

Article

First Robin Eigenvalue of the Laplacian on Kähler and Quaternionic Kähler Manifolds

Shaoheng Zhang *  and Weijie Zhu

School of Mathematical Sciences, Soochow University, Suzhou 215006, China; 20234207012@stu.suda.edu.cn

* Correspondence: 20234007008@stu.suda.edu.cn

Abstract

We investigate the first Robin eigenvalue of the Laplacian on Kähler and quaternionic Kähler manifolds. First, we establish Cheng-type eigenvalue comparison theorems for Kähler manifolds under lower bounds on holomorphic sectional curvature and orthogonal Ricci curvature and for quaternionic Kähler manifolds under scalar curvature lower bounds. For compact Kähler and quaternionic Kähler manifolds, our second result derives lower bounds for the first Robin eigenvalue when the Robin parameter α is positive, with corresponding upper bounds when it is negative.

Keywords: eigenvalue comparison; Kähler manifold; quaternionic Kähler manifold

MSC: 35P15; 53C26

1. Introduction

Let (M^m, g) be an m -dimensional compact Riemannian manifold with a smooth nonempty boundary ∂M . Denote by Δ the Laplacian of the metric g . We consider the Robin eigenvalue problem:

$$\begin{cases} -\Delta u = \lambda u, & \text{in } M, \\ \frac{\partial u}{\partial \nu} + \alpha u = 0, & \text{on } \partial M, \end{cases} \quad (1)$$

where ν is the outward normal vector field along ∂M , and $\alpha \in \mathbb{R}$ is the Robin parameter. The eigenvalues $\lambda_k(M, \alpha)$ satisfy

$$\lambda_1(M, \alpha) < \lambda_2(M, \alpha) \leq \dots \rightarrow \infty,$$

where each eigenvalue is repeated according to its multiplicity.

The Robin eigenvalue problem (1) interpolates between the Neumann ($\alpha = 0$) and Dirichlet ($\alpha \rightarrow +\infty$) cases, while $\lambda = 0$ gives the Steklov problem. Recent studies include [1–5] and related references. The first Robin eigenvalue admits the variational characterization:

$$\lambda_1(M, \alpha) = \inf \left\{ \int_M |\nabla u|^2 d\mu_g + \alpha \int_{\partial M} |u|^2 dA : u \in W^{1,2}(M), \int_M |u|^2 d\mu_g = 1 \right\},$$

where $d\mu_g$ and dA are volume and area measures, respectively.

The classical Cheng-type eigenvalue comparison theorem [6] states that for a geodesic ball $B_R(x_0)$ in M^m with $\text{Ric} \geq (m-1)\kappa$, for some $\kappa \in \mathbb{R}$, its first Dirichlet eigenvalue satisfies



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$$\lambda_1^D(B_R(x_0)) \leq \lambda_1^D(V(\kappa, R)),$$

where $V(\kappa, R)$ is a geodesic ball of radius R in $\mathbb{M}^m(\kappa)$, the space form of constant sectional curvature κ . The reversed inequality holds if the sectional curvature $\text{Sect} \leq \kappa$ on $B_R(x_0)$ and R is less than the injectivity radius at x_0 . Savo [5] established Cheng's eigenvalue theorem for Robin eigenvalues of the Laplacian via Green's formula; Li and Wang [3] later extended Savo's result to the p -Laplacian ($p > 1$) via Picone's identity.

For the closed or Neumann eigenvalues, Li and Wang [7,8] obtained lower bounds for the first nonzero eigenvalues on compact Kähler and quaternionic Kähler manifolds. These results were later extended to the p -Laplacian by Wang and the first author in [9,10]. Additionally, Rutkowski and Seto [11] and Li and Wang [8] established explicit lower bounds of the first nonzero closed or Neumann eigenvalues on Kähler and quaternionic Kähler manifolds using Wirtinger's inequality.

Our first result establishes the first Robin eigenvalue estimates for the Laplacian on Kähler and quaternionic Kähler manifolds.

Theorem 1. *Let (M^m, g, J) be a complete Kähler manifold of complex dimension m , and let $B_R(x_0) \subset M$ be a geodesic ball of radius R centered at x_0 with holomorphic sectional curvature $H \geq 4\kappa$ and orthogonal Ricci curvature $\text{Ric}^\perp \geq 2(m-1)\kappa$ for $\kappa \in \mathbb{R}$. Then*

$$\begin{aligned} \lambda_1(B_R(x_0), \alpha) &\leq \lambda_1(V_{\mathbb{C}}(\kappa, R), \alpha), \quad \text{if } \alpha > 0, \\ \lambda_1(B_R(x_0), \alpha) &\geq \lambda_1(V_{\mathbb{C}}(\kappa, R), \alpha), \quad \text{if } \alpha < 0, \end{aligned} \quad (2)$$

where $V_{\mathbb{C}}(\kappa, R)$ is a geodesic ball of radius R in $M_{\mathbb{C}}^m(\kappa)$, the complex space form with constant holomorphic sectional curvature $H = 4\kappa$, and complex dimension is m .

