

Article

A Note on the Joint Numerical Range of a Triple of 4-by-4 Hermitian Matrices

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Abstract

We present the degrees 14, 26 and 30 of the boundary generating surfaces of the joint numerical ranges for triples of 4-by-4 Hermitian matrices. This result completes the missing degrees from a previous study.

Keywords: joint numerical range; boundary generating surface; Davis-Wielandt shell

MSC: 15A60; 14J17

1. Introduction

Let $\mathcal{H} = (H_1, H_2, H_3)$ be a triple system of $n \times n$ Hermitian matrices. The joint numerical range of $\mathcal{H}$ is defined by

$$W(\mathcal{H}) = \{(\zeta^* H_1 \zeta, \zeta^* H_2 \zeta, \zeta^* H_3 \zeta) \in \mathbb{R}^3 : \zeta \in \mathbb{C}^n, \zeta^* \zeta = 1\}.$$

It is known that the range $W(\mathcal{H})$ is a compact convex set when $n \geq 3$ [1]. For additional results on the convexity of the joint numerical range, we refer the reader to ref. [2] and the references therein. Szymański et al. [3] classified interesting shapes of the joint numerical ranges of triples of 3×3 Hermitian matrices. The Kippenhahn polynomial of $\mathcal{H}$ is defined and denoted by

$$F_{\mathcal{H}}(t, x, y, z) = \det(tI_n + xH_1 + yH_2 + zH_3)$$

(cf. [4]). Plaumann et al. [5] provided an algebraic-geometric method to generate the boundary of the joint numerical range $W(\mathcal{H})$. Following their method, we consider the dual algebraic surface $G_{\mathcal{H}}(T, X, Y, Z) = 0$ of the algebraic surface

$$\mathcal{V}_{\mathbb{C}}(F_{\mathcal{H}}) = \{[(t, x, y, z)] \in \mathbb{CP}^3 : F_{\mathcal{H}}(t, x, y, z) = 0\},$$

based on the pairing $tT + xX + yY + zZ = 0$ of the dual complex projective 3-spaces $\mathbb{CP}^3$, where $[(t, x, y, z)]$ is the equivalence class containing the point $(t, x, y, z) \in \mathbb{C}^4 \setminus \{(0, 0, 0, 0)\}$ with respect to the equivalence relation $(t, x, y, z) \equiv (kt, kx, ky, kz)$ for $k \in \mathbb{C} \setminus \{0\}$. The partial derivatives of the Kippenhahn polynomial $F_{\mathcal{H}}(t, x, y, z)$ with respect to the variables t, x, y, z are denoted by $F_{\mathcal{H}t}(t, x, y, z)$, $F_{\mathcal{H}x}(t, x, y, z)$, $F_{\mathcal{H}y}(t, x, y, z)$ and $F_{\mathcal{H}z}(t, x, y, z)$ respectively. A point $(t_0, x_0, y_0, z_0) \in \mathbb{C}^4 \setminus \{(0, 0, 0, 0)\}$ is called a singular point of $\mathcal{V}_{\mathbb{C}}(F_{\mathcal{H}})$ if

$$F_{\mathcal{H}t}(t_0, x_0, y_0, z_0) = F_{\mathcal{H}x}(t_0, x_0, y_0, z_0) = F_{\mathcal{H}y}(t_0, x_0, y_0, z_0) = F_{\mathcal{H}z}(t_0, x_0, y_0, z_0) = 0.$$



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By the Euler relation,

$$\begin{aligned} nF_{\mathcal{H}}(t_0, x_0, y_0, z_0) &= t_0 F_{\mathcal{H}t}(t_0, x_0, y_0, z_0) + x_0 F_{\mathcal{H}x}(t_0, x_0, y_0, z_0) \\ &\quad + y_0 F_{\mathcal{H}y}(t_0, x_0, y_0, z_0) + z_0 F_{\mathcal{H}z}(t_0, x_0, y_0, z_0), \end{aligned}$$

a singular point automatically lies in the surface $\mathbb{V}_{\mathbb{C}}(F_{\mathcal{H}})$. The homogeneous polynomial $G_{\mathcal{H}}(T, X, Y, Z)$ is defined, up to a constant, as the generator of the principal ideal of polynomials $G(T, X, Y, Z) \in \mathbb{C}[T, X, Y, Z]$ satisfying

$$G(F_{\mathcal{H}t}(t_0, x_0, y_0, z_0), F_{\mathcal{H}x}(t_0, x_0, y_0, z_0), F_{\mathcal{H}y}(t_0, x_0, y_0, z_0), F_{\mathcal{H}z}(t_0, x_0, y_0, z_0)) = 0$$

