable Lie algebra is isomorphic to a subalgebra of upper triangular matrices. Any solvable Lie algebra $\mathfrak{n}$ has a vector space decomposition as $\mathfrak{n} = \mathfrak{n}_o \oplus \mathfrak{c}$, where $\mathfrak{n}_o$ is the maximal nilpotent ideal and $\mathfrak{c}$ is a vector space complement: $\mathfrak{c}$ is a subalgebra if and only if it is abelian. A general inequality asserts the following:

$$dim(n_0) \geq \frac{1}{2} dim(n). \tag{1}$$

In dimension one, there is only one Lie algebra up to isomorphism, which is abelian, and in dimension two, there are two, abelian and non-abelian, the latter that we denote by $A_{2,1}$. In dimension three, Equation (1) gives two cases. There is a unique nilpotent algebra $\dim(\mathfrak{n}_o) = 3$, the Heisenberg Lie algebra denoted by $A_{3,1}$, and for the others, $\dim(\mathfrak{n}_o) = 2$. In dimension four, Equation (1) gives three cases. There is a unique nilpotent algebra $\dim(\mathfrak{n}_o) = 4$ denoted by $A_{4,1}$, and a unique algebra for which $\dim(\mathfrak{n}_o) = 2$ denoted by $A_{4,12}$; for the other algebras, $\dim(\mathfrak{n}_o) = 3$. In dimension five, Equation (1) again gives three cases. However, this time there are six nilpotent algebras $A_{5,j,(1\leq j\leq 6)}$, seven cases for which $\dim(\mathfrak{n}_o) = 2$, and 26 where $\dim(\mathfrak{n}_o) = 1$. We also see the first example of a non-trivial Levi decomposition Lie algebra $A_{5,40}$. Following [8] and [9], the algebras of dimensions five and less are denoted by $A_{i,j}$, where i pertains to the dimension of the Lie algebra and j to the number in the list. The same notation is employed for the six-dimensional nilpotent Lie algebras, [10], although they will not be needed here.

We shall make two changes from the list of five-dimensional Lie algebras given in [8]. First of all, we should like to regard the algebras $A_{5.14}$ and $A_{5.16}$ as being a single case depending on two parameters that are not subject to any restriction. Secondly, for algebra $A_{5.33}$, the parameters a and b should satisify $ab \neq 0$; if either of a and b is zero, the algebra is decomposable.

Since this article is concerned with five-dimensional Lie groups, we have discussed, first of all, Lie algebras of dimension up to and including five. However, our goal is to give the Lie algebra of the space of Killing vector fields in each case, and so we have to go beyond dimension five. For six-dimensional solvable indecomposable Lie algebras, Equation (1) yields four cases: $\dim(\mathfrak{n}_o) = 6, 5, 4, 3$. The nilpotent algebras were classified by Morozov [10] and the six-dimensional solvables for which $\dim(\mathfrak{n}_o) = 1$, by Mubarakzyanov [11], and he indexes the algebras as $g_{6,j,(1\leq j\leq 99)}$. In that regard, we mention also [12], where an attempt was made to remove the many imperfections of [11]. The six-dimensional solvables for which $\dim(\mathfrak{n}_o) = 2$ were classified by Turkowski [13], and he indexes them as $N_{6,j,(1\leq j\leq 40)}$. The last case where $\dim(\mathfrak{n}_o) = 3$, leads to a decomposable algebra so need not be considered.

The final class of Lie algebra that we shall need are the Levi decomposition Lie algebras. Such algebras have been classified by Turkowski [14] and [15], in dimension up to and including nine. These algebras are denoted by $L_{i,j}$. The case $L_{5,1}$ is identical to $A_{5,40}$ that we shall investigate in Section 4.

Many of the low-dimensional Lie algebras belong to continuous families, so besides the notation we have discussed, most algebras are indexed by parameters. In certain cases we find that there are extra Killing vector fields when these parameters assume certain specific values. Below is a list of the Lie algebras that we encounter when classifying the

Lie algebras of the space of Killing vector fields for Lie groups of dimension five and less. We also recommend [16] as an updated reference and for an explanation of Equation (1).

- dim $L \leq 4$; Mubarakzyanov et al.; $A_{i,j}$
- dim $L \leq 5$; Mubarakzyanov et al.; $A_{i,j,(1\leq j\leq 40)}$
- dim $L = 6$, dim $n_0 = 6$ nilpotent; Morozov; $A_{6,j,(1\leq j\leq 24)}$
- dim $L = 6$, dim $n_0 = 5$ solvable; Mubarakzyanov; $g_{6,j,(1\leq j\leq 99)}$
- dim$L = 6$, dim $n_0 = 4$ solvable; Turkowski; $N_{6,j,(1\leq j\leq 40)}$
- $5 \leq$ dim $L \leq 9$ Levi decomposition algebra; Turkowski; $L_{i,j}$.

3. Some Structure Theorems

3.1. Classical Theorems

In this Section, we state several general structure theorems concerning the dimension of the Lie algebra of Killing vector fields. The first three results, which are classical, pertain to Riemannian manifolds in general and not just the narrower context of invariant metrics. We shall use σ to denote the dimension of the Lie algebra of Killing vector fields on a pseudo-Riemannian manifold.

Theorem 1. *On an n-dimensional pseudo-Riemannian manifold,* $\sigma \leq \frac{1}{2}n(n+1)$*, and this maximum is attained if and only if the space is of constant curvature.*

See for example [17].

Theorem 2 ((Wang) [18])**.** *On a pseudo-Riemannian manifold of dimension n where* $n \neq 4$*, if* $\sigma > \frac{n(n-1)}{2} + 1$ *then the manifold is of constant curvature and so* $\sigma = \frac{n(n+1)}{2}$*.*

Theorem 3 ((Egorov) [19])**.** *On a Riemannian manifold of dimension n, which is not an Einstein space,* $\sigma \leq \frac{n(n-1)}{2} + 2$*. and equality holds only if the space is Einstein.*

We point out that, taking the last two theorems together, the only case in which $\sigma = \frac{n(n-1)}{2} + 2$ can be attained, is for $n = 4$ and then $\frac{n(n-1)}{2} + 2 = 8$. Furthermore, for $n = 4$ the two cases in which the dimension of the Lie algebra of Killing vector fields is 8, give rise to Einstein spaces (see [1] and [20]). By contrast, for $n = 5$ we did not find any examples of Einstein spaces among the invariant Riemannian metrics.

3.2. Invariant Metrics

Turning now to invariant metrics we have the following:

Proposition 1. *Any left-invariant vector field is Killing for any right-invariant Riemannian metric, and similarly any right-invariant vector field is Killing for any left-invariant Riemannian metric.*

Proof. The proof is immediate from the fact that any right-invariant vector field commutes with any left-invariant vector field and conversely. □

Proposition 2. *A right-invariant Riemannian metric g on a Lie group G is bi-invariant if and only if every left- and every right-invariant vector field is Killing for g. In fact, a de Rham indecomposable Riemannian metric g on a Lie group G is bi-invariant if and only if g is a multiple of the Killing form and G is compact simple.*

See, for example, ref. [21] and for the following result.

Theorem 4. *A Lie group admits a flat invariant metric if and only if its Lie algebra splits as a vector space orthogonal direct* $\mathfrak{b} \bigoplus \mathfrak{u}$ *sum in which* $\mathfrak{b}$ *is an abelian subalgebra,* $\mathfrak{u}$ *is an abelian ideal, and finally every (invariant) vector field in* $\mathfrak{b}$ *is a Killing field.*

In terms of the classification of the five-dimensional Lie algebras, flat metrics occur only for a special case of $A_{5.17}$.

The following theorems form a natural sequel to Theorem 4. See [22] for details.

Theorem 5. *Necessary and sufficient conditions for* $\{e_1, e_2, ..., e_r, e_{r+1}, e_{r+2}, ..., e_n\}$ *to be an orthonormal basis for a flat right-invariant metric on a Lie group G are that the Lie algebra* $\mathfrak{g}$ *should be of the form* $[e_i, e_a] = C^j_{ia} e_j$*, where* $1 \leq i \leq r, r+1 \leq a \leq n$*, and the matrices* C^j_{ia} *(with indices i and j) should be skew-symmetric in i and j and that the* C^j_{ia} *for* $r+1 \leq a \leq n$ *should pairwise commute.*

Theorem 6. *An invariant metric on a Lie group G is of constant non-zero curvature if and only if the Lie algebra* $\mathfrak{g}$ *of G is solvable with a codimension one abelian nilradical and such that the linear transformation S, the self-adjoint part of* $ad(b)$ *restricted to* $\mathfrak{nil}(\mathfrak{g})$*, where b spans a complement to* $\mathfrak{nil}(\mathfrak{g})$*, is a multiple of the identity.*

In concrete terms the Lie algebra occurring in the theorem can be written as follows where a is non-zero:

$$\begin{aligned} &[e_1, e_n] = ae_1 + b_1 e_2, \;\; [e_2, e_n] = ae_2 - b_1 e_2, \;\; [e_3, e_n] = ae_3 + b_2 e_4, \\ &[e_4, e_n] = ae_4 - b_2 e_3, \cdots, [e_{2p-1}, e_n] = ae_{2p-1} + b_p e_{2p}, \;\; [e_{2p}, e_n] = ae_{2p} - b_p e_{2p-1}, \\ &[e_{2p+1}, e_n] = ae_{2p+1}, \;\; [e_{2p+2}, e_n] = ae_{2p}, \cdots, [e_{n-1}, e_n] = ae_{n-1}. \end{aligned} \tag{2}$$

In fact, by scaling e_n we can even assume that $a = 1$.

The metric corresponding to (2) is given in adapted coordinates by the following:

$$\begin{aligned} &\sum_{i=1}^{p} (dx_{2i-1} - (ax_{2i-1} + b_i x_{2i}) dx_n)^2 + (dx_{2i} - (ax_{2i} - b_i x_{2i-1}) dx_n)^2 \\ &+ \sum_{j=2p+1}^{n} (dx_j - ax_j dx_n)^2 + \epsilon^2 (dx_n)^2. \end{aligned} \tag{3}$$

In (3) ϵ can assume any non-zero value: the curvature of the metric in (3) is $-\frac{a^2}{\epsilon^2}$. We may assume for simplicity that $\epsilon = 1$. The five dimensional cases occur only for $A_{5.7,(a=b=c=1)}$, $A_{5.13,(a=p=1)}$ and $A_{5.17,(q=p)}$.

4. Case $A_{5.40}$

4.1. Classical Derivation of Integrability Conditions

Lie algebra $A_{5.40}$ is the lowest dimensional algebra that has a non-trivial Levi decomposition. As such, it is possible to find many inequivalent representations of the algebra and of the corresponding Lie group obtained by exponentiation. However, because of the Levi factor $\mathfrak{sl}(2, \mathbb{R})$, all of them will lead to expressions for the right or left Maurer–Cartan forms that are complicated. Accordingly, an invariant metric, right or left, is also complicated, and MAPLE was unable to provide a basis for the Killing vector fields. Suspecting that there are no extra Killing vector fields, we are compelled to take a different line of attack.

