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On Vertex Magic 3-Regular Graphs with a Perfect Matching

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Abstract

Let $G = (V, E)$ be a finite simple graph with $p = |V|$ vertices and $q = |E|$ edges, without isolated vertices or isolated edges. A vertex magic total labeling is a bijection f from $V \cup E$ to the consecutive integers$1, 2, \ldots, p + q$, with the property that, for every vertex $u \in V$, one has $f(u) + \sum_{uv \in E} f(uv) = k$ for some magic constant k. The vertex magic total labeling is called E-super if furthermore $f(E) = \{1, 2, \ldots, q\}$. A graph is called ($E$-super) vertex magic if it admits an (E-super) vertex magic total labeling. In this paper, we verify the existence of E-super vertex magic total labeling for a class of 3-regular graphs with a perfect matching, and we confirm the existence of such a labeling for general regular graphs of odd degree containing particular classes of 3-factors, which provides us with known and new examples. Note that Harary graphs are among the popular models used in communication networks. In 2012, G. Marimuthu and M. Balakrishnan raised a conjecture that if $n > 4$, $n \equiv 0 \ (mod\ 4)$ and m is odd, then the Harary graph $H_{m,n}$ admits an E-super vertex magic labeling. Among others, we are able to verify this conjecture except for one case while $m = 3$ and $n \equiv 4 \ (mod\ 8)$.

Keywords: vertex magic total labeling; E-super; quasi-prism; factor; 3-regular graph; perfect matching; Harary graph

MSC: 05C21; 05C25; 05C78

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1. Introduction and Background

All graphs in this paper are finite, simple, undirected, and possibly disconnected, but without any isolated vertex or any isolated edge. For graph theoretic notations not defined here, we follow [1]. Over the past few decades, many kinds of graph labelings have been studied intensively, in particular the labeling of magic type [2]. An excellent survey of graph labeling of various types can be found in Gallian's paper [3]. In 2002, MacDougall et al. [4] introduced the concept of vertex magic total labeling as follows: If G is a finite simple undirected graph with p vertices and q edges, then a vertex magic total labeling is a bijection f from $V(G) \cup E(G)$ to the integers $1, 2, \ldots, p + q$ with the property that for every $u \in V(G)$, the sum $f(u) + \sum_{uv \in E(G)} f(uv)$ is a constant. MacDougall et al. studied the basic properties of vertex magic graphs and showed some families of graphs having a vertex magic total labeling. In 2004, MacDougall et al. [5] further introduced the related concept of super vertex magic total labeling if $f(V(G)) = \{1, 2, \ldots, p\}$. Swaminathan and Jeyanthi [6] introduced dually the E-super vertex magic total labeling in case $f(E(G)) = \{1, 2, \ldots, q\}$. Here, we give the definitions as follows:

Definition 1. *Let G be a finite simple graph with $p = |V(G)|$ vertices and $q = |E(G)|$ edges. A* vertex magic total labeling *is a bijection from $V(G) \cup E(G)$ to the consecutive integers* $1, 2, \ldots,$

$p+q$, with the property that, for every vertex u in $V(G)$, one has $f(u)+\sum_{uv\in E(G)} f(uv) = k$ for some constant k. Such a labeling is called E-super vertex magic *if $f(E(G)) = \{1,2,\ldots,q\}$. A graph G is called* (E-super) vertex magic *if it admits an (E-super) vertex magic labeling. See Figure 1 for an example of an (E-super) vertex magic labeling of the cycle C_5 with magic constant 14.*

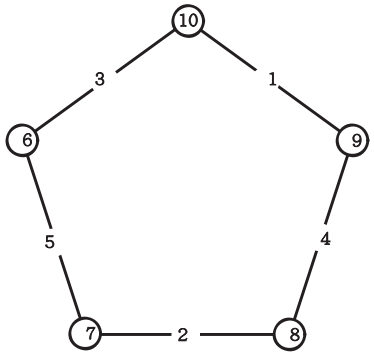


Figure 1. An E-super vertex magic labeling of C_5 with magic constant 14.

Note that Marimuthu, G. et al. [7,8] and Wang, T.-M. et al. [9,10] studied and established E-super vertex magic labeling of some families of graphs. Since the introduction of this notion on vertex magic total labeling, various results for particular classes of graphs have been found. For example, MacDougall et al. [4] proved that cycle C_n for $n \geq 3$, path P_n for $n \geq 2$, complete graph K_n for odd n, and complete bipartite graph $K_{n,n}$ for $n > 1$ have vertex magic total labelings. Bača, Miller and Slamin [11] proved that for $n \geq 3$, $1 \leq m \leq \lfloor \frac{n-1}{2} \rfloor$, every generalized Petersen graph $P(n,m)$ has a vertex magic total labeling with the magic constant $k = 9n+2$, $k = 10n+2$, and $k = 11n+2$. In 2007, Gray [12] studied such labeling for regular graphs. A more detailed survey of the known results on vertex magic total labeling of graphs can be found in [3,7,12,13]. Also note that in the literature, one article [14] introduces the conditional matching preclusion number of Cayley graphs as a robustness parameter under edge failures, which is closely related to the current one in the context of network reliability and graph labelings.