for points $(t_0, x_0, y_0, z_0) \in \mathbb{V}_{\mathbb{C}}(F_{\mathcal{H}})$. The joint numerical range $W(\mathcal{H})$ is the closed convex hull of real non-singular points of the affine surface $G_{\mathcal{H}}(1, X, Y, Z) = 0$ (cf. [5]). The polynomial $G_{\mathcal{H}}(T, X, Y, Z)$ is called the boundary generating polynomial of the joint numerical range $W(\mathcal{H})$, and its degree is denoted by $c_{\mathcal{H}}$. We recall an algebraic procedure from ref. [6] for obtaining the boundary-generating polynomial $G_{\mathcal{H}}(T, X, Y, Z)$ of $W(\mathcal{H})$. This polynomial $G_{\mathcal{H}}(T, X, Y, Z)$ can be computed by applying the Wolfram Mathematica function Resultant twice. Starting at the pairing $xX + yY + zZ + 1 = 0$, we solve it as $z = -(1/Z)(xX + yY + 1)$ and substitute into the polynomial $F_{\mathcal{H}}(1, x, y, z)$. Define the polynomial

$$H(X, Y, Z : x, y) = Z^4 F_{\mathcal{H}}(1, x, y, -(1/Z)(xX + yY + 1))$$

in 5 indeterminates X, Y, Z, x, y . To eliminate two indeterminates x, y from the simultaneous three equations

$$H(X, Y, Z : x, y) = 0, H_x(X, Y, Z : x, y) = 0, H_y(X, Y, Z : x, y) = 0,$$

we firstly delete the indeterminate y by computing the resultant $R(H, H_y : y)$ of H, H_y with respect to y , and then taking its main simple factor $K(X, Y, Z : x)$ in $\mathbb{Z}[X, Y, Z, x]$. Secondly we delete the indeterminate x by differentiating $K(X, Y, Z : x)$ with respect to x and computing the resultant $R(K, K_x : x)$. Then the boundary generating polynomial $G_{\mathcal{H}}(1, X, Y, Z) \in \mathbb{Z}[X, Y, Z]$ appears as the main simple factor of the resultant $R(K, K_x : x)$.

In a previous paper [6], the authors of this note present the degrees of the boundary generating polynomials $G_{\mathcal{H}}(T, X, Y, Z)$ when $\mathcal{H} = (\Re(T), \Im(T), T^*T)$ for some 4×4 matrices T , in this case, $W(\mathcal{H})$ is called the Davis-Wielandt shell of T ; see refs. [7,8]. The possible degrees of the boundary generating polynomials of $W(\mathcal{H})$ can be any even number between 6 and 36 except 14, 26 and 30 under the assumption that the number of singular points of $\mathcal{V}_{\mathbb{C}}(F_{\mathcal{H}})$ is finite. Also most of the degrees of the boundary generating polynomials satisfy the degree equation

$$c_{\mathcal{H}} = 36 - 2o_{\mathcal{H}} - 4b_{\mathcal{H}}, \quad (1)$$

where $o_{\mathcal{H}}$ is the number of ordinary double points of the surface $F_{\mathcal{H}}(t, x, y, z) = 0$, and $b_{\mathcal{H}}$ the number of biplanar double points. In this note, we present the missing degrees 14, 26 and 30 of the boundary generating polynomials of the joint numerical ranges for triples of 4-by-4 Hermitian matrices that are not the Davis-Wielandt shells of any matrix.

2. Boundary Generating Polynomial of Degree 26

We consider the joint numerical range $W(\mathcal{H})$ of the following triple system $\mathcal{H} = (H_1, H_2, H_3)$ of 4×4 Hermitian matrices with vanishing diagonal entries:

$$H_1 = \begin{pmatrix} 0 & 4 & 5 & 0 \\ 4 & 0 & 4 & 3 \\ 5 & 4 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{pmatrix}, H_2 = \begin{pmatrix} 0 & -4i & 5i & 0 \\ 4i & 0 & -4i & -3i \\ -5i & 4i & 0 & 0 \\ 0 & 3i & 0 & 0 \end{pmatrix}, H_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \quad (2)$$

This triple of Hermitian matrices produces a boundary generating polynomial of $W(\mathcal{H})$ of degree 26.

Theorem 1. Let $\mathcal{H} = (H_1, H_2, H_3)$ be the triple of Hermitian matrices given in (2). Then the algebraic surface $F_{\mathcal{H}}(t, x, y, z) = 0$ has five ordinary double points in the complex projective space $\mathbb{CP}^3$, and the degree of the boundary generating polynomial is $c_{\mathcal{H}} = 26$.