Theorem 2. *Let (M^m, g, J_1, J_2, J_3) be a complete quaternionic Kähler manifold of quaternionic dimension $m \geq 2$, and let $B_R(x_0) \subset M$ be a geodesic ball of radius R centered at x_0 with scalar curvature $S \geq 16m(m+2)\kappa$ for $\kappa \in \mathbb{R}$. Then*

$$\begin{aligned} \lambda_1(B_R(x_0), \alpha) &\leq \lambda_1(V_{\mathbb{H}}(\kappa, R), \alpha), \quad \text{if } \alpha > 0, \\ \lambda_1(B_R(x_0), \alpha) &\geq \lambda_1(V_{\mathbb{H}}(\kappa, R), \alpha), \quad \text{if } \alpha < 0, \end{aligned} \quad (3)$$

where $V_{\mathbb{H}}(\kappa, R)$ is a geodesic ball of radius R in $M_{\mathbb{H}}^m(\kappa)$, the quaternionic space form with constant quaternionic sectional curvature $Q = 12\kappa$ and quaternionic dimension m .

Remark 1 (Sharpness). *The conclusions of Theorems 1 and 2 are sharp, as equality is attained in the corresponding space forms. For relevant background on these space forms for Kähler and quaternionic Kähler manifolds, we refer to Section 2.1.*

Remark 2 (Rigidity). *The eigenvalue comparisons established in Theorem 1 is rigid. Specifically, in Theorem 1, the equality $\lambda_1(B_R(x_0), \alpha) = \lambda_1(V_{\mathbb{C}}(\kappa, R), \alpha)$ if and only if the geodesic ball $B_R(x_0)$ in the Kähler manifold (M^m, g, J) is holomorphic-isometric to the geodesic ball $V_{\mathbb{C}}(\kappa, R)$ in the complex space form, see, e.g. (Corollary 3.1, [12]).*

For $\kappa, \Lambda \in \mathbb{R}$, denote by $C_{\kappa, \Lambda}$ the unique solution to the initial value problem:

$$\begin{cases} C_{\kappa, \Lambda}''(t) + \kappa C_{\kappa, \Lambda}(t) = 0, \\ C_{\kappa, \Lambda}(0) = 1, \quad C_{\kappa, \Lambda}'(0) = -\Lambda, \end{cases}$$

and define

$$T_{\kappa,\Lambda}(t) := \frac{C'_{\kappa,\Lambda}(t)}{C_{\kappa,\Lambda}(t)}. \quad (4)$$

In our setting, t is the distance to the boundary, so $t \in [0, R]$ with R being the inradius introduced below. The solution $C_{\kappa,\Lambda}$ is real-analytic (hence C^∞) on $\mathbb{R}$. Note that the definition of $T_{\kappa,\Lambda}$ in (4) has the opposite sign of that in [7,8]. Let R be the inradius of M , i.e.,

$$R = \sup_{x \in M} d(x, \partial M).$$

In the Riemannian setting, the following comparison theorem for the first Dirichlet eigenvalue is well-known:

Theorem 3. *Let M^m be a compact Riemannian manifold with smooth boundary $\partial M \neq \emptyset$. Suppose the Ricci curvature of M is bounded from below by $(m-1)\kappa$ and the mean curvature of ∂M is bounded from below by $(m-1)\Lambda$ for some $\kappa, \Lambda \in \mathbb{R}$. Then, the first Dirichlet eigenvalue of M satisfies*

$$\lambda_1^D(M) \geq \bar{\lambda}_1(m, \kappa, \Lambda, R),$$

where $\bar{\lambda}_1(m, \kappa, \Lambda, R)$ is the first eigenvalue of the one-dimensional eigenvalue problem:

$$\begin{cases} \varphi'' + (m-1)T_{\kappa,\Lambda}\varphi' = -\lambda\varphi, \\ \varphi(0) = 0, \quad \varphi'(R) = 0. \end{cases}$$

Here, R is the inradius of M .

Theorem 3 was proven by Li and Yau [13] for $\kappa = \Lambda = 0$ and by Kasue [14] for $\kappa, \Lambda \in \mathbb{R}$. Subsequent work by Li and Wang [7,8] developed analogous results in Kähler and quaternionic Kähler geometries, with further generalizations to the p -Laplacian by Wang and the first author [9,10].

Our second result establishes first Robin eigenvalue estimates with an associated one-dimensional eigenvalue problem.