In 2012, G. Marimuthu and M. Balakrishnan [7] (E-super vertex magic labelings of graphs, Discrete Applied Math., 160, 2012, pp. 1766–1774) raised a conjecture, which claims if $n > 4$, $n \equiv 0 \ (mod\ 4)$ and m is odd, then the Harary graph $H_{m,n}$ has an E-super vertex magic labeling. Note that the Harary graphs are among the popular models used in communication networks. For reference [15], a Harary graph $H_{m,n}$ is defined as follows (see Figure 2 for examples):

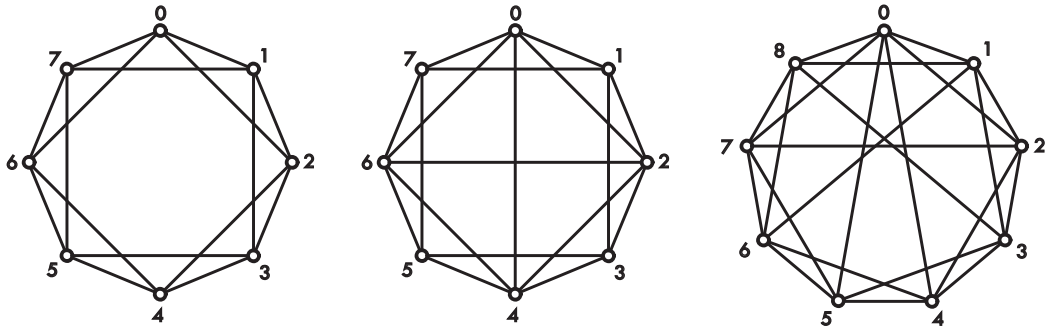


Figure 2. Harary graphs $H_{4,8}$, $H_{5,8}$, $H_{5,9}$.

Definition 2. *The graph $H_{m,n}$ on n vertices has exactly $\lceil mn/2 \rceil$ edges. The structure of $H_{m,n}$ depends on the parities of m and n; there are three cases.*

Case 1. *m even. Let $m = 2r$. Then, $H_{2r,n}$ is constructed as follows: It has vertices $0, 1, \ldots, n-1$ and two vertices i and j are joined if $i - r \leq j \leq i + r$ (where addition is taken modulo n).*

Case 2. *m odd, n even. Let $m = 2r + 1$. Then, $H_{2r+1,n}$ is constructed by first drawing $H_{2r,n}$ and then adding edges joining vertex i to vertex $i + \frac{n}{2}$ for $1 \leq i \leq \frac{n}{2}$.*

Case 3. *m odd, n odd. Let $m = 2r + 1$. Then, $H_{2r+1,n}$ is constructed by first drawing $H_{2r,n}$ and then adding edges joining vertex 0 to vertices $(n-1)/2$ and $(n+1)/2$ and vertex i to vertex $i + (n+1)/2$ for $1 \leq i < (n-1)/2$.*

In this paper, we show that an odd-regular graph containing either of two types of particular 3-factors, which we call type I and type II, admits a super vertex magic total labeling. As a byproduct, we verify the conjecture raised by G. Marimuthu and M. Balakrishnan in 2012 [7], except for one case. Also, we provide more examples by way of showing the following more general situation for the disjoint union of multiple copies of Harary graphs $H_{m,n}$:

1. Let t be a positive integer. For odd $m \geq 5$, and $n > 4$, $n \equiv 0 \ (mod\ 4)$, the disjoint union of t copies of the Harary graph $tH_{m,n}$ admits an E-super vertex magic labeling.
2. Let t be a positive integer. For $m = 3$, $n > 4$, $n \equiv 0 \ (mod\ 8)$, the disjoint union of t copies of the Harary graph $tH_{m,n}$ admits an E-super vertex magic labeling.
3. Let t be a positive integer. For odd $m \geq 5$, and $n > 4$, $n \equiv 2 \ (mod\ 4)$, the disjoint union of even number of Harary graph copies $2tH_{m,n}$ admits an E-super vertex magic labeling.

Notice that the method employed here is also valid for those graphs with multiple edges and loops. The significance of our technique is that we use the 3-regular spanning subgraph structures to guarantee the existence of such a labeling for general odd-regular graphs, instead of giving a precise construction of the required labeling. More examples and open problems will be provided in the Concluding Remarks Section.

2. Preliminaries

We first state a well-known result regarding the factorization of regular graphs of even degree, which is needed later and can be found in [16]:

Theorem 1 ([17]). *Every $2r$-regular graph has a $2k$-factor for every integer $k, 0 < k < r$.*

Notice that after removing the edges of the 2-factor by the Petersen Theorem, we will obtain an even-regular graph again and again. Thus, an even-regular graph has a 2-factorization. Also, we need another result pointed out in [6], and we give a proof here for completeness.

Theorem 2 ([6]). *Let $G = (V, E)$ be a graph with $p = |V|$ vertices and $q = |E|$ edges. Then, there exists an E-super vertex magic labeling f for G if and only if there exists a one-to-one edge labeling $g : E \to \{1, 2, \ldots, q\}$ such that the set of the vertex sums $\{w(u) = \sum_{v \in N(u)} g(uv) : u \in V\}$ is a set of consecutive integers.*

Proof. If f is an E-super vertex magic labeling, then $f(E) = \{1, 2, \ldots, q\}$ and $f(u) + \sum_{v \in N(u)} f(uv)$ is constant k for each $u \in V$. Then, obviously $g(uv) = f(uv)$, for each $uv \in E$, is an edge labeling as desired. The reason is that the set of vertex sums is simply $\{k - (q+1), k - (q+2), \ldots, k - (q+p)\}$, which is a set of consecutive integers, due to the fact $f(V) = \{q+1, q+2, \ldots, q+p\}$. Conversely, if G admits an edge labeling $g : E \to \{1, 2, \ldots, q\}$ such that the set of the vertex sums $\{w(u) = \sum_{v \in N(u)} g(uv) : u \in V\}$ is a set of consecutive integers $\{a, a+1, \ldots, a+p-1\}$, then one may extend g to an E-super vertex magic labeling f by the following rule. Let $f(e) = g(e)$ for $e \in E$, and then assign

$f(u) = p+q+a-\sum_{v\in N(u)} g(uv)$ for each $u \in V$. It is seen that f is an E-super vertex magic labeling since $f(u)+\sum_{v\in N(u)} f(uv)$ is constant with $f(V) = \{q+1, q+2, \ldots, q+p\}$. □