Suppose then that ∇ is a symmetric linear connection on an n-manifold M. Using a semicolon to denote a covariant derivative, Killing's equation for a form of degree one can be written in a local coordinate system (x^i) as follows:

$$K_{i;j} + K_{j;i} = 0. \tag{4}$$

In classical terminology K_p is a "covariant vector field". Now, as has been demonstrated many times before, by taking covariant derivatives and elimination, one may obtain the following:

$$(\delta_i^m R_{jkl}^p - \delta_j^m R_{ikl}^p - \delta_l^m R_{kij}^p + \delta_k^m R_{lij}^p) K_{m;p} + K_p (R_{kij;l}^p - R_{lij;k}^p) = 0. \tag{5}$$

It should be noted that the discussion so far pertains to a general symmetric connection. In particular it is valid if ∇ is the Levi-Civita connection of a metric g: we then obtain the (contravariant) Killing vector field by "raising an index with g". In any case, we may regard (5) as a new set of first-order conditions that should be appended to (4) unless, of course, they happen to be identically satisfied, which, as is well known, occurs if and only if g is of constant sectional curvature.

4.2. Structure of the Integrability Conditions

A simplification in the conditions (5) comes about as a result of the following Lemma whose proof was given in [1].

Lemma 1. *Equation (5) is skew-symmetric in the indices i, j and k, l, and satisfies a Bianchi identity in the indices j, k, l. This latter fact follows from the first and second Bianchi identities for R^i_{jkl} Hence the number of independent conditions in Equation (5) is at most $\frac{n^2(n^2-1)}{12}$.*

Briefly stated, the Lemma says that Equation (5) have exactly the same symmetries as the fully covariant Riemann tensor. It has an important practical consequence: for example, for $n = 3$, we need only consider a system of six, rather than nine, linear equations, and for $n = 4$, only 20 linear equations. For $n = 5$, the case of interest here, we only have to consider 50 conditions. In fact, we can encode condition (5) as a homogeneous linear system, for which the matrix of coefficients, which we denote by M, is of size 50×15. We take the variables in the linear system in the order $K_{1;2}, .., K_{1;5}, , K_{2;3}, .., K_{2;5}, ..., K_{4;5}, K_1, ..., K_5$. In the case of a Lie group, we know that for a right-invariant metric, each left-invariant vector field is Killing. Assuming that we are not in the case where the metric is of constant curvature, (5) may be interpreted as expressing linear dependence of the last five columns on the first ten.

4.3. Algebra $A_{5.40}$

Let us now turn to the case of interest, that is, Algebra $A_{5.40}$. An important point here is that we must find the matrix M, relative to the *Lie algebra frame*. When Maple tries to solve Killing's equations in the coordinate frame, it is confronted with hopelessly complicated expressions. By contrast, the invariant frame, which coincides with the coordinate frame at the identity, is simply a numerical matrix, since $A_{5.40}$ does not involve any parameters. In order to find the matrix M, we make use of the following formulas, which give the connection coefficients Γ^i_{jk} and curvature components R^i_{jkm} starting from the structure constants C^i_{jk} of the Lie algebra:

$$\Gamma^m_{ij} = \frac{1}{2}(C^i_{mj} + C^j_{mi} + C^m_{ij}). \tag{6}$$

$$R^i_{ljk} = \Gamma^m_{jl}\Gamma^i_{km} - \Gamma^m_{kl}\Gamma^i_{jm} + \Gamma^i_{ml}C^m_{jk}. \tag{7}$$

Armed with these formulas and the expression for $R^i_{ljk;l}$, we may find M, which we give at the end of this Section. As promised, we have the following relations between the columns of M:

- $4C_1 - C_4 - 2C_{10} + 2C_{11} = 0$;
- $C_2 - 2C_{12} = 0$;
- $C_1 - 4C_4 - C_{10} - 2C_{13} = 0$;
- $2C_5 + C_7 + C_9 + C_{14} = 0$;
- $C_3 + C_6 - 2C_8 + 2C_{15} = 0$.

As such, we may remove the last five columns and consider the resulting 50×10 matrix. If we construct a 10×10 submatrix using rows $1, 2, 3, 4, 6, 7, 8, 9, 14, 21$, we find that it is non-singular, and thus the rank of M is 10. Hence, there are no extra Killing fields besides the left-invariant ones if we use a right-invariant metric.

We remark finally that if the rank of M had been less than 10, we would be able to reach no conclusion: there may or may not be an extra Killing field. The only recourse is to compute yet more derivatives, but as a practical matter, such a procedure presents severe computational difficulties.

Finally, we give $8M$ rather than M, so as to avoid irksome fractions:

$$8M = \begin{bmatrix}
0 & 32 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16 & 0 & 0 & 0 \\
-16 & 0 & 0 & -56 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 104 & 0 & 0 \\
0 & 0 & 0 & 0 & -40 & 0 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 35 & 0 \\
0 & 0 & -14 & 0 & 0 & 0 & 0 & -16 & 0 & 0 & 0 & 0 & 0 & 0 & -9 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 22 & 0 & 0 & 16 & 0 & 14 & 0 & 0 & 0 & 0 & 0 & 0 & -5 \\
0 & 0 & 0 & 0 & -10 & 0 & 22 & 0 & 16 & 0 & 0 & 0 & 0 & -9 & 0 \\
0 & 0 & 0 & 0 & -14 & 0 & 6 & 0 & 8 & 0 & 0 & 0 & 0 & 7 & 0 \\
0 & 0 & -2 & 0 & 0 & -8 & 0 & -14 & 0 & 0 & 0 & 0 & 0 & 0 & -9 \\
0 & -8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -2 & 0 & 0 & 16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -7 \\
0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 40 & 0 & 0 & 0 & 0 & -19 & 0 \\
56 & 0 & 0 & 16 & 0 & 0 & 0 & 0 & 0 & 0 & -104 & 0 & -4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 6 & 0 & 0 & 0 & 0 & -2 & 0 \\
0 & 0 & 6 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \\
0 & 0 & -40 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 19 \\
0 & 0 & 0 & 0 & 0 & 0 & -16 & 0 & 2 & 0 & 0 & 0 & 0 & 7 & 0 \\
8 & 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & -12 & 0 & -12 & 0 & 0 \\
0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
18 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 4 & -33 & 0 & 3 & 0 & 0 \\
0 & 24 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 0 & 0 & 0 \\
2 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & -3 & 0 & 0 \\
0 & 0 & 0 & 0 & 10 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & -12 & 0 \\
0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
-6 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & -4 & 11 & 0 & -5 & 0 & 0 \\
-6 & 0 & 0 & -6 & 0 & 0 & 0 & 0 & 0 & 0 & 9 & 0 & 9 & 0 & 0 \\
0 & -24 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -12 & 0 & 0 & 0 \\
0 & 0 & -26 & 0 & 0 & -2 & 0 & 14 & 0 & 0 & 0 & 0 & 0 & 0 & 28 \\
0 & -32 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -16 & 0 & 0 & 0 \\
0 & 0 & -16 & 0 & 0 & -22 & 0 & -10 & 0 & 0 & 0 & 0 & 0 & 0 & 9 \\
0 & 0 & 0 & 0 & 14 & 0 & -16 & 0 & -22 & 0 & 0 & 0 & 0 & 5 & 0 \\
0 & 0 & 0 & 0 & 16 & 0 & 0 & 0 & -14 & 0 & 0 & 0 & 0 & -9 & 0 \\
0 & 0 & 0 & 0 & 0 & 10 & 0 & 40 & 0 & 0 & 0 & 0 & 0 & 0 & 35 \\
0 & -8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-4 & 0 & 0 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 6 & 0 & 0 \\
-2 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & -4 & 5 & 0 & -11 & 0 & 0 \\
0 & -6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 \\
0 & 0 & -14 & 0 & 0 & -10 & 0 & 20 & 0 & 0 & 0 & 0 & 0 & 0 & 32 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-2 & 0 & 0 & -18 & 0 & 0 & 0 & 0 & 0 & 4 & -3 & 0 & 33 & 0 & 0 \\
0 & 0 & 0 & 0 & -20 & 0 & -10 & 0 & -14 & 0 & 0 & 0 & 0 & 32 & 0 \\
0 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\
0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & -24 & -10 & 0 & 4 & 0 & 0 \\
0 & 0 & 0 & 0 & 14 & 0 & 2 & 0 & 26 & 0 & 0 & 0 & 0 & -28 & 0 \\
0 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\
0 & 0 & -2 & 0 & 0 & -2 & 0 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 12 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.$$

5. Five-Dimensional Right-Invariant Metrics with No Extra Killing Fields

$A_{5.2}$, $[e_2, e_5] = e_1, [e_3, e_5] = e_2, [e_4, e_5] = e_3$:

$$S = \begin{bmatrix} 1 & w & \frac{w^2}{2} & \frac{w^3}{6} & q \\ 0 & 1 & w & \frac{w^2}{2} & x \\ 0 & 0 & 1 & w & y \\ 0 & 0 & 0 & 1 & z \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant metric: $(dq - xdw)^2 + (dx - ydw)^2 + (dy - zdw)^2 + dz^2 + dw^2$
Left-invariant vector fields: $-D_q, D_x + wD_q, -(D_y + wD_x + \frac{w^2}{2}D_q), D_z + wD_y + \frac{w^2}{2}D_x + \frac{w^3}{6}D_q, D_w$.
$A_{5.5}$, $[e_3, e_4] = e_1, [e_2, e_5] = e_1\ [e_3, e_5] = e_2$:

$$S = \begin{bmatrix} 1 & q & w + \frac{q^2}{2} & x \\ 0 & 1 & q & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant one metric: $(dx - ydq - zdw)^2 + (dy - zdq)^2 + dz^2 + dw^2 + dq^2$
Left-invariant vector fields: $D_x, D_y + qD_x, D_z + qD_y + (\frac{q^2}{2} + w)D_x, -D_w, -D_q$.
$A_{5.6}$, $[e_3, e_4] = e_1, [e_2, e_5] = e_1, [e_3, e_5] = e_2, [e_4, e_5] = e_3$:

$$S = \begin{bmatrix} 1 & 2w & w^2 - z & y - zw + \frac{w^3}{3} & q \\ 0 & 1 & w & \frac{w^2}{2} & x \\ 0 & 0 & 1 & w & y \\ 0 & 0 & 0 & 1 & z \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant metric: $(dq - zdy + ydz + (z^2 - 2x)dw)^2 + (dx - ydw)^2 + (dy - zdw)^2 + dz^2 + dw^2$
Left-invariant vector fields: $2D_q, D_x + 2wD_q, D_y + wD_x + (w^2 - z)D_q, D_z + (y - zw + \frac{w^3}{3})D_q + \frac{w^2}{2}D_x + wD_y, -D_w$.
$A_{5.10}$, $[e_2, e_5] = e_1, [e_3, e_5] = e_2, [e_4, e_5] = e_4$:

$$S = \begin{bmatrix} e^w & 0 & 0 & 0 & q \\ 0 & 1 & w & \frac{w^2}{2} & x \\ 0 & 0 & 1 & w & y \\ 0 & 0 & 0 & 1 & z \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant metric: $(dx - ydw)^2 + (dy - zdw)^2 + (dq - qdw)^2 + dz^2 + dw^2$
Left-invariant vector fields: $D_x, D_y + wD_x, D_z + wD_y + \frac{1}{2}w^2D_x, e^wD_q, -D_w$.
$A_{5.11c(c\neq 0)}$, $[e_1, e_5] = e_1, [e_2, e_5] = e_1 + e_2, [e_3, e_5] = e_2 + e_3, [e_4, e_5] = ce_4$:

$$S = \begin{bmatrix} e^{cw} & 0 & 0 & 0 & q \\ 0 & e^w & we^w & \frac{w^2e^w}{2} & x \\ 0 & 0 & e^w & we^w & y \\ 0 & 0 & 0 & e^w & z \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant metric: $(dx-(x+y)dw)^2+(dy-(y+z)dw)^2+(dz-zdw)^2+(dq-cqdw)^2+dw^2$
Left-invariant vector fields: $e^wD_x, e^w(D_y+wD_x), e^w(D_z+wD_y+\frac{1}{2}w^2D_x), e^{cw}D_q, -D_w$.
$A_{5.12}$, $[e_1,e_5]=e_1, [e_2,e_5]=e_1+e_2, [e_3,e_5]=e_2+e_3, [e_4,e_5]=e_3+e_4$:

$$S=\begin{bmatrix} e^w & we^w & \frac{w^2e^w}{2} & \frac{w^3e^w}{6} & q \\ 0 & e^w & we^w & \frac{w^2e^w}{2} & x \\ 0 & 0 & e^w & we^w & y \\ 0 & 0 & 0 & e^w & z \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant metric: $(dq-(q+x)dw)^2+(dx-(x+y)dw)^2+(dy-(y+z)dw)^2+(dz-zdw)^2+dw^2$
Left invariant vector fields: $e^wD_q, e^w(D_x+wD_q), e^w(D_y+wD_x+\frac{1}{2}w^2D_q), e^w(D_z+wD_y+\frac{1}{2}w^2D_x+\frac{1}{6}w^3D_q), -D_w$.
$A_{5.18p(p\geq 0)}$, $[e_1,e_5]=pe_1-e_2, [e_2,e_5]=e_1+pe_2, [e_3,e_5]=e_1+pe_3-e_4, [e_4,e_5]=e_2+e_3+pe_4$:

$$S=\begin{bmatrix} e^{pw}\cos w & e^{pw}\sin w & we^{pw}\cos w & we^{pw}\sin w & x \\ -e^{pw}\sin w & e^{pw}\cos w & -we^{pw}\sin w & we^{pw}\cos w & y \\ 0 & 0 & e^{pw}\cos w & e^{pw}\sin w & z \\ 0 & 0 & -e^{pw}\sin w & e^{pw}\cos w & q \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant metric: $(dx-(px+q+z)dw)^2+(dy-(py+q-x)dw)^2+(dz-(pz+q)dw)^2+(dq-(pq-z)dw)^2+dw^2$
Left-invariant vector fields: $e^{pw}(\cos w\, D_x-\sin w\, D_y), e^{pw}(\sin w\, D_x+\cos w\, D_y), e^{pw}(\cos w\,(D_z+wD_x)-\sin w\,(D_q+wD_y), e^{pw}(\cos w\,(D_q+wD_y)+\sin w\,(D_z+wD_x)), -D_w$.
$A_{5.19ab(b\neq 0)}$, $[e_2,e_3]=e_1, [e_1,e_5]=(a+1)e_1, [e_2,e_5]=e_2, [e_3,e_5]=ae_3, [e_4,e_5]=be_4$

$$S=\begin{bmatrix} e^{(a+1)w} & e^wx & 0 & z \\ 0 & e^w & 0 & y \\ 0 & 0 & e^{bw} & q \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant metric: $(dz-ydx+(axy-(a+1)z)dw)^2+(dy-ydw)^2+(dx-axdw)^2+(dq-bqdw)^2+dw^2$
Left-invariant vector fields: $e^{(a+1)w}D_z, e^w(D_y+xD_z), e^{aw}D_x, e^{bw}D_q, -D_w$.
$A_{5.20(a\neq -1}$, $[e_2,e_3]=e_1$, $[e_1,e_5]=(a+1)e_1$, $[e_2,e_5]=e_2$, $[e_3,e_5]=ae_3$, $[e_4,e_5]=e_1+(a+1)e_4$:

$$S=\begin{bmatrix} e^{(a+1)w} & e^wx & we^{(a+1)w} & z \\ 0 & e^w & 0 & y \\ 0 & 0 & e^{(a+1)w} & q \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant metric: $(dz-ydx+(axy-q-(a+1)z)dw)^2+(dy-ydw)^2+(dx-axdw)^2+(dq-(a+1)qdw)^2+dw^2$
Left-invariant vector fields: $e^{(a+1)w}D_z, -e^w(D_y+xD_z), e^{aw}D_x, e^{(a+1)w}(D_q+wD_z), -D_w$.

$A_{5.21}$, $[e_2, e_3] = e_1, [e_1, e_5] = 2e_1, [e_2, e_5] = e_2 + e_3, [e_3, e_5] = e_3 + e_4, [e_4, e_5] = e_4$:

$$S = \begin{bmatrix} e^{2w} & 0 & ze^w & (z - y + zw)e^w & q \\ 0 & e^w & we^w & \frac{w^2 e^w}{2} & x \\ 0 & 0 & e^w & we^w & y \\ 0 & 0 & 0 & e^w & z \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant one metric: $\frac{1}{4}(dq + zdy - (y + z)dz - 2qdw)^2 + (dz - zdw)^2 + (dy - (y + z)dw)^2 + (dx - (x + y)dw)^2 + dw^2$
Left-invariant vector fields: $2e^w D_q, e^w(D_z + wD_y + \frac{1}{2}w^2 D_x + (z - y + wz)D_q), e^w(D_y + wD_x + zD_q), e^w D_x, -D_w$.
$A_{5.22}$, $[e_2, e_3] = e_1$, $[e_2, e_5] = e_3$, $[e_4, e_5] = e_4$:

$$S = \begin{bmatrix} e^z & 0 & 0 & 0 & q \\ 0 & 1 & w & \frac{w^2}{2} & x \\ 0 & 0 & 1 & w & y \\ 0 & 0 & 0 & 1 & z \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant metric: $(dx - ydw)^2 + (-dy + zdw)^2 + (dq - qdz)^2 + dw^2 + dz^2$
Left-invariant vector fields: $-D_x, D_w, -(D_y + wD_x), e^z D_q, -(D_z + wD_y + \frac{1}{2}w^2 D_x)$.
$A_{5.23b(b\neq 0)}$, $[e_2, e_3] = e_1, [e_1, e_5] = 2e_1, [e_2, e_5] = e_2 + e_3, [e_3, e_5] = e_3, [e_4, e_5] = be_4$:

$$S = \begin{bmatrix} e^{bw} & 0 & 0 & 0 & q \\ 0 & e^{2w} & -ze^w & (y - zw)e^w & x \\ 0 & 0 & e^w & we^w & y \\ 0 & 0 & 0 & e^w & z \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant one-forms: $\frac{1}{4}(dx - zdy + ydz + (z^2 - 2x)dw)^2 + (dy - (y + z)dw)^2 + (dz - zdw)^2 + (dq - bqdw)^2 + dw^2$
Left-invariant vector fields: $2e^w D_q, e^w(D_z + wD_y + \frac{1}{2}w^2 D_x + (z - y + wz)D_q), e^w(D_y + wD_x + zD_q), e^w D_x, -D_w$.
$A_{5.24}$, $[e_2, e_3] = e_1, [e_1, e_5] = 2e_1, [e_2, e_5] = e_2 + e_3, [e_3, e_5] = e_3, [e_4, e_5] = e_1 + 2e_4$:

$$S = \begin{bmatrix} 1 & w & ze^w & xe^{2w} \\ 0 & 1 & ye^w & \left(\frac{y^2}{2} + q\right)e^{2w} \\ 0 & 0 & e^w & ye^{2w} \\ 0 & 0 & 0 & e^{2w} \end{bmatrix}.$$

Right-invariant metric: $(-dx + ydz + (-2x + q + yz - \frac{1}{2}y^2)dw)^2 + (dy + ydw)^2 + (-dz + (y - z)dw)^2 + (dq + 2qdw)^2 + dw^2$
Left-invariant vector fields: $e^{-2w} D_x, e^{-w}(D_y + zD_x + wD_z), e^{-w} D_z, e^{-2w}(D_q + wD_x), D_w$.

$A_{5.25ab(ab\neq 0)}$, $[e_2,e_3]=e_1$,$[e_1,e_5]=2ae_1$, $[e_2,e_5]=ae_2-e_3$, $[e_3,e_5]=ae_3+e_2$, $[e_4,e_5]=be_4$:

$$S=\begin{bmatrix} \mathrm{e}^{2\,aw} & \mathrm{e}^{aw}(y\cos w+x\sin w) & \mathrm{e}^{aw}(y\sin w-x\cos w) & 0 & z \\ 0 & \mathrm{e}^{aw}\cos w & \mathrm{e}^{aw}\sin w & 0 & x \\ 0 & -\mathrm{e}^{aw}\sin w & \mathrm{e}^{aw}\cos w & 0 & y \\ 0 & 0 & 0 & \mathrm{e}^{bw} & q \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant metric: $\frac{1}{4}(dz+ydx-xdy-(2az+x^2+y^2)dw)^2+(dx-(ax+y)dw)^2+(dy-(ay-x)dw)^2+(dq-bqdw)^2+dw^2$
Left-invariant vector fields: $-2e^{2aw}D_z, e^{aw}(\cos wD_x-\sin wD_y+(x\sin w+y\cos w)D_z)$, $e^{aw}(\sin wD_x+\cos wD_y+(y\sin w-x\cos w)D_z), e^{bw}D_q, -D_w$.
$A_{5.26a\epsilon(a\neq 0,\epsilon=\pm 1)}$, $[e_2,e_3]=e_1,[e_1,e_5]=2ae_1,[e_2,e_5]=ae_2-e_3,[e_3,e_5]=ae_3+e_2,[e_4,e_5]=\epsilon e_1+2ae_4$:

$$S=\begin{bmatrix} \mathrm{e}^{2\,aw} & \mathrm{e}^{aw}(y\cos w+x\sin w) & \mathrm{e}^{aw}(y\sin w-x\cos w) & 2\,\epsilon\, w\mathrm{e}^{2\,aw} & z \\ 0 & \mathrm{e}^{aw}\cos w & \mathrm{e}^{aw}\sin w & 0 & x \\ 0 & -\mathrm{e}^{aw}\sin w & \mathrm{e}^{aw}\cos w & 0 & y \\ 0 & 0 & 0 & \mathrm{e}^{2\,aw} & q \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant metric: $\frac{1}{4}(dz+ydx-xdy-(2az+x^2+y^2+2\epsilon q)dw)^2+(dx-(ax+y)dw)^2+(dy-(ay-x)dw)^2+(dq-2aqdw)^2+dw^2$
Left-invariant vector fields: $-2e^{2aw}D_z, e^{aw}(\cos wD_x-\sin wD_y+(y\cos w+x\sin w)D_z)$, $e^{aw}(\sin wD_x+\cos wD_y+(y\sin w-x\cos w)D_z)), e^{2aw}(D_q+2\epsilon wD_z), -D_w$.
$A_{5.27}$, $[e_2,e_3]=e_1,[e_1,e_5]=e_1,[e_3,e_5]=e_3+e_4,[e_4,e_5]=e_1+e_4$:

$$S=\begin{bmatrix} \mathrm{e}^{w} & w\mathrm{e}^{w} & \left(q+\frac{1}{2}w^2\right)\mathrm{e}^{w} & z \\ 0 & \mathrm{e}^{w} & w\mathrm{e}^{w} & x \\ 0 & 0 & \mathrm{e}^{w} & y \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant metric: $(dz-ydq-(x+z)dw)^2+dq^2+(dy-ydw)^2+(dx-(x+y)dw)^2+dw^2$
Left-invariant vector fields: $e^wD_z, D_q, e^w(D_y+wD_x+\frac{1}{2}(2q+w^2)D_z), e^w(D_z+D_x), -D_w$.
$A_{5.28a}$, $[e_2,e_3]=e_1$, $[e_1,e_5]=ae_1$, $[e_2,e_5]=(a-1)e_2$, $[e_3,e_5]=e_3+e_4$,$[e_4,e_5]=e_4$:

$$S=\begin{bmatrix} e^{aw} & -ze^{(a-1)w} & 0 & xe^{w} & q \\ 0 & e^{(a-1)w} & 0 & 0 & (a-1)x \\ 0 & 0 & e^{w} & we^{w} & y \\ 0 & 0 & 0 & e^{w} & z \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant metric: $(dx-zdy+(yz-ax)dw)^2+(dz-(a-1)zdw)^2+(dy-ydw)^2+(dq-(y+q)dw)^2+dw^2$
Left-invariant vector fields: $-e^{aw}D_x, e^{(a-1)w}(D_z+yD_x), e^w(D_y+wD_q), e^wD_q, -D_w$.

$A_{5.29}$, $[e_2, e_4] = e_1$, $[e_1, e_5] = e_1$, $[e_2, e_5] = e_2$,$[e_4, e_5] = e_3$:

$$S = \begin{bmatrix} 1 & q & x\mathrm{e}^w & z \\ 0 & 1 & y\mathrm{e}^w & w \\ 0 & 0 & \mathrm{e}^w & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Right-invariant metric: $(dx - ydq + xdw)^2 + (dy + ydw)^2 + (dz - wdq)^2 + dq^2 + dw^2$
Left-invariant vector fields: $-e^{-w}D_x, e^{-w}(D_y + qD_x + D_y), D_z, D_q, D_w + qD_z$.
$A_{5.30a}$, $[e_2, e_4] = e_1, [e_3, e_4] = e_2$, $[e_1, e_5] = (a+1)e_1$, $[e_2, e_5] = ae_2$, $[e_3, e_5] = (a-1)e_3$,$[e_4, e_5] = e_4$

$$S = \begin{bmatrix} 1 & q & \frac{q^2}{2} & x \\ 0 & e^w & e^w q & e^w y \\ 0 & 0 & e^{2w} & e^{2w} z \\ 0 & 0 & 0 & e^{(a+1)w} \end{bmatrix}.$$

Right-invariant metric: $e^{-2(a+1)w}(dx - ydq)^2 + e^{-2aw}(dy - zdq)^2 + e^{2(1-a)w}dz^2 + e^{-2w}dq^2 + dw^2$
Left-invariant vector fields: $D_x, -(D_y + qD_x), \frac{1}{2}q^2D_x + qD_y + D_z, D_q, D_w + (a+1)xD_x + ayD_y + qD_q + (a-1)zD_z$.
$A_{5.31}$, $[e_2, e_4] = e_1$, $[e_3, e_4] = e_2$, $[e_1, e_5] = 3e_1$, $[e_2, e_5] = 2e_2$, $[e_3, e_5] = e_3$,$[e_4, e_5] = e_3 + e_4$:

$$S = \begin{bmatrix} e^{3w} & -ze^{2w} & \frac{1}{2}z^2e^w & \frac{1}{2}e^w(z^2w + x - yz + \frac{3z^2}{2}) & q \\ 0 & e^{2w} & -ze^w & e^w(y - z - zw) & x \\ 0 & 0 & e^w & we^w & y \\ 0 & 0 & 0 & e^w & z \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant metric: $\frac{1}{36}(2dq - zdx + z^2dy + (2x - yz - z^2)dz - 6qdw)^2 + \frac{1}{4}(dx - zdy + (y + z)dz - 2xdw)^2 + (dy - (y + z)dw)^2 + (dz - zdw)^2 + dw^2$
Left-invariant vector fields: $3e^{3w}D_q, 2e^{2w}(D_x - zD_q), \frac{1}{2}e^w(2D_y - 2zD_x + z^2D_q), e^w(D_z + wD_y + (y - z - wz)D_x + \frac{1}{4}(2x - 2yz + 3z^2 + 2z^2w)D_q), -D_w$.
$A_{5.32a}$, $[e_2, e_4] = e_1$, $[e_3, e_4] = e_2$, $[e_1, e_5] = e_1$, $[e_2, e_5] = e_2$, $[e_3, e_5] = ae_1 + e_3$:

$$S = \begin{bmatrix} 1 & q & \frac{1}{2}q^2 - aw & x \\ 0 & 1 & q & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & \mathrm{e}^w \end{bmatrix}.$$

Right-invariant metric: $e^{-2w}(dx + azdw - ydq)^2 + e^{-2w}(dy - zdq)^2 + e^{-2w}dz^2 + dq^2 + dw^2$
Left-invariant vector fields: $D_x, -D_y - qD_x, (\frac{1}{2}q^2 - aw)D_x + qD_y + D_z, D_q, D_w + xD_x + yD_y + zD_z$.

$A_{5.33ab(ab)\neq 0)}$, $[e_1, e_4] = e_1$,$[e_3, e_4] = be_3$, $[e_2, e_5] = e_2$, $[e_3, e_5] = ae_3$:

$$S = \begin{bmatrix} e^z & 0 & 0 & q \\ 0 & e^w & 0 & x \\ 0 & 0 & e^{aw+bz} & y \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant metric: $(dq - qdz)^2 + (dx - xdw)^2 + (dy - bydz - aydw)^2 + dz^2 + dw^2$
Left-invariant vector fields: $e^z D_q, e^w D_x, e^{aw+bz} D_y, -D_z, -D_w$.
$A_{5.34a}$, $[e_1, e_4] = ae_1$, $[e_2, e_4] = e_2$,$[e_3, e_4] = e_3$, $[e_1, e_5] = e_1$, $[e_3, e_5] = e_2$:

$$S = \begin{bmatrix} \mathrm{e}^{\alpha z+w} & 0 & 0 & q \\ 0 & \mathrm{e}^z & w & x \\ 0 & 0 & \mathrm{e}^z & y \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant metric $(dq - aqdz - qdw)^2 + (dx - xdz - ydw)^2 + (dy - ydz)^2 + dz^2 + dw^2$
Left-invariant vector fields: $e^{az+w} D_q, e^z D_x, e^z(D_y + wD_x), -D_z, -D_w$.
$A_{5.36}$, $[e_2, e_3] = e_1$, $[e_1, e_4] = e_1$, $[e_2, e_4] = e_2$, $[e_2, e_5] = -e_2$, $[e_3, e_5] = e_3$:

$$S = \begin{bmatrix} \mathrm{e}^w & \mathrm{e}^q x & z \\ 0 & \mathrm{e}^q & y \\ 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant metric: $(dz - ydx + (xy - z)dw - xydq)^2 + (dx - xdw + xdq)^2 + (dy - ydq)^2 + dw^2 + dq^2$
Left-invariant vector fields: $e^w D_z, e^{w-q} D_x, e^q(D_y + xD_z), -D_w, -D_q$.
$A_{5.38}$, $[e_1, e_4] = e_1$, $[e_2, e_5] = e_2$, $[e_4, e_5] = e_3$:

$$S = \begin{bmatrix} e^z & 0 & 0 & 0 & x \\ 0 & e^w & 0 & 0 & y \\ 0 & 0 & 1 & w & q \\ 0 & 0 & 0 & 1 & z \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant one metric: $(dx - xdz)^2 + (dy - ydw)^2 + (dq - zdw)^2 + dz^2 + dw^2$
Left-invariant vector fields: $e^z D_x, e^w D_y, -D_q, -D_z - wD_q, -D_w$.
$A_{5.40}$, $[e_1, e_2] = 2e_1$, $[e_1, e_3] = -e_2$, $[e_2, e_3] = 2e_3$, $[e_1, e_4] = e_5$, $[e_2, e_4] = e_4$, $[e_2, e_5] = -e_5$, $[e_3, e_5] = e_4$:

$$S = \begin{bmatrix} \mathrm{e}^z & x & w \\ y & (xy+1)\mathrm{e}^{-z} & q \\ 0 & 0 & 1 \end{bmatrix}.$$

Left-invariant vector fields: $e^{-z}((xy + 1)D_y + xD_z), D_z - xD_x + yD_y, e^z D_x, yD_q + e^z D_w$, $(xy + 1)e^{-z} D_q + xD_w$
Left-invariant one forms: $e^z(dy - ydz), -xdy + (xy + 1)dz, e^{-z}(dx - x^2 dy+, x(xy+1)dz), (xy$ $+1)e^{-z} dw - xdq, e^z dq - ydw$

Right-invariant vector fields: $wD_q + e^z D_y, -D_z - xD_x + yD_y + qD_q - wD_w), (xy+1)e^{-z}D_x + ye^{-z}D_z + qD_w, D_w, -D_q$
Right-invariant one forms: $e^{-z}(dy - y^2dx + y(xy+1)dz), ydx - (xy+1)dz, e^z(dx - xdz), dw + (yw - qe^z)dx, -dq + (yq - y^2we^{-z}dx + we^{-z})dy + (xy+1)(ywe^{-z} - q)dz$.