Proof. The Kippenhahn polynomial of $\mathcal{H}$ is given by

$$F_{\mathcal{H}}(t, x, y, z) = t^4 - t^2(66x^2 + 66y^2 + 2z^2 + 8xz) + t(160x^3 - 480xy^2 + 64x^2z - 64y^2z) \\ + 225(x^2 + y^2)^2 - 120x(x^2 + y^2)z - 14(x^2 + y^2)z^2 + 8xz^3 + z^4.$$

The algebraic surface

$$\mathcal{V}_{\mathbb{C}}(F_{\mathcal{H}}) = \{[(t, x, y, z)] \in \mathbb{CP}^3 : F_{\mathcal{H}}(t, x, y, z) = 0\}$$

defined as a subset of the 3-dimensional complex projective space has 5 ordinary double points at $(t, x, y, z) = (1, \frac{1}{5}, 0, 0)$, $(1, -\frac{1}{10}, \frac{\sqrt{3}}{10}, 0)$, $(1, -\frac{1}{10}, -\frac{\sqrt{3}}{10}, 0)$, $(1, 0, 0, -1)$, $(1, 0, 0, 1)$. In fact, we have

$$F_{\mathcal{H}}(1, \frac{1}{5} + x, y, z) = 84x^2 - 144y^2 - \frac{64}{25}z^2 + \frac{16}{5}xz + O([x^2 + y^2 + z^2]^{3/2}), \\ F_{\mathcal{H}}(1, -\frac{1}{10} + x, \frac{\sqrt{3}}{10} + y, z) = -87x^2 + 27y^2 - \frac{64}{25}z^2 - 114\sqrt{3}xy - 28xz \\ - \frac{52\sqrt{3}}{5}yz + O([x^2 + y^2 + z^2]^{3/2}), \\ F_{\mathcal{H}}(1, -\frac{1}{10} + x, -\frac{\sqrt{3}}{10} + y, z) = -87x^2 + 27y^2 - \frac{64}{25}z^2 + 114\sqrt{3}xy - 28xz \\ + \frac{52\sqrt{3}}{5}yz + O([x^2 + y^2 + z^2]^{3/2}), \\ F_{\mathcal{H}}(1, x, y, -1 + z) = -144x^2 - 16y^2 + 4z^2 + 16xz + O([x^2 + y^2 + z^2]^{3/2}), \\ F_{\mathcal{H}}(1, x, y, 1 + z) = -16x^2 - 144y^2 + 4z^2 + 16xz + O([x^2 + y^2 + z^2]^{3/2}).$$

Furthermore, the five real symmetric matrices corresponding to the bilinear forms in the above equations all have nonzero determinants:

$$\det \begin{pmatrix} 84 & 0 & 8/5 \\ 0 & -144 & 0 \\ 8/5 & 0 & -64/25 \end{pmatrix} = \frac{156672}{5}, \\ \det \begin{pmatrix} -87 & -57\sqrt{3} & -14 \\ -57\sqrt{3} & 27 & -26\sqrt{3}/5 \\ -14 & -26\sqrt{3}/5 & -64/25 \end{pmatrix} = \frac{39168}{5},$$

$$\det \begin{pmatrix} -87 & 57\sqrt{3} & -14 \\ 57\sqrt{3} & 27 & 26\sqrt{3}/5 \\ -14 & 26\sqrt{3}/5 & -64/25 \end{pmatrix} = \frac{39168}{5},$$

$$\det \begin{pmatrix} -144 & 0 & 8 \\ 0 & -16 & 0 \\ 8 & 0 & 4 \end{pmatrix} = 10240, \det \begin{pmatrix} -16 & 0 & 8 \\ 0 & -144 & 0 \\ 8 & 0 & 4 \end{pmatrix} = 18432.$$

Hence, the five singular points are ordinary double points. The joint numerical range $W(\mathcal{H})$ is contained in a triangular prism

$$\{(X, Y, Z) : -5 \leq X \leq 10, \frac{1}{\sqrt{3}}(X - 10) \leq Y \leq -\frac{1}{\sqrt{3}}(X - 10), -1 \leq Z \leq 1\}.$$

The five singular points of the surface $F_{\mathcal{H}}(x, y, z) = 0$ correspond to the five 2-dimensional faces of this prism:

$$X = -5; X - \sqrt{3}Y = 10; X + \sqrt{3}Y = 10; Z = 1; Z = -1.$$

The real affine part of the surface $G_{\mathcal{H}}(1, X, Y, Z) = 0$ intersects the above five faces of the prism at the respective five ellipses:

$$17Y^2 + 945(Z + \frac{2}{21})^2 = \frac{5100}{7}, \text{ on } X = -5,$$

$$1348X^2 - 15120XZ + 56700Z^2 - 6020X + 32400Z - 4175 = 0, \text{ on } X \pm \sqrt{3}Y = 10,$$

$$(X - 2)^2 + 10Y^2 = 40, \text{ on } Z = 1,$$

$$9(X + 2)^3 + 2Y^2 = 72, \text{ on } Z = -1.$$

We claim that the only singular points are the five mentioned above.