In fact, Theorem 2 allows us to study the E-super vertex magic total labeling by means of the mentioned edge labeling throughout this paper. Also, previously, V. Swaminathan and P. Jeyanthi [6] made the following observations and gave necessary conditions for admitting such labelings:

Lemma 1 ([6]). *If a graph G with p vertices and q edges admits an E-super vertex magic total labeling, then the magic constant is $k = q + \frac{p+1}{2} + \frac{q(q+1)}{p}$.*

Lemma 2 ([6]). *Let G be a regular graph of odd degree with p vertices and q edges. If G admits an E-super vertex magic total labeling, then $p \equiv 0 \pmod 4$.*

The following theorem was proved in [9], which guarantees the existence of the E-super vertex magicness by adding an arbitrary even factor. Therefore, we may use this theorem to confirm the existence of E-super vertex magic total labeling for general regular graphs of odd degree containing particular classes of 3-factors which admit E-super vertex magic total labeling.

Theorem 3 ([9]). *Let the graph G admit an E-super vertex magic total labeling, and let H be any 2-factor over $V(G)$. Then, $G \cup H$ still admits an E-super vertex magic total labeling.*

3. Two Types of 3-Regular Graphs with Perfect Matching

One knows that a 3-regular graph has an even number of vertices. We define a special type of 3-regular graphs with 1-factor and a 2-factor consisting of two 2-regular subgraphs of equal size as follows:

Definition 3. *A* quasi-prism *$G(U, V)$ is a 3-regular graph with $2k$ vertices $U = \{u_1, u_2, \ldots, u_k\}$ and $V = \{v_1, v_2, \ldots, v_k\}$, which consists of the edge-disjoint union of a 1-factor and two 2-regular subgraphs induced by U and V, respectively (Figure 3).*

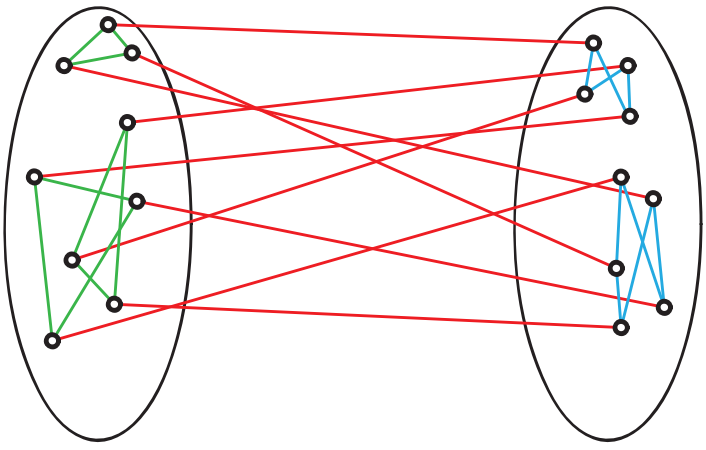


Figure 3. An example of a quasi-prism with green and blue edges inducing two 2-regular subgraphs while red edges inducing a 1-factor

The well-known generalized Petersen graphs $GP(n, k)$ are a particular subclass of quasi-prisms $G(U, V)$ for which 2-regular subgraphs induced by U and V are defined as follows:

Definition 4. *Let n, k be integers such that $n \geq 3$ and $1 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$. The generalized Petersen graph $GP(n,k)$ is defined by $V(GP(n,k)) = \{u_i, v_i | \ 1 \leq i \leq n\}$, and $E(GP(n,k)) = \{u_iu_{i+1}, u_iv_i, v_iv_{i+k} | \ 1 \leq i \leq n\}$ where the subscripts are taken modulo n (see Figure 4). We call $u_1, u_2, \ldots, u_n$ an outer cycle, and $v_1, v_2, \ldots, v_n$ an inner cycle.*

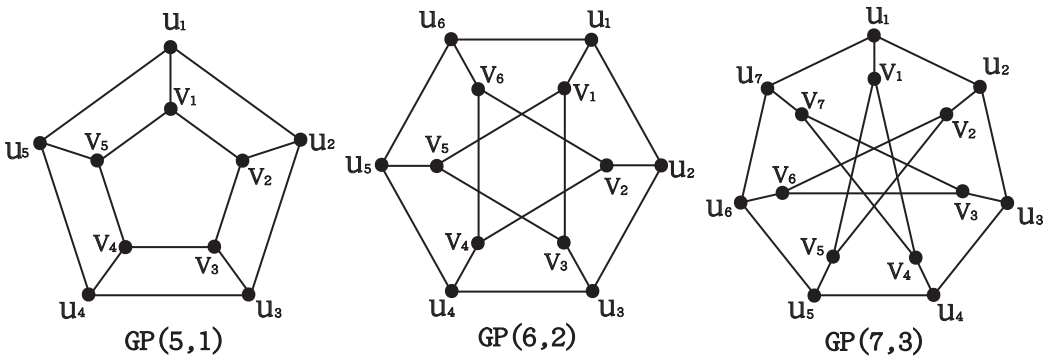


Figure 4. Examples of generalized Petersen graphs.

In 2015, T.-M. Wang et al. [10] proved the following result for one particular class of quasi-prisms admitting an E-super vertex magic total labeling as in the following:

Theorem 4 ([10]). *Let G be the quasi-prism $G(U,V)$ with $4m$ vertices, i.e., a 3-regular graph with vertex set $U \cup V$, where $U = \{u_1, u_2, \ldots, u_{2m}\}$ and $V = \{v_1, v_2, \ldots, v_{2m}\}$. Then, G admits an E-super vertex magic total labeling if one of the 2-regular subgraphs induced by U and V is bipartite; i.e., it consists of connected even cycles only.*

We say that such a G in the above theorem is a quasi-prism of type I. See Figure 5 for an example of a quasi-prism of type I with the E-super vertex magic total labeling.