6. Five-Dimensional Right-Invariant Metrics with Extra Killing Fields

$A_{5.1}$, $[e_3, e_5] = e_1, [e_4, e_5] = e_2$:

$$S = \begin{bmatrix} 1 & w & x & y \\ 0 & 1 & z & q \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Left-invariant vector fields: $D_x, D_y, D_z + wD_x, D_q + wD_y, -D_w$
Right-invariant metric: $(dx - zdw)^2 + (dy - qdw)^2 + dz^2 + dq^2 + dw^2$
Extra Killing vector field: $xD_y - yD_x + zD_q - qD_z$
Killing Lie algebra $g_{6.70}$, $(a = b = \delta = 0)$: COB $[e_6, e_5, e_2, e_3, e_4, -e_1]$
$[e_1, e_2] = e_3, [e_1, e_3] = -e_2, [e_1, e_5] = -e_6, [e_1, e_6] = e_5, [e_2, e_4] = e_6, [e_3, e_4] = e_5$.
$A_{5.3}$, $[e_3, e_4] = e_2, [e_3, e_5] = e_1$, $[e_4, e_5] = e_3$:

$$S = \begin{bmatrix} 1 & 0 & -\frac{z}{2} & \frac{y-zw}{2} & q \\ 0 & 1 & w & \frac{w^2}{2} & x \\ 0 & 0 & 1 & w & y \\ 0 & 0 & 0 & 1 & z \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant metric: $(dx - ydw)^2 + (dq - \frac{z}{2}dy + \frac{y}{2}dz + \frac{z^2}{2}dw)^2 + (dy - zdw)^2 + dz^2 + dw^2$
Left-invariant vector fields: $D_x, -D_q, -D_y - wD_x + \frac{z}{2}D_q, D_z + wD_y + \frac{w^2}{2}D_x + \frac{1}{2}(y - wz)D_q, D_w$.
Extra Killing vector field: $(wy + \frac{z^3}{6} - \frac{1}{2}zw^2 - 2x)D_q + (2q - yz + \frac{1}{3}w^3)D_x + (w^2 - z^2)D_y + 2wD_z - 2zD_w$
Killing Lie algebra $g_{6.80'}$, $(a = 0)$: $[e_1, e_6] = e_2, [e_3, e_4] = e_2, [e_3, e_5] = e_1, [e_2, e_6] = -e_1, [e_4, e_5] = e_3, [e_4, e_6] = -e_5, [e_5, e_6] = e_4$.
$A_{5.4}$, $[e_2, e_4] = e_1, [e_3, e_5] = e_1$:

$$S = \begin{bmatrix} 1 & z & w & q \\ 0 & 1 & 0 & x \\ 0 & 0 & 1 & y \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant metric: $(dq - xdz - ydw)^2 + dx^2 + dy^2 + dz^2 + dw^2$
Left-invariant vector fields: $D_q, D_z, D_w, D_x + zD_q, D_y + wD_q$
Extra Killing vector fields: $\frac{1}{2}(yD_x - xD_y + wD_z - zD_w), \frac{1}{2}((wz - xy)D_q + wD_x - xD_w + zD_y - yD_z), \frac{1}{2}(w^2 + x^2 - y^2 - z^2)D_q - zD_x + xD_z + wD_y - yD_w, \frac{1}{2}(w^2 - x^2 - y^2 + z^2)D_q + zD_x - xD_z + wD_y - yD_w$
Killing Lie algebra $L_{9.7(p=0)}$ COB $[e_6, e_7, e_8, e_3, e_4, e_5, -e_2, e_1, e_9][e_1, e_2] = e_3, [e_1, e_3] = -e_2, [e_1, e_4] = \frac{1}{2}e_7, [e_1, e_5] = \frac{1}{2}e_6, [e_1, e_6] = -\frac{1}{2}e_5, [e_1, e_7] = -\frac{1}{2}e_4, [e_2, e_3] = e_1, [e_2, e_4] = \frac{1}{2}e_5, [e_2, e_5] = -\frac{1}{2}e_4, [e_2, e_6] = \frac{1}{2}e_7, [e_2, e_7] = -\frac{1}{2}e_6, [e_3, e_4] = \frac{1}{2}e_6, [e_3, e_5] = -\frac{1}{2}e_7, [e_3, e_6] = -\frac{1}{2}e_4, [e_3, e_7] = \frac{1}{2}e_5, [e_4, e_6] = e_8, [e_4, e_9] = e_6, [e_5, e_7] = e_8, [e_5, e_9] = e_7, [e_6, e_9] = -e_4, [e_7, e_9] = -e_5$.

$A_{5.7abc(abc\neq 0,-1\leq c\leq b\leq a\leq 1)}$, $[e_1,e_5]=e_1,[e_2,e_5]=ae_2,[e_3,e_5]=be_3,[e_4,e_5]=ce_4$:

$$S=\begin{bmatrix} e^w & 0 & 0 & 0 & q \\ 0 & e^{aw} & 0 & 0 & x \\ 0 & 0 & e^{bw} & 0 & y \\ 0 & 0 & 0 & e^{cw} & z \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant metric: $(dq-qdw)^2+(dx-axdw)^2+(dy-bydw)^2+(dz-czdw)^2+dw^2$
Left-invariant vector fields: $e^wD_q, e^{aw}D_x, e^{bw}D_y, e^{cw}D_z, D_w$.
$(a-1)(a-b)(b-c)(c-a)\neq 0$:
No extra Killing fields.
$a=1,(b-1)(b-c)(c-1)\neq 0$:
Extra Killing field xD_q-qD_x
Killing Lie algebra: $N_{6,13\alpha\beta\gamma\delta(\alpha=0,\beta=b,\gamma=0,\delta=c)}$, COB $[e_6,-e_5,e_3,e_4,e_1,e_2]$ $[e_1,e_5]=e_6,[e_1,e_6]=-e_5,[e_2,e_3]=be_3,[e_2,e_4]=ce_4,[e_2,e_5]=e_5,[e_2,e_6]=e_6$.
$a=1,b=c\neq 1$:
Extra Killing fields xD_q-qD_x, yD_z-zD_y
Killing Lie algebra: $\mathbb{R}^4\ltimes\mathbb{R}^3$, COB $[e_1,e_2,e_3,e_4,e_5,e_6,e_7]$ $[e_1,e_5]=e_1,[e_1,e_6]=-e_2,[e_2,e_5]=e_2,[e_2,e_6]=e_1,[e_3,e_5]=be_3,[e_3,e_7]=e_4,[e_4,e_5]=be_4,[e_4,e_7]=-e_3$
$a=b=c=1$, refer to Theorem 6:
Killing Lie algebra basis:
$D_w, -e^wD_q, -e^wD_x, -e^wD_y, -e^wD_z, e^{-w}(\frac{1}{2}(q^2+x^2+y^2+z^2+1)D_q+qD_w), e^{-w}(\frac{1}{2}(q^2+x^2+y^2+z^2+1)D_x+xD_w), e^{-w}(\frac{1}{2}(q^2+x^2+y^2+z^2+1)D_y+yD_w), e^{-w}(\frac{1}{2}(q^2+x^2+y^2+z^2+1)D_z+zD_w), -qD_x+xD_q, yD_q-qD_y, zD_q-qD_z, yD_x-xD_y, zD_x-xD_z, zD_y-yD_z$
Killing Lie algebra: $\mathfrak{so}(5,1)$. $[e_1,e_2]=e_2,[e_1,e_3]=e_3,[e_1,e_4]=e_4,[e_1,e_5]=e_5,[e_1,e_6]=-e_6,[e_1,e_7]=-e_7,[e_1,e_8]=-e_8,[e_1,e_9]=-e_9,[e_2,e_6]=-e_1,[e_2,e_7]=e_{10},[e_2,e_8]=e_{11},[e_2,e_9]=e_{12},[e_2,e_{10}]=-e_3,[e_2,e_{11}]=-e_4,[e_2,e_{12}]=-e_5,[e_3,e_6]=-e_{10},[e_3,e_7]=-e_1,[e_3,e_8]=e_{13},[e_3,e_9]=e_{14},[e_3,e_{10}]=e_2,[e_3,e_{13}]=-e_4,[e_3,e_{14}]=-e_5,[e_4,e_6]=-e_{11},[e_4,e_7]=-e_{13},[e_4,e_8]=-e_1,[e_4,e_9]=e_{15},[e_4,e_{11}]=e_2,[e_4,e_{13}]=e_3,[e_4,e_{15}]=-e_5,[e_5,e_6]=-e_{12},[e_5,e_7]=-e_{14},[e_5,e_8]=-e_{15},[e_5,e_9]=-e_1,[e_5,e_{12}]=e_2,[e_5,e_{14}]=e_3,[e_5,e_{15}]=e_4,[e_6,e_{10}]=-e_7,[e_6,e_{11}]=-e_8,[e_6,e_{12}]=-e_9,[e_7,e_{10}]=e_6,[e_7,e_{13}]=-e_8,[e_7,e_{14}]=-e_9,[e_8,e_{11}]=e_6,[e_8,e_{13}]=e_7,[e_8,e_{15}]=-e_9,[e_9,e_{12}]=e_6,[e_9,e_{14}]=e_7,[e_9,e_{15}]=e_8,[e_{10},e_{11}]=e_{13},[e_{10},e_{12}]=e_{14},[e_{10},e_{13}]=-e_{11},[e_{10},e_{14}]=-e_{12},[e_{11},e_{12}]=e_{15},[e_{11},e_{13}]=e_{10},[e_{11},e_{15}]=-e_{12},[e_{12},e_{14}]=e_{10},[e_{12},e_{15}]=e_{11},[e_{13},e_{14}]=e_{15},[e_{13},e_{15}]=-e_{14},[e_{14},e_{15}]=e_{13}$.
$A_{5.8c}$ $(0<c\leq 1)$, $[e_2,e_5]=e_1$, $[e_3,e_5]=e_3$, $[e_4,e_5]=ce_4$:

$$S=\begin{bmatrix} e^{cw} & 0 & 0 & 0 & q \\ 0 & e^{w} & 0 & 0 & x \\ 0 & 0 & 1 & w & y \\ 0 & 0 & 0 & 1 & z \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant metric: $(dy-zdw)^2+dz^2+(dx-xdw)^2+(dq-cqdw)^2+dw^2$
Left-invariant vector fields: $D_y, D_z+wD_y, e^{cw}D_q, e^wD_x, -D_w$. $c=1$
Extra Killing vector field: qD_x-xD_q
Killing Lie algebra: $N_{6,4(\alpha=\beta=0)}$, COB $[-e_5,e_6,e_4,e_3,e_2,e_1]$,
$[e_1,e_3]=e_3,[e_1,e_4]=e_4,[e_1,e_5]=e_6,[e_2,e_3]=e_4,[e_2,e_4]=-e_3$.