Firstly, we verify that the surface $\mathcal{V}_{\mathbb{C}}(F_{\mathcal{H}})$ has no singular points on the hyperplane $t = 0$. we compute the Groebner basis of the ideal generated by the following three polynomials

$$F_{\mathcal{H}t}(0, x, y, z) = 32(5x^3 - 15xy^2 + 2x^2z - 2y^2z),$$

$$F_{\mathcal{H}x}(0, x, y, z) = 4(15x - 2z)(15x^2 + 15y^2 - 4xz - z^2),$$

$$F_{\mathcal{H}y}(0, x, y, z) = 4y(225x^2 + 225y^2 - 60xz - 7z^2).$$

If we delete the variables x, y from the basis, we get the generator z^7 . By deleting the variables x, z , we obtain the generator y^7 . Finally, by deleting the variables y, z , we obtain the generator x^7 . Hence the simultaneous equations $F_{\mathcal{H}t}(t, x, y, z) = 0, F_{\mathcal{H}x}(t, x, y, z) = 0, F_{\mathcal{H}y}(t, x, y, z) = 0$ have the unique solution $x = y = z = 0$ on the hyperplane $t = 0$, and thus there is no points satisfying the the simultaneous equations on the hyperplane $t = 0$ of the projective space $\mathbb{CP}^3$.

Secondly, we show that the affine surface

$$\{(x, y, z) \in \mathbb{C}^3 : F_{\mathcal{H}}(1, x, y, z) = 1 - (66x^2 + 66y^2 + 2z^2 + 8xz) + (160x^3 - 480xy^2 + 64x^2z - 64y^2z) + 225(x^2 + y^2)^2 - 120x(x^2 + y^2)z - 14(x^2 + y^2)z^2 + 8xz^3 + z^4 = 0\}$$

has no singular points other than $(x, y, z) = (1/5, 0, 0), (-1/10, \pm\sqrt{3}/10, 0), (0, 0, \pm 1)$. For this purpose we compute the Groebner basis of the ideal generated by the 4 polynomials $F_{\mathcal{H}}, F_{\mathcal{H}x}, F_{\mathcal{H}y}, F_{\mathcal{H}z}$ under the assumption $t = 1$. By deleting the variables x, y , the generator

of the ideal is given by $z(z-1)(z+1)$. Deleting the variables x, z , the generator is given by $y(100y^2 - 3)$. Deleting the variables y, z , the generator is given by $x(5x-1)(10x+1)$. It follows that every singular point (x_0, y_0, z_0) of the surface $F_{\mathcal{H}}(1, x, y, z) = 0$ satisfies

$$x_0 \in \{0, 1/5, -1/10\}, y_0 \in \{0, \sqrt{3}/10, -\sqrt{3}/10\}, z_0 \in \{0, 1, -1\}.$$

We examine each of the 27 candidate points individually and find that 5 of them are singular points, while the remaining 22 are not.

Next we compute the boundary generating surface $G_{\mathcal{H}}(T, X, Y, Z) = 0$ of the joint numerical range $W(\mathcal{H})$. Based on the pairing $Tt + Xx + Yy + Zz = 0$, the surface $\mathcal{V}_{\mathbb{C}}(G_{\mathcal{H}})$ is defined as the dual surface of the algebraic surface $\mathcal{V}_{\mathbb{C}}(F_{\mathcal{H}})$. The polynomial $G_{\mathcal{H}}$ is the generator of the principal ideal generated by the polynomials

$$L_0 = F_{\mathcal{H}}(t, x, y, z), L_1 = T - F_{\mathcal{H}t}(t, x, y, z), L_2 = X - F_{\mathcal{H}x}(t, x, y, z),$$

$$L_3 = Y - F_{\mathcal{H}y}(t, x, y, z), L_4 = Z - F_{\mathcal{H}z}(t, x, y, z)$$

with the elimination of the indeterminates t, x, y, z . So that the polynomial $G_{\mathcal{H}}$ can be theoretically obtained by the function

$$\text{GroebnerBasis}[\{L_0, L_1, L_2, L_3, L_4\}, \{T, X, Y, Z\}, \{t, x, y, z\}]$$

of the “Mathematica” (Version 14.3) software. However, this method is impractical due to the limitations of computational speed and memory capacity on a typical personal computer. Instead, we present a practical method for computing the polynomial $G_{\mathcal{H}}$.