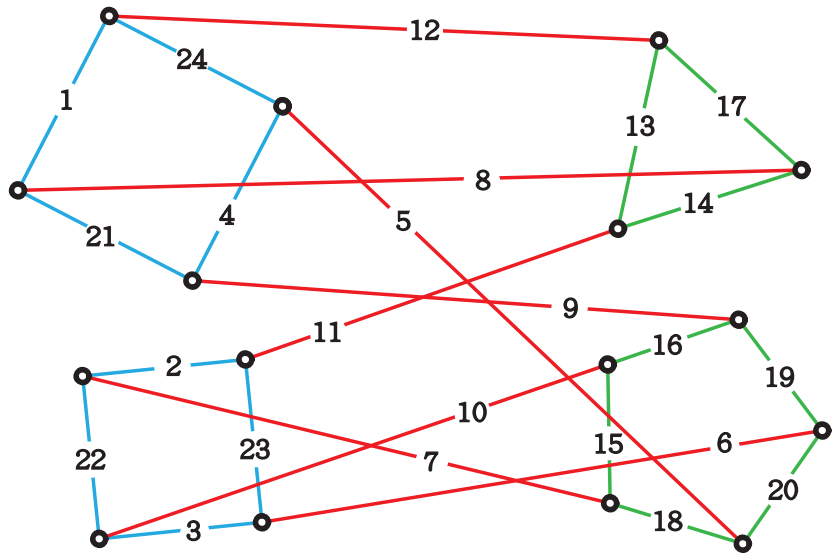


Figure 5. A quasi-prism of type I with an E-super vertex magic total labeling.

Corollary 1 ([10]). *The quasi-prism $tGP(n,k)$ is a quasi-prism of type I for positive integers t and even n. Thus, it admits an E-super vertex magic total labeling.*

In 2007 and 2008 [18,19], using the direct labeling construction method, Gomez solved the following open problems: if $n \equiv 0 \ (mod \ 4)$, $n > 4$, then tK_n has a super vertex magic total labeling for any positive integer t. By using Theorem 4, we have a different way to obtain the same result as follows. Note that for positive integers t and $n \equiv 0 \ (mod \ 4)$, $n > 4$,

the graph tK_n contains $tGP(\frac{n}{2}, 1)$ as a 3-regular spanning subgraph, which is a quasi-prism of type I, since one sees that $\frac{n}{2}$ is even and Corollary 1 applies. Then, by Theorem 3, we see that tK_n admits an E-super vertex magic total labeling due to the fact that tK_n is a sum of $tGP(\frac{n}{2}, 1)$, a 3-factor, and an $(n-4)$-factor. Therefore, we obtain the same results simply by checking the structure of subgraphs.

We use the following diagram to explain the above scenario. Note that the complete graph K_{4s} for $s \geq 2$ can be factored into as an edge-disjoint union of $GP(2s, 1)$ and a $(4s-4)$-regular graph (see Figure 6). Note that we use the spanning subgraph structure of a graph to distinguish whether a graph admits such a labeling, instead of giving a precise construction of the required labeling.

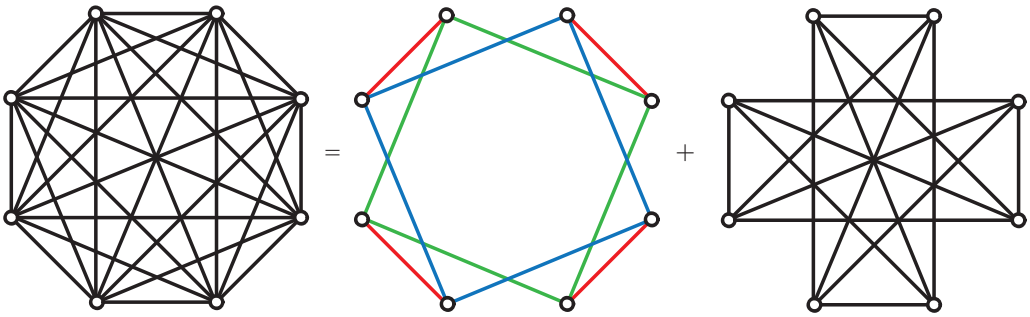


Figure 6. Edge partition of K_{4s} into $GP(2s, 1)$ and a $(4s-4)$-regular graph with $s = 2$.

Now, we turn to another class of quasi-prisms, called of type II, as follows:

Theorem 5. *Let G be a 3-regular graph consisting of the disjoint union of two quasi-prisms $G(U_a, V_a)$ and $G(U_b, V_b)$ with the same order. Then, G admits a super vertex magic total labeling. We call such G a quasi-prism of type II.*

Proof. Let G be the 3-regular graph consisting of the disjoint union of two quasi-prisms $G_a = G(U_a, V_a)$ and $G_b = G(U_b, V_b)$ with the same order. Assume that $V(G_a) = U_a \cup V_a$, where $U_a = \{u_1, \ldots, u_m\}$ and $V_a = \{v_1, \ldots, v_m\}$. Similarly assume that $V(G_b) = U_b \cup V_b$, where $U_b = \{u_{m+1}, \ldots, u_{2m}\}$ and $V_b = \{v_{m+1}, v_{m+2}, \ldots, v_{2m}\}$. We denote by $G[U_a]$, $G[V_a]$, $G[U_b]$, $G[V_a]$, respectively, the 2-regular subgraphs induced by U_a, V_a, U_b, and V_b. Hence, the total number of vertices is $4m$ and the total number of edges is $6m$. To obtain an E-super vertex magic total labeling, it suffices to use $\{1, 2, \ldots, 6m\}$ for the edge labels to obtain consecutive vertex sums in view of Theorem 2.