Theorem 4. *Let (M^m, g, J) be a compact Kähler manifold of complex dimension m and smooth boundary $\partial M \neq \emptyset$. Assume that the holomorphic sectional curvature $H \geq 4\kappa$, the orthogonal Ricci curvature $\text{Ric}^\perp \geq 2(m-1)\kappa$ on M for $\kappa \in \mathbb{R}$, and the second fundamental form of ∂M is bounded from below by $\Lambda \in \mathbb{R}$. Then*

$$\begin{aligned} \lambda_1(M, \alpha) &\geq \bar{\lambda}_1(m, \kappa, \Lambda, R, \alpha), & \text{if } \alpha > 0, \\ \lambda_1(M, \alpha) &\leq \bar{\lambda}_1(m, \kappa, \Lambda, R, \alpha), & \text{if } \alpha < 0, \end{aligned}$$

where $\lambda_1(M, \alpha)$ denotes the first Robin eigenvalue of the Laplacian on M , and $\bar{\lambda}_1(m, \kappa, \Lambda, R, \alpha)$ is the first eigenvalue of the one-dimensional eigenvalue problem

$$\begin{cases} \varphi'' + (2(m-1)T_{\kappa,\Lambda} + T_{4\kappa,\Lambda})\varphi' = -\lambda\varphi, \\ \varphi'(0) = \alpha\varphi(0), \quad \varphi'(R) = 0. \end{cases} \quad (5)$$

Here, R is the inradius of M .

Theorem 5. Let (M^m, g, J_1, J_2, J_3) be a compact quaternionic Kähler manifold of complex dimension m and smooth boundary $\partial M \neq \emptyset$. Assume that the scalar curvature $S \geq 16m(m+2)\kappa$ for $\kappa \in \mathbb{R}$, and the second fundamental form of ∂M is bounded from below by $\Lambda \in \mathbb{R}$. Then

$$\begin{aligned}\lambda_1(M, \alpha) &\geq \tilde{\lambda}_1(m, \kappa, \Lambda, R, \alpha), & \text{if } \alpha > 0, \\ \lambda_1(M, \alpha) &\leq \tilde{\lambda}_1(m, \kappa, \Lambda, R, \alpha), & \text{if } \alpha < 0,\end{aligned}$$

where $\lambda_1(M, \alpha)$ denotes the first Robin eigenvalue of the Laplacian on M , and $\tilde{\lambda}_1(m, \kappa, \Lambda, R, \alpha)$ is the first eigenvalue of the one-dimensional eigenvalue problem:

$$\begin{cases} \varphi'' + (4(m-1)T_{\kappa, \Lambda} + 3T_{4\kappa, \Lambda})\varphi' = -\lambda\varphi, \\ \varphi'(0) = \alpha\varphi(0), \quad \varphi'(R) = 0. \end{cases} \quad (6)$$

Here, R is the inradius of M .

Remark 3. Taking the limit $\alpha \rightarrow +\infty$ in Theorems 4 and 5, we obtain the first Dirichlet eigenvalue comparisons on Kähler and quaternionic Kähler manifolds, generalizing the results in [7,8].

Remark 4. For Theorems 2 and 5, concerning eigenvalue estimates on quaternionic Kähler manifolds, the scalar curvature lower bound of $16m(m+2)\kappa$ can be replaced by requiring $Q \geq 12\kappa$ and $\text{Ric}^\perp \geq 4(m-1)\kappa$ for the quaternionic sectional and orthogonal Ricci curvatures, respectively.

The main novelty of this manuscript lies in establishing the first sharp eigenvalue comparisons for the Robin Laplacian on Kähler and quaternionic Kähler manifolds. While Li and Wang [3] obtained such comparison results on Riemannian manifolds, our work extends this to the complex and quaternionic settings by exploiting the additional geometric structures. The proofs are achieved by following the method of Li and Wang [3] and applying it to the refined Laplacian comparison theorems for the distance function and the distance-to-boundary function on Kähler and quaternionic Kähler manifolds. This approach allows us to construct appropriate trial functions from the one-dimensional comparison models and, via Barta's inequality, obtain the sharp estimates announced.

The remainder of the paper is structured as follows: Section 2 recalls fundamental concepts of Kähler and quaternionic Kähler manifolds and introduces key lemmas. Theorems 1 and 2 are proven in Section 3. Section 4 is devoted to the proof of Theorems 4 and 5.

2. Preliminaries

2.1. Kähler Manifolds and Quaternionic Kähler Manifolds

In this subsection, we recall some basic notations and facts about Kähler and quaternionic Kähler manifolds.

Let M be a Kähler manifold. A plane $\sigma \subset T_p M$ is said to be holomorphic if it is invariant under the complex structure tensor J . Given a holomorphic plane σ spanned by X and JX , its holomorphic sectional curvature is defined as

$$H(\sigma) := H(X) = \frac{R(X, JX, X, JX)}{|X|^4}.$$

The orthogonal Ricci curvature, denoted by $\text{Ric}^\perp$, is defined for any $X \in T_p M$ by

$$\text{Ric}^\perp(X, X) := \text{Ric}(X, X) - H(X)|X|^2.$$

Denote by $M_{\mathbb{C}}^m(\kappa)$ the complex space form, i.e., a simply connected complete Kähler manifold of complex dimension m with constant holomorphic sectional curvature 4κ . Let $V_{\mathbb{C}}(\kappa, R)$ be a geodesic ball of radius R in $M_{\mathbb{C}}^m(\kappa)$. It is well known that the Ricci curvature of $M_{\mathbb{C}}^m(\kappa)$ is $2(m+1)\kappa$ (see, e.g., p. 168, [15]), implying constant orthogonal Ricci curvature $2(m-1)\kappa$. Thus, the equality case of (2) can be achieved, proving the sharpness of Theorem 1.