The key point of a practical method to compute the polynomial $G_{\mathcal{H}}(1, X, Y, Z) \in \mathbb{Z}[X, Y, Z]$ is successive eliminations of the variables and the Lagrange interpolation. As the first step, we substitute $z = -(1/Z)(xX + yY + 1)$ into the polynomial $F_{\mathcal{H}}(1, x, y, z)$. To remove the denominator, we multiply it by Z^4 , and obtain the polynomial $\Psi(x, y : X, Y, Z)$ with 66 terms as the following:

$$\begin{aligned} \Psi(x, y : X, Y, Z) &= Z^4 F_{\mathcal{H}}(1, x, y, -\frac{1}{Z}(xX + yY + 1)) \\ &= (Y^4 - 14Y^2Z^2 + 225Z^4)y^4 + b_3(x, X, Y, Z)y^3 + b_2(x, X, Y, Z)y^2 \\ &\quad + b_1(x, X, Y, Z)y + b_0(x, X, Y, Z). \end{aligned}$$

We then compute the partial derivative $\Psi_y(x, y : X, Y, Z)$ of $\Psi(x, y : X, Y, Z)$ with respect to y . Next, we compute the resultant $R(x, X, Y, Z)$ of the two polynomials $\Psi(x, y : X, Y, Z)$ and $\Psi_y(x, y : X, Y, Z)$ with respect to y , and factorize $R(x, X, Y, Z)$ in the polynomial ring $\mathbb{Z}[x, X, Y, Z]$. The resultant $R(x, X, Y, Z)$ contains a principal simple factor $R_1(x, X, Y, Z)$ which consists of 1659 terms given by

$$\begin{aligned} R_1(x, X, Y, Z) &= (60400X^{10}Y^2 + \dots - 4100625000000Z^{12} + \dots \\ &\quad - 16402500000XY^2Z^7)x^{10} + \dots + (-656100Z^{12} + \dots - 27225)x^0. \end{aligned}$$

In fact,

$$R(x, X, Y, Z) = -256R_1(x, X, Y, Z)(Y^4 - 14Y^2Z^2 + 225Z^4)Z^{12}.$$

The second step, using the Lagrange interpolation with respect to the variable Y , we compute the resultant $S(X, Y, Z)$ of $R(x, X, Y, Z)$ and $R_x(x, X, Y, Z)$ the partial derivative of $R(x, X, Y, Z)$ with respect to x . The polynomial $G_{\mathcal{H}}(1, X, Y, Z)$ is the principal simple factor of $S(X, Y, Z)$. The homogeneous polynomial $G_{\mathcal{H}}(T, X, Y, Z)$ has degree 26 and

consists of 1908 terms. Due to the symmetry $F_{\mathcal{H}}(t, x, -y, z) = F_{\mathcal{H}}(t, x, y, z)$, we also have $G_{\mathcal{H}}(T, X, -Y, Z) = G_{\mathcal{H}}(T, X, Y, Z)$. We may present the polynomial $G_{\mathcal{H}}(1, X, Y, Z)$ in the form

$$G_{\mathcal{H}}(1, X, Y, Z) = C_0(X, Y) + C_1(X, Y)Z + \cdots + C_{26}(X, Y)Z^{26},$$

where $C_j(X, Y) \in \mathbb{Z}[X, Y]$, $j = 0, 1, \dots, 26$. In particular,

$$C_{26}(X, Y) = 22393273747120359300000000000000 = 2^{14} \cdot 3^{18} \cdot 5^{14} \cdot 7^6 \cdot 17^3,$$

and

$$C_0(X, Y) = 28934010000X^{26} + 1425298518000X^{24}Y^2 + \cdots + 318362400000Y^{26} \\ + \cdots - 36421561596093750000X^4 - 36421561596093750000X^2Y^2.$$

Notice the degree $c_{\mathcal{H}} = 26$ satisfies the degree Equation (1). $\square$

We present several graphics of sections of the surface $G_{\mathcal{H}}(1, X, Y, Z) = 0$. The polynomial $G_{\mathcal{H}}(1, X, 0, Z) \in \mathbb{Z}[X, Z]$ has two factors: one factor, $G_{\mathcal{H}1}(X, Z)$, is simple, while the other, $G_{\mathcal{H}2}(X, Z)$, has multiplicity 2. Figure 1 displays the curve $G_{\mathcal{H}1}(X, Z) = 0$ which is the boundary generating curve of $W(H_1, H_3)$. Figure 2 shows the curve $G_{\mathcal{H}2}(X, Z) = 0$. Every point of the curve $G_{\mathcal{H}2}(X, Z) = 0$ with $Y = 0$ is a singular point of the surface $G_{\mathcal{H}}(1, X, Y, Z) = 0$. By the theorem of Plaumann et al. [5], the curve $G_{\mathcal{H}2}(X, Z) = 0$ can be disregarded in generating the joint numerical range $W(H_1, H_2, H_3)$. Figure 3 presents the curve $G_{\mathcal{H}}(1, X, Y, -2/5) = 0$.