Now, we are in a position to give the edge labeling on G. We split the edge set of G into the following parts, namely, the edge set for the perfect matching $E_{mat} = E^a_{mat} \cup E^b_{mat} = \{u_i v_i : 1 \leq i \leq m\} \cup \{u_i v_i : m+1 \leq i \leq 2m\}$ for G_a and G_b, respectively, and the edge sets for $G[U_a]$, $G[U_b]$, $G[V_a]$, and $G[V_b]$.

Case 1. Labeling edges first over the perfect matching E^a_{mat} and E^b_{mat} for G_a and G_b, respectively, using $\{1, 2, \ldots, m\}$ and $\{5m+1, 5m+2, \ldots, 6m\}$, respectively: For edges of $E^a_{mat} = \{u_i v_i : 1 \leq i \leq m\}$,

$$f(u_i v_i) = i,\ 1 \leq i \leq m.$$

For edges of $E^b_{mat} = \{u_i v_i : m+1 \leq i \leq 2m\}$,

$$f(u_i v_i) = 4m + i,\ m+1 \leq i \leq 2m.$$

Case 2. Labeling edges over $G[U_a]$ and $G[U_b]$:

First, we fix an orientation on each connected cycle of the 2-regular subgraphs $G[U_a]$ and $G[U_b]$ either clockwise or counterclockwise. Then, we denote by $e_{in}(u_i)$ the labeling of

the edge flowing into the vertex u_i, and denote by $e_{out}(u_i)$ the labeling of the edge flowing out of the vertex u_i according to the given orientation.

For $u_i \in U_a, 1 \leq i \leq m$,

$$e_{in}(u_i) = 5m + 1 - f(u_i v_i).$$

For $u_i \in U_b, m + 1 \leq i \leq 2m$,

$$e_{in}(u_i) = 7m + 1 - f(u_i v_i).$$

Note that the edges for $G[U_a]$ and $G[U_b]$ are labeled as $\{4m + 1, 4m + 2, \ldots, 5m\}$ and $\{m + 1, m + 2, \ldots, 2m\}$, respectively. Let $w(u_i)$ be the vertex sum at the vertex $u_i \in U_a \cup U_b$ via adding all incident edge labels. That is, $w(u_i) = e_{in}(u_i) + f(u_i v_i) + e_{out}(u_i)$. Since the set of all flowing-in edge labels is the same as the set of all flowing-out edge labels, one can see the vertex sums over vertices $u_i \in U_a \cup U_b$ are $\{8m + 2, 8m + 3, \ldots, 10m + 1\}$.

Case 3. Labeling edges over $G[V_a]$ and $G[V_b]$:

We fix an orientation on each connected cycle of the 2-regular subgraphs $G[V_a]$ and $G[V_b]$ either clockwise or counterclockwise. Then, we denote by $e_{in}(v_i)$ the labeling of the edge flowing into the vertex v_i, and denote by $e_{out}(v_i)$ the labeling of the edge flowing out of the vertex v_i according to the given orientation.

For $v_i \in V_a, 1 \leq i \leq m$,

$$e_{in}(v_i) = 4m + 1 - f(u_i v_i).$$

For $v_i \in V_b, m + 1 \leq i \leq 2m$,

$$e_{in}(v_i) = 8m + 1 - f(u_i v_i).$$

Note that the edges for $G[U_a]$ and $G[U_b]$ are labeled as $\{3m + 1, 3m + 2, \ldots, 4m\}$ and $\{2m + 1, 2m + 2, \ldots, 3m\}$, respectively. Let $w(v_i)$ be the vertex sum at the vertex $v_i \in V_a \cup V_b$ via adding all incident edge labels. That is, $w(v_i) = e_{in}(v_i) + f(u_i v_i) + e_{out}(v_i)$. Therefore, the vertex sums over vertices $v_i \in V_a \cup V_b$ are $\{7m + 2, 7m + 3, \ldots, 8m + 1\} \cup \{10m + 2, 10m + 3, \ldots, 11m + 1\}$, since the set of all flowing-in edge labels is the same as the set of all flowing-out edge labels.

Thus, it is clear from the above Case 2 and Case 3 that we have the desired edge labeling with consecutive vertex sums $\{7m + 2, 7m + 3, \ldots, 11m + 1\}$, and by Theorem 2, it is equivalent to have the E-super vertex magic total labeling. See Figure 7 for example. □

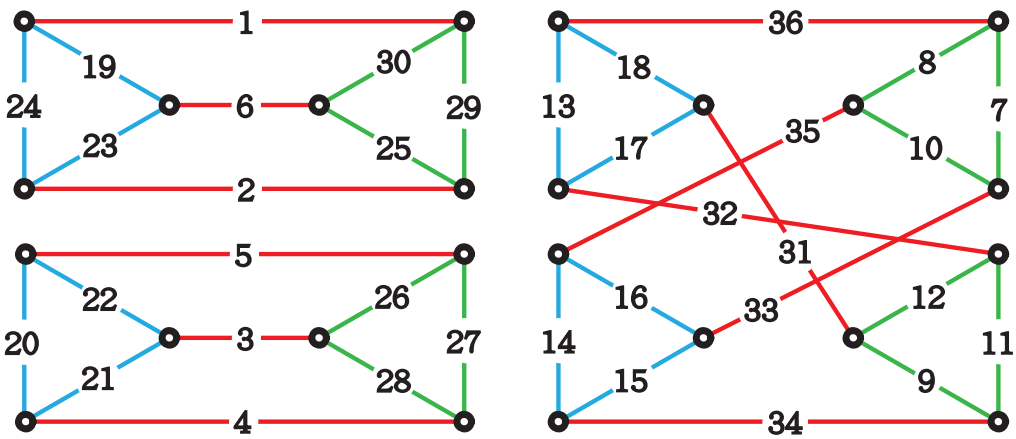


Figure 7. A quasi-prism of type II with an E-super vertex magic total labeling.