Next, we review some basic facts about quaternionic Kähler manifolds.

Definition 1. A quaternionic Kähler manifold (M^m, g) of quaternionic dimension m (the real dimension is $4m$) is a Riemannian manifold with a rank 3 vector bundle $V \subset \text{End}(TM)$, satisfying

1. In any coordinate neighborhood U of M , there exists a local basis $\{J_1, J_2, J_3\}$ of V such that

$$J_i J_{i+1} = -J_{i+1} J_i = J_{i+2} \quad (i \bmod 3),$$

and

$$J_i^2 = -1, \quad g(J_i X, J_i Y) = g(X, Y),$$

for all $X, Y \in TM$ and $i = 1, 2, 3$.

2. If $\phi \in \Gamma(V)$, then $\nabla_X \phi \in \Gamma(V)$ for all $X \in TM$.

Let M be a quaternionic Kähler manifold. As in [16] or [8], we define the quaternionic sectional curvature of M as

$$Q(X) := \sum_{i=1}^3 \frac{R(X, J_i X, X, J_i X)}{|X|^4},$$

and the orthogonal Ricci curvature of M is defined as

$$\text{Ric}^\perp(X, X) := \text{Ric}(X, X) - Q(X)|X|^2.$$

Note that the usual quaternionic sectional curvature $\bar{Q}$ is defined as follows (see Section 5, [17]): it is the constant $\bar{Q}(X)$ such that for any $Y_1, Y_2 \in \text{span}\{X, J_1 X, J_2 X, J_3 X\}$, the sectional curvature $K(Y_1, Y_2)$ equals $\bar{Q}(X)$.

Since all quaternionic Kähler manifolds of quaternionic dimension $m \geq 2$ are Einstein, there exists a constant δ such that $\text{Ric} = \delta g$. The manifold M has constant quaternionic sectional curvature $Q = \frac{3}{m+2}\delta$ and orthogonal Ricci curvature $\text{Ric}^\perp = \frac{m-1}{m+2}\delta$ (see, e.g., Proposition 2.1, [8]). Consequently, for quaternionic Kähler manifolds, the scalar curvature condition $S \geq 16m(m+2)\kappa$ holds if and only if $Q \geq 12\kappa$ and $\text{Ric}^\perp \geq 4(m-1)\kappa$.

Denote by $M_{\mathbb{H}}^m(\kappa)$ the quaternionic space form, i.e., a simply connected complete quaternionic Kähler manifold of quaternionic dimension m with $\bar{Q} = 4\kappa$. Let $V_{\mathbb{H}}(\kappa, R)$ be a geodesic ball of radius R in $M_{\mathbb{H}}^m(\kappa)$. On $M_{\mathbb{H}}^m(\kappa)$, we have $\text{Ric} = 4(m+2)\kappa$, and $Q = 12\kappa$, giving $\text{Ric}^\perp = 4(m-1)\kappa$. Hence, Theorem 2 is sharp.

2.2. Barta's Inequality

Barta's inequality is a classical tool to derive bounds for eigenvalues (see, for example, p. 70, [18]). It states that if $f \in C^2(M) \cap C^0(\bar{M})$ satisfies $f|_M > 0$ and $f|_{\partial M} = 0$, then

$$\inf_M \frac{-\Delta f}{f} \leq \lambda_1^D(M) \leq \sup_M \frac{-\Delta f}{f},$$

where $\lambda_1^D(M)$ is the first Dirichlet eigenvalue of the Laplacian, with equality iff f is a first Dirichlet eigenfunction. Kasue generalized Barta's inequality to continuous func-

tions (Lemma 1.1, [14]). Li and Wang recently extended it to the Robin eigenvalue of the p -Laplacian ($p > 1$) and proved the following lemma:

Lemma 1 (Theorem 3.1, [3]). *Let $v \in C^1(\overline{M})$ be a positive function.*

1. *Assume that v satisfies*

$$\begin{cases} -\Delta v \geq \lambda v, & \text{in } M, \\ \frac{\partial v}{\partial \nu} + \alpha v \geq 0, & \text{on } \partial M, \end{cases}$$

in the sense of distributions. Then

$$\lambda_1(M, \alpha) \geq \lambda.$$

2. *Assume that v satisfies*

$$\begin{cases} -\Delta v \leq \lambda v, & \text{in } M, \\ \frac{\partial v}{\partial \nu} + \alpha v \leq 0, & \text{on } \partial M, \end{cases}$$

in the sense of distributions. Then

$$\lambda_1(M, \alpha) \leq \lambda.$$

Moreover, equality holds iff v is an eigenfunction for $\lambda_1(M, \alpha)$.