$A_{5.9bc}$ $(0 \neq c \leq b)$, $[e_1, e_5] = e_1$, $[e_2, e_5] = e_1 + e_2$, $[e_3, e_5] = be_3$, $[e_4, e_5] = ce_4$:

$$S = \begin{bmatrix} e^{cw} & 0 & 0 & 0 & q \\ 0 & e^{bw} & 0 & 0 & x \\ 0 & 0 & e^{w} & we^{w} & y \\ 0 & 0 & 0 & e^{w} & z \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant metric: $(dy - (y+z)dw)^2 + (dz - zdw)^2 + (dx - bxdw)^2 + (dq - cqdw)^2 + dw^2$
Left-invariant vector fields: $e^w D_y, e^w(D_z + wD_y), e^{bw} D_x, e^{cw} D_q, -D_w$.
$b = c \neq 0$
Extra Killing vector field: $qD_x - xD_q$
Killing Lie algebra: $N_{6,7(\alpha=b,\beta=\gamma=0)}$ COB $[-e_5, e_6, e_4, -e_3, e_2, e_1]$, $[e_1, e_3] = be_3, [e_1, e_4] = be_4, [e_1, e_5] = e_5 + e_6, [e_1, e_6] = e_6, [e_2, e_3] = e_4, [e_2, e_4] = -e_3$.
$A_{5.13abc}$, $(0 < |a| \leq |b|)$ $[e_1, e_5] = ae_1, [e_2, e_5] = be_2, [e_3, e_5] = ce_3 - e_4, [e_4, e_5] = e_3 + ce_4$:

$$S = \begin{bmatrix} e^{aw} & 0 & 0 & 0 & q \\ 0 & e^{bw} & 0 & 0 & x \\ 0 & 0 & e^{cw}\cos w & e^{cw}\sin w & y \\ 0 & 0 & -e^{cw}\sin w & e^{cw}\cos w & z \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant metric: $(dq - aqdw)^2 + (dx - bxdw)^2 + (dy - (cy + z)dw)^2 + (dz + (y - cz)dw)^2 + dw^2$
Left-invariant vector fields: $e^{aw} D_q, e^{bw} D_x, e^{cw}(\cos w\, D_y - \sin w\, D_z), e^{cw}(\sin w\, D_y + \cos w\, D_z)$, $-D_w$
Extra Killing vector field: $zD_y - yD_z$
Killing Lie algebra: $N_{6,13}(\alpha = \gamma = 0, \beta = \frac{a}{c}, \delta = \frac{b}{c})$ COB $[e_5, \frac{1}{c}e_6, e_1, e2, e3, e4])$.
$(a = b \neq c)$
Extra Killing vector fields: $yD_z - zD_y, qD_x - xD_q$
Killing Lie algebra: $\mathbb{R}^4 \ltimes \mathbb{R}^3$, COB $[e_1, e_2, e_3, e_4, e_5 - e_6, e_7, e_6]$, $[e_1, e_5] = ae_1, [e_1, e_6] = e_2, [e_2, e_5] = ae_2, [e_2, e_6] = -e_1, [e_3, e_5] = ce_3, [e_3, e_7] = -e_4, [e_4, e_5] = ce_4, [e_4, e_7] = e_3$
$(a = c \neq b)$
Killing Lie algebra basis:
$q(\cos w\, D_z + \sin w\, D_y) - (z\cos w + y\sin w)D_q, q(\cos w\, D_y - \sin w\, D_z) - (y\cos w - z\sin w)D_q, yD_z - zD_y, e^{cw}(\sin w\, D_z - \cos w\, D_y), e^{cw}(\sin w\, D_y + \cos w\, D_z), e^{aw} D_q, e^{bw} D_x$, $b(yD_z - zD_y - D_w)$
Killing Lie algebra $L^p_{8.1(p=\frac{b}{a})}$:
$[e_1, e_2] = e_3, [e_1, e_3] = -e_2, [e_1, e_5] = e_6, [e_1, e_6] = -e_5, [e_2, e_3] = e_1, [e_2, e_4] = -e_6, [e_2, e_6] = e_4, [e_3, e_4] = e_5, [e_3, e_5] = -e_4, [e_4, e_8] = e_4, [e_5, e_8] = e_5, [e_6, e_8] = e_6, [e_7, e_8] = \frac{b}{a}e_7$.
$a = b = c \neq 0$: refer to Theorem (6). In this case change basis in $A_{5.13aaa}$, by replacing e_5 by $\frac{1}{a}e_5$ and a by $\frac{1}{\alpha}$, so as to obtain $[e_1, e_5] = e_1, [e_2, e_5] = e_2, [e_3, e_5] = e_3 - \alpha e_4, [e_4, e_5] = \alpha e_3 + e_4$:
Killing Lie algebra basis: $\alpha(yD_z - zD_y) - D_w, e^{-w}(\frac{1}{2}(q^2 + x^2 + y^2 + z^2 + 1)D_q + \alpha q(zD_y - yD_z) + qD_w), e^{-w}(\frac{1}{2}(((q^2 + x^2 + y^2 + z^2 + 2\alpha yz + 1)\sin\alpha w + 2\alpha z^2\cos\alpha w)D_y + (((q^2 + x^2 + y^2 + z^2) - 2\alpha yz + 1)\cos\alpha w - 2\alpha y^2\sin\alpha w)D_z) + (z\cos\alpha w + y\sin\alpha w)D_w), e^{-w}(\frac{1}{2}(q^2 + x^2 + y^2 + z^2 + 1)D_x + \alpha x(zD_y - yD_z) + xD_w), e^{-w}(\frac{1}{2}(((q^2 + x^2 + y^2 + z^2 + 2\alpha yz + 1)\cos\alpha w - 2\alpha z^2\sin\alpha w)D_y - ((q^2 + x^2 + y^2 + z^2 - 2\alpha yz + 1)\sin\alpha w + 2\alpha y^2\cos\alpha w)D_z) + (y\cos\alpha w - z\sin\alpha w)D_w), e^{w} D_q, e^{w}(\sin\alpha w D_y + \cos\alpha w D_z), -e^{w} D_x, e^{w}(\cos\alpha w D_y - \sin\alpha w D_z), (y\sin\alpha w + z\cos\alpha w)D_q - q\sin\alpha w D_y - q\cos\alpha w D_z, qD_x - xD_q, (-z\sin\alpha w + y\cos\alpha w)D_q - q\cos\alpha w D_y + q\sin\alpha w D_z, (y\sin\alpha w + z\cos\alpha w)D_x - x\sin\alpha w D_y - x\cos\alpha w D_z, -zD_y + yD_z, (z\sin\alpha w - y\cos\alpha w)D_x + x\cos\alpha w D_y - x\sin\alpha w D_z$.

Killing Lie algebra: $\mathfrak{so}(5,1)$: the Lie brackets are exactly the same as in $A_{5.7abc}$.
$A_{5.14/16ab}$, $[e_1, e_5] = ae_1, [e_2, e_5] = e_1 + ae_2, [e_3, e_5] = be_3 - e_4, [e_4, e_5] = e_3 + be_4$:

$$S = \begin{bmatrix} e^{aw} & we^{aw} & 0 & 0 & q \\ 0 & e^{aw} & 0 & 0 & x \\ 0 & 0 & e^{bw}\cos w & e^{bw}\sin w & y \\ 0 & 0 & -e^{bw}\sin w & e^{bw}\cos w & z \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant metric: $(dq - (aq + x)dw)^2 + (dx - axdw)^2 + (dy - (by + z)dw)^2 + (dz - (bz - y)dw)^2 + dw^2$
Left-invariant vector fields: $e^{aw}D_q, e^{aw}(D_x + wD_q), e^{bw}(\cos w\, D_y - \sin w\, D_z), e^{bw}(\sin w\, D_y + \cos w\, D_z), -D_w$.
Extra Killing vector field: $yD_z - zD_y$
$a \neq 0$
Killing Lie algebra: $N_{6,7\alpha\beta\gamma,\alpha=\frac{b}{a},\beta=0,\gamma=0}$, COB $[-\frac{1}{a}(e_5 + e_6), e_6, e_3, -e_4, e_2, \frac{1}{a}e_1]$, $[e_1, e_3] = \frac{b}{a}e_3, [e_1, e_4] = \frac{b}{a}e_4, [e_1, e_5] = e_5 + e_6, [e_1, e_6] = e_6, [e_2, e_3] = e_4, [e_2, e_4] = -e_3$.
$a = 0$ WLOG $b = 1$
Killing Lie algebra: $N_{6,23\alpha\epsilon,(\alpha=0,\epsilon=0}$, COB $[-(e_5 + e_6), e_6, e_3, e_4, e_2, e_1]$, $[e_1, e_3] = e_3, [e_1, e_4] = e_4, [e_1, e_5] = e_6, [e_1, e_6] = e_6, [e_2, e_3] = e_4, [e_2, e_4] = -e_3$.
$A_{5.15a(|a|\leq 1)}$, $[e_1, e_5] = e_1, [e_2, e_5] = e_1 + e_2, [e_3, e_5] = ae_3, [e_4, e_5] = e_3 + ae_4$:

$$S = \begin{bmatrix} e^{w} & we^{w} & 0 & 0 & q \\ 0 & e^{w} & 0 & 0 & x \\ 0 & 0 & e^{aw} & we^{aw} & y \\ 0 & 0 & 0 & e^{aw} & z \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant metric: $(dq - (q + x)dw)^2 + (dx - xdw)^2 + (dy - (ay + z)dw)^2 + (dz - azdw)^2 + dw^2$
Left-invariant vector fields: $e^{w}D_q, e^{w}(D_x + wD_q), e^{aw}D_y, e^{aw}(D_z + wD_y), -D_w$. $(a = 1)$
Extra Killing vector field: $xD_z - zD_x + qD_y - yD_q$
Killing Lie algebra: $N_{6,12\alpha\beta}(\alpha = \beta = 0)$ COB $[-e_5, -e_6, e_2, e_1, -e_4, -e_3]$,
$[e_1, e_3] = e_3 + e_4, [e_1, e_4] = e_4, [e_1, e_5] = e_5 + e_6, [e_1, e_6] = e_6, [e_2, e_3] = e_5, [e_2, e_4] = e_6, [e_2, e_5] = -e_3, [e_2, e_6] = -e_4$.
$A_{5.17ab\alpha\beta(\alpha=1,\beta>0)}$, $[e_1, e_5] = ae_1 - \alpha e_2, [e_2, e_5] = \alpha e_1 + ae_2, [e_3, e_5] = be_3 - \beta e_4, [e_4, e_5] = \beta e_3 + be_4$:

$$S = \begin{bmatrix} e^{aq}\cos\alpha q & e^{aq}\sin\alpha q & 0 & 0 & x \\ -e^{aq}\sin\alpha w & e^{aq}\cos\alpha q & 0 & 0 & y \\ 0 & 0 & e^{bq}\cos\beta q & e^{bq}\sin\beta q & z \\ 0 & 0 & -e^{bq}\sin\beta q & e^{bq}\cos\beta q & w \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Left-invariant metric: $e^{-2aq}(dx^2 + dy^2) + e^{-2bq}(dz^2 + dw^2) + dq^2$
Right-invariant vector fields: $D_x, D_y, D_z, D_w, D_q + axD_x + ayD_y + bzD_z + bwD_w$
Extra Killing vector fields: $yD_x - xD_y, wD_z - zD_w$
Killing Lie algebra WLOG $\alpha = 1$, $\mathbb{R}^4 \ltimes \mathbb{R}^3$: $[e_1, e_5] = ae_1 - \alpha e_2, [e_1, e_6] = -e_2, [e_2, e_5] = ae_2 + \alpha e_1, [e_2, e_6] = e_1, [e_3, e_5] = be_3 - \beta e_4, [e_3, e_7] = -e_4, [e_4, e_5] = be_4 + \beta e_3, [e_4, e_7] = e_3$.
$a = b = 0$: flat metric, refer to Theorem (5)
Killing Lie algebra basis:

$[xD_y - yD_x, xD_z - zD_x, xD_w - wD_x, xD_q - qD_x, yD_z - zD_y, yD_w - wD_y, yD_q - qD_y, zD_w - wD_z, zD_q - qD_z, wD_q - qD_w, D_x, D_y, D_z, D_w, D_q]$
Killing Lie algebra: $\mathfrak{so}(5) \rtimes \mathbb{R}^5$ (Euclidean group of $\mathbb{R}^5$),

$$[e_1,e_2] = -e_5, [e_1,e_3] = -e_6, [e_1,e_4] = -e_7, [e_1,e_5] = e_2, [e_1,e_6] = e_3, [e_1,e_7] = e_4, [e_1,e_{11}] = -e_{12}, [e_1,e_{12}] = e_{11},$$
$$[e_2,e_3] = -e_8, [e_2,e_4] = -e_9, [e_2,e_5] = -e_1, [e_2,e_8] = e_3, [e_2,e_9] = e_4, [e_2,e_{11}] = -e_{13}, [e_2,e_{13}] = e_{11},$$
$$[e_3,e_4] = -e_{10}, [e_3,e_6] = -e_1, [e_3,e_8] = -e_2, [e_3,e_{10}] = e_4, [e_3,e_{11}] = -e_{14}, [e_3,e_{14}] = e_{11}, [e_4,e_7] = -e_1,$$
$$[e_4,e_9] = -e_2, [e_4,e_{10}] = -e_3, [e_4,e_{11}] = -e_{15}, [e_4,e_{15}] = e_{11}, [e_5,e_6] = -e_8, [e_5,e_7] = -e_9, [e_5,e_8] = e_6,$$
$$[e_5,e_9] = e_7, [e_5,e_{12}] = -e_{13}, [e_5,e_{13}] = e_{12}, [e_6,e_7] = -e_{10}, [e_6,e_8] = -e_5, [e_6,e_{10}] = e_7, [e_6,e_{12}] = -e_{14},$$
$$[e_6,e_{14}] = e_{12}, [e_7,e_9] = -e_5, [e_7,e_{10}] = -e_6, [e_7,e_{12}] = -e_{15}, [e_7,e_{15}] = e_{12}, [e_8,e_9] = -e_{10}, [e_8,e_{10}] = e_9,$$
$$[e_8,e_{13}] = -e_{14}, [e_8,e_{14}] = e_{13}, [e_9,e_{10}] = -e_8, [e_9,e_{13}] = -e_{15}, [e_9,e_{15}] = e_{13}, [e_{10},e_{14}] = -e_{15}, [e_{10},e_{15}] = e_{14}.$$

$a = b \neq 0$ so WLOG $a = b = 1$: refer to Theorem 6. Killing Lie algebra basis:

$$-(D_q + xD_x + yD_y + zD_z + wD_w), D_y, D_z, D_w, D_x,$$
$$y(D_q + xD_x + zD_z + wD_w) - \frac{1}{2}(e^{2q} + w^2 + x^2 - y^2 + z^2)D_y,$$
$$z(D_q + xD_x + yD_y + wD_w) - \frac{1}{2}(e^{2q} + w^2 + x^2 + y^2 - z^2)D_z,$$
$$w(D_q + xD_x + zD_z + yD_y) - \frac{1}{2}(e^{2q} - w^2 + x^2 + y^2 + z^2)D_w,$$
$$x(D_q + yD_y + zD_z + wD_w) - \frac{1}{2}(e^{2q} + w^2 - x^2 + y^2 + z^2)D_x,$$
$$zD_y - yD_z, wD_y - yD_w, xD_y - yD_x, wD_z - zD_w, xD_z - zD_x, xD_w - wD_x.$$

Killing Lie algebra: $\mathfrak{so}(5,1)$: the Lie brackets are exactly the same as in $A_{5.7abc}$.
Right-invariant metric: $(dx - (ax + \alpha y)dq)^2 + (dy + (\alpha x - ay)dq)^2 + (dz - (bz + \beta w)dq)^2 + (dw + (\beta z - bq)dq)^2 + dq^2$
Left-invariant vector fields: $e^{aq}(\cos\alpha q\, D_x - \sin\alpha q\, D_y), e^{aq}(\sin\alpha q\, D_x + \cos\alpha q\, D_y), e^{bq}(\cos\beta q\, D_z - \sin\beta q\, D_q), e^{bq}(\sin\beta q\, D_z + \cos\beta q\, D_w), -D_q$
Extra Killing vector fields: $yD_x - xD_y, wD_z - zD_w$
Killing Lie algebra WLOG $\alpha = 1$, $\mathbb{R}^4 \ltimes \mathbb{R}^3$: $[e_1,e_5] = ae_1 - \alpha e_2, [e_1,e_6] = -e_2, [e_2,e_5] = ae_2 + \alpha e_1, [e_2,e_6] = e_1, [e_3,e_5] = be_3 - \beta e_4, [e_3,e_7] = -e_4, [e_4,e_5] = be_4 + \beta e_3, [e_4,e_7] = e_3$.
$a = b \neq 0$ so WLOG $a = b = 1$: refer to Theorem (6). Killing Lie algebra basis (labelled as $R'_A s, 1 \leq A \leq 15$, so as to aid the reader):

$$\begin{aligned}
R_1 &= e^{-q}((x\sin\alpha q + y\cos\alpha q)(D_q + \beta(wD_z - zD_w)) \\
&\quad + \tfrac{1}{2}\Big((w^2 + x^2 + y^2 + z^2 + 1 + 2\alpha xy)\sin\alpha q + 2\alpha y^2\cos\alpha q\Big)D_x \\
&\quad \tfrac{1}{2}\Big((w^2 + x^2 + y^2 + z^2 + 1 - 2\alpha xy)\cos\alpha q - 2\alpha x^2\sin\alpha q\Big)D_y) \\
R_2 &= e^{-q}((x\cos\alpha q - y\sin\alpha q)(D_q + \beta(wD_z - zD_w)) \\
&\quad \tfrac{1}{2}\Big((w^2 + x^2 + y^2 + z^2 + 1 + 2\alpha xy)\cos\alpha q - 2\alpha y^2\sin\alpha q\Big)D_x \\
&\quad -\tfrac{1}{2}\Big((w^2 + x^2 + y^2 + z^2 + 1 - 2\alpha xy)\sin\alpha q + 2\alpha x^2\cos\alpha q\Big)D_y) \\
R_3 &= e^{-q}((w\cos\beta q + z\sin\beta q)(D_q + \alpha(yD_x - xD_y)) \\
&\quad + \tfrac{1}{2}\Big((w^2 + x^2 + y^2 + z^2 + 1 + 2\beta zw)\sin\beta q + 2\beta w^2\cos\beta q\Big)D_z
\end{aligned}$$

$$
\begin{aligned}
&+\tfrac{1}{2}\Big((w^2+x^2+y^2+z^2+1-2\beta zw)\cos\beta q-2\beta z^2\sin\beta q\Big)D_w)\\
R_4 \quad &= e^{-q}((z\cos\beta q-w\sin\beta q)(D_q+\alpha(yD_x-xD_y))\\
&\tfrac{1}{2}\Big((w^2+x^2+y^2+z^2+1+2\beta zw)\cos\beta q-2\beta w^2\sin\beta q\Big)D_z\\
&-\tfrac{1}{2}\Big((w^2+x^2+y^2+z^2+1-2\beta zw)\sin\beta q+2\beta z^2\cos\beta q\Big)D_w)\\
R_5 \quad &= e^{q}\big(\cos\alpha q\,D_x-\sin\alpha q\,D_y\big)\\
R_6 \quad &= e^{q}\big(\sin\alpha q\,D_x+\cos\alpha q\,D_y\big)\\
R_7 \quad &= e^{q}(\cos\beta q\,D_z-\sin\beta q\,D_w)\\
R_8 \quad &= e^{q}(\sin\beta q\,D_z+\cos\beta q\,D_w)\\
R_9 \quad &= \cos((\alpha+\beta)q)(wD_y-yD_w-zD_x+xD_z)+\sin((\alpha+\beta)q)(wD_x-xD_w+zD_y-yD_z)\\
R_{10} \quad &= \cos((\alpha+\beta)q)(wD_x-xD_w+zD_y-yD_z)-\sin((\alpha+\beta)q)(wD_y-yD_w-zD_x+xD_z)\\
R_{11} \quad &= \cos((\alpha-\beta)q)(zD_x-xD_z+wD_y-yD_w)+\sin((\alpha-\beta)q)(wD_x-xD_w-zD_y+yD_z)\\
R_{12} \quad &= \cos((\alpha-\beta)q)(wD_x-xD_w-zD_y+yD_z)-\sin((\alpha-\beta)q)(zD_x-xD_z+wD_y-yD_w)\\
R_{13} \quad &= yD_x-xD_y\\
R_{14} \quad &= wD_z-zD_w\\
R_{15} \quad &= zD_w-yD_x+xD_y-wD_z-D_q.
\end{aligned}
$$

Killing Lie algebra: $\mathfrak{so}(5,1)$ but we do not report Lie brackets.
$A_{5.35ab(a^2+b^2\neq 0)}$, $[e_1,e_4]=be_1$, $[e_2,e_4]=e_2$,$[e_3,e_4]=e_3$, $[e_1,e_5]=ae_1$, $[e_2,e_5]=-e_3$,$[e_3,e_5]=e_2$:

$$
S=\begin{bmatrix} \mathrm{e}^{aw+bz} & 0 & 0 & q\\ 0 & \mathrm{e}^{z}\cos w & \mathrm{e}^{z}\sin w & x\\ 0 & -\mathrm{e}^{z}\sin w & \mathrm{e}^{z}\cos w & y\\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Right-invariant metric: $(dq-bqdz-aqdw)^2+(dx-xdz-ydw)^2+(dy-ydz+xdw)^2+dz^2+dw^2$
Left-invariant vector fields: $e^{aw+bz}D_q, e^z(\cos wD_x-\sin wD_y), e^z(\sin wD_x+\cos wD_y)$, $-D_z,-D_w$
$a=0, b\neq 1$
Extra Killing vector field: yD_x-xD_y:
Killing Lie algebra $A_{5.35ab(a=0)}\oplus\mathbb{R}$, COB $[e_1e_2,e_3,e_4,e_5,e_5-e_6]$.
$a\neq 0, b\neq 1$
Extra Killing vector field: yD_x-xD_y:
Killing Lie algebra $A_{4.12}\oplus A_{2.1}$, COB $[e_2,e_3,e_4-\frac{b}{a}(e_5-e_6),e_6,e_1,\frac{1}{a}(e_5-e_6)]$, $[e_1,e_3]=e_1,[e_1,e_4]=-e_2,[e_2,e_3]=e_2,[e_2,e_4]=e_1,[e_5,e_6]=e_5$.
Left-invariant metric: $e^{-2(aw+bz)}dq^2+e^{-2z}(dx^2+dy^2)+dz^2+dw^2$
Right-invariant vector fields: $D_q, D_x, D_y, D_z+xD_x+yD_y+bqD_q, D_w+yD_x-xD_y+aqD_q$
$a=0, b\neq 1$
Extra Killing vector field: yD_x-xD_y: Killing Lie algebra $A_{5.35ab(a=0)}\oplus\mathbb{R}$, COB $[e_1e_2,e_3,e_4,e_5,e_5-e_6]$.
$a\neq 0, b\neq 1$
Extra Killing vector field: yD_x-xD_y: Killing Lie algebra $A_{4.12}\oplus A_{2.1}$, COB $[e_2,e_3,e_4-\frac{b}{a}(e_5-e_6),e_6,e_1,$
$\frac{1}{a}(e_5-e_6)]$, $[e_1,e_3]=e_1,[e_1,e_4]=-e_2,[e_2,e_3]=e_2,[e_2,e_4]=e_1,[e_5,e_6]=e_5$. $a=0,b=1$