Corollary 2. *Let G be the disjoint union of even copies of generalized Petersen graphs $2tGP(2m+1,1)$ for a positive integer t and $m \geq 1$. Then, G admits an E-super vertex magic total labeling.*

Proof. Note that $2GP(2m+1,1)$ is a quasi-prism of type II; thus, so is $G = 2tGP(2m+1,1)$ for any positive integer t and $m \geq 1$. Therefore, by Theorem 5, G admits an E-super vertex magic total labeling. □

One may apply the above result to confirm the existence of the E-super vertex magic total labeling for general regular graphs of odd degree containing the quasi-prism of type II. We perform the following as an example.

Corollary 3. *Let K_{4m+2} be the complete graph on $4m+2$ vertices. Then, $2tK_{4m+2}$ admits an E-super vertex magic total labeling for positive integers t and $m \geq 1$.*

Proof. Note that K_{4m+2} contains $GP(2m+1,1)$ as a 3-regular spanning subgraph, and thus $2tK_{4m+2}$ contains $2tGP(2m+1,1)$ as a 3-regular spanning subgraph of type II by the previous Corollary 2. Again, $2tK_{4m+2}$ admits an E-super vertex magic total labeling for positive integers t and $m \geq 1$, in view of Theorem 3, since $2tK_{4m+2}$ can be treated as a sum of $2tGP(2m+1,1)$, a 3-factor, and a $(4m-2)$-factor. □

4. Conjecture for Harary Graphs

The Harary graph $H_{m,n}$ is a particular example of an m-connected graph with n vertices having the smallest possible number of edges. In 2012, G. Marimuthu and M. Balakrishnan [7] raised a conjecture, which claims if $n > 4$, $n \equiv 0 \ (mod\ 4)$ and m is odd, then the Harary graph $H_{m,n}$ has an E-super vertex magic labeling. We study the mentioned conjecture in this section by observing that $H_{m,n}$ contains 3-factors of type I and II as subgraphs. We first define circulant graphs as a subclass of regular graphs for later usage.

Definition 5. *A* circulant graph $CIR_n(S)$ *with n vertices, with respect to $S \subset \{1,2,\ldots,\lfloor \frac{n}{2} \rfloor\}$, is a graph with the vertex set $V = \{0,1,2,\ldots,n-1\}$, and the edge set $E = \{ij : i-j \equiv \pm s \ (mod\ n),\ s \in S\}$.*

Note that a circulant graph $CIR_n(S)$ is also called a Cayley graph of the finite cyclic group $\mathbb{Z}_n$ generated by S. For example, $CIR_n(\{a\})$ is isomorphic to the connected n-cycle C_n, if $gcd(n,a) = 1$. Also, $CIR_n(\{1,2,\ldots,\lfloor \frac{n}{2} \rfloor\})$ is isomorphic to the complete graph K_n, and $CIR_{2n}(\{1,n\})$ is isomorphic to the Möbius ladder graph. Note that $CIR_n(S)$ is regular of odd degree if n is even and $\frac{n}{2} \in S$, and $CIR_n(S)$ is regular of even degree otherwise.

Theorem 6. *We have the following facts:*
(1) For odd $m \geq 5$, $n > 4$, $n \equiv 0 \ (mod\ 4)$, the Harary graph $H_{m,n}$ admits an E-super vertex magic labeling.
(2) For $m = 3$, $n > 4$, $n \equiv 0 \ (mod\ 8)$, the Harary graph $H_{m,n}$ admits an E-super vertex magic labeling.

Proof.

(1) Note that for odd $m \geq 5$, $n > 4$, $n \equiv 0 \ (mod\ 4)$, the Harary graph $H_{m,n}$ contains $H_{5,n}$, and hence $H_{4,n}$, as a subgraph. In fact, we see that $H_{4,n}$ contains the generalized Petersen graph $GP(\frac{n}{2},1)$ as a subgraph by the earlier definitions (for example, see that the Harary graph $H_{4,8}$ in Fig. 2 contains $GP(4,1)$ as a subgraph). Thus, $H_{m,n}$ contains a 3-regular spanning subgraph $GP(\frac{n}{2},1)$, which is clearly of type I. Therefore, by Theorem 3, we see such $H_{m,n}$ admits an E-super vertex magic labeling.

(2) $m = 3$, $n > 4$, $n \equiv 0 \ (mod\ 8)$. Then, the Harary graph $H_{m,n}$ contains a 3-regular spanning subgraph $CIR_n(1, \frac{n}{2})$, namely the Möbius ladder. One can see that directly from the definition of $CIR_n(1, \frac{n}{2})$, there are $\frac{n}{4}$ copies of 4-cycles C_4s distributed evenly with a 1-factor, as shown in Figure 8 with examples $H_{3,8}$, $H_{3,16}$, in which red edges represent a 1-factor. The blue edges and the green edges represent the edges of two 2-regular subgraphs consisting of the same number of C_4s. Therefore, the 3-regular spanning subgraph $CIR_n(1, \frac{n}{2})$ is a quasi-prism of type I. By Theorem 3, we see that such $H_{m,n}$ admits an E-super vertex magic labeling.