2.3. Properties of One-Dimensional Eigenvalue Models

In this subsection, we present some properties of a one-dimensional eigenvalue problem that serves as a comparison model throughout this work. Suppose ω is a smooth function defined on $[0, R]$ satisfying $\omega > 0$ on $[0, R)$, we consider the following one-dimensional Robin eigenvalue problem

$$\begin{cases} \varphi'' + \frac{\omega'}{\omega} \varphi' = -\mu \varphi, & \text{in } (0, R), \\ \varphi'(0) = \alpha \varphi(0), \quad \varphi'(R) = 0, \end{cases} \quad (7)$$

The first eigenvalue of (7), denoted by $\bar{\mu}([0, R], \omega, \alpha)$, can be characterized variationally as

$$\begin{aligned} & \bar{\mu}([0, R], \omega, \alpha) \\ &= \inf \left\{ \frac{\int_0^R \varphi'(t)^2 \omega(t) dt + \alpha \varphi(0)^2 \omega(0)}{\int_0^R \varphi(t)^2 \omega(t) dt} : \varphi \in W^{1,2}([0, R], \omega dt), \varphi'(R) = 0 \right\}. \end{aligned}$$

Li and Wang established the following properties to the eigenvalue problem (7) in (Proposition 2.1, [3]):

Lemma 2. *Let $\varphi > 0$ be the first positive eigenfunction corresponding to $\bar{\mu}([0, R], \omega, \alpha)$.*

- (1) *If $\alpha > 0$, then $\varphi' > 0$ on $[0, R)$.*
- (2) *If $\alpha < 0$, then $\varphi' < 0$ on $[0, R)$.*

3. Proof of Theorems 1 and 2

For $\kappa \in \mathbb{R}$, let $\text{sn}_\kappa(t)$ be the unique solution of $\text{sn}_\kappa''(t) + \kappa \text{sn}_\kappa(t) = 0$ with $\text{sn}_\kappa(0) = 0$, $\text{sn}_\kappa'(0) = 1$. Explicitly, we have

$$\operatorname{sn}_{\kappa}(t) = \begin{cases} \frac{1}{\sqrt{\kappa}} \sin(\sqrt{\kappa}t), & \text{if } \kappa > 0, \\ t, & \text{if } \kappa = 0, \\ \frac{1}{\sqrt{-\kappa}} \sinh(\sqrt{-\kappa}t), & \text{if } \kappa < 0. \end{cases} \quad (8)$$

Denote by $\operatorname{ct}_{\kappa}(t) = \frac{\operatorname{sn}'_{\kappa}(t)}{\operatorname{sn}_{\kappa}(t)}$ and by $\operatorname{Cut}(x_0)$ the cut locus of x_0 . Next, we present the Laplacian comparison theorems for Kähler manifolds and quaternionic Kähler manifolds.

Lemma 3. 1. Let (M^m, g, J) be a complete Kähler manifold of complex dimension m , and let $r(x) = d(x_0, x)$. Assume that the holomorphic sectional curvature bound $H \geq 4\kappa$ and the orthogonal Ricci curvature bound $\operatorname{Ric}^{\perp} \geq (2m - 2)\kappa$ hold on M for $\kappa \in \mathbb{R}$. Then

$$\Delta r(x) \leq (2m - 2) \operatorname{ct}_{\kappa}(r) + \operatorname{ct}_{4\kappa}(r)$$

holds for all $x \in M \setminus \{x_0, \operatorname{Cut}(x_0)\}$, and distributionally on M .

2. Let (M^m, g, J_1, J_2, J_3) be a complete quaternionic Kähler manifold of quaternionic dimension $m \geq 2$, and let $r(x) = d(x_0, x)$. Assume that the scalar curvature bound $S \geq 16m(m + 2)\kappa$ on M for $\kappa \in \mathbb{R}$. Then

$$\Delta r(x) \leq (4m - 4) \operatorname{ct}_{\kappa}(r) + 3 \operatorname{ct}_{4\kappa}(r)$$

holds for all $x \in M \setminus \{x_0, \operatorname{Cut}(x_0)\}$, and distributionally on M .

Proof. For (1), see [12] or (Theorem 3.1, [16]). For (2), since on quaternionic Kähler manifolds, the scalar curvature condition $S \geq 16m(m + 2)\kappa$ is equivalent to $Q \geq 12\kappa$ and $\operatorname{Ric}^{\perp} \geq 4(m - 1)\kappa$, the result follows from Theorem 3.2 of [16]. $\square$