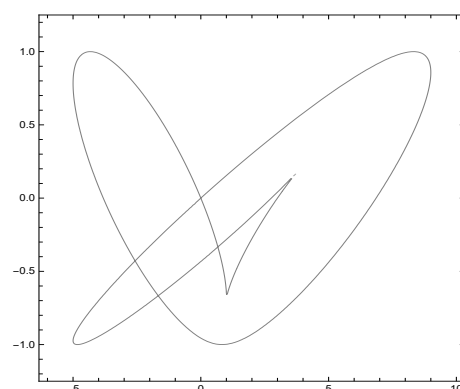


Figure 1. The curve $G_{\mathcal{H}1}(X, Z) = 0$ which is the boundary generating curve of $W(H_1, H_3)$.

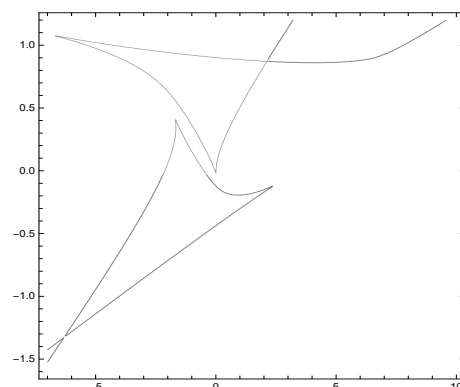


Figure 2. The curve $G_{\mathcal{H}2}(X, Z) = 0$.

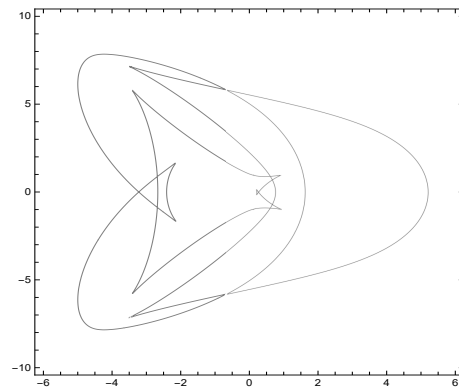


Figure 3. The curve $G_{\mathcal{H}}(1, X, Y, -2/5) = 0$.

3. Boundary Generating Polynomial of Degree 30

Let $\mathcal{H} = \{H_1, H_2, H_3\}$ be the triple of 4×4 Hermitian matrices, where H_1 and H_2 are the two given in (2), and H_3 is replaced by a new one:

$$H_3 = \begin{pmatrix} 0 & 1 & 2 & -3 \\ 1 & 0 & -1 & -2 \\ 2 & -1 & 0 & -1 \\ -3 & -2 & -1 & 0 \end{pmatrix}. \quad (3)$$

Theorem 2. Let $\mathcal{H} = (H_1, H_2, H_3)$ be the triple of Hermitian matrices, where H_1, H_2 are given in (2) and H_3 is given in (3). Then the algebraic surface $F_{\mathcal{H}}(t, x, y, z) = 0$ has three ordinary double points in the complex projective space $\mathbb{CP}^3$, and the degree of the boundary generating polynomial is $c_{\mathcal{H}} = 30$.

Proof. The proof follows a similar approach to that of Theorem 1. Computing the Kippenhahn polynomial of $\mathcal{H}$, we obtain that

$$\begin{aligned} F_{\mathcal{H}}(t, x, y, z) = & t^4 - t^2(66x^2 + 66y^2 + 8xz + 20z^2) + t(160x^3 - 480xy^2 \\ & - 32x^2z - 16y^2z + 72xz^2 + 16z^3) + 225(x^2 + y^2)^2 + 360x^3z - 120xy^2z \\ & - 228x^2z^2 + 688y^2z^2 - 144xz^3 + 48z^4. \end{aligned}$$

The algebraic surface $\mathcal{V}_{\mathbb{C}}(F_{\mathcal{H}})$ has 3 ordinary double points at the points

$$(t, x, y, z) = (1, \frac{1}{5}, 0, 0), (1, -\frac{1}{10}, \frac{\sqrt{3}}{10}, 0), (1, -\frac{1}{10}, -\frac{\sqrt{3}}{10}, 0)$$

and the surface $\mathcal{V}_{\mathbb{C}}(F_{\mathcal{H}})$ has no other singular points. The joint numerical range $W(H_1, H_2, H_3)$ is contained in the triangular prism

$$\{(X, Y, Z) : -5 \leq X \leq 10, \frac{1}{\sqrt{3}}(X - 10) \leq Y \leq -\frac{1}{\sqrt{3}}(X - 10), -1 - \sqrt{5} \leq Z \leq 1 + \sqrt{13}\}.$$