□

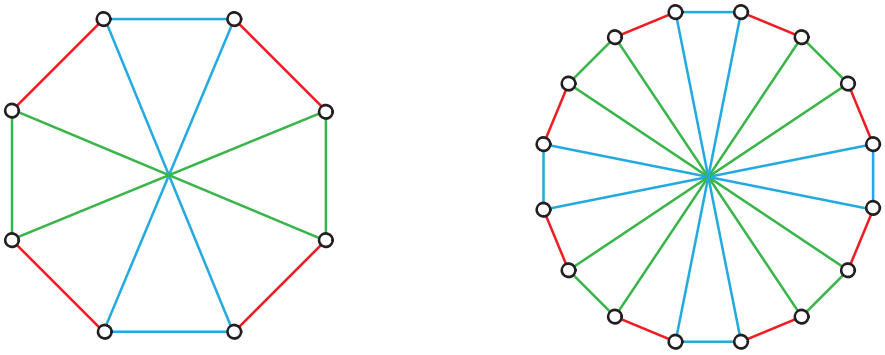


Figure 8. Harary graphs $H_{3,8}$ and $H_{3,16}$.

Moreover, we have the following more general situations.

Theorem 7. *We have the following facts:*

(1) For odd $m \geq 5$, $n > 4$, $n \equiv 0 \ (mod\ 4)$, the disjoint union of t copies of the Harary graph $tH_{m,n}$ admits an E-super vertex magic labeling.

(2) For $m = 3$, $n > 4$, $n \equiv 0 \ (mod\ 8)$, the disjoint union of t copies of the Harary graph $tH_{m,n}$ admits an E-super vertex magic labeling.

(3) For odd $m \geq 5$, $n > 4$, $n \equiv 2 \ (mod\ 4)$, the disjoint union of $2t$ copies of the Harary graph $2tH_{m,n}$ admits an E-super vertex magic labeling.

Proof.

(1) $m \geq 5$, $n > 4$, $n \equiv 0 \ (mod\ 4)$. Then, the disjoint union of t copies of the Harary graph $tH_{m,n}$ contains a spanning 3-regular subgraph (3-factor) $tGP(\frac{n}{2}, 1)$ of type I. Therefore, by Theorem 3, we see such $tH_{m,n}$ admits an E-super vertex magic labeling.

(2) $m = 3$, $n > 4$, $n \equiv 0 \ (mod\ 8)$. Then, the disjoint union of t copies of the Harary graph $tH_{m,n}$ contains a spanning 3-regular subgraph (3-factor) $tCIR_n(1, \frac{n}{2})$ (Mobius ladder) of type I (see Figure 8). Therefore, by Theorem 3, we see such $tH_{m,n}$ admits an E-super vertex magic labeling.

(3) $m \geq 5$, $n > 4$, $n \equiv 2 \ (mod\ 4)$. Then, the disjoint union of $2t$ copies of the Harary graph $2tH_{m,n}$ contains a spanning 3-regular subgraph (3-factor) $2tGP(\frac{n}{2}, 1)$ of type II, since $2tGP(\frac{n}{2}, 1) = tGP(\frac{n}{2}, 1) \cup tGP(\frac{n}{2}, 1)$, which is a disjoint union of two equal-size quasi-prisms. Therefore, by Theorem 3, we see that such $2tH_{m,n}$ admits an E-super vertex magic labeling.

□

Therefore, in summary, we are left with the following unsettled cases as open problems: Is it that for $n > 4$, $n \equiv 4 \ (mod\ 8)$, the Harary graph $H_{3,n}$ admits an E-super vertex magic labeling? Note that the above statement seems to be valid as it is supported by the following example Harary graph $H_{3,12}$ with the given labeling (see Figure 9).

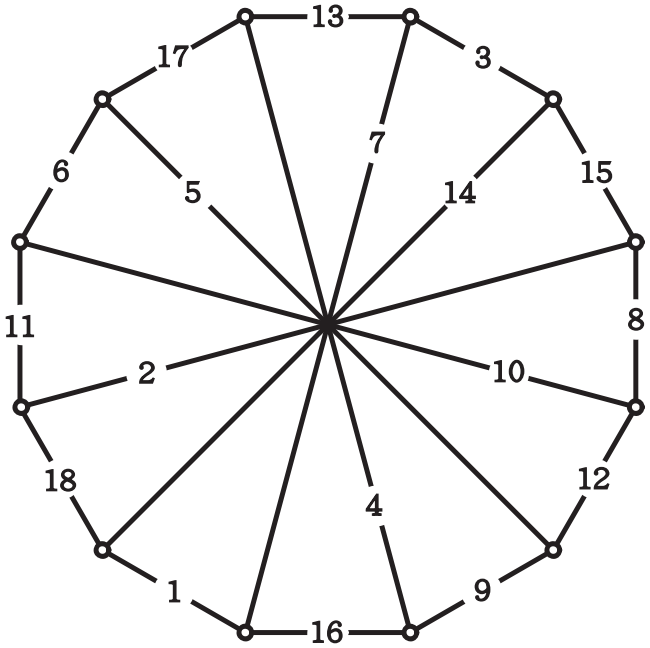


Figure 9. The Harary graph $H_{3,12}$ with an E-super vertex magic total labeling.

5. Concluding Remarks

In this paper, we see that an odd-regular graph containing a subgraph of either type I or type II admits a super vertex magic total labeling. Also, we provide more examples by way of showing more general situations for the disjoint union of multiple copies of Harary graphs. Inspired by the work, we pose the following open problems and conjectures. As we see from the previous section, only one case is left with the following cases for the conjecture for the existence of the E-super vertex magicness of Harary graphs, raised by G. Marimuthu and M. Balakrishnan in 2012, to be explored further:

Conjecture 1. *$n \geq 4$, $n \equiv 4 \pmod 8$ and $t \geq 2$. Then, the Harary graph $tH_{3,n}$ admits an E-super vertex magic labeling.*