Let (r, θ) be the geodesic polar coordinates of $V_{\mathbb{C}}(\kappa, R)$ with the origin o , namely

$$V_{\mathbb{C}}(\kappa, R) = \left\{ \exp_o(r\theta) : r \in [0, R], \theta \in \mathbb{S}^{2m-1}(1) \right\},$$

and the metric is given by $g = dr^2 + g_r$, where g_r is a Berger metric on $\mathbb{S}^{2m-1}(r)$ (cf. Section 2, [19]). Since the metric g is a warped product and $L^2(\mathbb{S}^{2m-1}(r))$ admits a Hilbert basis consisting of simultaneous eigenfunctions for $\Delta_{\mathbb{S}(r)}$ and $\Delta_{\mathbb{S}(1)}$ (cf. Theorem 3.6, [20] and Theorem 3.1, [19]), we can apply the separation of variables to find solutions of (1) (cf. pp. 40–42, [18]). Let $u(x) = \varphi(r)h(\theta)$ be a Robin eigenfunction of the Laplacian corresponding to eigenvalue λ on $V_{\mathbb{C}}(\kappa, R)$. The Laplacian is given by

$$\Delta = \frac{\partial^2}{\partial r^2} + ((2m - 2) \operatorname{ct}_{\kappa}(r) + \operatorname{ct}_{4\kappa}(r)) \frac{\partial}{\partial r} + \Delta_{\mathbb{S}(r)}.$$

Then, the Robin eigenvalue Equation (1) becomes

$$-\frac{\varphi''(r) + ((2m - 2) \operatorname{ct}_{\kappa}(r) + \operatorname{ct}_{4\kappa}(r)) \varphi'(r)}{\varphi(r)} - \frac{\Delta_{\mathbb{S}(r)} h(\theta)}{h(\theta)} = \lambda, \quad (9)$$

together with the following boundary condition:

$$\varphi'(0) = 0, \quad \varphi'(R) + \alpha \varphi(R) = 0.$$

For any fixed $r \in [0, R]$, $-\frac{\Delta_{\mathbb{S}(r)} h(\theta)}{h(\theta)}$ is independent of θ . Therefore

$$-\frac{\Delta_{\mathbb{S}(r)} h(\theta)}{h(\theta)} = \mu_k(\mathbb{S}(r))$$

for some $k \in \{1, 2, \dots\}$, where $\mu_k(\mathbb{S}(r))$ is the k -th eigenvalue of $\Delta_{\mathbb{S}(r)}$ on $\mathbb{S}(r)$. Then, (9) becomes

$$-\varphi''(r) - ((2m-2)\text{ct}_\kappa(r) + \text{ct}_{4\kappa}(r))\varphi'(r) + \mu_k(\mathbb{S}(r))\varphi(r) = \lambda\varphi(r), \quad 0 < r < R, \quad (10)$$

with $\varphi'(0) = 0, \varphi'(R) + \alpha\varphi(R) = 0$. Moreover, the first eigenvalue $\hat{\lambda}_1$ of (10) can be characterized variationally by

$$\hat{\lambda}_1 = \inf_{\varphi \in W^{1,2}([0,R], Jdr)} \left\{ \frac{\int_0^R (\varphi'(r)^2 + \mu_k(\mathbb{S}(r))\varphi(r)^2)J(r) dr + \alpha\varphi^2(R)J(R)}{\int_0^R \varphi(r)^2 J(r) dr} : \varphi'(0) = 0 \right\},$$

where $J(r) = \text{sn}_\kappa^{2m-2}(r) \text{sn}_{4\kappa}(r)$. When considering the first Robin eigenvalue $\lambda_1(V_{\mathbb{C}}(\kappa, R), \alpha)$, we have

$$\mu_k(\mathbb{S}(r)) = \mu_1(\mathbb{S}(r)) = 0,$$

and the corresponding eigenfunction $h(\theta)$ is a constant. Therefore, the first Robin eigenfunction on $V_{\mathbb{C}}(\kappa, R)$ is $u(x) = \varphi(r)$, which is radial and satisfies

$$\begin{cases} \varphi''(r) + ((2m-2)\text{ct}_\kappa(r) + \text{ct}_{4\kappa}(r))\varphi'(r) = -\lambda_1(V_{\mathbb{C}}(\kappa, R), \alpha)\varphi(r), & 0 < r < R, \\ \varphi'(0) = 0, \quad \varphi'(R) = -\alpha\varphi(R). \end{cases} \quad (11)$$

Similarly, the first Robin eigenfunction $u(x)$ on $V_{\mathbb{H}}(\kappa, R)$ is radial. Denote it by $u(x) = \varphi(r)$. Then, we have

$$\begin{cases} \varphi''(r) + ((4m-4)\text{ct}_\kappa(r) + 3\text{ct}_{4\kappa}(r))\varphi'(r) = -\lambda_1(V_{\mathbb{H}}(\kappa, R), \alpha)\varphi(r), & 0 < r < R, \\ \varphi'(0) = 0, \quad \varphi'(R) = -\alpha\varphi(R). \end{cases} \quad (12)$$

Proof of Theorem 1. Let $u > 0$ denote the first Robin eigenfunction corresponding to $\lambda_1(V_{\mathbb{C}}(\kappa, R), \alpha)$. The preceding discussion implies that $u(x) = \varphi(r)$ and $\varphi(r)$ satisfies Equation (11). Setting $\tilde{\varphi}(t) = \varphi(R-t)$, we have

$$\begin{cases} \tilde{\varphi}'' + \frac{\omega'}{\omega}\tilde{\varphi}' = -\lambda_1(V_{\mathbb{C}}(\kappa, R), \alpha)\tilde{\varphi}, \\ \tilde{\varphi}'(0) = \alpha\tilde{\varphi}(0), \quad \tilde{\varphi}'(R) = 0, \end{cases}$$

where $\omega(t) = \text{sn}_\kappa^{2m-2}(R-t) \text{sn}_{4\kappa}(R-t)$.