The boundary generating surface $G_{\mathcal{H}}(T, X, Y, Z) = 0$ of the joint numerical range $W(\mathcal{H})$ is defined as the dual surface of the algebraic surface $\mathcal{V}_{\mathbb{C}}(F_{\mathcal{H}})$. The polynomial $G_{\mathcal{H}}(1, X, Y, Z)$ has degree 30 and consists of 2856 terms. The polynomial $G_{\mathcal{H}}(1, X, Y, Z)$ is given by

$$\begin{aligned} G_{\mathcal{H}}(1, X, Y, Z) = & 536951446701904231229852928000000000000Z^{30} \\ & + \dots + 536951446701904231229852928000000000000. \end{aligned}$$

The 3 singular points of the surface $F_{\mathcal{H}}(t, x, y, z) = 0$ correspond to the 3 faces of the triangular prism:

$$X = -5; X - \sqrt{3}Y = 10; X + \sqrt{3}Y = 10.$$

The real affine part of the surface $G_{\mathcal{H}}(1, X, Y, Z) = 0$ meets the above 3 faces of the prism at respective 3 ellipses

$$532Y^2 + 4725\left(Z + \frac{2}{3}\right)^2 = 22800 \quad \text{on } X = -5,$$

$$18664\left(X - \frac{35}{14}\right)^2 + 7560\left(X - \frac{35}{14}\right)\left(Z - \frac{5}{42}\right) + 226800\left(Z - \frac{5}{42}\right)^2 - \frac{4185225}{7} = 0$$

$$\text{on } X \pm \sqrt{3}Y = 10.$$

The degree $c_{\mathcal{H}} = 30$ satisfies the degree Equation (1). $\square$

We present two graphics of the sections of the surface $G_{\mathcal{H}}(1, X, Y, Z) = 0$. The polynomial $G_{\mathcal{H}}(1, X, 0, Z) \in \mathbb{Z}[X, Z]$ is factored as

$$G_{\mathcal{H}}(1, X, 0, Z) = 9G_{\mathcal{H}1}(X, Z)G_{\mathcal{H}2}(X, Z)^2,$$

where

$$G_{\mathcal{H}1}(X, Z) = 938448Z^{10} + \dots + 833078601,$$

$$G_{\mathcal{H}2}(X, Z) = -7973355852000000Z^{10} + \dots - 3963809862852750000.$$

Figure 4 displays the curve $G_{\mathcal{H}1}(X, Z) = 0$ which is the boundary generating curve of the range $W(H_1, H_3)$. Figure 5 shows the graphic of the curve $G_{\mathcal{H}}(1, X, Y, 1/2) = 0$.

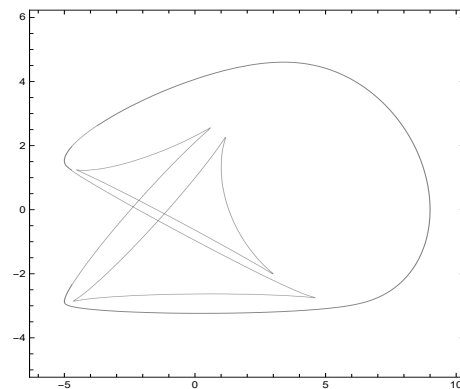


Figure 4. The curve $G_{\mathcal{H}1}(X, Z) = 0$ which is the boundary generating curve of the range $W(H_1, H_3)$.

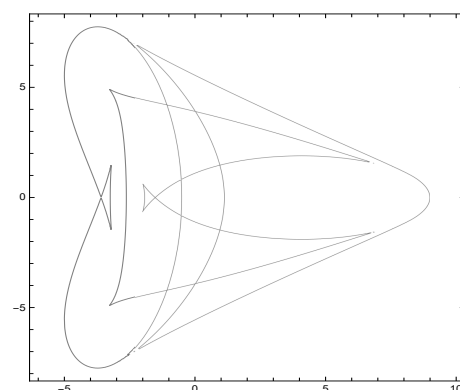


Figure 5. The curve $G_{\mathcal{H}}(1, X, Y, 1/2) = 0$.