Killing Lie algebra basis:
$-(D_z + qD_q + xD_x + yD_y), D_y, D_x, D_q, -\frac{1}{2}(q^2 + x^2 - y^2 + e^{2z})D_y + y(qD_q + xD_x + D_z)),$
$-\frac{1}{2}(q^2 - x^2 + y^2 + e^{2z})D_x + x(qD_q + yD_y + D_z), \frac{1}{2}(q^2 - x^2 - y^2 - e^{(2z}})D_q + q(xD_x + yD_y + D_z), xD_y - yD_x, qD_y - yD_q, qD_x - xD_q, D_w$
Killing Lie algebra: $\mathfrak{so}(4,1) \oplus \mathbb{R}$:
$[e_1, e_2] = e_2, [e_1, e_3] = e_3, [e_1, e_4] = e_4, [e_1, e_5] = -e_5, [e_1, e_6] = -e_6, [e_1, e_7] = -e_7, [e_2, e_5] = -e_1, [e_2, e_6] = e_8, [e_2, e_7] = e_9, [e_2, e_8] = -e_3, [e_2, e_9] = -e_4, [e_3, e_5] = -e_8, [e_3, e_6] = -e_1, [e_3, e_7] = e_{10}, [e_3, e_8] = e_2, [e_3, e_{10}] = -e_4, [e_4, e_5] = -e_9, [e_4, e_6] = -e_{10}, [e_4, e_7] = -e_1, [e_4, e_9] = e_2, [e_4, e_{10}] = e_3, [e_5, e_8] = -e_6, [e_5, e_9] = -e_7, [e_6, e_8] = e_5, [e_6, e_{10}] = -e_7, [e_7, e_9] = e_5, [e_7, e_{10}] = e_6, [e_8, e_9] = e_{10}, [e_8, e_{10}] = -e_9, [e_9, e_{10}] = e_8.$
$a = 1, b = 0$
Killing Lie algebra basis:
$xD_x + yD_y + D_z, -\frac{1}{2}(x^2 - y^2 + e^{2z})D_y + y(xD_x + D_z)), \frac{1}{2}(x^2 - y^2 - e^{2z})D_x + x(yD_y + D_z), D_y, D_x, xD_y - yD_x, D_w + qD_q, D_y, \frac{1}{2}(q^2 - e^{2w})D_q + qD_w$
Killing Lie algebra: $\mathfrak{so}(3,1) \oplus \mathfrak{sl}(2,\mathbb{R})$:
$[e_1, e_2] = e_2, [e_1, e_3] = e_3, [e_1, e_4] = -e_4, [e_1, e_5] = -e_5, [e_2, e_4] = -e_1, [e_2, e_5] = e_6, [e_2, e_6] = -e_3, [e_3, e_4] = -e_6, [e_3, e_5] = -e_1, [e_3, e_6] = e_2, [e_4, e_6] = -e_5, [e_5, e_6] = e_4, [e_7, e_8] = -e_8, [e_7, e_9] = e_9, [e_8, e_9] = e_7.$

The cases $a = 0, b = 1$ and $a = 1, b = 0$ are not surprising when one realizes that they each give de Rham decompositions of the metric. However, there is no similar decomposition of the right-invariant metric, and Maple is unable to give the Killing Lie algebra in closed form.
$A_{5.37}$, $[e_2, e_3] = e_1$, $[e_1, e_4] = 2e_1$,$[e_2, e_4] = e_2$, $[e_3, e_4] = e_3$, $[e_2, e_5] = -e_3$,$[e_3, e_5] = e_2$:

$$S = \begin{bmatrix} \mathrm{e}^{2q} & (y\cos w + x\sin w)\,\mathrm{e}^{q} & (y\sin w - x\cos w)\,\mathrm{e}^{q} & z \\ 0 & \cos w\,\mathrm{e}^{q} & \sin w\,\mathrm{e}^{q} & x \\ 0 & -\sin w\,\mathrm{e}^{q} & \cos w\,\mathrm{e}^{q} & y \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Right-invariant metric: $\frac{1}{4}(dz + ydx - xdy - (x^2 + y^2)dw - 2zdq)^2 + (dx - ydw - xdq)^2 + (dy + xdw - ydq)^2 + dq^2 + dw^2$
Left-invariant vector fields: $-2e^{2q}D_z, e^q(\cos wD_x - \sin wD_y + (y\cos w + x\sin w)D_z)$, $e^q(\sin wD_x + \cos wD_y + (y\sin w - x\cos w)D_z), -D_q, -D_w$.
Extra Killing vector field: $D_w + yD_x - xD_y$: Killing Lie algebra $A_{5.37} \oplus \mathbb{R}$.
$A_{5.39}$, $[e_1, e_4] = e_1$, $[e_2, e_4] = e_2$, $[e_1, e_5] = -e_2$,$[e_2, e_5] = e_1$, $[e_4, e_5] = e_3$:

$$S = \begin{bmatrix} \mathrm{e}^{w}\cos z & \mathrm{e}^{w}\sin z & 0 & 0 & x \\ -\mathrm{e}^{w}\sin z & \mathrm{e}^{w}\cos z & 0 & 0 & y \\ 0 & 0 & 1 & w & q \\ 0 & 0 & 0 & 1 & z \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant metric: $(dx - ydz - xdw)^2 + (dy + xdz - ydw)^2 + (-dq + zdw)^2 + dw^2 + dz^2$
Left-invariant vector fields: $e^w(\cos zD_x - \sin zD_y), e^w(\sin zD_x + \cos zD_y), D_q, -D_w, -D_z - wD_q$
Extra Killing vector field: $yD_x - xD_y$:
Killing Lie algebra $N_{6.23\alpha\epsilon,(\alpha=\epsilon=0)}$, COB $[-e_4, e_6, e_1, e_2, e_6 - e_5, e_3]$, $[e_1, e_3] = e_3, [e_1, e_4] = e_4, [e_1, e_5] = e_6, [e_2, e_3] = e_4, [e_2, e_4] = -e_3$.

7. Discussion

In this paper we have investigated Killing vector fields for right-invariant metrics on five-dimensional Lie groups using the symbolic manipulation system MAPLE. It forms a natural sequel to [1], which was concerned, for the most part, with the four-dimensional case. As far as we are aware, it is the first time that the five-dimensional case has been studied in a systematic way. One of the most important conclusions of the current work and [1], is that it is unrealistic to reduce an invariant metric on a Lie group by using the automorphism group of the Lie algebra: usually, the automorphism group is too small and the reduced metric still contains many parameters, so that the curvature tensor is hopelessly complicated. Accordingly, we have imposed the strong assumption that the metric is the sum of the squares of the Maurer–Cartan one-forms. We have considered every possible case corresponding to indecomposable five-dimensional Lie algebras and found several cases where extra Killing vector fields exist, sometimes because the parameters in the Lie algebra assume certain special values. We do not claim that our results are comprehensive. However, we have indicated the difficulty that one encounters in trying to obtain such a universal analysis by means of studying the algebra $A_{5.40}$, for which MAPLE is unable to furnish a solution, the defining conditions being too complicated. In the future, we anticipate being able to investigate Killing vector fields for right-invariant metrics on six-dimensional nilpotent Lie groups. At the moment, it is unclear how far it will be possible to extend the analysis; the 50×15 matrix that occurred in this paper must be replaced by a 105×21 matrix! On the other hand, it might be fruitful to explore some of the ideas presented in [5], since they might suggest ways of not confronting Killing's equations directly.

We do not claim to provide an exhaustive analysis of all possibilities for which there are additional Killing vector fields. As we said in [1], to attempt to do so would require an enormous amount of work to achieve largely null results. Nonetheless, we do hope that the examples that we are able to provide are sufficiently interesting to the community of researchers studying integrable systems.

Funding: This research received no external funding

Data Availability Statement: No new data were created or analyzed in this study. Data sharing is not applicable to this article.

Conflicts of Interest: The author declares no conflicts of interest.

Appendix A

Table A1. Summary of Results.

Lie Algebra	Special Values of Parameters	Lie Algebra of Killing Vector Fields	Metric Right or Left
$A_{5,1}$	NA	$g_{6.70ab\delta}, (a = b = \delta = 0)$	Right
$A_{5,3}$	NA	$g_{6.80}, (a = 0)$	Right
$g_{A_{5,4}}$	NA	$L_{9.7p}, (p = 0)$	Right
$A_{5,7ab}$	$(a-1)(a-b)(b-c)(c-1) \neq 0$	$A_{5,7ab}$	Right
$A_{5,7ab}$	$a = 1, (a-b)(b-c)(c-1) \neq 0$	$N_{6,13\alpha\beta\gamma\delta(\alpha=0,\beta=b,\gamma=0,\delta=c)}$	Right
$A_{5,7ab}$	$a = 1, b = c \neq 1$	$\mathbb{R}^4 \ltimes \mathbb{R}^3$	Right
$A_{5,7ab}$	$a = b = c = 1$	$\mathfrak{so}(5,1)$	Right
$A_{5,8c}$	$c = 1$	$N_{6.4\alpha\beta}, \alpha\beta = 0$	Right

Table A1. *Cont.*

Lie Algebra	Special Values of Parameters	Lie Algebra of Killing Vector Fields	Metric Right or Left
$A_{5,9bc}$	$b = c \neq 0$	$N_{6.7\alpha\beta\gamma}, \alpha = b, \beta = \gamma = 0$	Right
$A_{5,13abc}$	NA	$N_{6.13\alpha\beta\gamma\delta}, \alpha = \gamma = 0, \beta = \frac{a}{c}\delta = \frac{b}{c}$	Right
$A_{5,13abc}$	$a = b \neq c$	$\mathbb{R}^4 \ltimes \mathbb{R}^3$	Right
$A_{5,13abc}$	$a = c \neq b$	$L_{8.1p}, (p = \frac{a}{b})$	Right
$A_{5,13abc}$	$a = b = c$	$\mathfrak{so}(5,1)$	Right
$A_{5,14/16ab}$	NA	$N_{6.7\alpha\beta\gamma}, (\alpha = \frac{a}{b}\beta = 0, \gamma = 0)$	Right
$A_{5,15a}$	$a = 1$	$N_{6.12\alpha\beta}, \alpha = \beta = 0$	Right
$A_{5,17abc}$	NA	$\mathbb{R}^4 \ltimes \mathbb{R}^3$	Left
$A_{5,17abc}$	$a = b = 0$	$\mathfrak{so}(5) \rtimes \mathbb{R}^5$	Left
$A_{5,17abc}$	$a = b = 1$	$\mathfrak{so}(5,1)$	Left
$A_{5,35ab}$	$a = 0, b \neq 1$	$A_{5,35} \oplus \mathbb{R}$	Left
$A_{5,35ab}$	$a \neq 0, b \neq 1$	$A_{4.12} \oplus A_{2.1}$	Left
$A_{5,35ab}$	$a = 0, b = 1$	$\mathfrak{so}(4,1) \oplus \mathbb{R}$	Left
$A_{5,35ab}$	$a = 1, b = 0$	$\mathfrak{so}(3,1) \oplus \mathfrak{sl}(2,\mathbb{R})$	Left
$A_{5,37}$	NA	$A_{5,37} \oplus \mathbb{R}$	Right
$A_{5,39}$	NA	$N_{6.23\alpha\epsilon}, \alpha = \epsilon = 0$	Right

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