We first consider the case of $\alpha > 0$. Lemma 2 implies that $\varphi'(t) = -\tilde{\varphi}'(R-t) < 0$ for $t \in (0, R]$. Define the function $v(x)$ on $B_R(x_0)$ by

$$v(x) = \varphi(r(x)),$$

where $r(x) = d(x, x_0)$. Direct calculation gives

$$\begin{aligned} -\Delta v(x) &= -\varphi'' - \varphi' \Delta r(x) \\ &\leq -\varphi'' - (2(m-1)\text{ct}_\kappa(r) + \text{ct}_{4\kappa}(r))\varphi' \\ &= \lambda_1(V_{\mathbb{C}}(\kappa, R), \alpha)\varphi(r(x)) \\ &= \lambda_1(V_{\mathbb{C}}(\kappa, R), \alpha)v(x) \end{aligned}$$

for all $x \in M \setminus \{x_0 \cup \text{Cut}(x_0)\}$. Here, in the second line, we used Lemma 3. On the other hand, one can verify that $v(x)$ satisfies the Robin boundary condition

$$\frac{\partial v}{\partial \nu} + \alpha v = 0$$

on $\partial B_R(x_0)$. Since the cut locus has measure zero, a standard approximation argument (see, for example Lemma 4.1, [21]) yields that $v(x)$ satisfies

$$\begin{cases} -\Delta v \leq \lambda_1(V_{\mathbb{C}}(\kappa, R), \alpha)v, & \text{in } B_R(x_0), \\ \frac{\partial v}{\partial \nu} + \alpha v = 0, & \text{on } \partial B_R(x_0), \end{cases}$$

in the sense of distributions. Thus, by Lemma 1, we have

$$\lambda_1(B_R(x_0), \alpha) \leq \lambda_1(V_{\mathbb{C}}(\kappa, R), \alpha).$$

For $\alpha < 0$, Lemma 2 yields $\varphi'(t) > 0$. Following the same argument as in the $\alpha > 0$ case, $v(x) = \varphi(r(x))$ satisfies

$$\begin{cases} -\Delta v \geq \lambda_1(V_{\mathbb{C}}(\kappa, R), \alpha)v, & \text{in } B_R(x_0), \\ \frac{\partial v}{\partial \nu} + \alpha v = 0, & \text{on } \partial B_R(x_0), \end{cases}$$

in the distributional sense. Hence, Lemma 1 implies

$$\lambda_1(B_R(x_0), \alpha) \geq \lambda_1(V_{\mathbb{C}}(\kappa, R), \alpha),$$

which completes the proof. $\square$

Proof of Theorem 2. Since the proof is analogous, we will only specify the changes. Taking $\omega(t) = \text{sn}_{\kappa}^{4m-4}(R-t) \text{sn}_{4\kappa}^3(R-t)$, Equation (12) and Lemma 2 yield that the first positive eigenfunction $u(x) = \varphi(r)$ satisfies $\varphi' < 0$ when $\alpha > 0$. Hence, Theorem 2 follows from Lemma 1 and Lemma 3. For $\alpha < 0$, the proof is similar. $\square$

4. Proof of Theorems 4 and 5

This section is devoted to the proofs of Theorems 4 and 5. Following [3], we will use $d(x, \partial M)$ to construct subsolutions and supersolutions to the eigenvalue problem (1). To begin with, we present some lemmas that will be used.

Lemma 4. 1. Let (M^m, g, J) be a compact Kähler manifold of complex dimension m with smooth nonempty boundary ∂M . Suppose that the holomorphic sectional curvature bound $H \geq 4\kappa$ and the orthogonal Ricci curvature bound $\text{Ric}^{\perp} \geq 2(m-1)\kappa$ hold on M for $\kappa \in \mathbb{R}$, and that the second fundamental form of ∂M is bounded from below by $\Lambda \in \mathbb{R}$. Then, $d(x, \partial M)$ satisfies

$$\Delta d(x, \partial M) \leq 2(m-1)T_{\kappa, \Lambda}(d(x, \partial M)) + T_{4\kappa, \Lambda}(d(x, \partial M))$$

on $M \setminus \text{Cut}(\partial M)$.

2. Let (M^m, g, J_1, J_2, J_3) be a compact quaternionic Kähler manifold of quaternionic dimension $m \geq 2$ with smooth nonempty boundary ∂M . Suppose that the scalar curvature satisfies $S \geq 16m(m+2)\kappa$ on M for $\kappa \in \mathbb{R}$, and the second fundamental form of ∂M is bounded from below by $\Lambda \in \mathbb{R}$. Then, $d(x, \partial M)$ satisfies

$$\Delta d(x, \partial M) \leq 4(m-1)T_{\kappa, \Lambda}(d(x, \partial M)) + 3T_{4\kappa, \Lambda}(d(x, \partial M))$$

on $M \setminus \text{Cut}(\partial M)$.