Note that when $n = 4$, the Harary graph $H_{3,4} \cong K_4$. It is not hard to check that the complete graph on four vertices K_4 does not admit any E-super vertex magic total labeling. However, the existence of an E-super vertex magic labeling for multiple copies $tH_{3,4} \cong tK_4$, where $t > 1$, is an open problem. By the aid of running computer programs, we already see $tH_{3,4} \cong tK_4$, where $2 \leq t \leq 9$, admits an E-super vertex magic labeling. This makes us believe that the following should be valid:

Conjecture 2. *The disjoint union $tH_{3,4} \cong tK_4$ admits an E-super vertex magic labeling for each $t > 1$.*

On the other hand, note that for $n \equiv 2 \pmod 4$, the Harary graph $H_{3,n}$ does not admit any E-super vertex magic labeling since it does not satisfy the necessary condition for being E-super vertex magic. However, the disjoint union of an even number of such Harary graph $H_{3,n}$ copies with $n \equiv 2 \pmod 4$ does satisfy the necessary condition. Therefore, we have the following conjecture:

Conjecture 3. *For $n > 4$ and $n \equiv 2 \pmod 4$, the Harary graph $2tH_{3,n}$ admits an E-super vertex magic labeling.*

Remark 1. *Note that the following example* $2H_{3,6}$ *is a piece of supporting evidence for the above conjecture (Figure 10).*

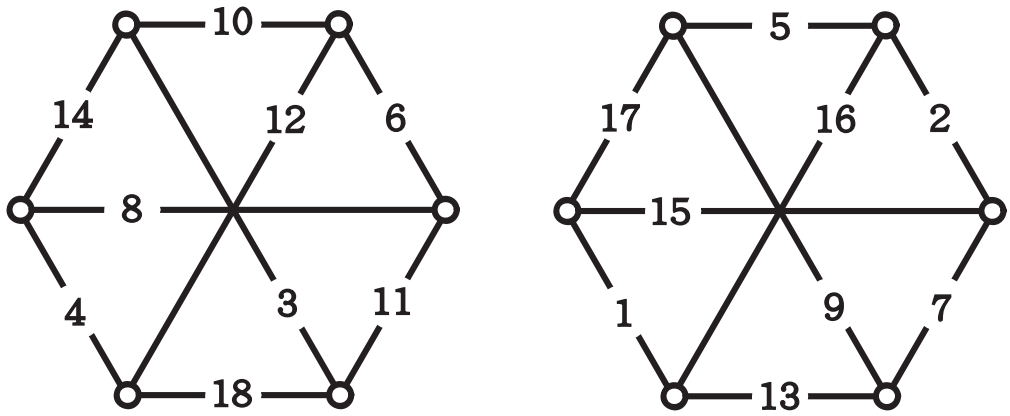


Figure 10. A Disjoint union of Harary graphs $2H_{3,6}$ with an *E*-super vertex magic total labeling

Note that also in the following Figure 11, we have an example of a 3-regular graph without any perfect matching, which admits an *E*-super vertex magic labeling. Therefore, one can not expect to have a certain type of characterization of the existence of *E*-super vertex magic labeling via the existence of the structure of 3-regular graphs containing a perfect matching. In view of these results in this paper and examples created by computer programs, it seems that all simple cubic graphs except K_4 admit an *E*-super vertex magic labeling as long as they satisfy the necessary condition.

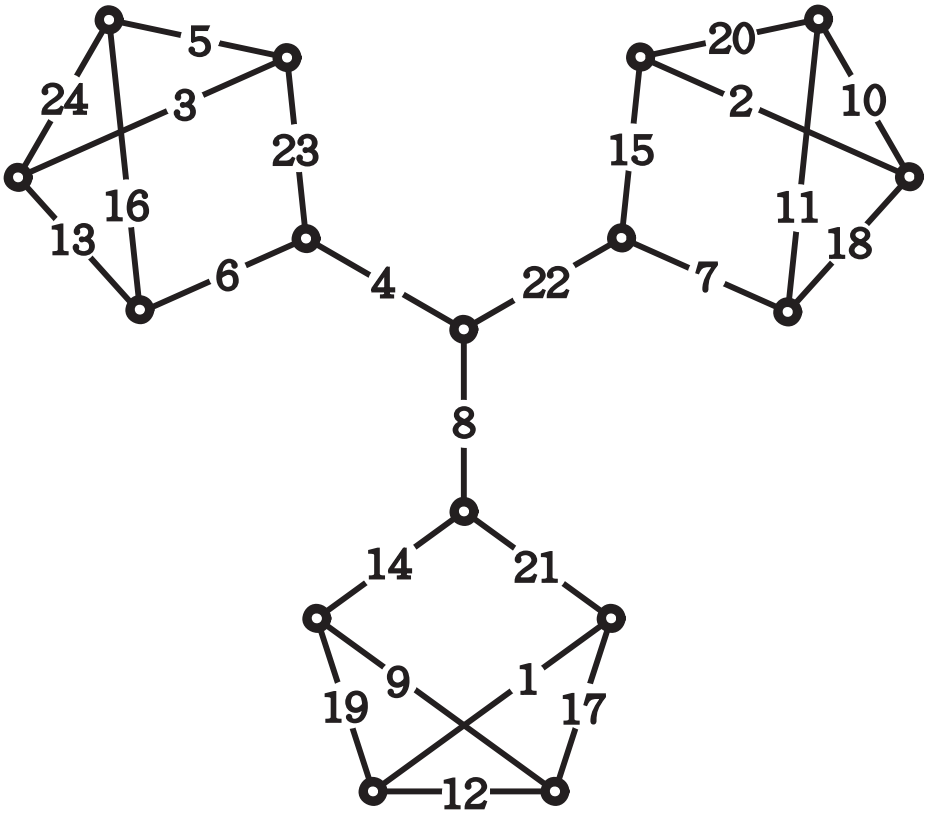


Figure 11. A 3-regular graph without perfect matching and an *E*-super vertex magic total labeling.

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