4. Boundary Generating Polynomial of Degree 14

We consider a triple system $\mathcal{H} = (H_1, H_2, H_3)$ of 4×4 Hermitian matrices given by

$$H_1 = \begin{pmatrix} -1 & 1 & 0 & -1 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & i \\ -1 & 1 & -i & 0 \end{pmatrix}, H_2 = \begin{pmatrix} 0 & -i & 0 & -i \\ i & 0 & 0 & i \\ 0 & 0 & 0 & 1 \\ i & -i & 1 & 0 \end{pmatrix}, H_3 = \text{diag}(0, 0, 0, -2). \quad (4)$$

This triple system is provided in ref. [9] (E6) for producing single face

$$\{(X, Y, 0) : -2 \leq X \leq -1/2, (X+1)^2 + Y^2 \leq 1\} \cup$$

$$\{(X, Y, 0) : -1/2 \leq X \leq 1, -1/\sqrt{3}(1-X) \leq Y \leq 1/\sqrt{3}(1-X)\}$$

on the boundary of $W(\mathcal{H})$.

Theorem 3. Let $\mathcal{H} = (H_1, H_2, H_3)$ be the triple of Hermitian matrices given in (4). Then the degree of the boundary generating polynomial is $c_{\mathcal{H}} = 14$.

Proof. The Kippenhan polynomial $F_{\mathcal{H}}(t, x, y, z)$ of this triple system is given by

$$F_{\mathcal{H}}(t, x, y, z) = t^4 - t^3(x + 2z) + t^2(-5x^2 - 4y^2 + 2xz) + t(-xy^2 + 4x^2z + 2y^2z) \\ + x^2y^2 + y^4 + 2xy^2z.$$

The degree of $F_{\mathcal{H}}(t, x, y, z)$ with respect to z is 1, and the surface $F_{\mathcal{H}}(1, x, y, z) = 0$ has a rational expression

$$z = \frac{x^2y^2 + y^4 - xy^2 - 5x^2 - 4y^2 - x + 1}{2(x+1)(-y^2 - 2x + 1)}$$

which gives a rational parametrization of the surface $F_{\mathcal{H}}(1, x, y, z) = 0$ as $z = z(x, y)$. In this sense, the surface $\mathcal{V}_{\mathbb{C}}(F_{\mathcal{H}})$ is a rational surface. This surface has a triple point $(t, x, y, z) = (0, 0, 0, 1)$, an ordinary double point $(t, x, y, z) = (0, 1, 0, 0)$, and a pair of ordinary points $(t, x, y, z) = (1, 1, \pm i, 3/2)$. The surface $\mathcal{V}_{\mathbb{C}}(F_{\mathcal{H}})$ has no singular points other than these 4 points. The boundary generating surface $G_{\mathcal{H}}(1, X, Y, Z) = 0$ of the joint numerical range $W(\mathcal{H})$ is given as the dual surface of the algebraic curve $F_{\mathcal{H}}(1, x, y, z) = 0$. The degree of the polynomial $G_{\mathcal{H}}(T, X, Y, Z)$ is 14 and consists of 368 terms. The surface $G_{\mathcal{H}}(1, X, Y, Z) = 0$ also has a rational parametrization as $X = -z_x / (xz_x + yz_y - z)$, $Y = -z_y / (xz_x + yz_y - z)$, $Z = 1 / (xz_x + yz_y - z)$. $\square$

The polynomial $G_{\mathcal{H}}(1, X, Y, 0)$ can be factored as

$$G_{\mathcal{H}}(1, X, Y, 0) = 49152(-\sqrt{3}Y + X + 1)^4(-\sqrt{3}Y - X + 1)^4((X - 1)^2 + Y^2)((X + 1)^2 + Y^2 - 1)^2.$$

The two lines $Y = \pm(X + 1)$ are tangents of the circle $(X + 1)^2 + Y^2 = 1$. The polynomial $G_{\mathcal{H}}(1, X, 0, Z)$ is factored as

$$G_{\mathcal{H}}(1, X, 0, Z) = G_{\mathcal{H}1}(X, Z)G_{\mathcal{H}2}(X, Z)^2,$$

where

$$G_{\mathcal{H}1}(X, Z) = 48X^4 + 64X^3Z + 168X^2Z^2 + 48XZ^3 + 175Z^4 + 96X^3 + 240X^2Z \\ + 24XZ^2 + 780Z^3 - 144X^2 - 240XZ + 1212Z^2 - 192X + 800Z + 192$$

and

$$G_{\mathcal{H}2}(X, Z) = 32X^5 + \cdots - 57Z^5 + 32X - 16Z.$$

The curve $G_{\mathcal{H}1}(X, Z) = 0$ is the boundary generating curve of the range $W(H_1, H_3)$. Every point of the curve $G_{\mathcal{H}2}(X, Z) = 0$ with $Y = 0$ is a singular point of the surface $G_{\mathcal{H}}(1, X, Y, Z) = 0$.

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