Proof. The Kähler case follows from Theorem 6.1 in [7] by setting $\alpha = \beta = 1$ and $\varphi(s) = s$. The quaternionic Kähler case was proven in (Corollary 5.8, [8]). $\square$

To establish the partial differential inequality in the distributional sense, we need the following lemma.

Lemma 5 (Lemma 2.5, [22]). *Let M be a smooth complete Riemannian manifold with boundary ∂M . Then, there exists a sequence $\{\Omega_k\}_{k=1}^\infty$ of closed subsets of M satisfying the following properties:*

1. *For every $k_1 < k_2$, $\Omega_{k_1} \subset \Omega_{k_2}$;*
2. *$M \setminus \text{Cut}(\partial M) = \bigcup_{k=1}^\infty \Omega_k$;*
3. *For every k , on $\partial\Omega_k \setminus \partial M$, there exists the unit outer normal vector field v_k for Ω_k such that $\langle v_k, \nabla d(x, \partial M) \rangle \geq 0$;*
4. *For every k , the set $\partial\Omega_k$ is a smooth hypersurface in M and satisfies $\partial\Omega_k \cap \partial M = \partial M$.*

Proof of Theorem 4. Let $\varphi > 0$ be the first eigenfunction of the one-dimensional model (5) corresponding to the eigenvalue $\bar{\lambda}_1$ and define $\psi(x) = \varphi \circ d(x, \partial M)$. It can be verified that ψ satisfies the Robin boundary condition $\frac{\partial \psi}{\partial \nu} + \alpha \psi = 0$ on ∂M . For $\alpha > 0$, by taking $\omega = C_{\kappa, \Lambda}^{2m-2} C_{4\kappa, \Lambda}$, Lemma 2 implies $\varphi' > 0$. Thus, we have

$$\Delta \psi = \varphi'' + \varphi' \Delta d(x, \partial M) \leq \varphi'' + (2(m-1)T_{\kappa, \Lambda} + T_{4\kappa, \Lambda})\varphi' = -\bar{\lambda}_1 \psi, \quad (13)$$

where we used Lemma 4. Let $\{\Omega_k\}_{k=1}^\infty$ be the subsets of M described in Lemma 5. For any non-negative function $\zeta \in C^1(M)$, the divergence theorem yields

$$\int_{\Omega_k} \langle \nabla \psi, \nabla \zeta \rangle = - \int_{\Omega_k} \zeta \Delta \psi + \int_{\partial\Omega_k} \frac{\partial \psi}{\partial \nu_k} \zeta.$$

Combining with (13), we obtain

$$\begin{aligned} \int_{\Omega_k} \langle \nabla \psi, \nabla \zeta \rangle &\geq \bar{\lambda}_1 \int_{\Omega_k} \zeta \psi + \int_{\partial\Omega_k} \frac{\partial \psi}{\partial \nu_k} \zeta \\ &= \bar{\lambda}_1 \int_{\Omega_k} \zeta \psi + \int_{\partial\Omega_k \cap \partial M} \frac{\partial \psi}{\partial \nu_k} \zeta + \int_{\partial\Omega_k \setminus \partial M} \varphi' \langle \nabla d(x, \partial M), v_k \rangle \zeta. \end{aligned}$$

Lemma 5 shows

$$\begin{aligned} \int_{\Omega_k} \langle \nabla \psi, \nabla \zeta \rangle &\geq \bar{\lambda}_1 \int_{\Omega_k} \zeta \psi + \int_{\partial M} \frac{\partial \psi}{\partial \nu_k} \zeta \\ &= \bar{\lambda}_1 \int_{\Omega_k} \zeta \psi - \alpha \int_{\partial M} \psi \zeta, \end{aligned}$$

where we used the Robin boundary condition. Taking $k \rightarrow \infty$, we obtain

$$\int_M \langle \nabla \psi, \nabla \zeta \rangle \geq \bar{\lambda}_1 \int_M \zeta \psi - \alpha \int_{\partial M} \psi \zeta.$$

Thus, we conclude that ψ satisfies

$$\begin{cases} -\Delta \psi \geq \bar{\lambda}_1 \psi, & \text{in } M, \\ \frac{\partial \psi}{\partial \nu} + \alpha \psi \geq 0, & \text{on } \partial M, \end{cases}$$

in the distributional sense. By virtue of Lemma 1, we have $\lambda_1(M, \alpha) \geq \bar{\lambda}_1$. The $\alpha < 0$ case follows similarly. $\square$

Proof of Theorem 5. The proof is similar to that of Theorem 4, except that we replace the function ω with $\omega = C_{\kappa, \Lambda}^{4m-4} C_{4\kappa, \Lambda}^3$. $\square$

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